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CHARACTERISATIONS OF SET-INDEXED BROWNIAN MOTION AND ASSOCIATED CONDITIONS FOR FINITE-DIMENSIONAL CONVERGENCE

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Two characterisations are given of the finite-dimensional laws of Brownian motion indexed by an arbitrary class of subsets of the d-dimensional unit cube. There are associated conditions for convergence of finite-dimensional laws of a sequence of set-indexed additive processes. These conditions have a more explicit form in the case of set-indexed partial-sum processes based on mixing random variables.

1. Introduction. Let $\mathscr{A}$ be any subset of the class $\mathscr{B}$ of Borel subsets of $[0,1]^d$. A *process* X on $\mathscr{A}$ is a collection $\{X(A)\}_{A\in\mathscr{A}}$ of real-valued random variables. It is *additive* if whenever $A, B, A\cup B, A\cap B \in \mathscr{A}$,

$$X(A\cup B) = X(A) + X(B) - X(A\cap B) \quad \text{a.s.} \tag{1.0.1}$$

A *standard Wiener process*, or *Brownian motion*, on $\mathscr{A}$ is a Gaussian process W with the properties $EW(A) = 0$, $\operatorname{cov}(W(A), W(B)) = |A\cap B|$ for $A, B\in\mathscr{A}$. It is necessarily additive, in the present limited sense.

In this paper we characterise W and weak convergence to it of additive processes in the Cartesian product space $\mathbb{R}^{\mathscr{A}}$ equipped with the σ-algebra $\mathscr{B}^{\mathscr{A}}$ generated by finite-dimensional cylinders. That is, we study convergence of finite-dimensional laws. In a companion paper [7] we restrict $\mathscr{A}$, requiring that it satisfy a metric entropy condition, in order that our processes and their Wiener limit can live in a space of continuous functions. For convergence of finite-dimensional laws, however, the elements of $\mathscr{A}$ and their finite intersections need to be approximable by unions of uniformly small similar subsets of $[0,1]^d$, e.g., $1/n$-cubes. The fact that finite intersections of elements of $\mathscr{A}$ have to be approximable means that for finite-dimensional calculations our space cannot be assumed to satisfy a metric entropy bound. In this paper, therefore, we shall work mostly on a large (in metric entropy terms) space $\mathscr{R}$, to be defined below, consisting of all finite unions of left-open right-closed intervals.

Section 2 contains our core results: characterisations of W on $\mathscr{R}$, and weak convergence to it. We write conditions which are as weak as we can manage; in corollaries some more concise sufficient conditions are developed. We have two methods of proof, leading to stabilising conditions on variances expressed, respectively, in terms of fixed intervals, and asymptotically negligible intervals, in $[0,1]^d$. In Section 3 we deduce characterisation and convergence results relative

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to an arbitrary class $\mathscr{A}$ of Borel subsets of $[0,1]^d$, from uniform continuity-in-probability conditions.

In Section 4 we apply the preceding results to the case of set-indexed partial-sum processes based on lattice-indexed summands subject to mixing conditions. We obtain new multidimensional central limit theorems for mixing random fields, and need only *second-moment* assumptions on the summands, a *logarithmic* mixing rate, and convergence of second moments of certain special partial sums. How close are these conditions to optimal? In the classical case where the index set is the natural numbers ($\{X_n\}_{n\in\mathbb{N}}$ strictly stationary and ϕ-mixing, $EX_n \equiv 0$, $S_n := X_1 + \cdots + X_n$, $\sigma_n^2 := ES_n^2 < \infty$) it is known that a logarithmic mixing rate suffices for S_n/σ_n to satisfy the central limit theorem and invariance principle (Ibragimov [12]; Peligrad [15, 16]) and that ϕ-mixing alone is not enough (Herrndorf [10], Example 4.1), but that no mixing rate is needed if one knows $\sigma_n^2 \to \infty$ and $\liminf \sigma_n^2/n > 0$ (Peligrad [17]) or if one suitably strengthens other conditions (cf. [12], [10]). State-of-the-art surveys of this case will appear in [5]. Only in this classical case are there theorems and counterexamples, not too far apart, by which one can judge closeness to optimality in the particular dependence setting in use. Thus for the set-indexed case one has to extrapolate, and we conjecture that our assumption of a mixing *rate* can be dispensed with if the variances of *all* the set-indexed partial sums are assumed to converge appropriately, but that without this or some other strengthening of the conditions of our theorems, some mixing rate is needed.

Our results in Section 4 should be compared with other central limit theorems for mixing random fields. Recent work (Bolthausen [2]; Bulinskii and Zurbenko [3]; Gorodetskii [8]; Guyon and Richardson [9]; Malysev [14]; Takahata [19, 20]) has allowed sums over arbitrary sets rather than just rectangular blocks. All these authors assume *polynomial* mixing rate, though often with a less restrictive form of mixing.

In [7] our results are combined with a tightness lemma to give weak convergence of continuous-path processes.

To proceed formally, for any $\mathbf{a} \equiv (a_1,\dots,a_d)$, $\mathbf{b} \equiv (b_1,\dots,b_d)$ in $[0,1]^d$ the half-open interval $(\mathbf{a},\mathbf{b}]$ is $\{(x_1,\dots,x_d)\colon a_i < x_i \le b_i,\ i = 1,\dots,d\}$, and the class of all such sets is $\mathscr{J}$. Let $\mathscr{R}$ denote the ring of all finite unions of elements of $\mathscr{J}$. For $m = 1,2,\dots$ let J_m be the set of all points $(j_1/m,\dots,j_d/m)$ where $j_i \in \{1,2,\dots,m\}$. Let $\mathbf{1} := (1,\dots,1)$ and

$$C_{m,\mathbf{j}} := (\mathbf{j} - m^{-1}\mathbf{1},\mathbf{j}], \qquad \mathbf{j} \in J_m,\ m = 1,2,\dots .$$

For sets $E, F \subseteq \mathbb{R}^d$ the *separation distance* is

$$\rho(E,F) := \inf_{\mathbf{x}\in E,\,\mathbf{y}\in F} \|\mathbf{x} - \mathbf{y}\|,$$

where $\|\ \|$ is sup norm in $\mathbb{R}^d$ and $\inf \varnothing = \infty$. For $E \subseteq \mathbb{R}^d$ and $\varepsilon > 0$ let

$$E^\varepsilon := \{\mathbf{x} \in \mathbb{R}^d\colon \|\mathbf{x} - \mathbf{y}\| \le \varepsilon \text{ for some } \mathbf{y} \in E\}. \tag{1.0.2}$$

Lebesgue measure is denoted $|\ |$ and on $\mathscr{B}$ we use the Lebesgue-disjunction pseudometric $d_L(A,B) := |A \vartriangle B|$.

