The Annals of Probability
1986, Vol. 14, No. 4, 1339–1346

ON THE RATE AT WHICH THE SAMPLE EXTREMES BECOME INDEPENDENT

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In this paper we compute exact rates at which the sample extremes become independent uniformly over all Borel sets.

Let P be a probability measure on the real line with distribution function ($\equiv$ df) F. Denote by P^n the n-fold independent product of P and by $Z_{i:n}$ the ith order statistic according to the sample of size n.

The asymptotic independence of the sample extremes under P^n was first observed by Gumbel (1946) who proved in particular that $Z_{k:n}$ and $Z_{n-m+1:n}$ may be dealt with as independent variables, provided that n is large, k and m are small, and that P is a continuous unlimited distribution of exponential type.

This exponential type assumption on P essentially means that P satisfies the well-known von Mises (1936) condition on a distribution to belong to the domain of attraction of $\exp(-e^{-x})$ (see, e.g., Chapter 2.7 of the book by Galambos (1978)). Consequently, Gumbel's (1946) proof runs via the approximation of the df of $(Z_{k:n}, Z_{n-m+1:n})$ by the product of the according limiting extreme value df's.

For such distributions which belong to the domain of attraction of an extreme value distribution, the following general result can be proved (see Theorem 2.9.1 of Galambos (1978)): If the asymptotic (nondegenerate) distribution of each of $Z_{k:n}$ and $Z_{n-m+1:n}$ exists (m and k fixed), when suitably normalized, then, with the same normalization, the joint distribution of $Z_{k:n}$ and $Z_{n-m+1:n}$ tends weakly to the product of the marginal limiting distributions.

However, since not every probability measure P belongs to the domain of attraction of an extreme value distribution (take for example $F(x) = 1 - 1/\log(x)$, $x \geq e$), the question still remained, under which conditions on P the sample extremes become independent as the sample size increases.

Concerning the maximum distance of the respective df's, a complete answer was given by Rossberg (1965, Satz 1; 1967) (see also page 270 of the book by David (1981)), which implies that

$$\sup_{x, y \in \mathbf{R}} \left| P^n\{Z_{k:n} \leq x, Z_{n-m+1:n} \leq y\} - P^n\{Z_{k:n} \leq x\} P^n\{Z_{n-m+1:n} \leq y\} \right| \underset{n\to\infty}{\to} 0$$

for any distribution P if $(k+m)/n \to_{n\to\infty} 0$.

For the case that F has a density f, Ikeda and Matsunawa (1970) proved that, if $(k+m)/n \to_{n\to\infty} 0$,

$$\begin{aligned} \Delta_P(n,k,m) := \sup_{B \in \mathscr{B}^{k+m}} \Big| & P^n * (Z_{1:n}, \dots, Z_{k:n}, Z_{n-m+1:n}, \dots, Z_{n:n})(B) \\ & - \left(P^n * (Z_{i:n})_{i=1}^k\right) \times \left(P^n * (Z_{j:n})_{j=n-m+1}^n\right)(B) \Big| \underset{n\to\infty}{\to} 0, \end{aligned} \tag{1}$$

Received November 1984.
AMS 1980 *subject classification*. Primary 62G30.
Key words and phrases. Extreme order statistics, asymptotic expansion.

where $\mathscr{B}^d$ denotes the Borel σ-algebra of $\mathbb{R}^d$, $P_1 \times P_2$ the independent product of P_1 and P_2 and $P * g$ the measure induced by P and a measurable function g (see also Ikeda (1963)).

Now, the equality $P = Q * F^{-1}$, where Q denotes the uniform distribution on $(0,1)$ and F^{-1} the generalized inverse of F, implies $\Delta_P(n,k,m) \le \Delta_Q(n,k,m)$ for any distribution P and thus, the above result by Ikeda and Matsunawa (1970) holds for arbitrary P.

In the following result we can even show that the value of $\Delta_P(n,k,m)$ does not depend on P if F is continuous and therefore, the exact rates of convergence of $\Delta_Q(n,k,m)$ to zero which we will compute in the following are also exact for any nonatomic probability measure P.

