

LEVEL CROSSINGS OF A CAUCHY PROCESS¹

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The asymptotic distribution as $t \rightarrow \infty$ is obtained for the number of jumps of a symmetric Cauchy process across level x up to time t , jointly as x varies. This result is related to the asymptotic joint distribution of windings of a planar Brownian motion about several points.

1. Introduction. Let $C = (C_s, s \geq 0)$ be a symmetric Cauchy process on the real line \mathbb{R} , that is a process with stationary independent increments with the Cauchy distribution,

$$P(C_{s+t} - C_s \in dx) = \frac{t dx}{\pi(t^2 + x^2)}, \quad s, t > 0, \quad x \in \mathbb{R}.$$

Except where otherwise mentioned, it will be supposed that $C_0 = 0$, and it will always be assumed that the paths of the process are right continuous with left limits.

Let $N^x(s)$ denote the number of times $t \leq s$ that the Cauchy process jumps across level x . More formally,

$$N^x(s) = \# \{S^x \cap (0, s]\},$$

where

$$S^x = \{t: (C_{t-} - x)(C_t - x) < 0\}$$

is the random set of times when jumps across x occur.

The purpose of this paper is to describe the asymptotic joint distribution of the level crossing numbers $N^x(s)$ as s tends to ∞ and the level x varies. Some results have analogues for the symmetric stable process with index α , $0 < \alpha < 2$. See Kesten (1963). But the main result concerns the asymptotic dependence structure of $N^x(s)$ as x varies, a phenomenon which is of interest only for a process which is recurrent without hitting points. Among symmetric stable processes, which are recurrent iff $1 \leq \alpha \leq 2$, and hit points iff $1 < \alpha \leq 2$, this happens only in the Cauchy case $\alpha = 1$.

To analyse the dependence of the Cauchy level crossings as the level varies, it is important to classify crossings according to the magnitude of the jumps involved. Fix numbers a and b with $0 < a < b < \infty$, and call a crossing of x that

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occurs at time t

$$\begin{aligned} \textit{small} & \quad \text{if } 0 < |C_t - x| \leq a, \\ \textit{medium} & \quad \text{if } a < |C_t - x| \leq b, \\ \textit{large} & \quad \text{if } b < |C_t - x| < \infty. \end{aligned}$$

Then

$$N^x(s) = N_{\textit{small}}^x(s) + N_{\textit{medium}}^x(s) + N_{\textit{large}}^x(s),$$

where for example $N_{\textit{small}}^x(s)$ is the number of small crossings of x up to time s . Instead of classifying by the size of the overshoot $|C_t - x|$, the crossings could also be classified by the total size of the jump $|C_t - C_{t-}|$, or the distance from x to the left limit $|C_{t-} - x|$. It is easily shown that the following two theorems, to be proved in Sections 3 and 4, respectively, hold regardless of which classification is used.

THEOREM 1. *For each $x \neq 0$, as $s \rightarrow \infty$, the distribution of the triple*

$$\frac{\pi^2}{\log s} \left[\frac{N_{\textit{small}}^x(s)}{\log s}, \frac{N_{\textit{medium}}^x(s)}{\log(b/a)}, \frac{N_{\textit{large}}^x(s)}{\log s} \right]$$

converges to the distribution of the triple

$$[\sigma_-, L, \sigma_+]$$

defined as follows in terms of the path of a standard Brownian motion $(B_t, t \geq 0)$ up to its hitting time of 1,

$$\begin{aligned} \sigma_+ &= \inf\{t: B_t = 1\}, \\ \sigma_- &= \int_0^{\sigma_+} 1(B_t < 0) dt, \end{aligned}$$

the time that B spends negative before σ_+ ,

$$\sigma_+ = \int_0^{\sigma_+} 1(B_t \geq 0) dt,$$

the time that B spends positive before σ_+ , and

$$L = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{\sigma_+} 1(0 \leq B_t \leq \epsilon) dt,$$

the local time of B at zero up to time σ_+ .

Note that apart from constants the normalisation of the count of medium crossings is by $\log s$ while for the counts of large and small crossings the normalisation is by $\log^2 s$. Also, the normalisation of these latter counts does not involve the values a and b determining the types of crossing. Thus large and small crossings occur an order of magnitude more often than medium crossings, and it is really only the *very large* and *very small* crossings which count in the

limit so far as the total $N^x(s)$ is concerned. In particular, Theorem 1 implies

$$\frac{\pi^2 N^x(s)}{\log^2 s} \xrightarrow{d} \sigma_- + \sigma_+ = \sigma_1,$$

where \xrightarrow{d} denotes convergence in distribution, and the limit distribution of σ_1 is the stable distribution with index $\frac{1}{2}$.

