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HUNT'S HYPOTHESIS (H) AND GETOOR'S CONJECTURE

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A large class of Markov processes satisfying Hunt's hypothesis (H) is displayed. In particular, if Φ is a Lévy–Khinchin exponent, then the Lévy process with exponent Φ^α $(0 < \alpha < 1)$ satisfies (H). That is, every semipolar set is polar.

Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a standard Markov process on an LCCB state space E with Borel field $\mathscr{E}$ ([2], Chapter 1). A great deal of effort has been expended in finding conditions which guarantee that X satisfies Hunt's hypothesis (H):

(H) Every set $A \in \mathscr{E}$ which is semipolar for X is polar for X.

Before we recall what conditions are known, we remind the reader of the importance of (H). In this paragraph, we assume that X is in duality with another standard process $\hat{X}$ on E with respect to a sigma finite excessive measure m ([2], Chapter 6). Then (H) holds for X if and only if every natural additive functional of X is in fact a continuous additive functional. There are also several analytic conditions equivalent to (H). Assume all α-excessive functions are lower semicontinuous, and let $u(x, y)$ be the potential density for X and $\hat{X}$ chosen as in ([2], Chapter 6-1). Then (H) is equivalent to the bounded maximum principle (M*) [1]:

(M*) Let μ be a finite measure on E with compact support K, and set $U\mu(x) = \int u(x, y)\mu(dy)$. If $U\mu$ is bounded, then $\sup\{U\mu(x): x \in E\} = \sup\{U\mu(x): x \in K\}$.

If $U^\alpha f$ is continuous whenever $\alpha > 0$ and f is bounded, then (M*) is also equivalent to the bounded continuity principle (I*) [1]:

(I*) Let μ be a finite measure with compact support K. If $U\mu$ is bounded and its restriction to K is continuous, then $U\mu$ is continuous.

These conditions should give the reader some idea of the importance of hypothesis (H). In spite of its importance, (H) has been verified in few situations. We briefly summarize what is known. (We continue to assume X and $\hat{X}$ are in duality.)

If $u(x, y) = u(y, x)$ (i.e., $X = \hat{X}$), then (H) holds. More generally, if the fine and cofine topologies agree, then (H) holds [1].

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If X is a Lévy process with Lévy exponent Φ, the Kanda–Forst [5], [4] condition states that X satisfies (H) if $1 + \mathrm{Re}(\Phi) \geq M \cdot |\mathrm{Im}(\Phi)|$ for some constant $M > 0$. This is still quite far from proving Getoor's conjecture which states that essentially all Lévy processes satisfy (H) (except in certain obvious cases where a translation component interferes). For a short proof of the Kanda–Forst theorem, see Rao [6]. This condition was extended to the nonsymmetric Dirichlet space setting by Silverstein [7].

These are all of the general conditions we know which imply that various classes of Markov processes satisfy (H). Of course, various individual processes have been proved to satisfy (H).

We now give a large class of Markov processes which satisfy (H). A subordinator T_t is a right continuous process with stationary independent increments, $T_0 = 0$, and increasing paths. It is well-known that if X and T are independent, then the "subordinated process" $X(T_t)$ is again a standard process [2], [3]. No duality or absolute continuity conditions are assumed in the following theorem.

THEOREM 1. *Let* $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ *be a standard process on an LCCB state space* E. *Let* $T = (W, \mathscr{G}, \mathscr{G}_t, T_t, \theta_t, Q^0)$ *be an independent subordinator satisfying Hunt's hypothesis* (H). *Then* $X(T_t)$ *satisfies* (H).

PROOF. Let $A \in \mathscr{E}$, and set

$$G(\omega, w) = \left\{t\colon X_{T_t(w)}(\omega) \in A\right\} = \{t\colon T_t(w) \in \{s\colon X_s(\omega) \in A\}\}.$$

By Fubini's theorem,

$$\begin{aligned} &P^x \times Q^0[G \text{ is either null or uncountable}] \\ &\quad = \int_\Omega \int_W 1_{\{G(\omega, w) \text{ is either null or uncountable}\}} Q^0(dw) P^x(d\omega). \end{aligned}$$

For every $\omega \in \Omega$, $J(\omega) = \{s\colon X_s(\omega) \in A\} \in \mathscr{B}(R)$. Since T_t satisfies (H),

$$\begin{aligned} &Q^0[G(\omega, \cdot) \text{ is either null or uncountable}] \\ &\quad = Q^0[\{t\colon T_t \in J(\omega)\} \text{ is either null or uncountable}] = 1. \end{aligned}$$

It follows that $\{t\colon X(T_t) \in A\}$ is either null or uncountable a.s. $P^x \times Q^0$. Assume A is semipolar for $X(T_t)$. Then $X(T_t)$ can visit A at most countably often. But in this case, $X(T_t)$ must not visit A at all by the discussion above, so A is polar for $X(T_t)$. Therefore, $X(T_t)$ satisfies (H). □

Blumenthal and Getoor [1] verify that the stable subordinator of index $0 < \alpha < 1$ verifies (H). These subordinators have potential densities

$$\begin{aligned} u(x, y) &= \Gamma(\alpha)^{-1}(y - x)^{\alpha - 1} && \text{if } x < y \\ &= 0 && \text{if } x \geq y. \end{aligned}$$

Their proof also works for other subordinators which do not hit points and for which $y \to u(x, y)$ is decreasing on (x, ∞). As far as we know, which subordinators satisfy (H) is unknown in general. Clearly, $T_t = t$ is a subordinator which does not satisfy (H).

Theorem 1 brings us a step closer to verifying Getoor's conjecture.

COROLLARY 2. *If Φ is a Lévy–Khinchin exponent, then the Lévy process X^α with exponent Φ^α satisfies* (H) *for every* $0 < \alpha < 1$.

PROOF. Let X be the Lévy process corresponding to Φ, and let T_t be an independent stable subordinator of index α. Then $X^\alpha = X(T_t)$ has Lévy exponent Φ^α. □

This is a large class of Lévy processes satisfying (H), but unfortunately not every Lévy process can be obtained this way. For example, if X is a Lévy process on R^2 with Lévy measure concentrated on the x and y axes, then X cannot be obtained as a nontrivial subordinate of another process. It would be interesting to characterize the class of Lévy processes which can be obtained from subordinators satisfying (H).

COROLLARY 3. *If X is a Lévy process with potential V and $V^2 = U$ is the potential of a transient Lévy process Y, then X satisfies* (H).

PROOF. If T_t is the stable subordinator of index $\frac{1}{2}$, then $Z_t = Y(T_t)$ has potential W with $W^2 = U$. By uniqueness of square roots, $W = V$ so Z and X are identical in law. Therefore, X satisfies (H). □

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