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THE MALLIAVIN CALCULUS FOR PURE JUMP PROCESSES AND APPLICATIONS TO LOCAL TIME

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A Malliavin calculus is developed whose scope includes point processes, pure jump Markov processes, and purely discontinuous martingales. An integration by parts formula for functionals of Poisson point processes is proved. This is used to develop a criterion for pure jump Markov processes to have a density in L^p. The integration by parts formula is then used to give conditions for a purely discontinuous martingale to have a jointly continuous local time L_t^x that is an occupation time density with respect to Lebesgue measure.

1. Introduction. The Malliavin calculus is a powerful tool in determining when the distribution of a stochastic process X_t has a smooth density, a question that comes up in studying hypoelliptic operators (cf. Bismut (1981)) and local times (cf. Bass (1983)). In this paper we establish a Malliavin calculus whose scope includes point processes, pure jump Markov processes, and purely discontinuous martingales. Our methods also permit some simplification in the diffusion case as well.

We obtain three main results: an integration by parts formula, conditions for a pure jump Markov process to have a density in L^p, and conditions for a purely discontinuous martingale to have a local time.

After some preliminaries on point processes and stochastic calculus in Section 2, in Section 3 we develop an integration by parts formula for functionals of Poisson point processes. The technique used is that of the calculus of variations together with a use of the Girsanov formula, an approach first introduced by Bismut (1981) for diffusions. See Williams (1981) for a good heuristic explanation of Bismut's work. We also derive an integration by parts formula for functionals of a family of independent Poisson point processes and Brownian motions.

In Section 4 we show by means of the chain rule that the integration by parts formulas of Section 3 are all that are needed to study densities of diffusions and pure jump Markov processes. The comment is often heard that the Malliavin calculus is too technical to be a truly useful tool. Although our approach does not eliminate the technicalities altogether, it does reduce them to relatively routine calculations and estimates.

Our second result, in Sections 4, 5, 6, and 7, concerns the existence of densities for pure jump Markov processes and conditions for these densities to be in L^p

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and C^α. The first such theorems were by Bismut who studied densities of processes that are essentially Lévy processes with nondeterministic drift in Bismut (1983) and of processes that arise as the restriction to a boundary of a diffusion in Bismut (1984).

Bichteler and Jacod (1983) considered processes that are the solution of the stochastic differential equation

$$X_t = x_0 + \int_0^t \int \sigma(X_{s-}, z)(\mu - \gamma)(ds, dz) + \int_0^t b(X_{s-})\, ds, \tag{1.1}$$

where μ is a Poisson point process with compensator γ. We extend and improve their results by giving conditions for the existence of a density for X_t satisfying (1.1) where σ is only Lipschitz in x and z and giving conditions for the density to be in L^p and C^α. Our theorems about densities for X_t can be used to obtain results in partial differential equations that are new. Suppose L is an integral operator that is the infinitesimal generator of a Markov process X_t. The existence of a density for X_t is then equivalent to the existence of a fundamental solution for the operator $\partial/\partial t - L$.

We then turn to the third and principal result of this paper: conditions for a purely discontinuous martingale X_t to have a local time. This problem was first posed by Meyer (1976). By a local time L_t^x, we mean an occupation time density with respect to Lebesgue measure: For all B Borel, $t \geq 0$,

$$\int_B L_t^x \, dx = \int_0^t 1_B(X_s)\, ds. \tag{1.2}$$

Until recently, the only conditions known were for Markov processes, and even there the conditions were intractable except in the case of Lévy processes (Kesten (1969)). In Bass (1984), it was shown that if the local characteristics of X_t satisfy a rather stringent condition, then X_t will have a local time, and that this stringent condition, a condition on the size of the jump component of the local characteristics, is essentially best possible.

In Sections 8–12, we show that if the local characteristics of X_t are smooth enough, only a minimal condition on the number of jumps of X_t is needed for X_t to have a local time. We consider X_t's that arise as solutions to stochastic differential equations of the form

$$X_t = x_0 + \int_0^t \int \sigma(s, z)(\mu(dz, ds) - dz\, ds),$$

where σ is predictable and a functional of the past of X, and μ is a random measure generated by a Poisson point process whose characteristic measure is Lebesgue measure. The class of such X_t is quite large. It includes, for example, Markov processes whose weak infinitesimal generator is of the form

$$Af(x) = \int [f(x+y) - f(x) - f'(x)y]\, n(x, dy), \qquad f \in C^2.$$

(See the example of Section 8.) Roughly, our results say that if σ is three times differentiable as a functional of X for each z and if $\sigma(s, z)$ does not tend to 0 too quickly as $z \to \infty$, then X_t will have a local time.

In Section 8 we state our theorem and also give an example to illustrate the hypotheses. In Sections 9, 10, and 11 we prove the theorem in a special case. Section 9 applies the Malliavin calculus for point processes to show that if

$$S_\lambda(A) = E\int_0^\infty e^{-\lambda t} 1_A(X_t)\,dt,$$

then $S_\lambda(dx)$ has a density with respect to Lebesgue measure that is bounded and Hölder continuous. Some of the needed estimates are proved in Section 10. Section 11 shows how the estimates we obtain lead to the existence and joint continuity of local time. In Section 12 we show that the proof of the special case considered leads to the proof of our main theorem; we also discuss some extensions and generalizations of our theorem.

We will use the following notation. For any process Y_t, let $Y_t^* = \sup_{s \le t}|Y_s|$. Let $\| \ \|$ denote sup norm, $\| \ \|_p$ the L^p norm with respect to Lebesgue measure, and $|A|$ the Lebesgue measure of A. We will use both σ_z and $\partial_z \sigma$ to denote the partial derivative of σ, as convenient. We will use c to denote constants whose values may change from place to place.

2. Preliminaries. In this section we recall briefly some facts about point processes and semimartingales. For details, see Jacod (1979), Meyer (1976), and Dellacherie and Meyer (1980).

Let $\mathcal{F}_t$ be a filtration satisfying the usual conditions. Let $\mathscr{Z}$ be a measurable space. A point process with state space $\mathscr{Z}$ is a countable collection of adapted r.v.'s $(Z_i, T_i) \in \mathscr{Z} \times \mathbb{R}^+$. Given a point process, one usually works with the associated random measure μ defined by

$$\mu(A \times [0, t])(\omega) = \sum_{T_i(\omega) \le t} 1_A(Z_i(\omega)). \tag{2.1}$$

A random measure μ has a random measure γ as compensator if γ is predictable and $\mu(A \times [0, t]) - \gamma(A \times [0, t])$ is a local martingale in t for all Borel sets A such that $E\gamma(A \times [0, t]) < \infty$ for all t.

A point process is a Poisson point process with characteristic measure ν if for each Borel set A with $\nu(A) < \infty$ and for each t, the r.v. $\mu(A \times [0, t])$ is Poisson with parameter $\nu(A)t$. It is then a consequence that μ has independent increments and that $\mu(A \times [0, s])$ is independent of $\mu(B \times [0, t])$ if $A \cap B = \varnothing$.

An important result (see Jacod (1979), page 92) is that if the compensator $\gamma(dz, ds)$ of the random measure μ associated to a point process is of the form

$$\gamma(dz, ds) = \nu(dz)\,ds \tag{2.2}$$

for some σ-finite measure ν on $\mathscr{Z}$, then the point process is a Poisson point process with characteristic measure ν, and hence the law of the point process is uniquely determined.

If $h(s, z, \omega)$ is predictable and a simple integrand, i.e., $h(s, z, \omega) = 1_{(t_0, t_1]}(s)1_A(z)H(\omega)$, where H is bounded and adapted to $\mathcal{F}_{t_0-}$ and $E\gamma(A \times [0, t_1]) < \infty$, define the stochastic integral of h with respect to $\mu - \gamma$ by the

Stieltjes integral

$$\int_0^t \int h(s,z,\omega)(\mu-\gamma)(dz,ds) = H(\omega)(\mu-\gamma)(A \times (t_0 \wedge t, t_1 \wedge t]). \tag{2.3}$$

One then extends the definition by linearity and L^2 limits to h in $\mathscr{M}^2 = \{h\colon h$ predictable, $E\int_0^t \int h^2(s,z,w)\gamma(dz,ds) < \infty\}$.

A purely discontinuous martingale is one where $EM_t^2 = E\Sigma_{s\le t}\Delta M_s^2$, where $\Delta M_s = M_s - M_{s-}$. In this case, let $[M,M]_t = \Sigma_{s\le t}\Delta M_s^2$. One can show $M_t = \int_0^t \int h\,d(\mu-\gamma)$ is a purely discontinuous (local) martingale with $[M,M]_t = \int_0^t \int h^2\,d\mu$ for $h \in \mathscr{M}^2$ by first considering simple h's. In particular,

$$E\left[\int_0^t \int h\,d(\mu-\gamma)\right]^2 = E\int_0^t \int h^2\,d\mu = E\int_0^t \int h^2\,d\gamma. \tag{2.4}$$

The last equality follows by monotone convergence and the fact that $\int_0^t \int (h^2 \wedge n)\,d(\mu-\gamma)$ is a mean zero martingale for each n.

Define

$$\mathscr{E}(K)_t = \exp(K_t)\prod_{s\le t}\left[(1+\Delta K_s)e^{-\Delta K_s}\right]. \tag{2.5}$$

If K_t is a semimartingale whose martingale part is purely discontinuous and H_t is a process of bounded variation, then it follows by Itô's lemma and integration by parts (Meyer (1976), pages 301–303) that

$$Z_t = \mathscr{E}(K)_t \int_0^t \mathscr{E}(K)_{s-}^{-1}(1+\Delta K_s)^{-1}\,dH_s \tag{2.6}$$

solves the stochastic differential equation

$$dZ_t = Z_{t-}\,dK_t + H_t. \tag{2.7}$$

In particular (take $H_{0-} = 0$, $H_t \equiv 1$ for $t \ge 0$),

$$d\mathscr{E}(K)_t = \mathscr{E}(K)_t\,dK_t.$$

We will need Burkholder's inequality: For $p > 0$ there is a $c(p) > 0$ such that for M_t a martingale,

$$E(M_t^*)^p \le c(p)E[M,M]_t^{p/2}.$$

Define

$\mathscr{M}_\infty^2 = \{h\colon h$ is predictable, there exists a bounded deterministic function $H(z)$ with $\int H^2(z)\nu(dz) < \infty$ such that $|h(s,z,w)| \le H(z)$ for all s, z, a.s.$\}$.

We also need

LEMMA 2.1. *Suppose $h \in \mathscr{M}_\infty^2$ and*

$$L_t = \int_0^t \int h(s,z)(\mu-\gamma)(dz,ds),$$

where γ *is given by* (2.2). *Then for all* λ,

(i) $$E\exp(\lambda[L, L]_t) \le c(\lambda, t, H, \nu) < \infty \quad \textit{and}$$

(ii) $$E\exp(\lambda L_t^*) \le c(\lambda, t, H, \nu) < \infty.$$

Proof. Fix $\lambda > 0$ and choose A so that $\nu(A) < \infty$. Let $M_t = \int_0^t \int_A h\, d(\mu - \gamma)$ and $N_t = \int_0^t \int_{A^c} h\, d(\mu - \gamma)$.

If S and T are stopping times bounded by t, we have

$$\begin{aligned}
E[[N, N]_T - [N, N]_{S-} | \mathscr{F}_S] &= \Delta N_S^2 + E[[N, N]_T - [N, N]_S | \mathscr{F}_S] \\
&\le \Delta N_S^2 + E\left[\int_S^T \int_{A^c} h^2(s, z) \mu(dz, ds) | \mathscr{F}_S\right] \\
&\le \Delta N_S^2 + E\left[\int_S^T \int_{A^c} H^2(z) \nu(dz)\, ds | \mathscr{F}_S\right] \\
&\le \sup_{z \in A^c} H^2(z) + t \int_{A^c} H^2(z) \nu(dz).
\end{aligned} \tag{2.8}$$

Then, provided A is chosen appropriately so that the right side of (2.8) is $< 1/8\lambda$, we have

$$E\exp(2\lambda[N, N]_t) \le c_1 < \infty$$

by Dellacherie and Meyer (1980, page 193).

We also have

$$\begin{aligned}
E[|N_T - N_{S-}| | \mathscr{F}_S] &\le |\Delta N_S| + \left(E\left[(N_T - N_S)^2 | \mathscr{F}_S\right]\right)^{1/2} \\
&\le \sup_{z \in A^c} |H(z)| + (E[[N, N]_T - [N, N]_S | \mathscr{F}_S])^{1/2},
\end{aligned} \tag{2.9}$$

and by Dellacherie and Meyer again,

$$E\exp(2\lambda N_t^*) \le c_2 < \infty,$$

provided A is chosen appropriately.

Now,

$$|M_s| \le \int_0^s \int_A |h(u, z)| \mu(dz, du) \le \sup_z |H(z)| R,$$

where

$$R = \int_0^t \int_A \mu(dz, ds)$$

is a Poisson r.v. with parameter $t\nu(A)$, and hence

$$E\exp(2\lambda M_t^*) \le E\exp\left(2\lambda \sup_z |H(z)| R\right) \le c_3 < \infty.$$

We also have

$$[M, M]_t = \int_0^t \int_A h^2(s, z) \mu(dz, ds) \le \sup_z H^2(z) R,$$

and so

$$E\exp(2\lambda[M, M]_t) \le c_4 < \infty.$$

To conclude the proof, choose A so that $\sup_{z\in A^c}|H(z)|$ and $\int_{A^c} H^2(z)\nu(dz)$ are sufficiently small (depending on λ) but $\nu(A) < \infty$. Since $L_t^* \le M_t^* + N_t^*$ and $[L, L]_t = [M, M]_t + [N, N]_t$, (i) and (ii) follow by Cauchy–Schwarz. □

Finally, we need the Girsanov theorem for martingales (Meyer (1976), page 377):

THEOREM 2.2. *If X_t is a martingale with respect to P, M_t a positive uniformly integrable P-martingale with $M_0 = 1$, Q a probability measure defined by $dQ/dP|_{\mathscr{F}_t} = M_t$, then $X_t - B_t$ is a martingale with respect to Q, where $B_t = \int_0^t M_s^{-1}\, d[X, M]_s$.*

3. Integration by parts formula. In this section we derive the integration by parts formula Theorem 3.4 which is basic to our work.

We will suppose throughout the remainder of this paper that $\mathscr{X} = \mathbb{R}$ and ν is Lebesgue measure. Suppose μ is the random measure associated to a Poisson point process with characteristic measure ν; the compensator of μ is then $\gamma(dz, ds) = \nu(dz)\, ds$. Suppose $l \in \mathscr{M}_\infty^2$ with $|l(s, z)| \le \frac{1}{2}$, a.s., and let

$$v(s, z) = \int_0^z l(s, y)\, dy.$$

Let τ, a random measure associated to a point process, be defined by

$$\tau(B \times [0, t]) = \int_0^t \int 1_B(z + v(s, z))\mu(dz, ds).$$

Suppose $\nu(A) < \infty$. Define the P-martingales

$$X_t = \int_0^t \int (1 + l(s, z))1_A(z + v(s, z))(\mu - \gamma)(dz, ds), \tag{3.1}$$

$$L_t = \int_0^t \int l(s, z)(\mu - \gamma)(dz, ds), \tag{3.2}$$

and

$$M_t = \mathscr{E}(L)_t = \exp(L_t) \prod_{s \le t} [(1 + \Delta L_s)\exp(-\Delta L_s)]. \tag{3.3}$$

Since $|\Delta L_s| \le \frac{1}{2}$, M is positive. Define a probability measure Q by

$$\left.\frac{dQ}{dP}\right|_{\mathscr{F}_t} = M_t.$$

THEOREM 3.1. *Under Q*

$$Y_t = \tau(A \times [0, t]) - t\nu(A)$$

is a local martingale.

PROOF. Since

$$\frac{\Delta M_t}{M_t} = 1 - \frac{M_{t-}}{M_t} = 1 - (1 + \Delta L_t)^{-1},$$

we have

$$\begin{aligned}\int_0^t M_s^{-1}\, d[X, M]_s &= \sum_{s \le t} \frac{\Delta X_s\, \Delta M_s}{M_s} \\ &= \sum_{s \le t} \Delta X_s \big(1 - (1 + \Delta L_s)^{-1}\big) \\ &= \int_0^t \int (1 + l(s, z)) 1_A(z + v(s, z)) \\ &\quad \times \big(1 - (1 + l(s, z))^{-1}\big) \mu(dz, ds).\end{aligned}$$

The last equality follows since both sides are pure jump processes with the same size jumps. Since

$$\int 1_A(z + v(s, z))(1 + l(s, z))\, dz = \int 1_A(y)\, dy = \nu(A),$$

then

$$Y_t = X_t - \int_0^t M_s^{-1}\, d[X, M]_s,$$

and our result follows by the Girsanov theorem, Theorem 2.2. □

COROLLARY 3.2. *The P-law of μ is equal to the Q-law of τ.*

PROOF. Under P, μ is a random measure with compensator $\nu(dz)\, ds$ which is associated to a point process; hence μ is the random measure associated to a Poisson point process with characteristic measure ν. By Theorem 3.1, under Q, τ is a random measure with compensator $\nu(dz)\, ds$ which is associated to a point process, and hence τ is also a random measure associated to a Poisson point process with characteristic measure ν. □

Now define an exponential martingale M_t^ε, a perturbed random measure μ^ε, and probabilities Q^ε by

$$M_t^\varepsilon = \mathscr{E}(\varepsilon L)_t = \exp(\varepsilon L_t) \prod_{s \le t} \left[(1 + \varepsilon \Delta L_s) \exp(-\varepsilon \Delta L_s)\right], \tag{3.4}$$

$$\mu^\varepsilon(B \times [0, t]) = \int_0^t \int 1_B(z + \varepsilon v(s, z)) \mu(dz, ds), \tag{3.5}$$

and

$$\left.\frac{dQ^\varepsilon}{dP}\right|_{\mathscr{F}_t} = M_t^\varepsilon. \tag{3.6}$$

For a functional G of μ, we have by Corollary 3.2 that the P-law of μ equals the Q^ε-law of μ^ε, and so

$$EG(\mu) = E_{Q^\varepsilon}G(\mu^\varepsilon) = E\left[G(\mu^\varepsilon)M_t^\varepsilon\right]. \tag{3.7}$$

Differentiating both sides of (3.7) with respect to ε and setting $\varepsilon = 0$ will give our integration by parts formula. First, a definition is needed.