This should be contrasted with the different limit distribution obtained by Kesten (1963) for the number $N_{\text{discrete}}^x(n)$ of crossings of level x up to time n of the discrete time Cauchy random walk C_1, C_2, \dots :

$$\frac{\pi^2 N_{\text{discrete}}^x(n)}{\log^2 n} \xrightarrow{d} \sigma_+,$$

the same limit as for the large count alone for the continuous time process. The explanation is that very large crossings of the continuous time process occur so rarely that they are almost invariably recorded as crossings by the embedded random walk, whereas when very small crossings occur in continuous time they tend to do so in such rapid succession that they make no significant contribution to the random walk crossings.

The starting point $x = 0$ is excluded in Theorem 1 because the number of small crossings of 0 is a.s. infinite in every interval $(0, s]$, $s > 0$. Kesten (1963) gave a closely related result with the distribution of σ_+ appearing as the asymptotic distribution as $\varepsilon \rightarrow 0$ of the number of downcrossings of zero in the time interval $[0, 1]$ for which the time elapsed since the previous upcrossing is at least ε , together with corresponding results for the other symmetric stable processes. As Kesten's result suggests, Theorem 1 remains valid when crossings are classified in size according to the time elapsed since the previous crossing.

The limit distribution appearing in Theorem 1 was encountered by Messulam and Yor (1982) in connection with the windings of planar Brownian motion. They found its Laplace transform using martingale calculus, as in the proof of Theorem 4.2 of Pitman and Yor (1986):

$$E \exp\left(-\alpha L - \frac{1}{2}\lambda^2\sigma_- - \frac{1}{2}\mu^2\sigma_+\right) = \phi(2\alpha + |\lambda|, \mu),$$

where

$$\phi(\alpha, v) = \left[\cosh v + \frac{\alpha}{v} \sinh v \right]^{-1},$$

which for $v = 0$ is defined by continuity as $(1 + \alpha)^{-1}$. The present results were discovered using the method of Spitzer (1958), representing the Cauchy process as a complex Brownian motion watched only as it hits the real line. Each jump of the Cauchy process across x then corresponds to a winding of either $\pm\pi$ around x by the complex Brownian motion. Asymptotic properties of the Cauchy level crossings are therefore related to the asymptotic properties of Brownian windings described in Pitman and Yor (1986) and Lyons and McKean (1984).

It should be noted that

$$(1.1a) \quad \frac{P(L \in dl)}{dl} = \frac{1}{2}e^{-l/2}, \quad l \geq 0,$$

and

$$(1.1b) \quad \sigma_- \text{ and } \sigma_+ \text{ are conditionally independent given } L.$$

These features of the distribution of $[\sigma_-, L, \sigma_+]$ can be seen from the Laplace transform above, but a deeper understanding is provided by excursion theory, as explained in Section 5 of Pitman and Yor (1986). It can also be seen either by excursion theory or transform calculations that the conditional distribution of σ_- given $L = l$ is stable with index $\frac{1}{2}$ and scale parameter $\frac{1}{2}l$. It follows that

$$\sigma_- \stackrel{d}{=} H^2\sigma_1,$$

where $\stackrel{d}{=}$ indicates equality in distribution, and H is a random variable independent of σ_1 and exponentially distributed with mean 1. Let Φ denote the standard normal distribution function. Burdzy (1984) notes that

$$P(\sigma_- \leq t) = 1 - 2e^{t/2}(1 - \Phi(\sqrt{t})),$$

and shows how this limit distribution arises more generally from counting excursions of complex Brownian motion.

The conditional distribution of σ_+ given $L = l$ is more complicated. Its Laplace transform can be calculated as

$$E\left(\exp\left(-\frac{1}{2}\mu^2\sigma_+\right) \mid L = l\right) = \frac{\mu}{\sinh \mu} \exp\left[\frac{l}{2}(1 - \mu \coth \mu)\right],$$

but this formula seems difficult to invert. However series formulae are available for the unconditional distribution of σ_+ . See, for example, Feller (1966), page 341, and Kesten (1963), page 402. Lévy (1951) and Pitman and Yor (1986) contain related formulae.