2. Characterisation and convergence on unions of intervals.

2.1 THEOREM. *Let W be an additive process on $\mathscr{R}$ that satisfies*

(i) $EW(C) = 0 \ \forall \ C \in \mathscr{J}$;

(ii) $EW^2(C) = |C| \ \forall \ C \in \mathscr{J}$;

(iii) $W(C_1), \dots, W(C_k)$ *are independent whenever* $C_1, \dots, C_k \in \mathscr{J}$ *and* $\rho(C_i, C_j) > 0$ *for* $i \neq j$;

(iv) $\forall \ \varepsilon > 0$, $\lim_{m \to \infty} \sum_{\mathbf{j} \in J_m} P(|W(C_{m,\mathbf{j}})| \geq \varepsilon) = 0$.

Then W is a standard Wiener process on $\mathscr{R}$.

2.2 THEOREM. *Let $\{Z_n\}$ be a sequence of additive processes on $\mathscr{R}$ such that*

(i) $EZ_n(C) \to 0 \ (n \to \infty) \ \forall \ C \in \mathscr{J}$;

(ii) $EZ_n^2(C) \to |C| \ (n \to \infty) \ \forall \ C \in \mathscr{J}$;

(iii) *whenever* $C_1, \dots, C_k \in \mathscr{J}$ *are such that* $\rho(C_i, C_j) > 0$ *for* $i \neq j$ *we have for all real* $z_1, \dots, z_k$ *that*

$$P(Z_n(C_1) \leq z_1, \dots, Z_n(C_k) \leq z_k) - \prod_{i=1}^{k} P(Z_n(C_i) \leq z_i) \to 0, \qquad n \to \infty; \tag{2.2.1}$$

(iv) $\forall \ \varepsilon > 0$, $\lim_{m \to \infty} \limsup_{n \to \infty} \sum_{\mathbf{j} \in J_m} P(|Z_n(C_{m,\mathbf{j}})| \geq \varepsilon) = 0$;

(v) *for each* $C \in \mathscr{J}$ *the set* $\{Z_n^2(C)\}_{n \geq 1}$ *is uniformly integrable.*

Then Z_n on $\mathscr{R}$ converges weakly to a standard Wiener process.

Our second route to characterisation and convergence was suggested by the differential equations method of Rosén [18] and Billingsley [1, Section 19]. Here our conditions on the moments of W and Z_n are entirely local in character.

By a *null family* in $\mathscr{J}$ we mean a set $\{D_h\}_{0 < h \leq h_0}$ of elements of $\mathscr{J}$ such that $D_h \subseteq D_{h'}$ for $h \leq h'$ and $|D_h| \equiv h$.

2.3 THEOREM. *Let W be an additive process on $\mathscr{R}$ that satisfies*

(i) *for any* $C_1, \dots, C_k \in \mathscr{J}$, *any real* $u_1, \dots, u_k$, *and any null family* $\{D_h\}_{0 < h \leq h_0}$ *in* $\mathscr{J}$ *such that* $\rho(D_{h_0}, \cup_1^k C_j) > 0$,

$$\lim_{h \downarrow 0} E\left(\exp\left(i \sum_{j=1}^{k} u_j W(C_j)\right) \frac{W(D_h)}{|D_h|}\right) = 0, \tag{2.3.1}$$

$$\lim_{h \downarrow 0} E\left[\exp\left(i \sum_{j=1}^{k} u_j W(C_j)\right)\left(\frac{W^2(D_h)}{|D_h|} - 1\right)\right] = 0; \tag{2.3.2}$$

(ii) *for any null family* $\{D_h\}_{0 < h \leq h_0}$ *in* $\mathscr{J}$,

$$\lim_{\alpha \to \infty} \limsup_{h \downarrow 0} E\left(\frac{W^2(D_h)}{|D_h|} 1\left\{\frac{W^2(D_h)}{|D_h|} \geq \alpha\right\}\right) = 0.$$

Then W is a standard Wiener process on $\mathscr{R}$.

2.4 REMARK. Conditions (i) and (ii) are equivalent to the apparently stronger conditions in which $|D_h| \equiv h$ is replaced by the requirement that $|D_h|$ strictly decrease to 0 as $h \downarrow 0$. For then we can simply re-index the set $\{D_h\}$ to make $|D_h| \equiv h$. The same remark applies to conditions 2.6(i) and (ii) below.

2.5 REMARK. While (ii) is not equivalent to uniform integrability of the set $\{W^2(D_h)/|D_h|\}_{0<h\le h_0}$, it is not hard to see it is equivalent to the following sequential version:

(ii′) for any sequence $D_1, D_2, \dots$ in $\mathscr{I}\setminus\{\varnothing\}$ such that $D_l \downarrow$ and $|D_l| \downarrow 0$ as $l \to \infty$, the set $\{W^2(D_l)/|D_l|\}_{l\ge 1}$ is uniformly integrable.

2.6 THEOREM. *Let $\{Z_n\}$ be a sequence of additive processes on $\mathscr{R}$ such that*

(i) *for any $C_1, \dots, C_k \in \mathscr{I}$, any real $u_1, \dots, u_k$, and any null family $\{D_h\}_{0<h<h_0}$ in $\mathscr{I}$ such that $\rho(D_{h_0}, \cup_1^k C_j) > 0$,*

$$\lim_{h\downarrow 0}\limsup_{n\to\infty}\left|E\left(\exp\left(i\sum_{j=1}^k u_j Z_n(C_j)\right)\frac{Z_n(D_h)}{|D_h|}\right)\right| = 0, \tag{2.6.1}$$

$$\lim_{h\downarrow 0}\limsup_{n\to\infty}\left|E\left\{\exp\left(i\sum_{j=1}^k u_j Z_n(C_j)\right)\left(\frac{Z_n^2(D_h)}{|D_h|} - 1\right)\right\}\right| = 0; \tag{2.6.2}$$

(ii) *for any null family $\{D_h\}_{0<h\le h_0}$ in $\mathscr{I}$,*

$$\lim_{\alpha\to\infty}\limsup_{h\downarrow 0}\limsup_{n\to\infty} E\left[\frac{Z_n^2(D_h)}{|D_h|}1\left\{\frac{Z_n^2(D_h)}{|D_h|} \ge \alpha\right\}\right] = 0;$$

(iii) *for each $C \in \mathscr{I}$ the set $\{Z_n(C)\}_{n\ge 1}$ is uniformly integrable.*

Then Z_n on $\mathscr{R}$ converges weakly to a standard Wiener process.