DEFINITION 3.3. A functional G of μ will be called $L^p(P)$ smooth with derivative $D_lG(\mu) \in L^p(P)$ if for every $l \in \mathscr{M}_\infty^2$,

$$E\left|\varepsilon^{-1}\left[G(\mu^\varepsilon) - G(\mu) - \varepsilon D_lG(\mu)\right]\right|^p \to 0.$$

Of course, the notion of $L^p(P)$ smooth is related to that of Frechet derivative.

THEOREM 3.4. *Suppose G is a bounded $L^1(P)$ smooth functional of $\{\mu\colon s \le t\}$, suppose $l \in \mathscr{M}_\infty^2$, and suppose L_t is defined by* (3.2). *Then*

$$E\left[G(\mu)L_t\right] = -E\left[D_lG(\mu)\right]. \tag{3.8}$$

PROOF. First of all, since $l \in \mathscr{M}_\infty^2$, $\varepsilon|l| \le \frac{1}{2}$ for ε sufficiently small.
By (3.7), $E[G(\mu^\varepsilon)M_t^\varepsilon]$ is constant in ε, and since $M_t^0 = 1$,

$$\begin{aligned} &E\left[\varepsilon^{-1}(G(\mu^\varepsilon) - G(\mu))\right] + E\left[G(\mu^\varepsilon)\varepsilon^{-1}(M_t^\varepsilon - 1 - \varepsilon L_t)\right] \\ &\quad + EG(\mu^\varepsilon)L_t = 0. \end{aligned} \tag{3.9}$$

The first term converges to $E\,D_lG(\mu)$ as $\varepsilon \to 0$ since G is $L^1(P)$ smooth. The third term converges to $EG(\mu)L_t$ by dominated convergence and the fact that G is bounded. Since G is bounded, the use of Lemma 3.5 below completes the proof. □

LEMMA 3.5. $E|\varepsilon^{-1}(M_t^\varepsilon - 1 - \varepsilon L_t)|^p \to 0$ *for all* $p > 2$, *as* $\varepsilon \to 0$.

PROOF. If ε is sufficiently small, so that $|\varepsilon l(s, z)| \le \frac{1}{2}$ for all s and z, then

$$\begin{aligned} M_t^\varepsilon = \mathscr{E}(\varepsilon L)_t &\le \exp(\varepsilon|L_t|)\exp\left(\sum_{s \le t}\left[\ln(1 + \varepsilon\,\Delta L_s) - \varepsilon\,\Delta L_s\right]\right) \\ &\le \exp\left(\varepsilon L_t^* + 2\varepsilon^2[L, L]_t\right). \end{aligned}$$

Then by Lemma 2.1 and Hölder, for any q, $E(M_t^{\varepsilon *})^q$ is uniformly bounded, provided ε is sufficiently small.

Since $M_t^\varepsilon = \mathscr{E}(\varepsilon L)_t$,

$$M_t^\varepsilon = 1 + \int_0^t M_{s-}^\varepsilon\,\varepsilon\,dL_s, \tag{3.10}$$

which by Burkholder's and Hölder's inequalities and Lemma 2.1 gives us

$$\begin{aligned} E(M_t^\varepsilon - 1)^{*2p} &\le c(p)\varepsilon^{2p}E\Big(\int_0^t (M_{s-}^\varepsilon)^2\, d[L,L]_s\Big)^p \\ &\le c(p)\varepsilon^{2p}E(M_t^{\varepsilon *})^{2p}[L,L]_t^p \\ &\le c(p)\varepsilon^{2p}\big(E(M_t^{\varepsilon *})^{4p}\big)^{1/2}\big(E[L,L]_t^{2p}\big)^{1/2} \to 0 \end{aligned}$$

as $\varepsilon \to 0$.

Finally, by (3.10),

$$\varepsilon^{-1}(M_t^\varepsilon - 1 - \varepsilon L_t) = \int_0^t (M_{s-}^\varepsilon - 1)\, dL_s,$$

and by Burkholder and Hölder and Lemma 2.1 again

$$\begin{aligned} E\big|\varepsilon^{-1}(M_t^\varepsilon - 1 - \varepsilon L_t)\big|^p &\le c(p)E\Big(\int_0^t (M_{s-}^\varepsilon - 1)^2\, d[L,L]_s\Big)^{p/2} \\ &\le c(p)E(M_t^\varepsilon - 1)^{*p}[L,L]_t^{p/2} \\ &\le c(p)\big(E((M_t^\varepsilon - 1)^*)^{2p}\big)^{1/2}\big(E[L,L]_t^p\big)^{1/2} \to 0. \ \square \end{aligned}$$

REMARK. In the above, if $|l(s,z)|$ is also in $L^1(\nu)$ as well as $\mathscr{M}_\infty^2$, we could have let $v(s,z) = \int_{-\infty}^z l(s,y)\,dy$, modified the definition of μ^ε and $D_l G(\mu)$, and obtained a modification of Theorem 3.4.

We can state an analog of Theorem 3.4 for functionals of Brownian motion (cf. Williams (1981)). Let us say that a functional G of a Brownian motion W_t is $L^p(P)$ smooth with derivative $\hat{D}_k G(W)$ if $G(W) \in L^p(P)$ and if for every bounded and predictable k_s,

$$E\left|\varepsilon^{-1}\left[G\Big(W + \varepsilon\int_0^{\cdot} k_s\, ds\Big) - G(W) - \varepsilon\hat{D}_k G(W)\right]\right|^p \to 0.$$

It is often useful to require only $L^p(P)$ smoothness of G rather than the stronger Frechet differentiability.

THEOREM 3.6. *Suppose G is a bounded $L^1(P)$ smooth functional of $\{W_s, s \le t\}$, suppose k_s is bounded and predictable, and $K_t = \int_0^t k_s\, dW_s$. Then*

$$E[G(W)K_t] = E[\hat{D}_k G(W)]. \tag{3.11}$$

PROOF. Let

$$M_t^\varepsilon = \exp\Big(-\varepsilon K_t - \tfrac{1}{2}\varepsilon^2\int_0^t k_s^2\, ds\Big),$$

and define a probability measure Q^ε by

$$\left.\frac{dQ^\varepsilon}{dP}\right|_{\mathscr{F}_t} = M_t^\varepsilon.$$

It is a well-known consequence of Theorem 2.2 that under Q^ε, $W_t^\varepsilon = W_t + \varepsilon\int_0^t k_s\,ds$ is a Brownian motion, and hence the Q^ε-law of W^ε is the same as the P-law of W. Then

$$EG(W) = E_{Q^\varepsilon}G(W^\varepsilon) = E\left[G(W^\varepsilon)M_t^\varepsilon\right].$$

Differentiating with respect to ε and setting $\varepsilon = 0$ gives the result. The technical details of taking the derivative, which are very similar to the proof of Theorem 3.4, are left to the reader. □

We now look at a multivariate version of Theorems 3.4 and 3.6. Suppose $\mu_1, \dots, \mu_m$ are random measures associated to independent Poisson point processes each with characteristic measure ν. Suppose $W_1, \dots, W_n$ are independent Brownian motions that are independent of the μ_i's. Suppose G is a bounded functional of $\{\mu_1, \dots, \mu_m, W_1, \dots, W_n;\ s \le t\}$ that is $L^1(P)$ smooth in each variable. Suppose $l_i \in \mathscr{M}^2_\infty$ for each i, $L_t^{(i)} = \int_0^t\int l_i(s,z)(\mu_i(dz, ds) - \nu(dz)\,ds)$, k_j is bounded and predictable for each j, and $K_t^{(j)} = \int_0^t k_j(s)\,dW_j(s)$. Let D_iG denote the $L^1(P)$ derivative of G with respect to μ_i in the direction l_i, and let $\hat{D}_jG$ denote the $L^1(P)$ derivative of G with respect to W_j in the direction k_j.

Theorem 3.7.

(i) $$E[D_iG] = -E\left[GL_t^{(i)}\right];$$

(ii) $$E\left[\hat{D}_jG\right] = E\left[GK_t^{(j)}\right];$$

(iii) $$E\left[\sum_{i=1}^{n} \alpha_i D_iG + \sum_{j=1}^{n} \beta_j \hat{D}_jG\right] = E\left[G\left(-\sum_{i=1}^{m} \alpha_i L_t^{(i)} + \sum_{j=1}^{n} \beta_j K_t^{(j)}\right)\right];$$

and if $1 \le i_1 < i_2 < \cdots < i_a \le m$ *and* $1 \le j_1 < \cdots < j_b \le n$, *then*

(iv) $$E\left[D_{i_1}\cdots D_{i_a}\hat{D}_{j_1}\cdots\hat{D}_{j_b}G\right] = (-1)^a E\left[G\prod_{\alpha=1}^{a} L_t^{(i_\alpha)}\prod_{\beta=1}^{b} K_t^{(j_\beta)}\right].$$

Proof. Statements (iii) and (iv) follow from (i) and (ii) by taking linear combinations and iterating, respectively.

Suppose we define Q^ε by $dQ^\varepsilon/dP|_{\mathscr{F}_t} = M_t^\varepsilon = \mathscr{E}(\varepsilon L_t^{(1)})$. Since the martingales $\{\mu_i(A_i \times [0,t]) - \nu(A_i)^t, W_j(t), 2 \le i \le m, 1 \le j \le n\}$ are all orthogonal to the martingale M_t^ε, it is not hard to see that the Q^ε-law of $(\mu_1^\varepsilon, \mu_2, \dots, \mu_m, W_1, \dots, W_n)$ is the same as the P-law of $(\mu_1, \mu_2, \dots, \mu_m, W_1, \dots, W_n)$. With this observation, the proof of (i) and (ii) of Theorem 3.7 is similar to the proofs of Theorem 3.4 and 3.6 and is left to the reader. □

4. Densities. Once we have Theorem 3.4, we can develop a criterion for a functional of μ to have a density and for this density to be in L^p. Suppose t is fixed, and $Y = Y(\mu)$ is a functional of $\{\mu, s \le t\}$.

THEOREM 4.1. *Suppose Y is $L^1(P)$ smooth. If $D_lY(\mu)$ is strictly positive, a.s. for some $l \in \mathscr{M}^2_\infty$, then the measure $P[Y \in dy]$ is absolutely continuous with respect to Lebesgue measure, i.e., the distribution of Y has a density.*

PROOF. Let $h \in C^2$ with compact support, $G(\mu) = h(Y)$. Since $\varepsilon^{-1}(Y(\mu^\varepsilon) - Y(\mu)) \to D_lY(\mu)$ in $L^1(P)$, $\varepsilon^{-1}(Y(\mu^\varepsilon) - Y(\mu))$ is uniformly integrable, and so

$$\begin{aligned} &E\varepsilon^{-1}|h(Y(\mu^\varepsilon)) - h(Y) - h'(Y)(Y(\mu^\varepsilon) - Y)| \\ &\quad\le 2\|h'\|E\left[\varepsilon^{-1}|Y(\mu^\varepsilon) - Y|; |Y(\mu^\varepsilon) - Y| \ge 1\right] \\ &\qquad + \|h''\|E\left[\varepsilon^{-1}|Y(\mu^\varepsilon) - Y||Y(\mu^\varepsilon) - Y|; |Y(\mu^\varepsilon) - Y| < 1\right] \\ &\quad\to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Since

$$\begin{aligned} |h(Y(\mu^\varepsilon)) - h(Y) - \varepsilon h'(Y)\, D_lY| \le{}& |h(Y(\mu^\varepsilon)) - h(Y) - h'(Y)(Y(\mu^\varepsilon) - Y)| \\ &+ \|h'\||Y(\mu^\varepsilon) - Y - \varepsilon\, D_lY|, \end{aligned}$$

we conclude $h(Y)$ is $L^1(P)$ smooth with derivative $h'(Y)\, D_lY$.

By Theorem 3.4,

$$E\left[h'(Y)\, D_lY\right] = -E\left[h(Y)L_t\right]. \tag{4.1}$$

By (2.4),

$$E|L_t| \le \left(EL_t^2\right)^{1/2} \le c(l)t^{1/2},$$

and so if $R = D_lY/E\, D_lY$,

$$|E\left[h'(Y)R\right]| \le \|h\|c(l)t^{1/2}/E\, D_lY. \tag{4.2}$$

If we define a probability measure Q by $dQ/dP|_{\mathscr{F}_t} = R$, (4.2) becomes

$$E_Q h'(Y) \le c(l, t)\|h\|/E\, D_lY. \tag{4.3}$$

By a limit argument, (4.3) holds for $h'(x) = 1_A(x)$, $h(-\infty) = 0$, A a bounded Borel set. Then $\|h\|$ equals $|A|$, and we can conclude Y has a density g_Q relative to Q, and moreover $\|g_Q\| \le c(l, t)/E\, D_lY$. Since $R > 0$, a.s., P is equivalent to Q, and hence Y has a density with respect to P also. □

REMARK. Easy examples show that in general, $R \ne 0$, a.s. does not suffice for Q to be equivalent to P. Hence a bit more work is needed to make the argument of Bichteler and Jacod complete.

To get the density g_P to be in L^p, we have:

THEOREM 4.2. *Suppose Y is $L^1(P)$ smooth. Suppose $D_lY(\mu)$ is strictly positive, a.s., for some $l \in \mathscr{M}^2_\infty$ and $k = \|(D_lY)^{-1}\|_{L^{p-1}(P)} < \infty$ for some $p > 1$.*

Then $g_P(y) = P[Y \in dy]/dy$ *is in* $L^p(dy)$ *with*

$$\|g_P\|_p \le (kc(l,t))^{1-1/p},$$

where $c(l,t)$ *is the constant of* (4.3).

PROOF. By Theorem 4.1, Y has a density g_Q relative to Q with $\|g_Q\| \le c(l,t)/E\,D_lY$. Then if $p^{-1} + q^{-1} = 1$,

$$\begin{aligned}\int h(y)g_P(y)\,dy &= E[h(Y)] = E_Q[h(Y)R^{-1}]\\ &\le \left(E_Q[h^q(Y)]\right)^{1/q}\left(E_Q[R^{-p}]\right)^{1/p}\\ &\le \left(\int h^q(y)g_Q(y)\,dy\right)^{1/q}\left(E[R^{-p+1}]\right)^{1/p}\\ &\le (kc(l,t))^{1-1/p}\left(\int h^q(y)\,dy\right)^{1/q},\end{aligned} \tag{4.4}$$

and the result follows by the duality of L^p and L^q. □

Note that the bound on $\|g_P\|_p$ depends on Y only through k and $c(l,t)$. Consequently, we have:

COROLLARY 4.3. *Suppose* $Y_n(\mu)$ *are* $L^1(P)$ *smooth functionals of* μ *with* $Y_n \to Y$ *in law. If for some* $l \in \mathscr{M}^2_\infty$, *each* $D_lY_n(\mu)$ *is strictly positive, and* $\sup k_n < \infty$, *where* $k_n = \|(D_lY_n)^{-1}\|_{L^{p-1}(P)}$, *then* Y *has a density* g_P *and* $\|g_P\|_p \le ((\sup k_n)c(l,t))^{1-1/p}$.

PROOF. By (4.4), if h is continuous with compact support,

$$Eh(Y) = \lim_{n\to\infty} Eh(Y_n) \le ((\sup k_n)c(l,t))^{1-1/p}\|h\|_q. \tag{4.5}$$

A monotone class argument shows that (4.5) holds with $h = 1_A$, A Borel, from which we conclude Y has a density g_P which satisfies (4.4). The result is immediate. □

The point of the corollary is that even if D_lY does not exist, one may still be able to conclude that Y has a density in L^p.