Consider now the asymptotic joint distribution of the counts of various sizes across several levels:

THEOREM 2. *For each finite subset F of the line with $0 \notin F$, as $s \rightarrow \infty$, the joint distribution of*

$$\left\{ \frac{\pi^2}{\log s} \left[\frac{N_{\text{small}}^x(s)}{\log s}, \frac{N_{\text{medium}}^x(s)}{\log(b/a)}, \frac{N_{\text{large}}^x(s)}{\log s} \right], x \in F \right\}$$

converges to the joint distribution of

$$\{[\sigma_-, L, \sigma_+], x \in F\}$$

and

$$\left\{ \frac{\pi^2 N^x(s)}{(\log s)^2}, x \in F \right\} \stackrel{d}{\rightarrow} \{\sigma_- + \sigma_+, x \in F\},$$

where

(i) for each x , $[\sigma_-^x, L, \sigma_+]$ has the distribution of $[\sigma_-, L, \sigma_+]$ as defined in Theorem 1;

(ii) the random variables $(\sigma_-^x, x \in F)$ and σ_+ are mutually conditionally independent given L .

Each of the various counting processes $N(s)$ introduced above, such as

$$N(s) = N_{\text{medium}}^x(s), \quad s \geq 0,$$

has a compensator of the form

$$(1.2) \quad A(s) = \int_0^s f(C_t) dt,$$

where f is a nonnegative function defined on the line. Since time changing a counting process by its compensator yields a Poisson process,

$$(1.3) \quad \frac{N(s)}{A(s)} \rightarrow 1 \quad \text{a.s. as } s \rightarrow \infty.$$

Thus the statements of Theorems 1 and 2 are equivalent to modified statements with counting processes replaced by their compensators. As such, these results become statements about the asymptotic behaviour of certain continuous additive functionals of the Cauchy process.

According to the ratio ergodic theorem for additive functionals A_i of the Cauchy process obtained from functions f_i as in (1.2), as $s \rightarrow \infty$

$$(1.4) \quad \frac{A_1(s)}{A_2(s)} \rightarrow \frac{\int f_1(y) dy}{\int f_2(y) dy} \quad \text{a.s.}$$

See, for example, Maruyama and Tanaka (1959). The intensity $f_{\text{medium}}^x(y)$ for medium size jumps across x initiated when the path is at y is readily calculated using the Lévy measure of the Cauchy process $\mu(dx) = dx/\pi x^2$:

$$f_{\text{medium}}^x(y) = \frac{1}{\pi} \left[\frac{1}{a + |x - y|} - \frac{1}{b + |x - y|} \right].$$

For each x the integral of this function is $(2/\pi)\log(b/a)$. In view of these remarks, the asymptotic distribution given in Theorem 1 for $N_{\text{medium}}^x(s)$ amounts to the result of Kasahara (1982) that for additive functionals $A(s)$ of the form (1.2),

$$(1.5) \quad \frac{A(s)}{\log s} \xrightarrow{d} \frac{L}{2\pi} \int f(y) dy.$$

The corresponding functions for the large and small counts,

$$f_{\text{large}}^x(y) = \frac{1}{\pi} \left[\frac{1}{b + |x - y|} \right],$$

$$f_{\text{small}}^x(y) = \frac{1}{\pi} \left[\frac{1}{|x - y|} - \frac{1}{a + |x - y|} \right],$$

both fail to be integrable.

But in the same way, Theorem 2 specifies the asymptotic distribution with normalisation by $\log^2 s$ for the additive functionals obtained from positive finite linear combinations of functions of this form. It is a straightforward matter to push this result further to obtain limit theorems with convergence of finite dimensional distributions for such additive functionals or level crossing counts of the Cauchy process, analogous to (2.2) of Kasahara (1982). The limit is a process $Z^*(t)$ which is a linear combination of the time spent positive and the time spent negative by a Brownian motion B before the time σ_t that it first hits t .

As will be seen from the proof of Theorem 2, the same limit laws may be obtained from particular additive functionals of planar Brownian motion. But we do not know of any general result for Markov processes which would include these results in the way that Kasahara's Theorem 2.1 includes (1.5) and its analog for planar Brownian motion.

2. Spitzer's embedding and windings. Let $Z = (Z(t), t \geq 0)$ be a complex Brownian motion starting at a point $Z(0) = x_0$ on the real axis. That is,

$$Z(t) = X(t) + iY(t),$$

where X and Y are independent one-dimensional Brownian motions starting at $X(0) = x_0$ and $Y(0) = 0$. Let $S(t)$ be the local time of Y at 0 up to time t , and $(\tau_s, s \geq 0)$ the right-continuous inverse of $(S(t), t \geq 0)$. The following result was first exploited by Spitzer (1958):

PROPOSITION 2.1. *The process $(C(s), s \geq 0)$ defined by*

$$(2.1) \quad C(s) = X(\tau_s), \quad s \geq 0,$$

is a symmetric Cauchy process starting at x_0 .