2.7 COROLLARY. *For 2.6(i) it suffices that for any $C_1, \dots, C_k \in \mathscr{I}$ and any $D \in \mathscr{I}$ such that $\rho(D, \cup_1^k C_j) > 0$,*

$$\lim_{n\to\infty} E\left(\exp\left(i\sum_{j=1}^k u_j Z_n(C_j)\right)Z_n(D)\right) = 0,$$

$$\lim_{n\to\infty} E\left(\exp\left(i\sum_{j=1}^k u_j Z_n(C_j)\right)(Z_n^2(D) - |D|)\right) = 0.$$

2.8 COROLLARY. *For 2.6(i) it suffices that for any $C_1, \dots, C_k \in \mathscr{I}$ and any null family $\{D_h\}_{0<h\le h_0}$ in $\mathscr{I}$ such that $\rho(D_{h_0}, \cup_1^k C_j) > 0$,*

$$\lim_{h\downarrow 0}\limsup_{n\to\infty} E\left|\frac{E(Z_n(D_h)|Z_n(C_1), \dots, Z_n(C_k))}{|D_h|}\right| = 0, \tag{2.8.1}$$

$$\lim_{h\downarrow 0}\limsup_{n\to\infty} E\left|\frac{E(Z_n^2(D_h)|Z_n(C_1), \dots, Z_n(C_k))}{|D_h|} - 1\right| = 0. \tag{2.8.2}$$

2.9 COROLLARY. *For* 2.6(i) *it suffices that for any* $C_1, \dots, C_k \in \mathscr{J}$ *and any* $D \in \mathscr{J}$ *such that* $\rho(D, \cup_1^k C_j) > 0$,

$$E(Z_n(D)|Z_n(C_1), \dots, Z_n(C_k)) \to_{L_1} 0, \qquad n \to \infty,$$
$$E(Z_n^2(D)|Z_n(C_1), \dots, Z_n(C_k)) \to_{L_1} |D|, \quad n \to \infty.$$

2.10 REMARK. Corollaries to the characterisation Theorem 2.5 are obtained if in the above corollaries we replace Z_n by W and remove the limiting operation in n.

2.11 REMARK. A sufficient condition for 2.6(ii) and (iii) is that the set $\{Z_n^2(C)/|C|\}_{n \geq 1, C \in \mathscr{J} \setminus \varnothing}$ be uniformly integrable.

3. Convergence on an arbitrary class of sets.

3.1 THEOREM. *Let* $\{Z_n\}$ *be a sequence of processes on* $\mathscr{R} \cup \mathscr{A}$ *such that*

(i) Z_n *on* $\mathscr{R}$ *converges weakly to a standard Wiener process on* $\mathscr{R}$;

(ii) *for each* $A \in \mathscr{A}$ *there exists a nonincreasing sequence* $C_l \in \mathscr{R}$ *such that* $|(\cap_1^\infty C_l) \setminus A| = 0$ *and, for every* $\varepsilon > 0$,

$$\lim_{l \to \infty} \limsup_{n \to \infty} P(|Z_n(A) - Z_n(C_l)| \geq \varepsilon) = 0.$$

Then Z_n *on* $\mathscr{A}$ *converges to a standard Wiener process on* $\mathscr{A}$.

3.2 REMARK. The first half of condition (ii) does not impose any condition on $\mathscr{A}$. That is, for any $A \in \mathscr{B}$ we can find a nonincreasing sequence C_l in $\mathscr{R}$ such that $\cap_1^\infty C_l \supseteq A$ and $|(\cap_1^\infty C_l) \setminus A| = 0$. For proof see Section 5.7.

3.3 COROLLARY. *Let* $\{Z_n\}$ *be a sequence of additive processes on* $\mathscr{B}$ *such that the set* $\{Z_n^2(A)/|A|\}_{n \geq 1, A \in \mathscr{B}}$ *is uniformly integrable, and satisfying either* 2.2(i), (ii), (iii), *or* 2.6(i). *Then* Z_n *converges weakly to a standard Wiener process on* $\mathscr{B}$.

3.4 REMARK. We use the convention $0/0 = 0$. The uniform integrability condition above is then well-defined and includes, in particular, that for each $A \in \mathscr{B}$ of zero measure, the random variables $Z_n(A)$ are zero a.s.

3.5 REMARK. As in Section 2, if we replace Z_n by W and delete all limiting operations in n, we get characterisations of the standard Wiener process.

4. Partial sum processes. Let $\xi_{n,\mathbf{j}}$, $\mathbf{j} \in J_n$, be random variables and define the smoothed row-sum

$$Z_n(A) := \sum_{\mathbf{j} \in J_n} \frac{|A \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} \xi_{n,\mathbf{j}}, \qquad A \in \mathscr{B},$$

so Z_n is an additive process on $\mathscr{B}$.

We use the following mixing coefficients on the triangular array $\{\xi_{n,\mathbf{j}}\}$. The *maximal correlation coefficient* (for the nth row, at separation distance x) is, for

$n = 1, 2, \ldots$ and $x > 0$,

$$\rho_n(x) := \sup_{\substack{I, J \subseteq J_n \\ \rho(I, J) \geq x}} \sup_{\substack{X \in L_2(\sigma(\xi_{n,\mathbf{j}}, \mathbf{j} \in I)) \\ Y \in L_2(\sigma(\xi_{n,\mathbf{j}}, \mathbf{j} \in J))}} |\operatorname{corr}(X, Y)|,$$

where $L_2(\cdot)$ is the set of L_2 random variables measurable with respect to $(\cdot)$. (When X or Y is degenerate, define $\operatorname{corr}(X, Y) := 0$.) The *symmetric ϕ-mixing coefficient* is

$$\phi_n(x) := \sup_{\substack{I, J \subseteq J_n \\ \rho(I, J) \geq x}} \sup_{\substack{E \in \sigma(\xi_{n,\mathbf{j}}, \mathbf{j} \in I) \\ F \in \sigma(\xi_{n,\mathbf{j}}, \mathbf{j} \in J) \\ P(E) > 0,\, P(F) > 0}} \max(|P(E|F) - P(E)|, |P(F|E) - P(F)|).$$

Both coefficients are zero for $x \geq 1$. To see the way they should behave in n and x, consider the canonical example of a normed sum, when $X_{(i_1, \ldots, i_d)}$ is a fixed Z^d-indexed array on which we define the triangular array $\xi_{n,\mathbf{j}} := n^{-d/2} X_{(nj_1, \ldots, nj_d)}$. The maximal correlation coefficient for the Xs is

$$\rho(x) := \sup_{\substack{I, J \subseteq \mathbb{Z}^d \\ \rho(I, J) \geq x}} \sup_{\substack{X \in L_2(\sigma(X_{\mathbf{i}}, \mathbf{i} \in I)) \\ Y \in L_2(\sigma(X_{\mathbf{i}}, \mathbf{i} \in J))}} |\operatorname{corr}(X, Y)|.$$

Thus $\rho_n(x) \leq \rho(nx)$ for this example, the possibility of strict inequality arising because not all the Xs are used to define the nth row of the triangular array. Polynomial decay of $\rho(\cdot)$, for instance, corresponds to the requirement $\rho_n(x) = O((nx)^{-b})$, some $b > 0$. Similar considerations attach to the symmetric ϕ-mixing coefficient.