Now suppose X_t is the solution to

$$X_t = \int_0^t\int \sigma(X_{s-},s,z)(\mu-\gamma)(dz,ds) + \int_0^t b(X_{s-},s)\,ds, \tag{4.6}$$

where $\sigma\colon \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ and $b\colon \mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$. Then X_t is a functional of μ, and we will obtain a criterion for X_1 to have a density. We want to emphasize that we do *not* need to perform a perturbation argument on X; it suffices to use a slight extension of Theorems 4.1 and 4.2. This is quite unlike some of the other approaches to the Malliavin calculus. We still have some technicalities to deal with: computing $D_l X$ and estimating $(D_l X)^{-1}$. But in our approach, the technicalities are kept completely separate from the perturbation argument.

THEOREM 4.4. *Suppose X_t solves* (4.6), *where*

(i) $\sup_{x,s}|b(x,s)|$ *and* $\sup_{x,s}|b_x(x,s)|$ *are bounded;*
(ii) $\sup_{x,s}|\sigma(x,s,z)|$ *is in* $L^2(dz)$;
(iii) $\sup_{x,s}|\sigma_x(x,s,z)|$ *is bounded and in* $L^2(dz)$;
(iv) $\sup_{x,s}|\sigma_z(x,s,z)|$ *is bounded and in* $L^1(\nu)$, *and* $\sigma_z(x,s,z) \le 0$ *for all* x, s, *and* z; *and*
(v) *for some* $z_0 > 0$ *and* $p_0 > 0$,
  (a) $\sup_{x,s}\sup_{|z|>z_0}|\sigma_x(x,s,z)| < \frac{1}{2}$ *and* $\sup_{x,s}\sup_{|z|\le z_0}|\sigma_x(x,s,z)/\sigma_z(x,s,z)|$ *is bounded with the convention that* $0/0 = 0$, *and*
  (b) *for* $z \ge z_0$, $\inf_{x,s}(|\sigma_z(x,s,z)| + |\sigma_z(x,s,-z)|) \ge \exp(-z/p_0)$.

Then for $p < p_0 + 1$, X_1 *has a density in* $L^p(dx)$ *and the* L^p *norm of the density depends on* X *only through the constants of* (i), (ii), (iii), (iv), *and* (v).

REMARK 1. Under the hypotheses of Theorem 4.4, the existence and uniqueness of X follows by Skorokhod (1965, Chapter 3).

REMARK 2. As will be apparent from the proof (given in Section 6), we do not need σ_x and b_x to exist but only that σ and b be uniformly Lipschitz in x.

REMARK 3. The hypotheses of Theorem 4.4 are reasonably weak. Conditions such as (i), (ii), and (iii) are necessary just to guarantee uniqueness of the solution to (4.6). In view of (iii), (v)(a) is a very minor restriction. Condition (v)(b) says that at least one of $\sigma_z(x,s,z)$, $\sigma_z(x,s,-z)$ cannot be too small for z large.

REMARK 4. Of course, under (iv), σ is bounded, or X_t has bounded jumps. We could obtain a theorem to handle the unbounded jumps case by replacing the first part of (iv) by

$$|z|\sup_{x,s}|\sigma_z(x,s,z)| \quad \text{is in } L^2(\nu)$$

and replacing the second part of (v)(b) by

$$\sup_{x,s}\sup_{|z|\le z_0}\left|\frac{\sigma_x(x,s,z)}{z\sigma_z(x,s,z)}\right| \quad \text{is bounded.}$$

5. Derivatives of processes. In this section let X_t be the solution to (4.6), and let X_t^ε be the solution to (4.6) with μ replaced by μ^ε. Let $(D_l X)_t$ be the solution to

$$(5.1)\quad \begin{aligned}(D_lX)_t = &\int_0^t\int \sigma_x(X_{s-},s,z)(D_lX)_{s-}(\mu-\gamma)(dz,ds)\\ &+\int_0^t b_x(X_{s-},s)(D_lX)_{s-}\,ds + \int_0^t\int v(s,z)\sigma_z(X_{s-},s,z)\mu(dz,ds).\end{aligned}$$

THEOREM 5.1. *Suppose*

(i) *for all* x, s, z,

$$(1+|z|)^2|\sigma_{zz}(x,s,z)|,\ (1+|z|)\,|\sigma_{zx}(x,s,z)|,$$
$$(1+|z|)|\sigma_{xz}(x,s,z)|, \quad and \quad |\sigma_{xx}(x,s,z)|$$

are all $\le M(z)$, *where* $M(z)$ *is a bounded deterministic function in* $L^2(\nu)$.

(ii) $\sup_{x,s}|b(x,s)|$, $\sup_{x,s}|b_x(x,s)|$, $\sup_{x,s}|b_{xx}(x,s)|$ *are bounded; and*

(iii) $l \in \mathscr{M}^2_\infty$.

Then $\sup_{s\le t}\varepsilon^{-1}|X^\varepsilon_s - X_s - \varepsilon(D_lX)_s| \to 0$ *in* $L^p(P)$ *for all* p, t.

REMARK 1. Under the assumptions of Theorem 5.1, the solution to (5.1) can be shown to exist and be unique by the methods of Skorokhod (1965, Chapter 3) or Jacod (1979, Chapter 14).

REMARK 2. Our equation for D_lX_t is equivalent to that of Bichteler and Jacod if one adds and subtracts $\int_0^t\int\sigma_z v\,d\gamma$ and then integrates by parts.

We first prove the following lemma.

LEMMA 5.2. *Suppose* $h(s,z)$ *is predictable and* $|h(s,z)| \le K_sH(z)$, *where* H *is a deterministic bounded function that is in* $L^2(\nu)$. *Suppose* $Z_t = \int_0^t\int h(s,z)(\mu-\gamma)(dz,ds)$. *Then, if* $p = 2^n$, $n \ge 1$,

$$E(Z_t^*)^p \le c(p,t,H)\int_0^t E|K_s|^p\,ds.$$

PROOF. For $r \ge 1$, let

$$Z_t^{(r)} = \int_0^t\int h^r(s,z)(\mu-\gamma)(dz,ds).$$

By Burkholder's inequality and (2.4),

$$\begin{aligned} E\big(Z_t^{(r)*}\big)^2 &\le cE\int_0^t\int h^{2r}(s,z)\mu(dz,ds) \le cE\int_0^t\int K_s^{2r}H^{2r}(z)\nu(dz)\,ds \\ &\le c(H)E\int_0^t K_s^{2r}\,ds. \end{aligned} \tag{5.2}$$

By Burkholder's and Hölder's inequalities, if $q \ge 2$,

$$\begin{aligned} E\big(Z_t^{(r)*}\big)^q &\le c(q)E\bigg(\int_0^t\int h^{2r}(s,z)\mu(dz,ds)\bigg)^{q/2} \\ &\le c(q)E\big(Z_t^{(2r)*}\big)^{q/2} + c(q)E\bigg(\int_0^t\int h^{2r}(s,z)\nu(dz)\,ds\bigg)^{q/2} \\ &\le c(q)E\big(Z_t^{(2r)*}\big)^{q/2} + c(q,H,t)E\int_0^t K_s^{rq}\,ds. \end{aligned} \tag{5.3}$$

Using (5.2) and (5.3), the result now follows by induction. □

An immediate consequence is that if in addition to the hypotheses of Lemma 2.1, $H \in L^1(\nu)$, then

$$\begin{aligned}
E\sup_{s\le t}\left|\int_0^s\int h(r,z)\mu(dz,dr)\right|^p &\le c(p)E\left(\int_0^t\int |h(r,z)|(\mu-\gamma)(dz,dr)\right)^p \\
&\quad + c(p)E\left(\int_0^t\int |h(r,z)|\nu(dz)\,dr\right)^p \qquad (5.4)\\
&\le c(p,H,t)E\int_0^t K_r^p\,dr.
\end{aligned}$$

We also note that by Hölder,

$$E\sup_{s\le t}\left|\int_0^s g(r)\,dr\right|^p \le c(p,t)E\int_0^t |g(r)|^p\,dr. \qquad (5.5)$$

We need the following real variable lemma.

LEMMA 5.3. (i) *If f' is bounded and in $L^2(\nu)$ and $f(0)=0$, then $f(z)/(1+|z|)$ is also bounded and in $L^2(\nu)$.*

(ii) *If $(1+|z|)^2 f''(z)$ is bounded and in $L^2(\nu)$, then so are $(1+|z|)f'(z)$ and $f(z)$.*

PROOF. The proof of boundedness is clear. Suppose for the moment that f' has compact support. Writing $f^2(z) = 2\int_0^z f'(y)f(y)\,dy$, we have

$$\begin{aligned}
\int_0^\infty \frac{f^2(z)}{(1+|z|)^2}\,dz &= 2\int_0^\infty\int_0^z \frac{f(y)f'(y)}{(1+z)^2}\,dy\,dz \\
&= 2\int_0^\infty\int_y^\infty \frac{f(y)f'(y)}{(1+z)^2}\,dz\,dy \\
&= 2\int_0^\infty \frac{f'(y)f(y)}{(1+y)}\,dy \\
&\le 2\|f'\|_2\left(\int_0^\infty \frac{f^2(y)}{(1+y)^2}\,dy\right)^{1/2}.
\end{aligned}$$

Then $(\int_0^\infty f^2(z)/(1+|z|)^2\,dz)^{1/2} \le 2\|f'\|_2$ if f' has compact support; a similar argument for $z<0$ and then a limit argument proves (i). The proof of (ii) is similar, starting with $(f'(z))^2 = 2\int_z^\infty f'(y)f''(y)\,dy$ for $z>0$. □

We are now ready to prove Theorem 5.1.

PROOF OF THEOREM 5.1. First, we must show X_t, X_t^ε, and $(D_l X)_t$ are all in $L^p(P)$ for all p. We will do $(D_l X)_t$, the others being similar. Note that $v(s,z)\sigma_z(x,s,z) = (v(s,z)/(1+|z|))(1+|z|)\sigma_z(x,s,z))$ is bounded by Lemma

5.3. Let $T_N = \inf\{s: (D_l X)_s \geq N\}$. For $s \leq T_N$, $(D_l X)_{s-} \leq N$, and hence inspection of (5.1) shows that the jump of $(D_l X)_s$ at time T_N is bounded. Hence $(D_l X)^*_{T_N}$ is bounded, a.s. By Lemma 5.2 applied to the first term on the right of (5.1), (5.5) applied to the second term, and (5.4) applied to the third, we have

$$
\begin{aligned}
E(D_l X)^p_{t\wedge T_N} &\leq c(p,t) + c(p,t) E\int_0^{t\wedge T_N} |(D_l X)_{s-}|^p\, ds \\
&\leq c(p,t) + c(p,t)\int_0^t E(D_l X)^{*p}_{s\wedge T_N}\, ds.
\end{aligned} \tag{5.6}
$$

An application of Gronwall's lemma gives

$$E(D_l X)^{*p}_{t\wedge T_N} \leq c(p,t),$$

independent of N; letting $N \to \infty$ shows $(D_l X)^*_t \in L^p(P)$ by Fatou.

Next, if $W_t^\varepsilon = X_t^\varepsilon - X_t$, we will show that if p is a power of 2, then

$$E(W_t^{\varepsilon *})^{2p} = O(\varepsilon^{2p}). \tag{5.7}$$

We have

$$
\begin{aligned}
W_t^\varepsilon = &\int_0^t \int [\sigma(X_{s-}^\varepsilon, s, z) - \sigma(X_{s-}, s, z)]\, d(\mu - \gamma) \\
&+ \int_0^t \int [\sigma(X_{s-}^\varepsilon, s, z)]\, d(\mu^\varepsilon - \mu) \\
&+ \int_0^t [b(X_{s-}^\varepsilon, s) - b(X_{s-}, s)]\, ds.
\end{aligned} \tag{5.8}
$$

Since $|b(X_{s-}^\varepsilon, s) - b(X_{s-}, s)| \leq c(\|b_x\|) W_{s-}^\varepsilon$, then by (5.5),

$$E\left(\int_0^t [b(X_{s-}^\varepsilon, s) - b(X_{s-}, s)]\, ds\right)^{*2p} \leq c\int_0^t E(W_s^{\varepsilon *})^{2p}\, ds. \tag{5.9}$$

Since $|\sigma(X_{s-}^\varepsilon, s, z) - \sigma(X_{s-}, s, z)| \leq M(z) W_{s-}^\varepsilon$, then by Lemma 5.2,

$$E\left(\int_0^t \int [\sigma(X_{s-}^\varepsilon, s, z) - \sigma(X_{s-}, s, z)]\, d(\mu - \gamma)\right)^{*2p} \leq c\int_0^t E(W_s^{\varepsilon *})^{2p}\, ds. \tag{5.10}$$

Note that if $|\varepsilon l(s,z)| \leq \frac{1}{2}$ and $z > 0$,

$$
\begin{aligned}
&\varepsilon^{-1} |\sigma(x, s, z + \varepsilon v(s,z)) - \sigma(x, s, z)| \\
&\quad\leq |v(s,z)| \sup_{y \geq z/2} |\sigma_z(x, s, y)| \\
&\quad\leq 2\left|v(s,z)(1+z)^{-1}\right| \left|(1+z/2)\int_{z/2}^\infty |\sigma_{zz}(x,s,y)|\, dy\right|,
\end{aligned} \tag{5.11}
$$

with a similar inequality for $z < 0$. Then by Lemma 5.3, the left-hand side of (5.11) is bounded and in $L^2(v)$ with bounds independent of ε. By the definition of μ^ε,

$$
\begin{aligned}
&\int_0^t \int [\sigma(X_{s-}^\varepsilon, s, z)]\, d(\mu^\varepsilon - \mu) \\
&\quad= \int_0^t \int [\sigma(X_{s-}^\varepsilon, s, z + \varepsilon v(s,z)) - \sigma(X_{s-}, s, z)]\, d\mu,
\end{aligned}
$$

and so by (5.4),

$$E\left(\int_0^t\int[\sigma(X^\varepsilon_{s-}, s, z)]\, d(\mu^\varepsilon - \mu)\right)^{*2p} \le c\varepsilon^{2p}. \tag{5.12}$$

Adding (5.9), (5.10), and (5.12) gives

$$E(W_t^{\varepsilon *})^{2p} \le c\varepsilon^{2p} + c\int_0^t E(W_s^{\varepsilon *})^{2p}\, ds, \tag{5.13}$$

and Gronwall applied to (5.13) yields (5.7).

Finally, let

$$Y_t^\varepsilon = X_t^\varepsilon - X_t - \varepsilon(D_l X)_t,$$

and we will show that if p is a power of 2,

$$E(Y_t^{\varepsilon *})^p = O(\varepsilon^{2p}). \tag{5.14}$$

From the definition of Y_t^ε,

$$\begin{aligned} Y_t = &\int_0^t\int[\sigma(X^\varepsilon_{s-}, s, z) - \sigma(X_{s-}, s, z) - \varepsilon\sigma_x(X_{s-}, s, z)\, D_l X_{s-}]\, d(\mu - \gamma) \\ &+ \int_0^t\int[\sigma(X^\varepsilon_{s-}, s, z + \varepsilon v(s, z)) - \sigma(X^\varepsilon_{s-}, s, z) \\ &\qquad - \varepsilon\sigma_z(X_{s-}, s, z) v(s, z)]\, d\mu \\ &+ \int_0^t [b(X^\varepsilon_{s-}, s) - b(X_{s-}, s) - \varepsilon b_x(X_{s-}, s)\, D_l X_{s-}]\, ds. \end{aligned} \tag{5.15}$$

By Taylor's theorem,

$$\begin{aligned} &|\sigma(X^\varepsilon_{s-}, s, z) - \sigma(X_{s-}, s, z) - \varepsilon\sigma_x(X_{s-}, s, z)\, D_l X_{s-}| \\ &\quad \le M(z)|Y^\varepsilon_{s-}| + M(z)(W^\varepsilon_{s-})^2 \end{aligned}$$

and

$$|b(X^\varepsilon_{s-}, s) - b(X_{s-}, s) - \varepsilon b_x(X_{s-}, s)\, D_l X_{s-}| \le c|Y^\varepsilon_{s-}| + c(W^\varepsilon_{s-})^2.$$

Then as in (5.9) and (5.10),

$$\begin{aligned} &E\bigg(\int_0^t\int[\sigma(X^\varepsilon_{s-}, s, z) - \sigma(X_{s-}, s, z) \\ &\qquad - \varepsilon\sigma_x(X_{s-}, s, z)\, D_l X_{s-}]\, d(\mu - \gamma)\bigg)^{*p} \\ &\quad \le c\int_0^t\left[E(Y_s^{\varepsilon *})^p + E(W_s^{\varepsilon *})^{2p}\right] ds \\ &\quad \le c\varepsilon^{2p} + c\int_0^t E(Y_s^{\varepsilon *})^p\, ds \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} &E\left(\int_0^t [b(X^\varepsilon_{s-}, s) - b(X_{s-}, s) - \varepsilon b_x(X_{s-}, s)\, D_l X_{s-}]\right)^{*p} \\ &\quad \le c\varepsilon^{2p} + c\int_0^t E(Y_s^{\varepsilon *})^p\, ds. \end{aligned} \tag{5.17}$$

Since

$$|\sigma(X^\varepsilon_{s-}, s, z+\varepsilon v(s,z)) - \sigma(X^\varepsilon_{s-}, s, z) - \varepsilon\sigma_z(X_{s-}, s, z)v(s,z)|$$
$$\le \varepsilon|v(s,z)||\sigma_z(X^\varepsilon_{s-}, s, z) - \sigma_z(X_{s-}, s, z)| + \varepsilon^2 v^2(s,z)\sup_{x,s}|\sigma_{zz}(x,s,z)|$$
$$\le \frac{\varepsilon|v(s,z)|}{1+|z|}(1+|z|)\sup_{x,s}|\sigma_{zx}(x,s,z)||W^\varepsilon_{s-}|$$
$$+ \frac{\varepsilon^2 v^2(s,z)}{(1+|z|)^2}(1+|z|)^2\sup_{x,s}|\sigma_{zz}(x,s,z)|,$$

by (5.7) and Lemma 5.3, we have

$$E\bigg(\int_0^t\int[\sigma(X^\varepsilon_{s-}, s, z+\varepsilon v(s,z)) - \sigma(X^\varepsilon_{s-}, s, z)$$
$$-\varepsilon\sigma_z(X_{s-}, s, z)v(s,z)]\,d\mu\bigg)^{*p} \tag{5.18}$$
$$\le c\varepsilon^{2p} + c\varepsilon^p\int_0^t E(W^\varepsilon_s)^p\,ds$$
$$\le c\varepsilon^{2p}.$$

Adding (5.16), (5.17), and (5.18) yields

$$E(Y^{\varepsilon *}_t)^p \le c\varepsilon^{2p} + c\int_0^t E[Y^{\varepsilon *}_s]^p\,ds,$$

which with Gronwall gives (5.14). □

REMARK. If $|l(s,z)|$ is bounded above by a bounded deterministic function in $L^1(\nu)$, we could let $v(s,z) = \int_{-\infty}^z l(s,y)\,dy$ as in the remark following Lemma 3.5. The estimates are easier in this case.