The conclusion of Proposition 2.1 is better known with τ_s replaced by the hitting time σ_s of the point s by the Brownian motion Y . See, for example, Feller (1966), page 348. According to a well-known result of Lévy, the processes $(\sigma_s, s \geq 0)$ and $(\tau_s, s \geq 0)$ determined by Y are identical in law, both stable subordinators with index $\frac{1}{2}$, so the result holds for τ_s as well as for σ_s . Roughly speaking, the process C is Z watched only when it touches the real axis, with a new time parameter. Each excursion of Z away from the real axis corresponds to a unique local time s . The excursion starts at $C(s -)$ and finishes at $C(s)$. Spitzer used this representation of the Cauchy process to calculate the distribution of $C(S_{x,y})$, the position of C at the time $S_{x,y}$ of its first exit from $[x, y]$ for $x < x_0 < y$. For

$$C(S_{x,y}) = Z(T_{x,y}),$$

where

$$T_{x,y} = \inf\{t: Z_t \in (-\infty, x) \cup (y, \infty)\},$$

and the distribution of $Z(T_{x,y})$ may be obtained by solving a Dirichlet boundary value problem in the plane with boundary $(-\infty, x) \cup (y, \infty)$.

Our object here is to use Spitzer's embedding to obtain more detailed information about the sequence of times $(S_n^x, n = 1, 2, \dots)$ when C crosses level $x \neq x_0$, and the associated overshoot sequence $(V_n^x, n = 1, 2, \dots)$ defined by

$$V_n^x = |C(S_n^x) - x|.$$

For simplicity we shall take $x = 0$ and $x_0 = 1$, since results for a general level x and starting point x_0 can easily be derived from this case by simple scaling arguments. The superscript x will be simply omitted in the notation to indicate $x = 0$.

Suppose then that Z starts at $Z(0) = 1$, so C starts at $C(0) = 1$. Let $T_0 = 0$. Let T_1 be the first time that Z hits the negative axis, T_2 the first subsequent time that Z hits the positive axis, T_3 the next time on the negative axis, and so on. Then the corresponding local times $S_n = S(T_n)$ are the successive times at which C jumps across 0, and the successive overshoots are just

$$(2.2) \quad V_n = |C(S_n)| = |Z(T_n)|.$$

Between times T_n and T_{n+1} the planar Brownian motion Z must wind through an angle of either $+\pi$ or $-\pi$ around the origin. Indeed, T_{n+1} is the first time t after T_n that the angle between the points $Z(T_n)$ and $Z(t)$ reaches $\pm\pi$. Consider therefore the continuous total angle $\Phi(t)$ swept around the origin by Z up to time t , a process which is a.s. well defined for all $t \geq 0$ because Z never hits 0 a.s. Then

$$(2.3) \quad T_n = \inf\{t > T_{n-1} : |\Phi(t) - \Phi(T_{n-1})| = \pi\}.$$

More formally, $(\Phi(t), t \geq 0)$ is the imaginary part of the a.s. unique continuous determination of the process $(\log Z(t), t \geq 0)$ starting at $\log Z(0) = 0$. According to the theorem of Lévy (1948) on the conformal invariance of Brownian motion [see also Gettoor and Sharpe (1972) and Pitman and Yor (1986)],

$$\log Z(t) = \zeta_{U(t)},$$

where $U(t)$ is the radial clock

$$(2.4) \quad U(t) = \int_0^t |Z_s|^{-2} ds$$

and

$$\zeta_u = \beta_u + i\theta_u, \quad u \geq 0,$$

where β and θ are two independent Brownian motions starting at zero. Thus

$$(2.5) \quad \log|Z(t)| = \beta_{U(t)}, \quad \Phi(t) = \theta_{U(t)}, \quad t \geq 0.$$

The precise form of the time change (2.4) is quite unimportant here. All that matters is that $U(t)$ is strictly increasing and continuous. So if we define

$$U_n = U(T_n),$$

then from (2.2) and (2.5)

$$(2.6a) \quad \log V_n = \log|Z(T_n)| = \beta(U_n),$$

where

$$(2.6b) \quad U_0 = 0, \quad U_n = \inf\{u: u > U_{n-1}; |\theta_u - \theta(U_{n-1})| = \pi\},$$

$$n = 1, 2, \dots$$

Using the strong Markov property of β and θ at the times U_n , we immediately obtain the following proposition:

PROPOSITION 2.2. (i) *The successive overshoots V_n of zero by a Cauchy process C started at $C_0 = 1$ are such that*

$$(\log V_n, n = 1, 2, \dots)$$

is a random walk with independent and identically distributed increments.