The connection between $\phi_n(x)$ and $\rho_n(x)$ is that, by Peligrad's inequality ([16]),

$$\rho_n(x) \leq 2\phi_n(x), \qquad n = 1, 2, \ldots,\ x > 0. \tag{4.0.1}$$

We now state Peligrad's inequality, as it will be needed for other purposes. For σ-algebras of events $\mathscr{F}, \mathscr{G}$, let

$$\phi_{\mathscr{G}|\mathscr{F}} := \sup_{F \in \mathscr{F},\, G \in \mathscr{G},\, P(F) > 0} |P(G|F) - P(G)|.$$

Let X be $\mathscr{F}$-measurable, Y be $\mathscr{G}$-measurable, and p, q satisfy $1 \leq p \leq \infty$, $p^{-1} + q^{-1} = 1$. The inequality is

$$|E(XY) - EXEY| \leq 2\phi_{\mathscr{G}|\mathscr{F}}^{1/p} \|X\|_p \phi_{\mathscr{F}|\mathscr{G}}^{1/q} \|Y\|_q.$$

It strengthens the inequality of Doob–Cogburn–Ibragimov [4, 11] by including the $\phi_{\mathscr{F}|\mathscr{G}}^{1/q}$ term.

4.1 THEOREM. *Let the $\xi_{n,\mathbf{j}}$ and the smoothed partial-sum processes Z_n satisfy*

(i) $E\xi_{n,\mathbf{j}} = 0 \ \forall\ n, \mathbf{j}$;
(ii) *the set* $\{n^d \xi_{n,\mathbf{j}}^2\}_{\mathbf{j} \in J_n,\, n \in \mathbb{N}}$ *is uniformly integrable*;
(iii) $\sup_{n \in \mathbb{N}} \sum_{j=1}^{\infty} \rho_n^{1/2}(n^{-1} 2^j) < \infty$,

and either 2.2(ii) *or* 2.6(i). *Then* Z_n *on* $\mathscr{B}$ *converges weakly to a standard Wiener process.*

4.2 REMARK. For (iii) to hold it suffices that there exist $\delta > 0$, $K < \infty$ such that $\rho_n(x) \le K(\log(nx))^{-(2+\delta)}$ $(n = 1, 2, \dots,\ n^{-1} < x < 1)$.

4.3 THEOREM. *Let the* $\xi_{n,\mathbf{j}}$ *and the smoothed partial-sum processes* Z_n *satisfy*

(i) $E\xi_{n,\mathbf{j}} = 0\ \forall\ n, \mathbf{j}$;
(ii) *the set* $\{n^d \xi^2_{n,\mathbf{j}}\}_{\mathbf{j} \in J_n,\, n \ge 1}$ *is uniformly integrable*;
(iii) $\sup_{n \in \mathbb{N}} \sum_{j=1}^{\infty} \phi_n^{1/2}(n^{-1}2^j) < \infty$;
(iv) *for any null family* $\{D_h\}_{0 < h \le h_0}$ *of elements of* $\mathscr{J}$,

$$\lim_{h \downarrow 0} \limsup_{n \to \infty} \left| \frac{EZ_n^2(D_h)}{|D_h|} - 1 \right| = 0.$$

Then Z_n *on* $\mathscr{B}$ *converges weakly to a standard Wiener process.*

5. Proofs. Theorem 2.1 is based on

5.1 THEOREM (Gnedenko and Kolmogorov [6], page 126). *For each* $m = 1, 2, \dots$ *let* $\xi_{m,1}, \dots, \xi_{m,k_m}$ *be independent. If* $\sum_{k=1}^{k_m} \xi_{m,k} \Rightarrow X$ *and*

$$\lim_{m \to \infty} \sum_{m=1}^{k_m} P(|\xi_{m,k}| \ge \varepsilon) = 0, \qquad \forall\ \varepsilon > 0,$$

then X *is normally distributed.*

5.2 PROOF OF THEOREM 2.1. First, if $C_1, \dots, C_k \in \mathscr{J}$ are such that $C_i \cap C_j = \varnothing$ for $i \ne j$, then in each C_i we take subintervals $C_i^{(m)}$, $m = 1, 2, \dots,$ with $C_i^{(m)}$ similar to C_i, of the same centre, and with measure $(1 - 1/m)|C_i|$. Since $\rho(C_i^{(m)}, C_j^{(m)}) > 0$ for $i \ne j$ we have $W(C_1^{(m)}), \dots, W(C_k^{(m)})$ independent. Now $C_i \setminus C_i^{(m)}$ is the union of a fixed number of disjoint intervals $I_{i,m,j}$, $j = 1, \dots, j_0$ say, of total area $m^{-1}|C_i|$. We have by additivity and (ii) that

$$\begin{aligned} \|W(C_i) - W(C_i^{(m)})\|_2 &= \|W(I_{i,m,1}) + \cdots + W(I_{i,m,j_0})\|_2 \\ &\le |I_{i,m,1}|^{1/2} + \cdots + |I_{i,m,j_0}|^{1/2} \\ &\le j_0 (m^{-1}|C_i|)^{1/2} \to 0, \qquad m \to \infty. \end{aligned}$$

Thus independence of $W(C_1), \dots, W(C_k)$ follows from that of the $W(C_i^{(m)})$. That is, (iii) extends to (iii′), the corresponding property for disjoint but not necessarily separated intervals. From this property, additivity, and (i) and (ii), we have $EW(A) = 0$ and $EW^2(A) = |A|$ for all $A \in \mathscr{R}$.

Choose $C \in \mathscr{I}$. Let $K_m := \{\mathbf{j} \in J_m: C_{m,\mathbf{j}} \subseteq C\}$, then $\Sigma_{\mathbf{j} \in K_m} W(C_{m,\mathbf{j}})$ converges in L_2 to $W(C)$, since the difference is $W(C \setminus U_{\mathbf{j} \in K_m} C_{m,\mathbf{j}})$ whose argument is in $\mathscr{R}$. By Theorem 5.1 and (iii′), $W(C)$ is normal. By (i) and (ii), it is $N(0, |C|)$.