PROPOSITION 2. *For* $k, m \in \{1,\dots,[n/2]\}$, $n \in \mathbb{N}$,

(i) $$\Delta_P(n,k,m) = \Delta_Q(n,k,m) \quad \text{if } F \text{ is continuous},$$

(ii) $$\Delta_P(n,k,m) \le \Delta_Q(n,k,m) \quad \text{for arbitrary } F.$$

PROOF. The assertions follow immediately from the equalities $P = Q * F^{-1}$ and, for continuous F, $Q = P * F$. □

Notice that $\Delta_P(n,k,m) = 0$ if P is concentrated on a single point.

Denote by p_k the Lebesgue density of the negative gamma distribution with parameter k, i.e., $p_k(x) := (-x)^{k-1}\exp(x)/(k-1)!$, $x \le 0$, $k \in \mathbb{N}$. Then our main result is the following one: ($x \wedge y := \min(x,y)$ and $x \vee y := \max(x,y)$).

THEOREM 3. *Let* $k = k(n)$, $m = m(n) \in \{1,\dots,n\}$, *satisfy*

$$(k+m)/n \to_{n\in\mathbb{N}} 0.$$

Then,

$$\Delta_Q(n,k,m) = \frac{1}{2n}\int_{-k-n^{1/2}}^{0\wedge(-k+n^{1/2})} |x+k|\,p_k(x)\,dx \int_{-m-n^{1/2}}^{0\wedge(-m+n^{1/2})} |y+m|\,p_m(y)\,dy + O\left(\frac{(k+m)^2}{n^2}\right),$$

where this expansion holds uniformly for all sequences k, m.

Notice that

$$\begin{aligned}\int_{-k-n^{1/2}}^{0\wedge(-k+n^{1/2})} |x+k|\,p_k(x)\,dx &= 2k^k\frac{\exp(-k)}{(k-1)!} \\ &\quad -(k+n^{1/2})^k\frac{\exp(-k-n^{1/2})}{(k-1)!} \\ &\quad -\left(0\vee(k-n^{1/2})\right)^k\frac{\exp(-k+n^{1/2})}{(k-1)!} \\ &= 2k^k\frac{\exp(-k)}{(k-1)!} + o(k^{1/2})\end{aligned} \tag{4}$$

and, if $k \to \infty$,

$$2k^k \frac{\exp(-k)}{(k-1)!} = \left(\frac{2}{\pi}\right)^{1/2} k^{1/2} + o(k^{1/2}),$$

yielding the following consequences to the above theorem.

COROLLARY 5.

$$\lim_{n\to\infty} \Delta_Q(n, k, m) = 0 \quad \text{if } (k+m)/n \underset{n\to\infty}{\to} 0.$$

COROLLARY 6. *For k and m fixed we have*

$$\Delta_Q(n, k, m) = C(k, m)n^{-1} + O(n^{-2}),$$

where

$$C(k, m) := 2k^k m^m \frac{\exp(-m-k)}{\{(k-1)!(m-1)!\}}.$$

COROLLARY 7. *If $k, m = O(n^{2/3})$ both tend to infinity as n increases, then*

$$\frac{n}{(km)^{1/2}} \Delta_Q(n, k, m) \underset{n\to\infty}{\to} \pi^{-1}.$$

Corollary 6 can in particular be utilized to determine approximately the least sample size $n_{k,m}(\delta)$ such that $\Delta_Q(n, k, m) \le \delta$, $\delta \in (0, 1)$.

For the case $k = m = 1$ and $\Delta_Q(n, k, m)$ replaced by the maximum difference of the respective df's, this problem was investigated by Walsh (1970), who also computed the exact smallest sample sizes for δ ranging between 0.007 and 0.0902.

Corollary 6 immediately implies that for k and m fixed

$$\lim_{\delta\to 0} \frac{n_{k,m}(\delta)}{\{C(k, m)/\delta\}} = 1 \tag{8}$$

and hence, $n_{k,m}(\delta)$ is given approximately by $C(k, m)/\delta$, at least if δ is small.

In particular for $k = m = 1$ we get

$$n_{1,1}(\delta) \sim 2\frac{\exp(-2)}{\delta} \tag{9}$$

yielding approximations for $n_{1,1}(\delta)$ if $\delta \in (0.007, 0.0902)$ which are nearly exactly twice the minimum sample sizes according to the maximum difference of the respective df's given in the table on page 861 of Walsh (1970).