(ii) $\log V_1 \stackrel{d}{=} \beta(U_1)$ where $U_1 = \inf\{u: |\theta_u| = \pi\}$, and β and θ are independent Brownian motions. In particular, the distribution of $\log V_1$ is symmetric with median zero and finite moments of all orders.

Put another way, the expression

$$V_n = V_1 \left(\frac{V_2}{V_1} \right) \cdots \left(\frac{V_n}{V_{n-1}} \right), \quad n \geq 2,$$

represents V_n as a product of n i.i.d. random variables. It is quite easy to see this by induction, using the strong Markov property and the symmetric stable property of the Cauchy process. Indeed this argument shows that part (i) of the proposition holds for a symmetric stable process with index α for any $0 < \alpha < 2$.

Let us write simply V for V_1 . From part (ii) of the proposition we easily obtain the Fourier transform of $\log V$:

$$(2.7a) \quad Ee^{it \log V} = Ee^{it\theta(U_1)} = Ee^{-(1/2)t^2 U_1} = \frac{1}{\cosh t\pi},$$

whence [see Feller (1966), page 503]

$$(2.7b) \quad P(\log V \in dx) = \frac{dx}{2\pi \cosh(x/2)},$$

that is,

$$(2.7c) \quad P(V \in dv) = \frac{dv}{\pi\sqrt{v}(1+v)}.$$

This formula was obtained by Ray (1958) who also gave the corresponding formula for a symmetric stable process.

3. Proof of Theorem 1. This section outlines a proof of Theorem 1 based solely on Proposition 2.2 without further appeal to the theory of windings. By a simple scaling argument, there is no loss of generality in just treating the case $x_0 = 1, x = 0$. Consider then a Cauchy process $(C_s, s \geq 0)$ started at $C_0 = 1$, and let S_n be the time of the n th zero crossing, $V_n = |C(S_n)|$, the n th overshoot.

Put $G_n = \log V_n$. By Proposition 2.2,

$$(G_1, G_2, \dots)$$

may be regarded as a random walk embedded in a Brownian motion β :

$$G_n = \beta(U_n),$$

where U_1, U_2, \dots is a renewal process, independent of β , with $EU_n = n\pi^2$.

Consider first the total number $N(s)$ of zero crossings of C by time s . Let

$$\nu(s) = \inf\{n: V_n \geq s\}.$$

By a simple scaling argument [cf. Pitman and Yor (1986), (3.g)]

$$(3.1) \quad N(s) - \nu(s) \xrightarrow{d} D \quad \text{as } s \rightarrow \infty,$$

where D is defined as follows in terms of a Cauchy process Y starting at 0:

$$\begin{aligned} D &= N(\hat{S}, 1] \quad \text{if } \hat{S} \leq 1 \\ &= -N(1, \hat{S}) \quad \text{if } \hat{S} > 1, \end{aligned}$$

where $N(S, T]$ is the number of zero crossings of Y in the interval $(S, T]$, and

$$\hat{S} = \inf\{s: |Y_s| > 1 \text{ and } Y_s Y_{s-} < 0\}.$$

Now,

$$\nu(s) = \inf\{n: G_n \geq \log s\}$$

and

$$U_{\nu(s)} = \inf\{u: u = U_n \text{ for some } n \text{ and } \beta(U_n) > \log s\}.$$

Let $\sigma_{\log s} = \inf\{u: \beta_u = \log s\}$. See Figure 1.