Choose $A_1, \dots, A_k \in \mathscr{R}$; then we can find a finite collection $\{C_1, \dots, C_m\}$ of disjoint intervals such that each A_i is the union of some of the C_j. By (iii′) the $W(C_j)$ are independent $N(0, |C_j|)$; hence, by additivity the joint law of $W(B_1), \dots, W(B_k)$ is multinormal with the correct mean and covariances. □

5.3 Proof of Theorem 2.2. Let S be a countable dense set in $[0,1]$, containing the rationals. Let $\mathscr{I}_S$ be the set of all intervals $(\mathbf{a}, \mathbf{b}]$ with $\mathbf{a}, \mathbf{b} \in S^d$. Note $C_{m,\mathbf{j}} \in \mathscr{I}_S$ for all $m, \mathbf{j}$. Let $\mathscr{R}_S$ be the ring of all finite unions of elements of $\mathscr{I}_S$. By (v), for each $C \in \mathscr{I}_S$ the sequence $\{Z_n(C)\}_{n \geq 1}$ is tight. By additivity the same holds for any $C \in \mathscr{R}_S$. Since $\mathscr{I}_S$ is countable we can by selection and diagonalisation find from any subsequence $Z_{n''}$ a further subsequence $Z_{n'}$ such that $Z_{n'}$ on $\mathscr{R}_S$ converges weakly, i.e., the finite-dimensional laws on $\mathscr{R}_S$ converge, to those of W, say. We shall check that W satisfies the hypotheses of that version of Theorem 2.1 in which $\mathscr{I}$, $\mathscr{R}$ are everywhere replaced by $\mathscr{I}_S$, $\mathscr{R}_S$. The proof of Theorem 2.1 then applies, with little alteration, to show W is a standard Wiener process on $\mathscr{R}_S$. This is enough to show Z_n on $\mathscr{R}_S$ converges weakly to a standard Wiener process on $\mathscr{R}_S$. But that suffices to show the finite-dimensional laws of Z_n on $\mathscr{R}$ converge to those of a standard Wiener process, as the theorem claims.

Each of properties (i)–(iv), and additivity (on $\mathscr{I}_S$, $\mathscr{R}_S$ instead of $\mathscr{I}$, $\mathscr{R}$), implies the corresponding property in 2.1, utilising (v) in the cases of (i) and (ii). □

5.4 Proof of Theorem 2.3. First we prove the following strengthened form of (i):

(i′) for any $C_1, \dots, C_k \in \mathscr{I}$, any real $u_1, \dots, u_k$, and any null family $\{B_h\}_{0 < h \leq h_0}$ in $\mathscr{I}$ such that $B_{h_0} \cap (\bigcup_1^k C_j) = \varnothing$,

$$\lim_{h \downarrow 0} E\left(\exp\left(i \sum_{j=1}^k u_j W(C_j)\right) \frac{W(B_h)}{|B_h|}\right) = 0, \tag{5.4.1}$$

$$\lim_{h \downarrow 0} E\left(\exp\left(i \sum_{j=1}^k u_j W(C_j)\right)\left[\frac{W^2(B_h)}{|B_h|} - 1\right]\right) = 0. \tag{5.4.2}$$

Given C_i, u_i, and B_h we shall construct a family $\{D_h\}$ satisfying the conditions of (i) (in the form given in Remark 2.4), and such that

$$E\left|\frac{W(B_h)}{|B_h|} - \frac{W(D_h)}{|D_h|}\right| \to 0, \qquad h \downarrow 0, \tag{5.4.3}$$

$$E\left|\frac{W^2(B_h)}{|B_h|} - \frac{W^2(D_h)}{|D_h|}\right| \to 0, \qquad h \downarrow 0. \tag{5.4.4}$$

Then (i′) will follow from (i).

Observe first that (ii) implies for any null family $\{D_h\}$ in $\mathscr{I}$ that for some $h_1 > 0$

$$\sup_{0<h\le h_1} E\frac{\left(W^2(D_h)\right)}{|D_h|} < \infty. \tag{5.4.5}$$

Let D_h be a geometrically similar subinterval of B_h, of the same centre, and such that $|B_h \setminus D_h| = h^3$. Then $|D_h| = h - h^3 \downarrow 0$. By construction D_h decreases with h, and $\rho(D_{h_0}, \bigcup_1^k C_j) > 0$. Now we can write $B_h \setminus D_h$, for all h, as the union of a fixed number of m (depending only on the dimension of the space) disjoint intervals $I_h^{(1)}, \dots, I_h^{(m)}$. From (5.4.5), for each $i = 1, \dots, m$, $\sup_{h<h_1} \|W(I_h^{(i)})\|_2/|I_h^{(i)}|^{1/2} < \infty$. Since $W(B_h \setminus D_h) = \sum_{\iota=1}^m W(I_k^{(i)})$ and $|B_h \setminus D_h| = \sum_{\iota=1}^m |I_h^{(i)}|$, by Minkowski's inequality we find

$$\sup_{h\le h_1} \|W(B_h \setminus D_h)\|_2/|B_h \setminus D_h|^{1/2} =: K < \infty$$

Then

$$\begin{aligned} E\left|\frac{W(B_h)}{|B_h|} - \frac{W(D_h)}{|D_h|}\right| &= E\left|\frac{W(B_h \setminus D_h)}{|B_h|} - W(D_h)\frac{|B_h \setminus D_h|}{|B_h|\,|D_h|}\right| \\ &\le \frac{\|W(B_h \setminus D_h)\|_2}{|B_h \setminus D_h|^{1/2}}\frac{|B_h \setminus D_h|^{1/2}}{|B_h|} + \|W(D_h)\|_2\frac{|B_h \setminus D_h|}{|B_h|\,|D_h|} \\ &\le Kh^{1/2} + \left(\sup_{h\le h_1}\frac{\|W(D_h)\|_2}{|D_h|^{1/2}}\right)\frac{h^2}{(h-h^3)^{1/2}} \\ &\to 0, \qquad h \downarrow 0, \end{aligned}$$

and similarly for (5.4.4). This establishes (i′).

Denote $Z := u_1W(C_1) + \cdots + u_kW(C_k)$. Fix $B \in \mathscr{I}$ such that $B \cap (\bigcup_1^k C_j) = \varnothing$, and let $b := |B|$. Let $\{B_t\}_{0\le t\le b}$ be a family of elements of $\mathscr{I}$ such that $B_t \subseteq B_{t'}$ and $B_{t'} \setminus B_t \in \mathscr{I}$ for $t \le t'$, $|B_t| = t$ for all t, and $B_b = B$. Since B is aligned with the coordinate axes, this can be achieved by cutting B by a hyperplane with normal in the first coordinate direction, and displacing the hyperplane in the direction of its normal. Define

$$\psi(t, u) := E(e^{iZ}e^{iuW(B_t)}), \qquad 0 \le t \le b,\ u \in \mathbb{R}.$$

For $h > 0$, $W(B_{t+h}) - W(B_t) = W(B_{t+h} \setminus B_t)$ which tends in L_2 to 0 as $h \downarrow 0$, by (5.4.5); hence, $\psi(t, u)$ is continuous in t to the right, at each t, u with $t < b$. Similarly it is continuous in t to the left. Joint continuity of ψ in t and u, throughout the region of definition, is then clear.