6. Estimates of derivatives. In this section we obtain estimates of $(D_l X)^{-1}$, and then prove Theorem 4.4. First we need a lemma.

Suppose

$$A_t = \int_0^t\int_0^\infty h(z)\mu(dz, ds), \tag{6.1}$$

where $h(z)$ is a bounded nonnegative deterministic function in $L^1(\nu)$. A_t is a subordinator (nonincreasing Lévy process), and recall that if $n(dx)$ is its Lévy measure,

$$E\exp(-KA_t) = \exp\bigg(t\int_0^\infty(e^{-Kx}-1)n(dx)\bigg). \tag{6.2}$$

LEMMA 6.1. *Suppose A_t is given by* (6.1) *and $h(z) = \exp(-z/p_0)$ for $z \ge z_0$. Then for $p < p_0$,*

$$\|A_1^{-1}\|_{L^p(P)} \le c(p, h, p_0, z_0) < \infty.$$

Proof. Let us first estimate the Lévy measure $n(dx)$ of A_t. If $f \in C^2$ with compact support,

$$\begin{aligned} Ef(A_t) - f(0) &= E\sum_{s \le t} [f(A_s) - f(A_{s-})] \\ &= E\int_0^t\int_0^\infty [f(A_{s-} + h(z)) - f(A_{s-})]\mu(dz, ds). \\ &= E\int_0^t\int_0^\infty [f(A_{s-} + h(z)) - f(A_{s-})]\, dz\, ds. \end{aligned}$$

Dividing by t and letting $t \to 0$ gives

$$Lf(0) = \int_0^\infty [f(x) - f(0)]n(dx) = \int_0^\infty [f(h(z)) - f(0)]\, dz, \tag{6.3}$$

where L is the infinitesimal generator of A_t. By a limit argument, (6.3) holds for $f(x) = 1_{[u,\infty)}(x)$, and so given $\varepsilon > 0$ and u_0 sufficiently small, then

$$\begin{aligned} n[u,\infty) = \int_0^\infty 1_{[u,\infty)}(h(z))\, dz &\ge \int_{z_0}^\infty 1_{[u,\infty)}(h(z))\, dz \ge -p_0 \ln u - z_0 \\ &\ge -p_0(1-\varepsilon)\ln u, \qquad u \le u_0. \end{aligned} \tag{6.4}$$

Next we estimate $P(A_1^{-1} > \lambda)$. Using (6.2) and integration by parts,

$$\begin{aligned} P\left(A_1^{-1} > \lambda\right) &= P\left(e^{-KA_1} \ge e^{-K\lambda^{-1}}\right) \\ &\le e^{K\lambda^{-1}} E e^{-KA_1} \\ &\le \exp\left(K\lambda^{-1} + \int_0^{u_0} (e^{-Kx} - 1) n(dx)\right) \\ &\le \exp\left(K\lambda^{-1} + n[u_0,\infty) - K\int_0^{u_0} n[x,\infty)e^{-Kx}\, dx\right) \\ &\le c(u_0, h)\exp\left(K\lambda^{-1} + Kp_0(1-\varepsilon)\int_0^{u_0} e^{-Kx}\ln x\, dx\right) \\ &\le c(u_0, h)\exp\left(K\lambda^{-1} - p_0(1-\varepsilon)^2 \ln K\right), \end{aligned} \tag{6.5}$$

provided K sufficiently large If we now take ε small so that $p_0(1-\varepsilon)^2 > p$, if we let $K = \lambda$, and if we multiply (6.5) by λ^{p-1} and then integrate from 0 to ∞, we get our result. □

We now proceed to the proof of Theorem 4.4.

Proof of Theorem 4.4. Let us suppose for the time being that σ and b satisfy the hypotheses of Theorem 5.1 as well as those of Theorem 4.4.

Define a function $\tilde{\sigma}$ by

$$\tilde{\sigma}(x, s, z) = \left(-\sigma_x(x, s, z) - \tfrac{1}{2}1_{(-\infty, -1/2]}(\sigma_x(x, s, z))\right). \tag{6.6}$$

Note $\tilde{\sigma} + \sigma_x \ge -\frac{1}{2}$, $|\tilde{\sigma}| \le |\sigma_x|$, and $\sup_{x,s}|\tilde{\sigma}(x, s, z)/\sigma_z(x, s, z)|$ is bounded, by N, say, with support $[-z_0, z_0]$ by condition (v)(a). Let $m(z)$ be a bounded negative

C^1 function such that $|m'(z)| \le (1 + z^2)^{-1}$ and $m(-\infty) = 0$. Let $u(z)$ be a C^1 function such that $u(-\infty) = 0$, $|u'(z)| \le 1$ for all z, u' is supported on $[-2N - z_0, -z_0]$, and

$$-2N \le u(z) \le -N1_{[-z_0, z_0]}(z) \tag{6.7}$$

for all z.

Let Z_t be the (unique) solution to the stochastic differential equation

$$\begin{aligned} Z_t = &\int_0^t \int \sigma_x(X_{s-}, s, z) Z_{s-}(\mu - \gamma)(dz, ds) + \int_0^t b_x(X_{s-}, s) Z_{s-}\, ds \\ &+ \int_0^t \int \sigma_z(X_{s-}, s, z) u(z) Z_{s-} \mu(dz, ds) \\ &+ \int_0^t \int \sigma_z(X_{s-}, s, z) m(z) \mu(dz, ds). \end{aligned} \tag{6.8}$$

By arguments analogous to those of the first part of the proof of Theorem 5.1, $Z_t^* \in L^p(P)$ and

$$EZ_t^{*p} \le c(p, \sigma, b), \tag{6.9}$$

where $c(p, \sigma, b)$ depends on the bounds on σ and b imposed by the hypotheses of Theorem 4.4 but not on those imposed by Theorem 5.1. Let

$$l(s, z) = u'(z) Z_{s-} + m'(z). \tag{6.10}$$

If we define L_t by

$$L_t = \int_0^t \int l(s, z)(\mu - \gamma)(dz, ds), \tag{6.11}$$

we have

$$EL_t^2 = E\int_0^t \int l^2(s, z)\, dz\, ds \le 2tNEZ_t^{*2} + t\int m'^2(z)\, dz. \tag{6.12}$$

Let $Z_s^{(n)} = \operatorname{sgn}(Z_s)(|Z_s| \wedge n)$, and note that $E(Z_t - Z_t^{(n)})^{*2} \to 0$ as $n \to \infty$. Let

$$l_n(s, z) = u'(z) Z_{s-}^{(n)} + m'(z),$$

let

$$L_t^{(n)} = \int_0^t \int l_n(s, z)(\mu - \gamma)(dz, ds), \tag{6.13}$$

and note that $E(L_t - L_t^{(n)})^{*2} \to 0$ as $n \to \infty$.

By (5.1), (6.10), and the remark following the proof of Theorem 5.1,

$$\begin{aligned} (D_{l_n}X)_t = &\int_0^t \int \sigma_x(X_{s-}, s, z)(D_{l_n}X)_{s-}(\mu - \gamma)(dz, ds) \\ &+ \int_0^t b_x(X_{s-}, s)(D_{l_n}X)_{s-}\, ds \\ &+ \int_0^t \int \sigma_z(X_{s-}, s, z) u(z) Z_{s-}^{(n)} \mu(dz, ds) \\ &+ \int_0^t \int m(z) \sigma_z(X_{s-}, s, z) \mu(dz, ds). \end{aligned} \tag{6.14}$$

Subtracting (6.14) from (6.8) and using Burkholder's and Hölder's inequalities and Gronwall's lemma as in Section 5,

$$E\big(Z_t - (D_{l_n}X)_t\big)^{*2} \le cE\big(Z_t - Z_t^{(n)}\big)^{*2},$$

which tends to 0 as $n \to \infty$.

Let $G(\mu) = h(X_t)$, where h is C^2 with compact support. As in the proof of Theorem 4.1, $h(X_t)$ is $L^1(P)$ smooth since X_t is. Applying Theorem 3.4 and the remark following Lemma 3.5 with l_n, we get

$$E\big[h(X_t)L_t^{(n)}\big] = -E\big[h'(X_t)(D_{l_n}X)_t\big],$$

and letting $n \to \infty$ gives

$$E\big[h(X_t)L_t\big] = -E\big[h'(X_t)Z_t\big]. \tag{6.15}$$

We next estimate Z_t^{-1}. Since Z_t solves

$$Z_t = \int_0^t Z_{s-}\,dK_s + H_s,$$

where

$$H_t = \int_0^t\int \sigma_z(X_{s-}, s, z)m(z)\mu(dz, ds)$$

and

$$K_t = \int_0^t\int \sigma_x(X_{s-}, s, z)(\mu - \gamma)(dz, ds) + \int_0^t b_x(X_{s-}, s)\,ds + \int_0^t\int \sigma_z(X_{s-}, s, z)u(z)\mu(dz, ds),$$

then by (2.6),

$$Z_t = \mathscr{E}(K)_t\int_0^t(1 + \Delta K_s)^{-1}\mathscr{E}(K)_{s-}^{-1}\,dH_s. \tag{6.16}$$

Since $u\sigma_z + \sigma_x \ge \tilde{\sigma} + \sigma_x \ge -\frac{1}{2}$, then $1 + \Delta K_s \ge \frac{1}{2}$ for all s, and so $\mathscr{E}(K)_s > 0$ for all s, a.s. This implies, since $\sigma_z(X_{s-}, s, z)m(z) \ge 0$ and hence all the jumps of H_t are positive, that $Z_t \ge 0$.

Now let us examine $\mathscr{E}(K)_t$. Since the jumps of K_t are bounded above and bounded below by $-\frac{1}{2}$, since $\ln(1 + x) \le x$ for $x \ge \frac{1}{2}$, and since $|\ln(1 + x) - x| \le x^2/2$ for $|x| \le \frac{1}{2}$, we see that both $\mathscr{E}(K)_t$ and $\mathscr{E}(K)_t^{-1}$ are bounded by $\exp(K_t^*)\exp(c[K, K]_t)$ for all t. An application of Cauchy–Schwarz and Lemma 2.1 then shows that $\mathscr{E}(K)_t^*$ and $(\mathscr{E}(K)_t^{-1})^*$ are both in $L^q(P)$ for all q.

It is now easy to show $Z_1^{-1} \in L^p(P)$ for $p < p_0$, and in particular, $Z_1 > 0$, a.s. Since $(1 + \Delta K_s)^{-1}\mathscr{E}(K)_{s-}^{-1} = \mathscr{E}(K)_s^{-1}$, we have

$$Z_t \ge \mathscr{E}(K)_t\Big(\inf_{s\le t}\mathscr{E}(K)_s^{-1}\Big)H_t$$

or

$$Z_t^{-1} \le \mathscr{E}(K)_t^{-1}\mathscr{E}(K)_t^*H_t^{-1}. \tag{6.17}$$

If $p < p_0$ and ε is sufficiently small, $\sigma_z(x, s, z)m(z) + \sigma_z(x, s, -z)m(-z) \geq \exp(-z/p_0(1-\varepsilon))$ for $z \geq z_0$, and so $H_t \geq A_t$, where A_t is defined by (6.1). By Lemma 6.1, $\|H_1^{-1}\|_{L^p(P)} \leq \|A_1^{-1}\|_{L^p(P)} < \infty$. It follows by (6.17) that for all $p < p_0$, $Z_1^{-1} \in L^p(P)$. Moreover, tracing through the estimates shows that the $L^p(P)$ norm depends on σ and b only through the hypotheses of Theorem 4.4, not those of Theorem 5.1.

That X_1 has a density in $L^{p+1}(dx)$ for $p < p_0$ now follows by the proofs of Theorems 4.1 and 4.2, the L^p bound on Z_1^{-1}, (6.15), (6.12), and (6.9).

Finally, if σ and b do not satisfy the hypotheses of Theorem 5.1, let σ_i and b_i be smooth approximations that do. If $X_t^{(i)}$ is the solution to (4.6) with σ and b replaced by σ_i and b_i, then arguments using the methods of Section 5 show that $E(X_t^{(i)} - X_t)^{*2} \to 0$, and in particular, $X_1^{(i)} \to X_1$ in law. Then just as in the proof of Corollary 4.3, we can conclude X_1 has a density in $L^{p+1}(dx)$ for $p < p_0$. □

Remark. The proof for the case when X_t has unbounded jumps (see Remark 4 following the statement of Theorem 4.4) is essentially the same, except that one chooses u' so that $u(z) \leq -N|z|$ on $[-z_0, z_0]$, $u(0) = 0$.

7. Higher derivatives. If Y is a functional of $\{\mu, s \leq t\}$, then further smoothness of Y will imply that Y has a density that is bounded and even Hölder continuous.

If $l \in \mathscr{M}_\infty^2$, let $D_l^2 Y = D_l(D_l Y)$, and let

$$S_1 = L_t(D_l Y)^{-1} + (D_l^2 Y)(D_l Y)^{-2}, \tag{7.1}$$

where L_t is defined by (3.2).

Theorem 7.1. *Suppose Y and $D_l Y$ are $L^1(P)$ smooth for all $l \in \mathscr{M}_\infty^2$. Suppose there exists $l \in \mathscr{M}_\infty^2$ such that $D_l Y > 0$, a.s. Then if $S_1 \in L^1(P)$, Y has a bounded density. If $S_1 \in L^p(P)$, $p > 1$, Y has a density in $C^{1-1/p}(\mathbb{R})$ (Hölder continuous of order $1 - 1/p$).*

Proof. Let h be in C^2 with compact support, and let $\psi_n(x)$ be smooth approximations to $|x|^{-1}$ with ψ_n, ψ_n' tending monotonically to x^{-1}, $-x^{-2}$, respectively, for $x > 0$.

If $G_n(\mu) = h(Y)\psi_n(D_l Y)$, then

$$D_l G_n(\mu) = h'(Y)(D_l Y)\psi_n(D_l Y) + h(Y)\psi_n'(D_l Y)(D_l^2 Y).$$

If we now apply Theorem 3.4 to $G_n(\mu)$ and let $n \to \infty$, then monotone convergence gives

$$Eh'(Y) = E[h(Y)S_1]. \tag{7.2}$$

If $S_1 \in L^1(P)$, we then get

$$|Eh'(Y)| \leq \|S_1\|_{L^1(P)}\|h\|,$$

and proceeding as in the proof of Theorem 4.1, we see Y has a density g that is bounded by $\|S_1\|_{L^1(P)}$.

If $S_1 \in L^p(P)$ for $p > 1$ and $1/p + 1/q = 1$, then

$$(7.3)\qquad \begin{aligned} \left|\int h'(y)g(y)\,dy\right| &= |Eh'(Y)| \le \|h(Y)\|_{L^q(P)}\|S_1\|_{L^p(P)} \\ &= \|S_1\|_{L^p(P)}\left(\int h^q(y)g(y)\,dy\right)^{1/q} \\ &= \|S_1\|_{L^p(P)}\|g\|^{1/q}\|h\|_q. \end{aligned}$$

This implies that g has a weak derivative in $L^p(dy)$. If g were actually differentiable, then by Hölder,

$$|g(y) - g(x)| = \left|\int_x^y g'(u)\,du\right| \le \|g'\|_p|y - x|^{1/q}.$$

But an easy argument involving approximation of g by differentiable functions then shows that in any case g is Hölder of index $1/q$. □

To get the existence of g', still higher derivatives are needed. For example, replacing h by h' in (7.2),

$$(7.4)\qquad \begin{aligned} Eh''(Y) &= E[h'(Y)S_1] \\ &= E\left[D_l\left(h(Y)S_1(D_lY)^{-1}\right)\right] - E\left[h(Y)D_l\left(S_1(D_lY)^{-1}\right)\right] \end{aligned}$$

Using Theorem 3.4 again with (a suitable approximation to) $G(\mu) = h(Y)S_1(D_lY)^{-1}$,

$$(7.5)\qquad \begin{aligned} Eh''(Y) &= E\left[h(Y)S_1(D_lY)^{-1}L_t\right] - E\left[h(Y)D_l\left(S_1(D_lY)^{-1}\right)\right] \\ &= E[h(Y)S_2], \end{aligned}$$

where $S_2 = S_1(D_lY)^{-1}L_t - D_l(S_1(D_lY)^{-1})$.