By renewal theory and the strong Markov property of β at time $\sigma_{\log s}$,

$$(3.2) \quad U_{\nu(s)} - \sigma_{\log s} \xrightarrow{d} \alpha,$$

where

$$\alpha = \inf\{u: u = U_n^* \text{ for some } n \text{ and } \beta_u > 0\}.$$

(U_1^*, U_2^*, \dots) is a renewal process independent of β with the stationary delay distribution

$$P(U_1^* \in dt) = \frac{P(U_1 > t) dt}{\pi^2},$$

and $(U_{n+1}^* - U_n^*, n \geq 1)$ i.i.d. like U_1 . By the strong law of large numbers,

$$(3.3) \quad \frac{\nu(s)}{U_{\nu(s)}} \rightarrow \frac{1}{\pi^2} \quad \text{a.s.}$$

and by Brownian scaling

$$(3.4) \quad \frac{\sigma_{\log s}}{\log^2 s} \xrightarrow{d} \sigma_1.$$

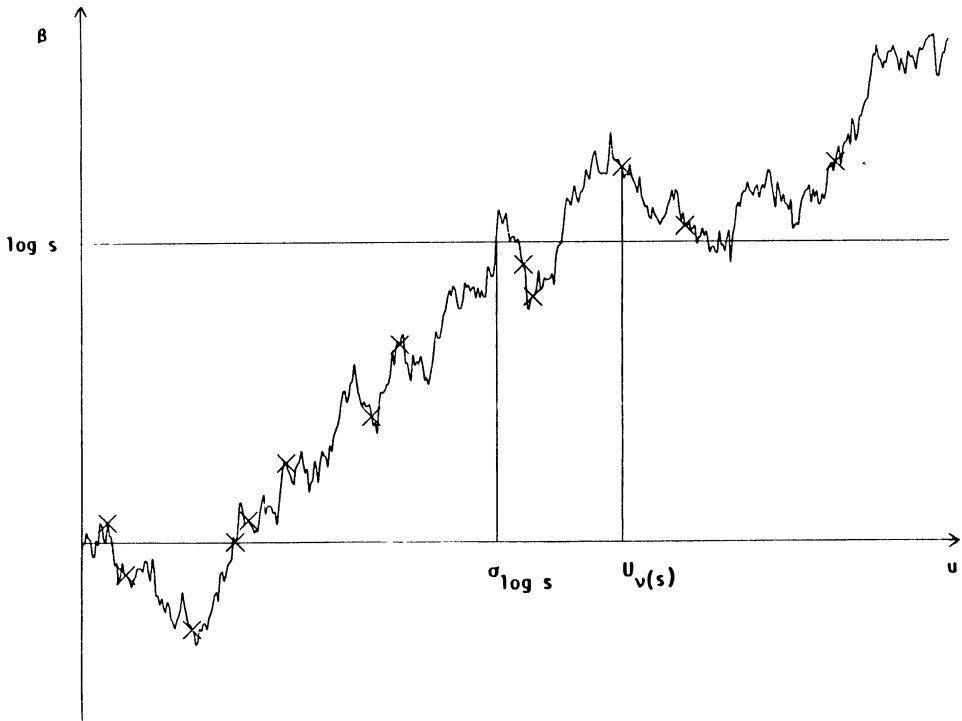


FIG. 1.

Putting together (3.1), (3.2), (3.3), and (3.4), it is now plain that

$$(3.5) \quad \frac{\pi^2 N(s)}{\log^2 s} \xrightarrow{d} \sigma_1.$$

Consider now $N_{\text{small}}(s)$, $N_{\text{medium}}(s)$, and $N_{\text{large}}(s)$. Noting that $\nu(s) = N(S_{\nu(s)})$ it is plain from (3.1) that it suffices to obtain the result of Theorem 1 with $\nu_*(s)$ substituted for $N_*(s)$, where for $*$ = small, medium, or large

$$\nu_*(s) = N_*(S_{\nu(s)}).$$

Now define intervals I_* by

$$\begin{aligned} I_{\text{small}} &= (-\infty, \log a], \\ I_{\text{medium}} &= (\log a, \log b], \\ I_{\text{large}} &= (\log b, \infty). \end{aligned}$$

The number $\nu_*(s)$ is the number $\#_*(s)$ of renewal instants $\{U_1, U_2, \dots\}$ falling in the random set $\mathcal{U}_*(s) = \{u: u < \sigma_{\log s}, \beta_u \in I_*\}$, plus a contribution from renewals in the interval $[\sigma_{\log s}, U_{\nu(s)}]$, which may be safely neglected by a variant of (3.2). Now since the random set $\mathcal{U}_*(s)$ is independent of the renewal process