Define $\Delta_{t,h} := W(B_{t+h} - B_t)$ and $c(z) := e^{iz} - 1 - iz + \frac{1}{2}z^2$. As noted in [1, page 162] we have both $|c(z)| \le z^2$ and $|c(z)| \le |z|^3$, for real z. Therefore for each real u and $0 \le t < b$,

$$h^{-1}E|c(u\Delta_{t,h})| \le |u|^3\alpha^{3/2}h^{1/2} + u^2E\left(h^{-1}\Delta_{t,h}^2 1\{h^{-1}\Delta_{t,h}^2 \ge \alpha\}\right)$$

whence, on letting $h \downarrow 0$ then $\alpha \to \infty$, by (ii),

$$\lim_{h \downarrow 0} h^{-1} E|c(u\Delta_{t,h})| = 0. \tag{5.4.6}$$

We now have

$$\psi(t+h, u) - \psi(t, u) = E\{e^{iZ} e^{iuW(B_l)}[iu\Delta_{t,h} - \tfrac{1}{2}u^2\Delta_{t,h} + c(u\Delta_{t,h})]\};$$

hence,

$$\begin{aligned} &|h^{-1}[\psi(t+h,u) - \psi(t,u)] + \tfrac{1}{2}u^2\psi(t,u)| \\ &\quad \le h^{-1}|u|\,|E(e^{iZ}e^{iuW(B_l)}\Delta_{t,h})| + \tfrac{1}{2}h^{-1}u^2|E\{e^{iZ}e^{iuW(B_l)}(\Delta_{t,h}^2 - h)\}| \\ &\qquad + h^{-1}E|c(u\Delta_{t,h})|. \end{aligned}$$

By (5.4.1), (5.4.2), and (5.4.6) the right-hand side tends to 0 as $h \downarrow 0$, so

$$\frac{\partial}{\partial t}\psi(t,u) = -\tfrac{1}{2}u^2\psi(t,u), \qquad u \in \mathbb{R}, 0 \le t < b,$$

there being only a right-hand partial derivative here. Billingsley's [1, page 155] argument, that because of continuity the derivative is in fact two-sided, applies. The solution of the p.d.e. is $\psi(t,u) = e^{-tu^2/2}a(u)$, continuous on the definition strip, and on setting $t := 0$ we identify $a(u)$ as Ee^{iZ}. Then taking $t := b$ we have shown that $W(B)$ has a $N(0, |B|)$ law and is independent of $(W(C_1), \dots, W(C_k))$. By induction it follows that $W(B_1), \dots, W(B_l)$ are independent and, respectively, $N(0, |B_i|)$ whenever the B_i are disjoint elements of $\mathscr{I}$. The argument at the end of Section 5.2 now gives the result. □

5.5 Proof of Theorem 2.6. Let $Z_{n''}$ be any subsequence of Z_n. Let $S := [0,1] \cap \mathbb{Q}$, let $\mathscr{I}_S$ be the class of intervals in $\mathscr{I}$ whose bounding points have only rational coordinates, and let $\mathscr{R}_S$ be the ring of all finite unions of elements of $\mathscr{I}_S$. As in Section 5.3, by (iii) we may find a subsequence $Z_{n'}$ of $Z_{n''}$ that converges weakly on $\mathscr{R}_S$. But in this proof we shall argue that because of (ii), $Z_{n'}$ must then converge weakly on $\mathscr{R}$ itself. We replace n' by n.

Note first that (ii) yields for the null family $\{D_h\}$ in question the existence of constants h_1, $K > 0$ such that $\limsup_{n\to\infty} EZ_n^2(D_h) \le K|D_h|$ for all $h \le h_1$.

So, assuming Z_n on $\mathscr{R}_S$ converges weakly to W, choose $A_1, \dots, A_k \in \mathscr{I}$, $\eta > 0$, and $u_1, \dots, u_k$ real nonzero. Take $\varepsilon := \eta/(8\sum_1^k |u_j|)$. For each $j := 1, \dots, k$ we can find $B_j \in \mathscr{I}_S$ such that B_j is a subinterval of A_j and so close to A_j that

$$\limsup_{n\to\infty} EZ_n^2(A_j \setminus B_j) < \varepsilon^2\eta/(8k).$$

This is because $A_j \setminus B_j$ is the union of a fixed (depending only on d) collection of disjoint intervals to each of which applies the above consequence of (ii). We then have, by Chebychev's inequality and additivity,

$$\limsup_{n\to\infty} P\big(|Z_n(A_j) - Z_n(B_j)| \ge \varepsilon\big) < \eta/(8k), \qquad j = 1, \dots, k. \tag{5.5.1}$$

Consider the joint characteristic function $\phi_n(\mathbf{u}) := E\exp(i\Sigma_{j=1}^k u_j Z_n(A_j))$. Now

$$\begin{aligned}
&|\phi_m(\mathbf{u}) - \phi_n(\mathbf{u})| \\
&\quad \le E\left|\exp\left(i\sum_j u_j Z_m(A_j)\right) - \exp\left(i\sum_j u_j Z_m(B_j)\right)\right| \\
&\qquad + E\left|\exp\left(i\sum_j u_j Z_n(A_j)\right) - \exp\left(i\sum_j u_j Z_n(B_j)\right)\right| \\
&\qquad + \left|E\exp\left(i\sum_j u_j Z_m(B_j)\right) - E\exp\left(i\sum_j u_j Z_n(B_j)\right)\right| \\
&\quad \le E\left|\exp\left(i\sum_j u_j (Z_m(A_j) - Z_m(B_j))\right) - 1\right| \\
&\qquad + E\left|\exp\left(i\sum_j u_j (Z_n(A_j) - Z_n(B_j))\right) - 1\right| + \tfrac{1}{4}\eta \quad (\text{for large } m, n) \\
&\quad \le 2\sum_{j=1}^k P\left(|Z_m(A_j) - Z_m(B_j)| \ge \varepsilon\right) + 2\sum_{j=1}^k P\left(|Z_n(A_j) - Z_n(B_j)| \ge \varepsilon\right) \\
&\qquad + 2\sum_{j=1}^k |u_j|\varepsilon + \tfrac{1}{4}\eta \qquad (\text{since } |e^{iz} - 1| \le |z|) \\
&\quad \le 4\sum_{j=1}^k \eta/(8k) + \tfrac{1}{4}\eta + \tfrac{1}{4}\eta \qquad (\text{for large } m, n, \text{ by } (5.5.1)) \\
&\quad = \eta.
\end{aligned}$$