If $S_2 \in L^1(P)$, (7.5) yields $|Eh''(Y)| \le \|S_2\|_{L^1(P)}\|h\|$, from which it is not hard to show that g' exists and is bounded. We can repeat the procedure: replace h by h' in (7.5), etc., to get bounds on g'', g''', etc.

When the functional Y being considered is X_1, where X_t solves (4.6) (for simplicity we take $b \equiv 0$), one can show as in Section 5 that D_l^2X satisfies

$$(7.6)\qquad \begin{aligned} (D_l^2X)_t = &\int_0^t\int\left[\sigma_{xx}(X_{s-}, s, z)(D_lX)_{s-}^2 + \sigma_x(X_{s-}, s, z)(D_l^2X)_{s-}\right] \\ &\times(\mu - \gamma)(dz, ds) \\ &+\int_0^t\int\left[2\sigma_{zx}(X_{s-}, s, z)(D_lX)_{s-}v(s, z)\right. \\ &\left.+\sigma_{zz}(X_{s-}, s, z)v^2(s, z) + \sigma_z(s, z)D_lv(s, z)\right]\mu(dz, ds) \end{aligned}$$

under appropriate hypotheses on σ, and one can obtain similar equations for D_l^3X, D_l^4X, etc. Note that $(D_lL)_t = \int_0\int D_ll(s, z)(\mu - \gamma)(dz, ds) + \int_0^t\int vl_z(s, z)\mu(dz, ds)$. If $(D_lX)_1^{-1} \in L^p(P)$ for all p and if we have suitable bounds on D_l^2X, D_l^3X, an examination of S_2 shows that S_2 is in $L^1(P)$, and hence

X_1 has a differentiable density. Repeating the argument shows that if σ is C^∞ with suitable bounds on the derivatives and $(D_l X)_1^{-1} \in L^p(P)$ for all p, then X_1 has a C^∞ density.

8. Local times. For the remainder of the paper we consider the existence of local times for purely discontinuous martingales. We consider X_t satisfying

$$X_t = x_0 + \int_0^t \int \sigma(s, z)(\mu(dz, ds) - dz\, ds), \tag{8.1}$$

where μ is the random measure associated to a Poisson point process whose characteristic measure is Lebesgue measure. Here σ is predictable and $\sigma(s, z) \in \mathscr{F}_{s-}$. Occasionally we will write $\sigma(X, s, z)$ to emphasize that σ is a functional of the past of X. We will often write $D_l X_t^*$ for $(D_l X)_t^*$, $D_l X_{t\wedge\rho}$ for $(D_l X)_{t\wedge\rho}$ where ρ is a stopping time, $\partial_z D_l \sigma(s, z)$ for $(\partial_z(D_l\sigma))(s, z)$, $D_l \sigma_z(s, z)$ for $(D_l(\sigma_z))(s, z)$, and D_l^2 for $D_l(D_l)$.

We will assume the following about σ:

(8.2)(a) (Monotonicity) $z\sigma_{zz}(s, z) \geq 0$ for all s, z;

(8.2)(b) (Smoothness) for each $l \in \mathscr{M}_\infty^2$
- (i) σ, $(1 + |z|)\sigma_z$, $D_l\sigma$, and $D_l^2\sigma$ are $L^1(P)$ smooth,
- (ii) there is a bounded deterministic function $M(z) \in L^2(dz)$ such that for each z, σ, $(1 + |z|)\sigma_z$, $D_l\sigma$, $D_l^2\sigma$, $(1 + |z|)\partial_z D_l\sigma$, and $(1 + |z|)D_l\sigma_z$ are Lipschitz as functionals of X with Lipschitz constant $M(z)$, uniformly in s,
- (iii) σ_z, $D_l\sigma$, $D_l\sigma_z$, and $D_l^2\sigma$ are twice continuously differentiable in z, uniformly in s and X;

(8.2)(c) (Boundedness) there is a bounded deterministic function $H(z) \in L^2(dz)$ such that for all $l, m \in \mathscr{M}_\infty^2$
- (i) $(1 + |z|)^2|\sigma_{zz}(s, z)| \leq H(z)$ for all s, z,
- (ii) $|D_l\sigma(s, z)| + (1 + |z|)(|\partial_z D_l\sigma(s, z)| + |D_l\sigma_z(s, z)|) \leq H(z) D_l X_{s-}^*$ for all s, z,
- (iii) $|D_l^2\sigma(s, z)| \leq H(z)[D_l^2 X_{s-}^* + (D_l X_{s-}^*)^2]$ for all s, z,
- (iv) $|D_l\sigma(s, z) - D_m\sigma(s, z)| \leq H(z)(D_l X - D_m X)_{s-}^*$ for all s, z.

REMARK. The conditions (b)(i) and (b)(iii) are the critical ones. As the example below shows, the other conditions are natural ones.

Let us say that σ is *ε-stable of index between α_- and α^+* if there exist z_0, α_0, α_1, and $\alpha_- < \alpha_0 < \alpha_1 < \alpha_0 + \varepsilon < \alpha^+$ such that

$$c_1|z|^{-1/\alpha_1 - 2} \leq |\sigma_{zz}(s, z)| \leq c_2|z|^{-1/\alpha_0 - 2} \tag{8.3}$$

for $|z| \geq z_0$.

The reason for the name is that if X_t is a symmetric stable process of index α, then $\sigma(s, z) = c(\operatorname{sgn} z)|z|^{-1/\alpha}$ (cf. (10.8)). Let us say that σ is *locally ε-stable of index between α_- and α^+* if there exist an increasing sequence of stopping times

$T_i \to \infty$, $T_0 = 0$ such that $\sigma(s, z)$ is ε-stable of index between α_- and α^+ for $T_i < s < T_{i+1}$, $i = 0, 1, 2, \dots$. Our main result is:

THEOREM 8.1. *Suppose for some $1 < \alpha_- < \alpha^+ < 2$, σ is locally ε-stable of index between α_- and α^+ and satisfies* (8.2). *Then X_t has a local time that is jointly continuous in t and x and that is an occupation time density.*

REMARK. The hypotheses of Theorem 8.1 are stronger than necessary. For example, the near symmetry that (8.3) imposes on σ can be avoided. Additionally, one does not need $D_l^2\sigma$ to be $L^1(P)$ smooth. See the results of Section 12.

EXAMPLE. Consider a Markov process with weak infinitesimal generator

$$Af(x) = \alpha(x) \int_{|y| \le 1} [f(x+y) - f(x)] |y|^{-(1+\alpha(x))}\, dy.$$

Then X_t can be thought of as a process that behaves like a symmetric stable process of index $\alpha(x)$ whenever X_t is at x, except that we have removed all jumps larger than 1 in absolute value. (If one wanted a different constant in front in place of $\alpha(x)$, one could achieve this by a time change and the results of Bass (1984, Section 6).)

It has been known for a long time that when $\alpha(x) \equiv \alpha$ is constant, X_t will have a local time if and only if $2 > \alpha > 1$. If $\alpha(x)$ is not constant and we do not require continuity of $\alpha(x)$, then X_t need not have a local time, even when $\inf_x \alpha(x) > 1$ (see Bass (1984, Section 8)). Theorem 8.1 says that if $2 > \sup_x \alpha(x) \ge \inf_x \alpha(x) > 1$ and $\sigma(x)$ is C^3 in x (actually C^2 will do by Section 12), then X_t will have a local time. To see this, first check (cf. the calculation of (10.8)) that X_t satisfies (8.1) with $\sigma(s, z) = w(X_{s-}, z)$, where

$$w(x, z) = \begin{cases} (\operatorname{sgn} z)|z|^{-1/\alpha(x)}, & |z| \ge 1, \\ 0, & |z| < 1. \end{cases}$$

Of course, w is not smooth in z for $|z| = 1$, but the techniques of Bass (1984, Section 6) allow one to show that X_t will have a local time if and only if $\hat{X}_t$ does, where $\hat{X}_t$ satisfies (8.1) with $\hat{\sigma}(s, z) = \hat{w}(\hat{X}_{s-}, z)$, $\hat{w}$ a smooth approximation to w (see also Section 12). If $\alpha(x)$ is C^3,

$$D_l\sigma(s, z) = \frac{\partial w}{\partial x}(X_{s-}, z)\, D_l X_{s-},$$

$$D_l^2\sigma(s, z) = \frac{\partial^2 w}{\partial x^2}(X_{s-}, z)(D_l X_{s-})^2 + \frac{\partial w}{\partial x}(X_{s-}, z)(D_l^2 X_{s-}),$$

and it is now easy to check that (8.2) is satisfied. If we fix ε and let $T_0 = 0$,

$$T_{i+1} = \inf\left\{t > T_i: \left|\alpha(X_t) - \alpha(X_{T_i})\right| > \varepsilon\right\},$$

then $T_i \to \infty$ and σ is locally ε-stable. Hence Theorem 8.1 applies.

9. Densities of potentials. We will suppose throughout this section and Sections 10 and 11 that X_t is given by (8.1), σ satisfies (8.2), and moreover, there exist $\alpha_0, \alpha_1, d_1, d_2$ such that

$$d_1\big(|z|^{-1/\alpha_1-2} \wedge 1\big) \le |\sigma_{zz}(s,z)| \le d_2\big(|z|^{-1/\alpha_0-2} \wedge 1\big) \quad \text{for all } z \tag{9.1}$$

and

$$\big(\alpha_1^{-1} + \tfrac{1}{2}\big)\big(2\alpha_1^{-1} - \alpha_0^{-1} + 1\big) > \big(2\alpha_0^{-1} + 1\big)\alpha_0^{-1}. \tag{9.2}$$

Since $1 < \alpha_0 < \alpha_1$, (9.2) will hold provided α_1 is sufficiently close to α_0.

A consequence of (9.1) is that

$$\left|\frac{\sigma_{zz}(s,z)}{\sigma_z(s,z)}\right| \le c(1+|z|)^{-1+\alpha_1^{-1}-\alpha_0^{-1}}. \tag{9.3}$$

Let $l(s,z) = -\operatorname{sgn}(z)(\frac{1}{2} \wedge |z|^{-1/2}|\ln|z||^{-1})$, let $v(s,z) = \int_0^z l(s,y)\,dy$, and let

$$L_t = \int_0^t \int l(s,z)(\mu(dz,ds) - dz\,ds). \tag{9.4}$$

Note that if $1 \le p' \le 2$,

$$\|L_t\|_{L^{p'}(P)} \le \|L_t\|_{L^2(P)} \le ct^{1/2} \tag{9.5}$$

since by (2.4),

$$EL_t^2 = E\int_0^t \int l^2(s,z)\,dz\,ds \le ct.$$

Note also that $v(s,z) \le 0$ for all s and z.

By virtue of (8.2) and the fact that v is deterministic, one can show exactly as in Section 5, that $D_l X$ and $D_l^2 X$ satisfy

$$\begin{aligned} D_l X_t = &\int_0^t \int D_l \sigma(s,z)(\mu(dz,ds) - dz\,ds) \\ &+ \int_0^t \int \sigma_z(s,z) v(s,z) \mu(dz,ds) \end{aligned} \tag{9.6}$$

and

$$\begin{aligned} D_l^2 X_t = &\int_0^t \int D_l^2 \sigma(s,z)(\mu(dz,ds) - dz\,ds) \\ &+ \int_0^t \int \big[(\partial_z D_l \sigma(s,z) + D_l \sigma_z(s,z)) v(s,z) \\ &\qquad + \sigma_{zz}(s,z) v^2(s,z)\big] \mu(dz,ds). \end{aligned} \tag{9.7}$$

Moreover, $D_l X_t^*$ and $D_l^2 X_t^*$ are in $L^p(P)$ for all p.

Let Θ_1, Θ_2 be iid random variables which are independent of $\sigma(X_t; 0 \le t < \infty)$ and which have a C^1 density whose support is $[1,2]$. Let F be the distribution function of Θ_1, Θ_2. We will need the following lemma.

LEMMA 9.1. *If R and H are independent of Θ_i and R is integrable, then*

$$E\left[R1_{(H \geq \Theta_i)}\right] = E[RF(H)].$$

PROOF. Since Θ_i has a continuous distribution and hence F is continuous, it suffices by dominated convergence to consider H discrete taking on the values $h_1, h_2, \ldots, h_n$. Then

$$\begin{aligned} E\left[R1_{(H \geq \Theta_i)}\right] &= E\left[E\left[R\sum_{j=1}^{n} 1_{(H=h_j)}1_{(h_j \geq \Theta_i)}|H, R\right]\right] \\ &= E\left[R\sum_j 1_{(H=h_j)}E\left[1_{(h_j \geq \Theta_i)}|H, R\right]\right] \\ &= E\left[R\sum_j 1_{(H=h_j)}F(h_j)\right] \quad \text{(by independence)} \\ &= E[RF(H)]. \ \square \end{aligned}$$

Choose $1 < \beta_l < \beta_L < \beta_U < \beta_u < \frac{3}{2}$ so that

$$\beta_L < \tfrac{1}{2} + \alpha_1^{-1}, \qquad \beta_U > \tfrac{1}{2} + \alpha_0^{-1},$$

and

$$\beta_l(2\beta_l - \beta_u + \tfrac{1}{2}) > \beta_u(2\beta_u - 1). \tag{9.8}$$

This is possible by (9.2). In what follows, the various constants that appear can be seen to depend on the process X_t only through the constants β_l, β_L, β_u, β_U, d_1, and d_2, and the function $H(z)$.

Let

$$J_t = \inf_{s \leq t}\left\{\frac{D_l X_s}{s^{\beta_u}}\right\}, \qquad K_t = \left(\frac{D_l X_t}{t^{\beta_l}}\right)^* = \sup_{s \leq t}\left\{\frac{D_l X_s}{s^{\beta_l}}\right\}. \tag{9.9}$$

Since J_t and K_t are not $L^1(P)$ smooth, we need to approximate them by quantities $J_t^{(n,r)}$, $K_t^{(m,r)}$ which are $L^1(P)$ smooth.

Let $l_r(s, z) = l(s, z)1_{[r,\infty)}(s)$, and define $D_{l_r}X_t$, $D_{l_r}^2 X_t$ by the equations (9.6) and (9.7) with l replaced by l_r. Again, for each r and for all p, $D_{l_r}X_t^*, D_{l_r}^2 X_t^* \in L^p(P)$. Define v_r by $v_r(s, z) = v(s, z)1_{[r,\infty)}(s)$.

Fix $r > 0$ and let

$$K_t^{(r)} = \left(\frac{D_{l_r}X_t}{t^{\beta_l}}\right)^*.$$

For each m, let $\Lambda_{m,k}(x_1, \ldots, x_m)$ be a smooth approximation to $\max(x_1, \ldots, x_m)$ such that

$$\Lambda_{m,k}(x_1, \ldots, x_m) \to \max(x_1, \ldots, x_m)$$

as $k \to \infty$, and for each k,

$$|\Lambda_{m,k}(x_1,\dots,x_m) - \Lambda_{m,k}(y_1,\dots,y_m)| \le 2 \max_{i=1,\dots,m} |x_i - y_i|.$$

Now let

$$K_t^{(m,r)} = \Lambda_{m,k_m}\left(\frac{D_{l_r}X_{t/m}}{(t/m)^{\beta_l}}, \frac{D_{l_r}X_{2t/m}}{(2t/m)^{\beta_l}}, \dots, \frac{D_{l_r}X_t}{t^{\beta_l}}\right),$$

where k_m is chosen tending to ∞ fast enough so that $K_t^{(m,r)} \to K_t^{(r)}$ a.s. as $m \to \infty$.

Similarly, let $\Xi_{n,k}$ be a smooth approximation to $\min(x_1,\dots,x_n)$, let

$$J_t^{(n,r)} = \Xi_{n,k_n}\left(\frac{D_{l_r}X_{t/n}}{(t/n)^{\beta_u}}, \dots, \frac{D_{l_r}X_t}{t^{\beta_u}}\right),$$

but this time choose $k_n \to \infty$ fast enough so that $J_t^{(n,0)} \to J_t$ a.s. as $n \to \infty$.

Note that each of $J_t^{(n,r)}$ and $K_t^{(m,r)}$ is $L^1(P)$ smooth since $D_l X_t$ and $D_{l_r}X_t$ are.

The principal technical estimates and results we need are given by the following three propositions.