U_1, U_2, \dots , whose increments have a smooth density, and for each value of $*$ the Lebesgue measure $l_*(s)$ of $\mathcal{U}_*(s)$ tends to ∞ as $s \rightarrow \infty$ by the recurrence of the Brownian motion β , it is to be expected that

$$(3.6a) \quad \frac{\#_*(s)}{l_*(s)} \xrightarrow{P} \frac{1}{\pi^2},$$

which gives

$$(3.6b) \quad \frac{N_*(s)}{l_*(s)} \xrightarrow{P} \frac{1}{\pi^2}.$$

On the other hand, Brownian scaling and the definition of local time as occupation density shows that as $s \rightarrow \infty$

$$(3.7) \quad \left[\frac{l_{\text{small}}(s)}{\log^2 s}, \frac{l_{\text{medium}}(s)}{\log(b/a)\log s}, \frac{l_{\text{large}}(s)}{\log^2 s} \right] \xrightarrow{d} [\sigma_1^-, L, \sigma_1^+],$$

with $[\sigma_1^-, L, \sigma_1^+]$ as in Theorem 1. Clearly (3.6b) and (3.7) yield Theorem 1.

To complete the argument, it only remains to verify (3.6a). Because each of the processes $l_*(s)$ increases to ∞ a.s. as $s \rightarrow \infty$, (3.6a) follows immediately from the following lemma by conditioning on the Brownian motion β :

LEMMA 3.1. *Let U_1, U_2, \dots be a random walk on the real line such that $E U_1 = \mu_1 > 0$, $E|U_1|^3 < \infty$, and the distribution of U_n has a nontrivial absolutely continuous component for some n .*

For a Borel set B let

$$\begin{aligned} \#(B) &= \#\{n \geq 0: U_n \in B\}, \\ l(B) &= \text{Lebesgue measure}(B). \end{aligned}$$

Then there exist constants l_0 and c , depending only on the distribution of U_1 , such that

$$E \left(\frac{\#(B)}{l(B)} - \frac{1}{\mu_1} \right)^2 \leq \frac{c}{l(B)}$$

for all Borel sets $B \subset [0, \infty)$ with $l(B) \geq l_0$.

PROOF. Let $\nu(B) = E\#(B)$. According to Stone (1966), for $B \subseteq \mathbb{R}$

$$\nu(B) = l(B \cap [0, \infty))/\mu_1 + \nu_0(B),$$

where ν_0 is a signed measure on $[0, \infty)$, with absolute mass $\|\nu_0\| < \infty$. But by the identity of Pitman (1974), page 41,

$$E\left\{ \frac{1}{2} \#(B) [\#(B) + 1] \right\} = \int_B \nu(du) \nu(B_u),$$

where $B_u = \{v: u + v \in B\}$. The right side is easily seen to be bounded above by

$$\frac{t^2}{2\mu_1^2} + \frac{2\|\nu_0\|t}{\mu_1} + \|\nu_0\|^2, \quad \text{where } t = l(B).$$

This estimate rearranges to yield the desired result with $c = 8\|v_0\|/\mu_1$, $l_0 = \mu_1(\|v_0\| + \frac{1}{2})$.

4. Proof of Theorem 2. The main difficulty in proving Theorem 2 is to show that for two different levels, say x and y , as $s \rightarrow \infty$, the counts $N_{\text{small}}^x(s)$, $N_{\text{small}}^y(s)$, and $N_{\text{large}}^x(s)$ are asymptotically conditionally independent given $N_{\text{medium}}^x(s)$. The fact that the large and medium counts do not depend on the level in the limit is an easy consequence of Theorem 1, (1.3), and the ratio ergodic theorem (1.4).

Our approach to Theorem 2 will exploit Spitzer’s embedding and its connection with Brownian windings much more fully. Suppose that the Cauchy process $(C(s), s \geq 0)$ is embedded as in Proposition 2.1 as

$$C(s) = Z(\tau_s), \quad s \geq 0,$$

where $C(0) = Z(0) = x_0$, and (τ_s) is the inverse of the local time (S_t) on the real axis. For x in the finite set F let $\Phi^x(t)$ be the continuous total angle wound by Z around x up to time t .

Let D_{small}^x be the open disc centered at x with radius a , D_{large}^x the complement of the open disc centered at x with radius b . For $* = \text{small or large}$ define

$$\Phi_*^x(t) = \int_0^t d\Phi^x(s) 1(Z_s \in D_*^x), \quad t \geq 0.$$

These are the processes of *small windings* and *large windings* about x , as considered in Messulam and Yor (1982) and Pitman and Yor (1986). As in (2.4) let

$$U_*^x(t) = \int_0^t |Z_s - x|^{-2} 1(Z_s \in D_*^x) ds, \quad t \geq 0,$$

the increasing process of the local martingale $(\Phi_*^x(t), t \geq 0)$. Finally, let

$$A(t) = \int_0^t dS_t 1(Z_t \in [0, 1]),$$

the additive functional of Z which measures local time on the unit interval.