Thus the sequence $\{\phi_n(\mathbf{u})\}_{n\ge1}$ is Cauchy and so converges, to $\phi(\mathbf{u})$, say. We must check that ϕ is continuous at $\mathbf{0}$. Again, take $\eta > 0$ and now $\varepsilon := \eta/(4k)$. Take $B_1, \dots, B_k \in \mathscr{J}$ such that (5.5.1) holds. Take $u_1, \dots, u_k$ so small that $|u_j| \le 1$ $(j = 1, \dots, k)$ and $|E\exp(i\Sigma_j u_j W(B_j)) - 1| < \eta/4$. Then

$$\begin{aligned}
|\phi_n(\mathbf{u}) - 1| &\le E\left|\exp\left(i\sum_j u_j Z_n(A_j)\right) - \exp\left(i\sum_j u_j Z_n(B_j)\right)\right| \\
&\quad + \left|E\exp\left(i\sum_j u_j Z_n(B_j)\right) - E\exp\left(i\sum_j u_j W(B_j)\right)\right| \\
&\quad + \left|E\exp\left(i\sum_j u_j W(B_j)\right) - 1\right|.
\end{aligned}$$

The last term is less than $\eta/4$, the middle term on the right is at most $\eta/4$ for large n, and the first term on the right is at most

$$2\sum_{j=1}^k P\left(|Z_n(A_j) - Z_n(B_j)| \ge \varepsilon\right) + \sum_{j=1}^k |u_j|\varepsilon \le \tfrac{1}{4}\eta + k\varepsilon = \tfrac{1}{2}\eta$$

for large n. Thus $|\phi(\mathbf{u}) - 1| \le \eta$; hence, $\phi(\mathbf{u}) \to 1$ as $\mathbf{u} \to \mathbf{0}$. So $(Z_n(A_1), \dots, Z_n(A_k))$ converges in law, whence Z_n converges weakly on $\mathscr{R}$.

We have now shown that the original arbitrary subsequence $Z_{n''}$ contains a further subsequence $Z_{n'}$ converging weakly on $\mathscr{R}$, to W, say. To show W is a standard Wiener process it suffices to verify that it satisfies the conditions of Theorem 2.3. First, it is additive. Write n for n' and set $Z_{n,h} := Z_n^2(D_h)/|D_h|$, $W_h := W^2(D_h)/|D_h|$. For each $\alpha > 0$ let p_α: $[0,\infty) \to [0,\infty)$ be continuous and nondecreasing, with $p_\alpha(x) = 0$ for $x \le \alpha$, $p_\alpha(x) = 1$ for $x \ge \alpha + 1$. Since $1\{x \ge \alpha + 1\} \le p_\alpha(x) \le 1\{x \ge \alpha\}$ for all $x \ge 0$, 2.6(ii) and 2.3(ii) are, respectively, equivalent to

$$\lim_{\alpha \to \infty} \limsup_{h \downarrow 0} \limsup_{n \to \infty} E\left[Z_{n,h} p_\alpha(Z_{n,h})\right] = 0, \tag{5.5.2}$$

$$\lim_{\alpha \to \infty} \limsup_{h \downarrow 0} E\left[W_h p_\alpha(W_h)\right] = 0. \tag{5.5.3}$$

But (5.5.2) implies (5.5.3), by Loève [13, Theorem A(i), page 185]. Thus 2.6(ii) implies 2.3(ii).

It remains to verify 2.3(i). Set $X_n := \exp(\sum_{j=1}^k i u_j Z_n(C_j))$, $X := \exp(\sum_{j=1}^k i u_j W(C_j))$. Now

$$\begin{aligned} &|E[X(W_h - 1)]| \\ &\quad \le |E[X_n(Z_{n,h} - 1)]| + E[(W_h + 1)p_\alpha(W_h)] + E[(Z_{n,h} + 1)p_\alpha(Z_{n,h})] \\ &\qquad + |E[X(W_h - 1)(1 - p_\alpha(W_h))] - E[X_n(Z_{n,h} - 1)(1 - p_\alpha(Z_{n,h}))]|. \end{aligned}$$

For fixed α, h, the last term tends to 0 as $n \to \infty$, being the difference in expectation between a particular bounded continuous function of $(X_n, Z_{n,h})$ and of its weak limit (X, W_h). Let $n \to \infty$, then $h \downarrow 0$ and $\alpha \to \infty$, then the other three terms on the right tend to 0, by (2.6.2), (5.5.3), and (5.5.2), respectively. This establishes (2.3.2), and (2.3.1) is immediate from (2.6.1) because of (iii). □

5.6 Proof of Theorem 3.1. Let W be the weak limit of Z_n on $\mathscr{R}$. Choose $A^{(1)}, \dots, A^{(k)} \in \mathscr{A}$ and let $(W_1, \dots, W_k)$ be normal with mean $\mathbf{0}$ and $\operatorname{cov}(W_i, W_j) := |A^{(i)} \cap A^{(j)}|$. Let $\{C_l^{(i)}\}_{l \ge 1}$ be a sequence for which (ii) holds relative to $A^{(i)}$. Choose $z_1, \dots, z_k \in \mathbb{R}$ and $\varepsilon, \eta > 0$. By (iv) there exists L such that $\limsup_{n \to \infty} P(|Z_n(A^{(i)}) - Z_n(C_l^{(i)})| \ge \varepsilon) \le \eta/k$ for all $l \ge L$ and $i = 1, \dots, k$. Then for $l \ge L$, with $z_i' := z_i + \varepsilon$,

$$\begin{aligned} &\limsup_{n \to \infty} P\left(Z_n(A^{(1)}) \le z_1, \dots, Z_n(A^{(k)}) \le z_k\right) \\ &\quad \le \limsup_{n \to \infty} P\left(Z_n(C_l^{(1)}) \le z_1', \dots, Z_n(C_l^{(k)}) \le z_k'\right) \\ &\qquad + \sum_{i=1}^k \limsup_{n \to \infty} P\left(\left|Z_n(A^{(i)}) - Z_n(C_l^{(i)})\right| \ge \varepsilon\right) \\ &\quad \le P\left(W(C_l^{(1)}) \le z_1', \dots, W(C_l^{(k)}) \le z_k'\right) + \eta. \end{aligned}$$