PROPOSITION 9.2. *There exist positive constants $\zeta < 1$, δ, t_0, and c independent of r such that if $t \le t_0$,*

$$E\left[|D_{l_r}^2 X_t|^{1+\delta};\, K_t^{(r)} \le 2\right] \le ct^{2\beta_u - \zeta}.$$

PROPOSITION 9.3. *There exist positive constants t_0, η, c, and $c(a)$, independent of r such that if $t \le t_0$,*

$$P[J_t < 2] \le ct^{\eta}$$

and

$$P\left[K_t^{(r)} > a\right] \le c(a)t^{\eta}.$$

PROPOSITION 9.4. *As $r \to 0$, $E(D_l X_t - D_{l_r}X_t)^{*2} \to 0$ and $K_t^{(r)} \to K_t$ in probability.*

We defer the proofs of these three propositions until Section 10 in order not to interrupt the main argument.

The use of the Malliavin calculus comes in the proof of the next proposition.

PROPOSITION 9.5. *If $h \in C^2$ has compact support, then*

$$\left|E\left[h'(X_t)1_{(J_t \ge \Theta_1,\, K_t < \Theta_2)}\right]\right| \le ct^{-\xi}\|h(X_t)\|_{L^p(P)}$$

for positive constants $p > 2$, $\xi < 1$, independent of h.

Proof. Let $g_t(x)$ be a C^2 function on $\mathbb{R}$ such that $g_t(x) = |x|^{-1}$ if $|x| > t^{-\beta_u}$, $\|g_t\| \le 2t^{-\beta_u}$, and $\|g_t'\| \le ct^{-2\beta_u}$. Define the functional $G_{n,m,r}(\mu)$ by

$$G_{n,m,r}(\mu) = h(X_t)g_t(D_{l_r}X_t)F\big(J_t^{(n,r)}\big)\big(1 - F\big(K_t^{(m,r)}\big)\big). \tag{9.10}$$

$G_{n,m,r}(\mu)$ is $L^1(P)$ smooth since $D_{l_r}X_t$, $J_t^{(n,r)}$, and $K_t^{(m,r)}$ are $L^1(P)$ smooth. We apply (3.8) to $G_{n,m,r}(\mu)$, and let $m \to \infty$, then $r \to 0$, and finally $n \to \infty$. Then writing $L_t^{(r)} = \int_0^t \int l_r(s,z)(\mu(dz,ds) - dz\,ds)$ and using (3.8) we have

$$\begin{aligned} I_1(n,m,r) &= E\big[G_{n,m,r}(\mu)L_t^{(r)}\big] \\ &= -E\big[h'(X_t)\,D_{l_r}X_t\,g_t(D_{l_r}X_t)F\big(J_t^{(n,r)}\big)\big(1 - F\big(K_t^{(m,r)}\big)\big)\big] \\ &\quad -E\big[h(X_t)g_t'(D_{l_r}X_t)\,D_{l_r}^2X_t\,F\big(J_t^{(n,r)}\big)\big(1 - F\big(K_t^{(m,r)}\big)\big)\big] \\ &\quad -E\big[h(X_t)g_t(D_{l_r}X_t)f'\big(J_t^{(n,r)}\big)\,D_{l_r}J_t^{(n,r)}\big(1 - F\big(K_t^{(m,r)}\big)\big)\big] \\ &\quad +E\big[h(X_t)g_t(D_{l_r}X_t)F\big(J_t^{(n,r)}\big)f'\big(K_t^{(m,r)}\big)\,D_{l_r}K_t^{(m,r)}\big] \\ &= I_2(n,m,r) + I_3(n,m,r) + I_4(n,m,r) + I_5(n,m,r). \end{aligned} \tag{9.11}$$

Using the bounds on g_t and F,

$$\begin{aligned} |I_1(n,m,r)| &\le ct^{-\beta_u}E\big[|h(X_t)|\,|L_t^{(r)}|\big] \\ &\le ct^{-\beta_u}\|h(X_t)\|_{L^p(P)}\|L_t^{(r)}\|_{L^{p'}(P)} \qquad (p^{-1} + p'^{-1} = 1) \\ &\le ct^{1/2-\beta_u}\|h(X_t)\|_{L^p(P)} \quad \big(\text{by analog for } L_t^{(r)} \text{ of } (9.5)\big). \end{aligned}$$

Recalling that $J_t^{(n,0)} \to J_t$, first let $m \to \infty$, then $r \to 0$, then $n \to \infty$ to obtain

$$\begin{aligned} I_2(n,m,r) &\to -E\big[h'(X_t)\,D_lX_t\,g_t(D_lX_t)F(J_t)(1 - F(K_t))\big] \quad \text{(Proposition 9.4)} \\ &= -E\Big[h'(X_t)\,D_lX_t\,g_t(D_lX_t)1_{(J_t \ge \Theta_1,\,K_t < \Theta_2)}\Big] \quad \text{(Lemma 9.1)} \\ &= -E\Big[h'(X_t)1_{(J_t \ge \Theta_1,\,K_t < \Theta_2)}\Big] \quad (\text{definition of } J_t \text{ and } g_t). \end{aligned}$$

Recalling that $1 - F(K_t^{(r)}) = 0$ if $K_t^{(r)} > 2$,

$$\begin{aligned} |I_3(n,m,r)| &\to \Big|E\big[h(X_t)g_t'(D_{l_r}X_t)\,D_{l_r}^2X_t\,F\big(J_t^{(n,r)}\big)\big(1 - F\big(K_l^{(r)}\big)\big)\big]\Big| \\ &\le c\|h(X_t)\|_{L^p(P)}t^{-2\beta_u}\Big(E\Big[\big(D_{l_r}^2X_t^*\big)^{p'};\,K_t^{(r)} \le 2\Big]\Big)^{1/p'} \\ &\qquad\qquad (p^{-1} + p'^{-1} = 1) \\ &\le c\|h(X_t)\|_{L^p(P)}t^{-\xi} \quad \text{(Proposition 9.2)} \end{aligned}$$

as $m \to \infty$, provided p is sufficiently large.

By the definitions of $D_{l_r}X_t$ and $D_{l_r}^2X_t$, $D_{l_r}^2X_t = 0$ if $t \le r$. By the Lipschitz bound on $\Lambda_{m,k}$, we get

$$|D_{l_r}K_t^{(m,r)}| \le 2\big(D_{l_r}^2X_t/t^{\beta_l}\big)^* \le 2r^{-\beta_l}D_{l_r}^2X_t^*, \tag{9.12}$$

which is in $L^p(P)$ for all p. Also,

$$
\begin{aligned}
&E\left[\left(\left(D_{l_r}^2 X_t/t^{\beta_u}\right)^*\right)^{p'};\, K_t^{(r)} \le 2\right] \\
&\quad \le \sum_{n=0}^{\infty} E\left[\left(\sup_{t2^{-(n+1)} \le s \le t2^{-n}} |D_{l_r}^2 X_s|/s^{\beta_u}\right)^{p'};\, K_t^{(r)} \le 2\right] \\
(9.13)\qquad &\quad \le c \sum_{n=0}^{\infty} t^{-\beta_u p'} 2^{n\beta_u p'} E\left[\left(D_{l_r}^2 X_{t2^{-n}}^*\right)^{p'};\, K_{t2^{-n}} \le 2\right] \\
&\quad \le c t^{(2-p')\beta_u - \zeta} \quad \text{(Proposition 9.2)},
\end{aligned}
$$

since $(2 - p')\beta_u - \zeta > 0$, provided p' is sufficiently close to 1. We also get

$$|D_{l_r} J_t^{(n,r)}| \le 2\left(D_{l_r}^2 X_t/t^{\beta_l}\right)^* \le c\left(D_{l_r}^2 X_t/t^{\beta_u}\right)^*,$$

which is in $L^p(P)$ for all p, the same way.

Then, provided p is sufficiently large, as $m \to \infty$,

$$
\begin{aligned}
|I_4(n, m, r)| &\to \left|E\left[h(X_t) g_t(D_{l_r} X_t) f'\left(J_t^{(n,r)}\right) D_{l_r} J_t^{(n,r)}\left(1 - F\left(K_t^{(r)}\right)\right)\right]\right| \\
&\le c\|h(X_t)\|_{L^p(P)} t^{-\beta_u}\left(E\left[\left(\left(D_{l_r}^2 X_t/t^{\beta_u}\right)^*\right)^{p'};\, K_t^{(r)} \le 2\right]\right)^{1/p'} \\
&\le c\|h(X_t)\|_{L^p(P)} t^{-\xi} \quad (\text{by } (9.13)).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&I_5(n, m, r) \\
(9.14)\qquad &\quad = E\left[h(X_t) g_t(D_{l_r} X_t) F\left(J_t^{(n,r)}\right)\left(f'\left(K_t^{(m,r)}\right) - f'\left(K_t^{(r)}\right)\right) D_{l_r} K_t^{(m,r)}\right] \\
&\qquad + E\left[h(X_t) g_t(D_{l_r} X_t) F\left(J_t^{(n,r)}\right) f'\left(K_t^{(r)}\right) D_{l_r} K_t^{(m,r)}\right].
\end{aligned}
$$

By (9.12) and dominated convergence, the first term on the right of (9.14) tends to 0 as m tends to ∞. Since $f'(K_t^{(r)}) = 0$ if $K_t^{(r)} > 2$, we then get

$$
\begin{aligned}
\limsup_{m \to \infty} |I_5(n, m, r)| &\le c\|h(X_t)\|_{L^p(P)} t^{-\beta_u}\left(E\left[\left(\left(D_{l_r}^2 X_t/t^{\beta_u}\right)^*\right)^{p'};\, K_t^{(r)} \le 2\right]\right)^{1/p'} \\
&\le c\|h(X_t)\|_{L^p(P)} t^{-\xi} \quad (\text{by } (9.13))
\end{aligned}
$$

Proposition 9.5 follows by substituting into (9.11). □

We are now ready to prove the main result of this section.

Let S_λ be the measure defined on Borel subsets of $\mathbb{R}$ by

$$S_\lambda(A) = E\int_0^\infty e^{-\lambda t} 1_A(X_t)\, dt.$$

Theorem 9.6. *The measure S_λ is absolutely continuous with respect to Lebesgue measure. The density s_λ satisfies*

(a) $\|s_\lambda\| \le c\lambda^{\xi - 1}$, *and*
(b) $|s_\lambda(y) - s_\lambda(x)| \le c|y - x|^\kappa$ *for some constant* $\kappa > 0$.

PROOF. The proof follows closely that of the analogous proposition in Bass (1983). Let

$$T_1 = \inf\{t\colon J_t < \Theta_1 \text{ or } K_t > \Theta_2\} \wedge 1.$$

By Proposition 9.3 with $a = 1$ and $r = 0$, it is easy to see that for some $\gamma < 1$,

$$E\exp(-\lambda T_1) \le \gamma, \tag{9.15}$$

provided λ is sufficiently large.

If we integrate the result of Proposition 9.5, we get

$$\begin{aligned} \left| E\int_0^{T_1} e^{-\lambda t} h'(X_t)\,dt \right| &\le c\int_0^1 e^{-\lambda t} t^{-\xi} \| h(X_t) \|_{L^p(P)}\,dt \\ &\le c\lambda^{\xi-1}\|h\|. \end{aligned} \tag{9.16}$$

Now let $\Theta_1^{(i)}, \Theta_2^{(i)}$ be iid random variables with the same distribution as Θ_1, Θ_2, and independent of X_t. Given T_i, define $X_t^{(i)} = X_{T_i + t}$, define $J_t(i)$, $K_t(i)$ analogously to J_t, K_t, and then define

$$T_{i+1} = \inf\{t > T_i\colon J_t(i) < \Theta_1^{(i)} \text{ or } K_t(i) > \Theta_2^{(i)}\} \wedge (T_i + 1).$$

If $Q_\omega^{(1)}$ is a regular conditional probability distribution (r.c.p.d.) for $E[\cdot | \mathscr{F}_{T_1}, \Theta_1, \Theta_2]$, it is not hard to see that $(X_t^{(1)}, Q_\omega^{(1)})$ satisfies the same hypotheses as (X_t, P), with the same constants (cf. Bass (1984)). Propositions 9.3 and 9.5 then give us the analogs of (9.15) and (9.16) with T_1 replaced by T_2, E replaced by $Q_\omega^{(1)}$, and X_t replaced by $X_t^{(1)}$. We then have

$$\begin{aligned} Ee^{-\lambda T_2} &= E\left[e^{-\lambda T_1} E\left[e^{-\lambda(T_2 - T_1)} | \mathscr{F}_{T_1}, \Theta_1, \Theta_2\right]\right] \\ &= E\left[e^{-\lambda T_1} Q_\omega^{(1)}\left[e^{-\lambda(T_2 - T_1)}\right]\right] \\ &\le \gamma E e^{-\lambda T_1} \\ &\le \gamma^2. \end{aligned} \tag{9.17}$$

Defining $Q_\omega^{(i)}$ by induction, we get, as in (9.17), that $Ee^{-\lambda T_n} \le \gamma^n \to 0$. Hence $T_n \to \infty$. Moreover,

$$\begin{aligned} \left| E\int_0^{T_2} e^{-\lambda t} h'(X_t)\,dt \right| &\le \left| E\int_0^{T_1} e^{-\lambda t} h'(X_t)\,dt \right| \\ &\quad + E\left[e^{-\lambda T_1} \left| Q_\omega^{(1)} \int_0^{T_2 - T_1} e^{-\lambda t} h'(X_t^{(1)})\,dt \right| \right] \\ &\le c\lambda^{\xi-1}\|h + c\lambda^{\xi-1}\|h\|\gamma. \end{aligned}$$

By induction,

$$\left| E\int_0^{T_n} e^{-\lambda t} h'(X_t)\,dt \right| \le c\lambda^{\xi-1}\|h\|(1 + \gamma + \cdots + \gamma^{n-1}) \le c\lambda^{\xi-1}\|h\|/(1-\gamma),$$

and by dominated convergence,

$$\left| E\int_0^{\infty} e^{-\lambda t} h'(X_t)\,dt \right| \le c\lambda^{\xi-1}\|h\|. \tag{9.18}$$

Using a limiting argument, (9.18) holds with $h'(x) = 1_A(x)$, $h(-\infty) = 0$, and $\|h\|$ is the Lebesgue measure of A, where A is a Borel set with compact support. The proof of (a) is now immediate.

Note by (9.16) that, provided q' is small enough so that $q'\xi < 1$ and $q^{-1} + q'^{-1} = 1$, then

$$
\begin{aligned}
\left| E\int_0^{T_1} e^{-\lambda t} h'(X_t)\,dt \right| &\le c\int_0^1 e^{-\lambda t} t^{-\xi} \left(E|h(X_t)|^p \right)^{1/p} dt \\
&\le c\left(E\int_0^1 e^{-\lambda t} t^{-\xi} |h(X_t)|^p\,dt \right)^{1/p} \\
&\qquad \text{(Hölder's inequality with the measure } e^{-\lambda t} t^{-\xi}\,dt) \\
&\le c\left(\int_0^1 t^{-\xi q'}\,dt \right)^{1/pq'} \left(E\int_0^\infty e^{-\lambda t} |h(X_t)|^{pq}\,dt \right)^{1/pq} \\
&\le c\left(\int |h(x)|^{pq} s_\lambda(x)\,dx \right)^{1/pq} \\
&\le c\|h\|_{pq}.
\end{aligned} \tag{9.19}
$$

In exactly the same manner as (9.17) was derived from (9.16), from (9.19) we derive

$$
\left| E\int_0^\infty e^{-\lambda t} h'(X_t)\,dt \right| \le c\|h\|_{pq}. \tag{9.20}
$$

If s'_λ existed and was continuous, we would get

$$
\left| \int s'_\lambda(x) h(x)\,dx \right| = \left| \int s_\lambda(x) h'(x)\,dx \right| \le c\|h\|_{pq},
$$

and hence that $\|s'_\lambda\|_{r'} \le c$, where $r'^{-1} + (pq)^{-1} = 1$. This would imply

$$
|s_\lambda(y) - s_\lambda(x)| \le \int_x^y |s'_\lambda(z)|\,dz \le c|y - x|^{1/pq}. \tag{9.21}
$$

Now s_λ need not be differentiable, but an easy argument approximating s_λ by C^1 functions shows that (9.21) still holds, and (b) follows. □

10. Estimates. In this section we prove Propositions 9.2, 9.3, and 9.4. The assumptions and definitions of Section 9 remain in force.

PROOF OF PROPOSITION 9.2. Fix $r > 0$ and set $\Gamma = [K_{t_0}^{(r)} \le 2]$. Choose γ so that $\beta_l/\beta_u > \gamma > (2\beta_u - 1)/(2\beta_l - \beta_u + \frac{1}{2})$; this is possible by (9.8). Recall $v_r(s, z) = v(s, z)1_{[r,\infty)}(s)$. Let $M = t^{-\gamma}$ and let $m(s, z) = l_r(s, z)1_{[M,\infty)}(|z|)$. Let $w(s, z) = \int_0^z m(s, y)\,dy$.

Our first goal is to show that for $\omega \in \Gamma$, $D_m X_t = D_{l_r} x_t$ and $D_m^2 X_t = D_{l_r}^2 X_t$. Suppose then that $\omega \in \Gamma$ and $t \le t_0$ for t_0 sufficiently small depending only on β_l, β_u, d_1, γ, and H.