According to Theorems 8.2 and 8.4 of Pitman and Yor (1986), as $s \rightarrow \infty$ the joint distribution of

$$\left\{ \left[\frac{\Phi_{\text{small}}^x(\tau_s)}{\log s}, \frac{U_{\text{small}}^x(\tau_s)}{(\log s)^2}, \frac{2\pi A(\tau_s)}{\log s}, \frac{\Phi_{\text{large}}^x(\tau_s)}{\log s}, \frac{U_{\text{large}}^x(\tau_s)}{(\log s)^2} \right], x \in F \right\}$$

converges to the joint distribution of

$$\{[\theta^x(\sigma_-), \sigma_-, L, \theta^\infty(\sigma_+), \sigma_+], x \in F\},$$

where $\{[\sigma^x, L, \sigma_+], x \in F\}$ has the distribution described in the statement of Theorem 2, and $(\theta^x, x \in F)$ and θ^∞ are further mutually independent Brownian motions. Theorem 2 is obtained by applying this result together with the facts that for $* = \text{small or large}$

$$(4.1) \quad \frac{N_*^x(s)}{U_*^x(\tau_s)} \xrightarrow{P} \frac{1}{\pi^2} \quad \text{as } s \rightarrow \infty,$$

and by a change of variables

$$A(\tau_s) = \int_0^s \mathbf{1}(C_v \in [0, 1]) \, dv,$$

so by (1.3) and (1.4)

$$(4.2) \quad \frac{N_{\text{medium}}^x(s)}{A(\tau_s)} \rightarrow \frac{2}{\pi} \log \frac{b}{a} \quad \text{a.s.}$$

Here (4.1) is established by a variation of the argument in Section 3. To make the connection with that section, take $x_0 = 1$, $x = 0$, and let $T_s = \inf\{t: |Z_t| = s\}$, so

$$U_*^0(T_s) = l_*(s),$$

where $l_*(s)$ was defined above (3.6), by a change of variables. Now (3.6b) gives (4.1) with T_s instead of τ_s . But $U_*^0(T_s) - U_*^0(\tau_s)$ converges in distribution as $s \rightarrow \infty$ by Brownian scaling, just as in Pitman and Yor (1986), (3.g).

REFERENCES

- BURDZY, K. (1984). Excursions of complex Brownian motion. Ph.D. thesis, Dept. Statist., Univ. California, Berkeley.
- FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications 2*. Wiley, New York.
- GETTOOR, R. K. and SHARPE, M. J. (1972). Conformal martingales. *Invent. Math.* **16** 271–308.
- KASAHARA, Y. (1982). A limit theorem for slowly increasing occupation times. *Ann. Probab.* **10** 728–736.
- KESTEN, H. (1963). Positivity intervals of stable processes. *J. Math. Mech.* **12** 391–410.
- LÉVY, P. (1948). *Processus Stochastiques et Mouvement Brownien*. Gauthier–Villars, Paris.
- LÉVY, P. (1951). Wiener's random function, and other Laplacian random functions. *Proc. Second Berkeley Symp. Math. Statist. Probab.* 171–187.
- LYONS, T. and MCKEAN, H. P. (1984). Windings of the plane Brownian motion. *Adv. in Math.* **51** 212–225.
- MARUYAMA, G. and TANAKA, H. (1959). Ergodic property of N -dimensional recurrent Markov process. *Mem. Fac. Sci. Kyushu Univ. Ser. A* **13**(2) 157–172.
- MESSULAM, P. and YOR, M. (1982). On D. Williams' "pinching method" and some applications. *J. London Math. Soc.* (2) **26** 348–364.
- PITMAN, J. W. (1974). An identity for stopping times of a Markov process. In *Studies in Probability and Statistics* (E. J. Williams, ed.) 41–57. Jerusalem Academic Press.
- PITMAN, J. W. and YOR, M. (1986). Asymptotic laws of planar Brownian motion. *Ann. Probab.* **14** 733–779.
- RAY, D. (1958). Stable processes with an absorbing barrier. *Trans. Amer. Math. Soc.* **89** 16–24.
- SPITZER, F. (1958). Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87** 187–197.
- STONE, C. (1966). On absolutely continuous components and renewal theory. *Ann. Math. Statist.* **37** 271–275.

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