Let $l \to \infty$; then since the multinormal laws with mean $\mathbf{0}$ and covariance $|\cdot \cap \cdot|$

are continuous in probability with respect to the d_L-pseudometric, the right-hand side converges to $P(W_1 \le z_1', \dots, W_k \le z_k') + \eta$. Thus

$$\limsup_{n\to\infty} P\big(Z_n(A^{(1)}) \le z_1, \dots, Z_n(A^{(k)}) \le z_k\big) \le P(W_1 \le z_1', \dots, W_k \le z_k')$$

and the reverse inequality for the lim inf is proved similarly. Finally, let $\varepsilon \downarrow 0$. □

5.7 PROOF OF REMARK 3.2. For each l we can find open $G_l \supset A$ such that $|G_l \setminus A| < 1/l$. Now G_l is the countable union of open intervals, each of which has a boundary of Lebesgue measure zero, hence so does G_l. Since the closure of G_l is compact, for each m we can find a set $B_{l,m}$ which contains G_l and which is a finite union of intervals $(\mathbf{x} - m^{-1}\mathbf{1}, \mathbf{x} - m^{-1}\mathbf{1}] \cap [0,1]^d$ for $\mathbf{x} \in G_l$. Now $B_{l,m} \setminus G_l \subseteq (\text{bndy}\, G_l)^{1/m}$ (notation from (1.0.2)), and the measure of the latter set tends to $|\text{bndy}\, G_l| = 0$ as $m \to \infty$; hence, by choosing $m = m(l)$ large we ensure $|B_{l,m(l)} \setminus G_l| < 1/l$. Then $|B_{l,m(l)} \setminus A| < 2/l$. Take $C_l := \bigcap_{i=1}^{l} B_{i,m(i)}$. □

5.8 PROOF OF COROLLARY 3.3. Because of Remark 2.11, what remains to be shown is that the uniform integrability implies 2.2(iv), (v) and 3.1(ii). Condition 2.2(v) is immediate, as is 3.1(ii), using additivity. For 2.2(iv), pick $\varepsilon, \eta > 0$; then we can make m so large that

$$E\left[\frac{Z_n^2(A)}{|A|} 1\left\{\frac{Z_n^2(A)}{|A|} \ge \varepsilon^2 m^d\right\}\right] \le \varepsilon^2\eta, \qquad n \ge 1,\ A \in \mathscr{B}.$$

Then

$$\sum_{\mathbf{j}\in\mathscr{J}_m} P\big(|Z_n(C_{m,\mathbf{j}})| \ge \varepsilon\big) \le \sum_{\mathbf{j}\in\mathscr{J}_m} (\varepsilon^2 m^d)^{-1} E\left[\frac{Z_n^2(C_{m,\mathbf{j}})}{|C_{m,\mathbf{j}}|} 1\left\{\frac{Z_n^2(C_{m,\mathbf{j}})}{|C_{m,\mathbf{j}}|} \ge \varepsilon^2 m^d\right\}\right]$$
$$\le \sum_{\mathbf{j}\in\mathscr{J}_m} \eta m^{-d} = \eta,$$

hence 2.2(iv). □

5.9 PROOF OF THEOREM 4.1. By [7] Theorem 4.1 we have uniform integrability of the set $\{Z_n^2(A)/|A|\}_{n\ge1, A\in\mathscr{B}}$. So Corollary 3.3 applies once we have checked 2.2(iii). For the *strong mixing coefficient*

$$\alpha_n(x) := \sup_{\substack{I,\mathscr{J}\subseteq\mathscr{J}_n\\ \rho(I,\mathscr{J})\ge x}} \ \sup_{\substack{E\in\sigma(\xi_{n,\mathbf{j}},\mathbf{j}\in I)\\ F\in\sigma(\xi_{n,\mathbf{j}},\mathbf{j}\in\mathscr{J})}} |P(E\cap F) - P(E)P(F)|,$$

obviously $\alpha_n(x) \le \rho_n(x)$. Because $\rho_n(x)$ is nonincreasing in x, condition 4.1(iii) implies $\rho_n(x) \to 0$ $(n \to \infty)$ for each fixed x. Thus $\alpha_n(x) \to 0$ $(n \to \infty)$. Clearly the left-hand side of (2.2.1) has absolute value at most $(k-1)\alpha_n(\rho)$ where $\rho > 0$ is the least separation distance between the sets $C_1, \dots, C_k$. Thus 2.2(iii) holds. □

5.10 PROOF OF THEOREM 4.3. We show 2.6(i), when the result follows by Theorem 4.1. Thus, pick $C_1, \dots, C_k \in \mathscr{I}$, with $|\bigcup_1^k C_k| < 1$, pick real $u_1, \dots, u_k$, and a null family $\{D_h\}_{0<h\le h_0}$ in $\mathscr{J}$ with $2x := \rho(D_{h_0}, \cup_1^k C_j) > 0$. Now $Z_n(D_h)$ and $(Z_n(C_1), \dots, Z_n(C_k))$ are measurable with respect to respective σ-algebras

$$\mathscr{F}_n := \sigma\left\{\xi_{n,\mathbf{j}}: \mathbf{j} \in D_{h_0}^{1/n}\right\}, \qquad \mathscr{G}_n := \sigma\left\{\xi_{n,\mathbf{j}}: \mathbf{j} \in \left(\cup_{j=1}^k C_j\right)^{1/n}\right\},$$

which are based on sets of $\xi_{n,\mathbf{j}}$ at separation distances $\ge 2x - 2/n$, and we keep n large enough that this is always $\ge x$. We may also, by reducing h_0 if need be, and on account of (iv), assume that $\limsup_{n\to\infty} EZ_n^2(D_h) < \infty$ for all $h \le h_0$. Write $U_n := \exp(i\sum_{j=1}^k u_j Z_n(C_j))$.

By Peligrad's inequality, whose proof is valid for complex-valued r.v.s,

$$|E(U_n Z_n(D_h))| \le 2\phi_n(x)\|U_n\|_2\|Z_n(D_h)\|_2 \le 2\phi_n(x)\|Z_n(D_h)\|_2.$$

For fixed h, the right-hand side tends to 0 as $n \to \infty$, hence (2.6.1). Likewise, writing $V_{n,h} := Z_n^2(D_h)/|D_h| - 1$,

$$\begin{aligned}|E(U_n V_{n,h})| &\le |E(U_n V_{n,h}) - EU_n EV_{n,h}| + |EV_{n,h}| \\ &\le 2\phi_n(x)\|U_n\|_\infty\|V_{n,h}\|_1 + |EV_{n,h}|,\end{aligned}$$

whence for each $h \le h_0$,

$$\limsup_{n\to\infty}|E(U_n V_{n,h})| \le \limsup_{n\to\infty}|EV_{n,h}|$$

and (2.6.2) follows. □

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