Since $D_{l_r}X_t^* \le 2t^{\beta_l}$, $\sup_{s \le t}|\Delta D_{l_r}X_s| \le 4t^{\beta_l}$. By (9.6),

$$|\Delta D_{l_r}X_s| = \left|\int \left[D_{l_r}\sigma(s,z) + \sigma_z(s,z)v_r(s,z)\right]\mu(dz,\{s\})\right|,$$

and by (8.2)(c)(ii),

$$\left|D_{l_r}\sigma(s,z)\right| \le H(z)D_l X_{s-}^* \le 2\|H\|t^{\beta_l};$$

so we have

$$\int \sigma_z(s,z)v_r(s,z)\mu(dz,\{s\}) \le (4 + 2\|H\|)t^{\beta_l}$$

for $s \le t$. On the other hand, for $|z| \le M$, and for $r \le s \le t_0$,

$$\sigma_z(s,z)v_r(s,z) \ge d_1\left(|z|^{-\beta_u} \wedge 1\right) \ge d_1 t_0^{\gamma\beta_u} > (4 + 2\|H\|)t_0^{\beta_l}$$

by the choice of γ and (9.1). Hence, on Γ, $\mu([-M,M] \times [r,t_0]) = 0$, and so $D_{l_r}X_t = D_m X_t$ and $D_{l_r}^2 X_t = D_m X_t$ for $t \le t_0$.

Now let

$$\rho = \inf\left\{t\colon |D_m X_t| > 2t^{\beta_l}\right\}.$$

On Γ, $\rho \ge t_0$. Note that

$$\begin{aligned}
|\Delta D_m X_{t\wedge\rho}| &\le \sup_{|z|\ge M}\left|D_m\sigma(t\wedge\rho,z) + \sigma_z(t\wedge\rho,z)v_r(t\wedge\rho,z)\right| \\
&\le \sup_{|z|\ge cM}|H(z)\|D_m X_{t\wedge\rho-}^*| + cM^{-\beta_l} \qquad (10.1)\\
&\le ct^{\beta_l\gamma}.
\end{aligned}$$

We can write $D_m^2 X_t$ as

$$\begin{aligned}
D_m^2 X_{t\wedge\rho} = {} & \int_0^{t\wedge\rho}\int_{|z|\ge M} D_m^2\sigma(s,z)(\mu(dz,ds) - dz\,ds) \\
& + \int_0^{t\wedge\rho}\int\left[\partial_z D_m\sigma(s,z) + D_m\sigma_z(s,z)\right] \\
& \quad \times w(s,z)(\mu(dz,ds) - dz\,ds) \qquad (10.2)\\
& + \int_0^{t\wedge\rho}\int\left[\partial_z D_m\sigma(s,z) + D_m\sigma_z(s,z)\right]w(s,z)\,dz\,ds \\
& + \int_0^{t\wedge\rho}\int_{|z|\ge M}\sigma_{zz}(s,z)v_r^2(s,z)\mu(dz,ds).
\end{aligned}$$

By (9.3), the last term of (10.2) is bounded in absolute value by

$$\begin{aligned}
& cM^{-1/2+\beta_u-\beta_l}\int_0^{t\wedge\rho}\int_{|z|\ge M}\sigma_z(s,z)v_r(s,z)\mu(dz,ds) \\
& \le cM^{-1/2+\beta_u-\beta_l}|D_m X_{t\wedge\rho}| \\
& \quad + cM^{-1/2+\beta_u-\beta_l}\left|\int_0^t\int_{|z|\ge M} D_m\sigma(s,z)(\mu(dz,ds) - dz\,ds)\right|.
\end{aligned}$$

Then by (2.4) and Hölder, we have

$$
\begin{aligned}
E\big(D_m^2 X_{t\wedge\rho}^*\big)^2 \le{}& cE\int_0^{t\wedge\rho}\int_{|z|\ge M}\big(D_m^2\sigma(s,z)\big)^2\,dz\,ds \\
&+cE\int_0^{t\wedge\rho}\int\big[\partial_z D_m\sigma(s,z)+D_m\sigma_z(s,z)\big]^2 w^2(s,z)\,dz\,ds \\
&+cE\int_0^{t\wedge\rho}\int\big[\partial_z D_m\sigma(s,z)+D_m\sigma_z(s,z)\big]^2 w^2(s,z)\,dz\,ds \qquad (10.3)\\
&+cM^{-1+2(\beta_u-\beta_l)}E\big(D_m X_{t\wedge\rho}\big)^2 \\
&+cM^{-1+2(\beta_u-\beta_l)}E\int_0^t\int_{|z|\ge M}\big(D_m\sigma(s,z)\big)^2\,dz\,ds \\
={}& I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

By (8.2)(c)(iii), if $s\le\rho$,

$$\big|D_m^2\sigma(s,z)\big|\le\big(cD_m^2X_{s\wedge\rho}^*+ct^{2\beta_l}\big)H(z),$$

and so

$$I_1\le cE\int_0^{t\wedge\rho}\big(D_m^2X_{s\wedge\rho}^*\big)^2\,ds+ct^{4\beta_l}.$$

By (8.2)(c)(ii),

$$(1+|z|)\big(|\partial_z D_m\sigma(s,z)|+|D_m\sigma_z(s,z)|\big)\le cH(z)s^{\beta_l}$$

if $s\le\rho$, so

$$I_2\le cM^{-1}\int_0^t\int_{|z|\ge M}H^2(z)s^{2\beta_l}\,dz\,ds\le ct^{2\beta_l+1+\gamma}.$$

The term I_3 is identical to I_2.

We have $|D_mX_{t\wedge\rho}|\le|D_mX_{t\wedge\rho-}|+|\Delta D_mX_{t\wedge\rho}|$, and so by (10.1)

$$I_4\le M^{-1+2(\beta_u-\beta_l)}\big(ct^{2\beta_l}+ct^{2\beta_l\gamma}\big)\le ct^{2\beta_l\gamma+\gamma+2(\beta_l-\beta_u)\gamma}.$$

Finally, by (8.2)(c)(ii) again,

$$
\begin{aligned}
I_5&\le cM^{-1+2(\beta_u-\beta_l)}\int_0^t\int H^2(z)s^{2\beta_l}\,dz\,ds=cM^{-1+2(\beta_u-\beta_l)}t^{2\beta_l+1}\\
&=ct^{2\beta_l+1+\gamma+2(\beta_l-\beta_u)\gamma}.
\end{aligned}
$$

Substituting the bounds for $I_1,\dots,I_5$ in (10.3), and recalling $\beta_l>1>\gamma$, we have

$$E\big(D_m^2X_{t\wedge\rho}^*\big)^2\le c\int_0^tE\big(D_m^2X_{s\wedge\rho}^*\big)^2\,ds+ct^{4\beta_l\gamma-2\beta_u\gamma+\gamma}. \qquad (10.4)$$

By Gronwall, we have

$$(10.5)\qquad E\left(D_m^2 X_{t\wedge\rho}^*\right)^2 \le ct^{4\beta_l\gamma-2\beta_u\gamma+\gamma},$$

and so if $0 < \delta < 1$,

$$E\left(D_m^2 X_{t\wedge\rho}^*\right)^{1+\delta} \le ct^{(2\beta_l+1/2-\beta_u)\gamma(1+\delta)}.$$

The proposition now follows by the definition of γ if we choose δ sufficiently small and ζ sufficiently close to 1. □

PROOF OF PROPOSITION 9.3. Let $s_n = 2^{-n+1}$. Choose γ so that $1 > \gamma > \beta_u - \frac{1}{2}$, and let $T = \inf\{s > 0; A_s/s^\gamma > 1\} \wedge s_N$, where

$$A_t = \int_0^t \int \sigma_z(s, z) v(s, z) \mu(dz, ds),$$

and $N > 0$. Note that A_t is nondecreasing. Then by (2.4),

$$\begin{aligned} E(D_l X_{T\wedge t} - A_{T\wedge t})^{*2} &\le cE\int_0^{T\wedge t}\int \left(D_l\sigma(s,z)\right)^2 dz\, ds \\ &\le cE\int_0^{T\wedge t}(D_l X_s - A_s)^{*2}\, ds + cE\int_0^{T\wedge t} A_s^2\, ds \\ &\qquad\qquad \text{(by (8.2)(c)(ii))} \\ &\le cE\int_0^t (D_l X_{s\wedge T} - A_{s\wedge T})^{*2}\, ds + ct^{2\gamma+1}, \end{aligned}$$

and so by Gronwall,

$$(10.6)\qquad E(D_l X_{T\wedge t} - A_{T\wedge t})^{*2} \le ct^{2\gamma+1}.$$

If $D_l X_s/s^{\beta_u} < 2$, either $|D_l X_s - A_s|/s^{\beta_u} \ge 1$ or $A_s/s^{\beta_u} < 3$. Then

$$\begin{aligned} &P\left[\inf_{s\le s_N}\left(D_l X_s/s^{\beta_u}\right) < 2\right] \\ (10.7)\qquad &\le P\left[\sup_{s\le s_N}\left(|D_l X_s - A_s|/s^{\beta_u}\right) \ge 1, T = s_N\right] \\ &\quad + P[T < s_N] + P\left[\inf_{s\le s_N}\left(A_s/s^{\beta_u}\right) < 3, T = s_N\right] \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Then

$$I_1 \le \sum_{n=N}^{\infty} P\left[\sup_{s_{n+1}\le s\le s_n} |D_l X_s - A_s|/s^{\beta_u} \ge 1, T = s_N\right]$$
$$\le \sum_{n=N}^{\infty} P\left[\left(D_l X_{s_n\wedge T} - A_{s_n\wedge T}\right)^* \ge s_{n+1}^{\beta_u}\right]$$
$$\le \sum_{n=N}^{\infty} cE\left(D_l X_{s_n\wedge T} - A_{s_n\wedge T}\right)^{*2} s_n^{-2\beta_u}$$
$$\le \sum_{n=N}^{\infty} c s_n^{2\gamma+1-2\beta_u} \quad (\text{by } (10.6)),$$

which is summable by the choice of γ. Since

$$EA_t = E\int_0^t \int \sigma_z(s,z)v(s,z)\,dz\,ds \le ct\int\left(|z|^{-\beta_l}\wedge 1\right)dz,$$

then

$$I_2 = P\left[\left(A_{s_N}/s_N^{\gamma}\right)^* > 1\right]$$
$$\le \sum_{n=N}^{\infty} P\left[\sup_{s_{n+1}\le s\le s_n} (A_s/s^{\gamma}) > 1\right]$$
$$\le \sum_{n=N}^{\infty} P\left[A_{s_n}^* > s_{n+1}^{\gamma}\right]$$
$$\le \sum_{n=N}^{\infty} cEA_{s_n}/s_n^{\gamma} \quad (A_t \text{ is increasing})$$
$$\le \sum_{n=N}^{\infty} c s_n^{1-\gamma},$$

which is summable by the choice of γ. Finally, we consider I_3. Let

$$B_t = \int_0^t \int_{|z|\ge 1} |z|^{-\beta_U}\mu(dz,ds),$$

and note that $A_t \ge cB_t$, B_t is a subordinator (nondecreasing Lévy process), and if $f \in C^2$,

$$t^{-1}E[f(B_t) - f(B_0)] = t^{-1}E\sum_{s\le t}[f(B_s) - f(B_{s-})]$$
$$= t^{-1}E\int_0^t\int_{|z|\ge 1}\left[f\left(B_{s-} + |z|^{-\beta_U}\right) - f(B_{s-})\right]\mu(dz,ds)$$

(10.8)
$$= t^{-1}E\int_0^t\int_{|z|\ge 1}\left[f\left(B_{s-} + |z|^{-\beta_U}\right) - f(B_{s-})\right]dz\,ds$$
$$\to \int_{|z|\ge 1}\left[f\left(|z|^{-\beta_U}\right) - f(0)\right]dz \quad (\text{as } t\to 0)$$
$$= c\int_{0<y\le 1}[f(y) - f(0)]\,y^{-1-1/\beta_U}\,dy.$$

Hence the Lévy measure of B_t is of the form $cy^{-1-1/\beta_U}\,dy$, $0 < y \le 1$, and so if $t \le 1$,

$$
\begin{aligned}
P[A_t < b] &\le P\left[\exp(-kB_t) > e^{-ckb}\right] \\
&\le e^{ckb} E \exp(-kB_t) \\
&= e^{ckb}\exp\left(-tc'\int_0^1 (1 - e^{-kx})x^{-1-1/\beta_U}\,dx\right) \\
&\le e^{ckb}e^{c''t}\exp\left(-c't\int_0^\infty (1 - e^{-kx})x^{-1-1/\beta_U}\,dx\right) \\
&\le c''e^{ckb}\exp\left(-c'tk\int_0^\infty e^{-kx}x^{-1/\beta_U}\,dx\right) \\
&= c''\exp(ckb - c'tk^{1/\beta_U}).
\end{aligned}
\tag{10.9}
$$

If we now choose $b = ct^{\beta_u}$ and $k = t^{-\beta_u}$, then for $t \le 1$ we get

$$P\left[A_t < ct^{\beta_u}\right] \le c'\exp(-c''t^{1-\beta_u/\beta_U}). \tag{10.10}$$

Then,

$$
\begin{aligned}
I_3 &\le \sum_{n=N}^{\infty} P\left(\inf_{s_{n+1}\le s\le s_n} A_s < 3s_n^{\beta_u}\right) \\
&\le \sum_{n=N}^{\infty} P\left(A_{s_{n+1}} < 3s_{n+1}^{\beta_u}2^{\beta_u}\right),
\end{aligned}
$$

which is summable by (10.10) and the fact that $\beta_u > \beta_U$.

Substituting in (10.7), we have

$$P\left[J_{s_N} < 2\right] \le c\sum_{n=N}^{\infty} s_n^{\eta}$$

for some $\eta > 0$, and the first part of Proposition 9.3 now follows easily.

To prove the second part of Proposition 9.3 is considerably easier. Fix r, and let $A_t^{(r)} = \int_0^t\int \sigma_z(s,z)v_r(s,z)\mu(dz,ds)$ and $S = \inf\{t\colon A_t^{(r)} > at^{\beta_l}\}$. As in (10.6),

$$E\left(D_{l_r}X_{S\wedge t} - A_{S\wedge t}^{(r)}\right)^{*2} \le ct^{2\beta_l+1}. \tag{10.11}$$

Note $A_t^{(r)} \le cC_t$, where

$$C_t = \int_0^t\int_{z>0} z^{-\beta_L}\mu(dz,ds).$$

The process C_t is a stable subordinator of index $1/\beta_L$ (cf. (10.8)). Since $1/\beta_L > \frac{2}{3} > \frac{1}{2}$, $EC_1^{1/2} < \infty$ and by scaling,

$$EA_t^{(r)1/2} \le cEC_t^{1/2} \le ct^{\beta_L/2}. \tag{10.12}$$

From this, (10.11), and $\beta_l/2 + 1/4 > \beta_L/2$,

$$
\begin{aligned}
E\left(D_{l_r}X_{S\wedge t}\right)^{*1/2} &\le E\left(D_{l_r}X_{S\wedge t} - A_{S\wedge t}\right)^{*1/2} + EA_{S\wedge t}^{1/2} \\
&\le \left(E\left(D_{l_r}X_{S\wedge t} - A_{S\wedge t}\right)^{*2}\right)^{1/4} + ct^{\beta_L/2} \\
&\le ct^{\beta_L/2}.
\end{aligned}
\tag{10.13}
$$

To estimate $P[K_t^{(r)} > a]$, we proceed with

$$P\Big[K_{s_N}^{(r)} > a\Big] \le P[S < s_N] + P\Big[\big(D_{l_r}X_{S\wedge s_N}/s_N^{\beta_l}\big)^* > a\Big] \tag{10.14}$$
$$= \mathbb{I}_1 + \mathbb{I}_2.$$

We have

$$\begin{aligned}
\mathbb{I}_1 &\le \sum_{n=N}^{\infty} P\Bigg[\sup_{s_{n+1}\le s\le s_n}\big(A_s^{(r)}/s^{\beta_l}\big) > a\Bigg]\\
&\le \sum_{n=N}^{\infty} P\Big[A_{s_n}^{(r)} > a s_{n+1}^{\beta_l}\Big]\\
&\le \sum_{n=N}^{\infty} cEA_{s_n}^{(r)1/2}/s_n^{\beta_l/2}\\
&\le \sum_{n=N}^{\infty} s_n^{(\beta_L-\beta_l)/2},
\end{aligned}$$

which is summable, and

$$\begin{aligned}
\mathbb{I}_2 &\le \sum_{n=N}^{\infty} P\Bigg[\sup_{s_{n+1}\le s\le s_n}\big(D_{l_r}X_{S\wedge s}/s^{\beta_l}\big) > a\Bigg]\\
&\le \sum_{n=N}^{\infty} P\Big[D_{l_r}X_{S\wedge s_n}^* > a s_{n+1}^{\beta_l}\Big]\\
&\le \sum_{n=N}^{\infty} cE\big(D_{l_r}X_{S\wedge s_n}\big)^{*1/2}/s_n^{\beta_l/2}\\
&\le \sum_{n=N}^{\infty} c s_n^{(\beta_L-\beta_l)/2}.
\end{aligned}$$

As before, substituting in (10.14) gives

$$P\Big[K_{s_N}^{(r)} > a\Big] \le c\sum_{n=N}^{\infty} s_n^{\eta}$$

for some $\eta > 0$, and the remainder of the proposition follows. □

Proof of Proposition 9.5. We may write

$$\begin{aligned}
D_lX_t - D_{l_r}X_t = &\int_0^t\!\!\int\big(D_l\sigma(s,z) - D_{l_r}\sigma(s,\cdot z)\big)\big(\mu(dz,ds) - dz\,ds\big)\\
&+\int_0^r\!\!\int\sigma_z(s,z)v(s,z)\big(\mu(dz,ds) - dz\,ds\big)\\
&+\int_0^r\!\!\int\sigma_z(s,z)v(s,z)\,dz\,ds,
\end{aligned}$$

and then by (2.4) and (8.2)(c)(iv),

$$E\left(D_l X_t - D_{l_r} X_t\right)^{*2} \le cE\int_0^t \int \left(D_l \sigma(s,z) - D_{l_r}\sigma(s,z)\right)^2 dz\,ds$$
$$+ cE\int_0^r \int \sigma_z^2(s,z) v^2(s,z)\,dz\,ds + cr^2$$
$$\le c\int_0^t E\left(D_l X_s - D_{l_r} X_s\right)^{*2} ds + cr + cr^2.$$

By Gronwall, since both $D_{l_r} X_t^*$ and $D_l X_t^*$ are in $L^p(P)$ for all p, if $r \le 1$,

$$E\left(D_l X_t - D_{l_r} X_t\right)^{*2} \le cr. \tag{10.15}$$

Letting $r \to 0$ gives the first part of the proposition.

For any $\varepsilon > 0$, if $t \le 1$,

$$\begin{aligned} P\left[|K_t - K_t^{(r)}| > \varepsilon\right] &\le P\left[\left(|D_l X_t - D_{l_r} X_t|/t^{\beta_l}\right)^* > \varepsilon\right] \\ &\le \sum_{n=0}^{\infty} P\left[\sup_{s_{n+1} \le s \le s_n} |D_l X_s - D_{l_r} X_s|/s^{\beta_l} > \varepsilon\right] \\ &\le \sum_{n=0}^{\infty} P\left[\left(D_l X_{s_n} - D_{l_r} X_{s_n}\right)^* > \varepsilon s_{n+1}^{\beta_l}\right]. \end{aligned} \tag{10.16}$$

Each term of the series goes to 0 as $r \to 0$ by (10.15). Our result follows by dominated convergence since

$$P\left[\left(D_l X_{s_n} - D_{l_r} X_{s_n}\right)^* > \varepsilon s_{n+1}^{\beta_l}\right] \le P\left[D_l X_{s_n}^* > \varepsilon s_{n+1}^{\beta_l}/2\right] + P\left[D_{l_r} X_{s_n}^* > \varepsilon s_{n+1}^{\beta_l}/2\right]$$
$$\le P\left[K_{s_n} > c\right] + P\left[K_{s_n}^{(r)} > c\right],$$

which is summable by Proposition 9.3. □

11. Construction of local time. Throughout this section, the assumptions of Section 9 still hold.

THEOREM 11.1. *Under the assumptions of Section* 9, X_t *has a jointly continuous local time that is an occupation time density.*

PROOF. Once we have Theorem 9.6, the proof that a local time L_t^x exists that is continuous in t and that is an occupation time density follows by Sections 4 and 5 of Bass (1984). To briefly summarize the construction, we let Q_{t_0} be a r.c.p.d. for $E[\cdot|\mathscr{F}_{t_0}]$. Then (X_{t_0+t}, Q_{t_0}) satisfies the hypotheses of Theorem 9.6, and so there exists $V_{t_0}(\lambda, x)(\omega)$ such that

$$\left|V_{t_0}(\lambda, x)\right| \le c\lambda^{\xi - 1}; \tag{11.1}$$

$$\left|V_{t_0}(\lambda, x) - V_{t_0}(\lambda, y)\right| \le c|y - x|^{\kappa}, \tag{11.2}$$

and for all Borels B,

$$\int_B V_{t_0}(\lambda, x)\,dx = Q_{t_0}\left[\int_0^\infty e^{-\lambda s} 1_B\left(X_{s+t_0}\right) ds\right]. \tag{11.3}$$

We then let $U_t(\lambda, x)$ be a regularized version of $V_t(\lambda, x)$, and show

$$|U_t(\lambda, x)| \le c\lambda^{\xi-1} \tag{11.4}$$

and

$$|U_t(\lambda, x) - U_t(\lambda, y)| \le c|y-x|^{\kappa}. \tag{11.5}$$

Then if $L_t^{\lambda}(x)$ is the increasing predictable part of the bounded supermartingale $e^{-\lambda t}U_t(\lambda, x)$, we set $L_t^x = \int_0^t e^{\lambda s}\, dL_s^{\lambda}(x)$. In the above reference it is shown that L_t^x is an occupation time density for X_t, and in view of (11.4), that L_t^x can be taken so as to be continuous in t. It remains to show that L_t^x can be chosen to be jointly continuous in x and t, and to do this, it suffices to consider $L_t^{\lambda}(x)$, since

$$L_t^x = e^{\lambda t}L_t^{\lambda}(x) - \int_0^t \lambda e^{\lambda s}L_s^{\lambda}(x)\, ds.$$

Fix x and y so that neither is in the null set of Lebesgue measure 0 that arises in the construction of $L_t^{\lambda}(\cdot)$. Suppose S and T are bounded stopping times. For ease of notation, let $R_t = L_t^{\lambda}(y) - L_t^{\lambda}(x)$ and

$$W_t = e^{-\lambda t}U_t(\lambda, x) - e^{-\lambda t}U_t(\lambda, y).$$

By the definition of $L_t^{\lambda}(x)$, R_t is predictable, and $W_t - R_t$ is a martingale. Then

$$\begin{aligned}
E\left[(R_T - R_S)^2|\mathscr{F}_S\right] &= E\left[2\int_S^T E[R_T - R_r|\mathscr{F}_r]\, dR_r|\mathscr{F}_S\right]\\
&= E\left[2\int_S^T E[W_T - W_r|\mathscr{F}_r]\, dR_r|\mathscr{F}_S\right]\\
&\le c|y-x|^{\kappa}E\int_S^T d\big(L_t^{\lambda}(y) + L_t^{\lambda}(x)\big) \quad (\text{by}(11.5))\\
&\le c|y-x|^{\kappa}\big(EL_T^{\lambda}(y) + EL_T^{\lambda}(x)\big)\\
&\le c|y-x|^{\kappa}.
\end{aligned}$$

The last inequality follows since the potential of $L_t^{\lambda}(y)$ is $e^{-\lambda t}U_t(\lambda, y)$, which is bounded by (11.4). Then,

$$E[|R_T - R_S|\,|\mathscr{F}_S] \le \left(E\left[(R_T - R_S)^2|\mathscr{F}_S\right]\right)^{1/2} \le c|y-x|^{\kappa/2},$$

and since R_t is continuous,

$$E\exp\left(\big(L_t^{\lambda}(y) - L_t^{\lambda}(x)\big)^*/8c|y-x|^{\kappa/2}\right) \le 2 \tag{11.6}$$

by Dellacherie and Meyer (1980, page 193).

Since $L_t^{\lambda}(x)$ can be taken to be continuous in t for each x, a.s., we can argue exactly as in Getoor and Kesten (1972, pages 285–286) that (11.6) implies $L_t^{\lambda}(x)$ can be taken to be jointly continuous in t and x. □

12. Localization and extensions. In this section we give the localization procedure that completes the proof of Theorem 8.1. Following the proof, we make some remarks on how Theorem 8.1 could be extended.

PROPOSITION 12.1. *Suppose σ satisfies* (8.2) *and that for some positive constants d_1, d_2, z_0, and $1 < \alpha_0 < \alpha_1 < 2$ satisfying* (9.2),

$$d_1\big(|z|^{-1/\alpha_1 - 2} \wedge 1\big) \le |\sigma_{zz}(s, z)| \le d_2\big(|z|^{-1/\alpha_0 - 2} \wedge 1\big)$$

for $|z|$ larger than z_0. Then X_t has a local time that is jointly continuous in t and that is an occupation time density.

REMARK. This is a weakening of the hypotheses of Sections 9, 10, and 11, since now we only require (9.1) to hold for $|z|$ larger than some z_0. Our first goal is to show that under these weaker hypotheses, X_t still has a jointly continuous local time that is an occupation time density.

PROOF. Let $S_0 = 0$, and

$$S_{i+1} = \inf\{t > S_i\colon \mu([-z_0, z_0] \times [0, t]) > \mu([-z_0, z_0] \times [0, S_i])\}.$$

Thus the S_i's are the times at which $\mu([-z_0, z_0] \times \{s\}) = 1$. Let $\hat{\sigma}(s, z)$ be a C^3 function such that $\hat{\sigma}_{zz}(s, z) = \sigma_{zz}(s, z)$ if $|z| \ge z_0$. Let $\hat{X}_t$ be the solution of (8.1) with σ replaced by $\hat{\sigma}$. Now $\hat{\sigma}$ satisfies the conditions of Sections 9, 10, and 11, so $\hat{X}_t$ has a jointly continuous local time. Since $\hat{X}_t = X_t$ for $t < S_1$, then X_t, $t < S_1$, has a local time, and hence X_t, $t \le S_1$ does also.

Let Q_ω be a r.c.p.d. for $E[\cdot|\mathscr{F}_{S_1}]$, and consider $(X_{t+S_1} - X_{S_1}, Q_\omega)$. This process satisfies the same hypotheses as (X_t, P) does, and so arguing as in the preceding paragraph, $X_{t+S_1} - X_{S_1}$, $t \le S_2 - S_1$, has a jointly continuous occupation time density a.s. (Q_ω) for almost every $\omega(P)$. An easy argument using Fubini shows that $X_{t+S_1} - X_{S_1}$, $t \le S_2 - S_1$, has a jointly continuous occupation time density a.s. (P). We then look at $X_{t+S_2} - X_{S_2}$, etc. By Bass (1984), Section 6, we conclude that X_t has a jointly continuous occupation time density. □

PROOF OF THEOREM 8.1. Since $1 < \alpha_- < \alpha^+ < 2$, we can take ε sufficiently small so that whenever $\alpha_- \le \alpha_0 < \alpha_1 < \alpha_0 + \varepsilon \le \alpha^+$, (9.2) holds. Let $T_0, T_1, T_2, \dots$ be a sequence of stopping times so that $\sigma(s, z)$ is ε-stable of index between α_- and α^+ if $T_i < s < T_{i+1}$, for each $i = 0, 1, 2, \dots$. Then there exists d_1, d_2, α_1, α_0, and z_0 so that

$$d_1\big(|z|^{-1/\alpha_1 - 2} \wedge 1\big) \le |\sigma_{zz}(s, z)| \le d_2\big(|z|^{-1/\alpha_0 - 2} \wedge 1\big)$$

if $0 < s < T_1$, $|z| \ge z_0$. For each z, let $\psi(z, r)$ be an odd C^∞ function satisfying

(12.1)(i) $\quad \psi(z, r) = r \quad$ if $|z| < z_0$;

(12.1)(ii) $\quad \psi(z, r) = r \quad$ if $|z| \ge z_0$, and

$$d_1\big(|z|^{-1/\alpha_1 - 2} \wedge 1\big) < |r| < d_2\big(|z|^{-1/\alpha_0 - 2} \wedge 1\big); \quad \text{and}$$

(12.1)(iii) $\quad \tfrac{1}{2} d_1\big(|z|^{-1/\alpha_1 - 2} \wedge 1\big) \le |\psi(z, r)| \le 2 d_2\big(|z|^{-1/\alpha_0 - 2} \wedge 1\big) \quad$ if $|z| \ge z_0$.

Let Y_t be the solution to

$$Y_t = x_0 + \int_0^t \int \tilde{\sigma}(Y, s, z)(\mu(dz, ds) - dz\, ds),$$

where $\tilde{\sigma}$ is chosen so that $\tilde{\sigma}_{zz}(s, z) = \psi(z, \sigma_{zz}(s, z))$, $\tilde{\sigma}_z(\pm\infty) = 0$, and $\tilde{\sigma}(\pm\infty) = 0$. Then $\tilde{\sigma}$ satisfies the conditions of Proposition 12.1, and so Y_t has a jointly continuous occupation time density. Since changing $\sigma(Y, 0, z)$ does not affect the value of Y_t, it follows that Y_t is equal to X_t if $t < T_1$. Therefore, X_t, $t < T_1$, hence X_t, $t \le T_1$, has a jointly continuous local time.

Just as in the proof of Proposition 12.1, we argue that $X_{t+T_i} - X_{T_i}$, $t \le T_{i+1} - T_i$, has a local time, and we then use Bass (1984), Section 6 again to conclude that X_t has a local time that is jointly continuous in t and x that is an occupation time density. □

REMARK 1. We can dispense with the near symmetry condition on σ. Suppose that (locally) there exist $1 < \alpha_- < \alpha_0 < \alpha_1 < \alpha_0 + \varepsilon < \alpha^+ < 2$ and $0 < \alpha_2 < \alpha_3 < (\alpha_2 + \varepsilon) \wedge \alpha_1$ for ε sufficiently small so that

$$c_1\left(|z|^{-1/\alpha_0 - 2} \wedge 1\right) \ge |\sigma_{zz}(s, z)| \ge c_2\left(|z|^{-1/\alpha_1 - 2} \wedge 1\right) \quad \text{if } z \ge z_0$$

and

$$c_3\left(|z|^{-1/\alpha_2 - 2} \wedge 1\right) \ge |\sigma_{zz}(s, z)| \ge c_4\left(|z|^{-1/\alpha_3 - 2} \wedge 1\right) \quad \text{if } z \le -z_0.$$

If σ also satisfies (8.2), then the conclusions of Theorem 8.1 still hold. To show this, let

$$N_t^{(r)} = \left(\int_0^t \int_{z<0} \sigma(s, z) l_r(s, z) \mu(dz, ds) / t^{\beta_n}\right)^*,$$

where we choose β_n to be less than but close to $\frac{1}{2} + \alpha_2^{-1}$. By considering $\mu([0, M] \times [0, t])$ and $\mu([-M, 0] \times [0, t])$ separately, we can show, as in Proposition 9.2, that $\mu([-M, M] \times [0, t]) = 0$ on the set $[K_t^{(r)} \le 2, N_t^{(r)} \le 2]$. Once we have that, we can modify the remainder of the proof of Proposition 9.2 to get

$$E\left[\left(D_{l_r}^2 X_t^*\right)^{1+\delta}; K_t^{(r)} \le 2, N_t^{(r)} \le 2\right] \le ct^{2\beta_u - \zeta}.$$

The proof of Proposition 9.3 needs no change. We need to show $P[N_t^{(r)} > \alpha] \to 0$ uniformly in r as $t \to 0$ and $N_t^{(r)} \to N_t^{(0)}$ in probability as $r \to 0$; the proofs are analogous to those of Propositions 9.3 and 9.4. Finally, if $N_t^{(k, r)}$ are smooth approximations to $N_t^{(r)}$, we define

$$G_{n, m, k, r}(\mu) = h(X_t) g_t(D_{l_r} X_t) F\left(J_t^{(n, r)}\right)\left(1 - F\left(K_t^{(m-r)}\right)\right)\left(1 - F\left(N_t^{(k, r)}\right)\right)$$

in place of (9.10) and conclude as in Proposition 9.5 that

$$\left|E\left[h'(X_t) 1_{(J_t > \Theta_1, K_t < \Theta_2, N_t < \Theta_3)}\right]\right| \le ct^{-\xi} \|h(X_t)\|_{L^p(P)},$$

where Θ_3 is identically distributed to Θ_1 and Θ_2 and independent of them and $\sigma(X_s; 0 \le s < \infty)$. With the obvious modification to the definition of T_i in Theorem 9.6, the proofs then proceed virtually identically.

REMARK 2. The only place we used the existence of $D_l^3\sigma$ was to get the existence of $D_l^2 X$. If we take approximations σ_n to σ, use (9.19) to get

$$\left|E\int_0^\infty e^{-\lambda t} h'\left(X_t^{(n)}\right) dt\right| \le c\|h\|_{pq},$$

and then let $n \to \infty$, we see that we only need $D_l^2\sigma$ to exist and satisfy (8.2)(c)(iii). Thus, in the example of Section 8, $\alpha(x) \in C^2$ suffices. We conjecture that for this particular example, $\alpha(x) \in C^{1+\delta}$ for some $\delta > 0$ suffices.

REMARK 3. If one wants to consider semimartingales X_t that may have a drift or unbounded jumps, one can proceed as in Bass (1984) Section 6 to extend Theorem 8.1 to a theorem covering these cases.

REMARK 4. Note that in the proof of Theorem 8.1 we do not actually need σ to be locally ε-stable for all ε, but only for $\varepsilon \geq \varepsilon_0$, where ε_0 depends on α_- and α^+.

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