Furthermore, we note that if one is interested only in the case of single order statistics, i.e., in

$$\begin{aligned}\Delta_P^*(n, k, m) := \sup_{B\in\mathscr{B}^2} \big|&P^n * (Z_{k:n}, Z_{n-m+1:n})(B) \\ &-(P^n * Z_{k:n}) \times (P^n * Z_{n-m+1:n})(B)\big|,\end{aligned} \tag{10}$$

then it is easy to see that again

$$\begin{aligned}
\Delta_P^*(n, k, m) &= \Delta_Q^*(n, k, m) \quad \text{if } F \text{ is continuous,} \\
\Delta_P^*(n, k, m) &\le \Delta_Q^*(n, k, m) \quad \text{for arbitrary } F.
\end{aligned} \tag{11}$$

Moreover, one can prove the following result along the lines of the proof of Theorem 3. (For an analogous result concerning asymptotic expansions of extreme order statistics we refer to Theorem 3.2 of Reiss (1981)).

THEOREM 12. *Let k, m satisfy $(k+m)/n \to_{n \in \mathbb{N}} 0$. Then,*

$$\left|\Delta_Q(n, k, m) - \Delta_Q^*(n, k, m)\right| = O\left(\frac{(k+m)^2}{n^2}\right),$$

where this bound holds uniformly for all sequences k, m.

Consequently, the preceding results from Theorem 3 up to (9) remain valid with $\Delta_Q(n, k, m)$ replaced by $\Delta_Q^*(n, k, m)$.

For the sake of completeness we finally mention that independence results also including central order statistics or sample means were established by Tiago de Oliveira (1961), Rosengard (1962), Rossberg (1965), and Ikeda and Matsunawa (1970), among others, and the case where the observations come from different underlying probability measures was dealt with by Walsh (1969). We remark that the following proof of Theorem 3 might be extended to cover these cases as well. For the sake of a clear presentation, however, we concentrate on the extremes only.

PROOF OF THEOREM 3. First observe that with

$$A_n := \left\{(\mathbf{u}, \mathbf{v}) \in (-n, 0)^{k+m}: u_1 > \cdots > u_k, v_1 < \cdots < v_m, u_k + v_1 > -n\right\}$$

and

$$B_n := \left\{(\mathbf{u}, \mathbf{v}) \in (-n, 0)^{k+m}: u_1 > \cdots > u_k, v_1 < \cdots < v_m\right\}$$

we have (see formula 2.2.3 of David (1981))

$$\begin{aligned}
\Delta_Q(n, k, m) = \sup_{B \in \mathscr{B}^{k+m}} \Big| & Q^n * (-nZ_{1:n}, \dots, -nZ_{k:n}, n(Z_{n-m+1:n} - 1), \dots, \\
& n(Z_{n:n} - 1))(B) \\
& - \left(Q^n * (-nZ_{i:n})_{i=1}^k\right) \cdot \left(Q^n * \left(n(Z_{j:n} - 1)\right)_{j=n-m+1}^n\right)(B)\Big| \\
= 2^{-1} \int_{\mathbb{R}^{k+m}} \Bigg| & \frac{n!}{\left(n^{m+k}(n-m-k)!\right)} \left(1 + \frac{x_k + y_1}{n}\right)^{n-m-k} 1_{A_n}(\mathbf{x}, \mathbf{y}) \\
& - \frac{n!^2}{\{n^{m+k}(n-k)!(n-m)!\}} \\
& \times \left(1 + \frac{x_k}{n}\right)^{n-k} \left(1 + \frac{y_1}{n}\right)^{n-m} 1_{B_n}(\mathbf{x}, \mathbf{y}) \Bigg| \, d(\mathbf{x}, \mathbf{y})
\end{aligned} \tag{13}$$

$$= 2^{-1}\int_{\mathbb{R}^{k+m}}\left|\left(n!\frac{(n-m-k)^{n-m-k}}{\{n^n(n-m-k)!\}}\right)\right.$$
$$\times\left(1+\frac{x_k+y_1+m+k}{n-m-k}\right)^{n-m-k}\cdot 1_{A_n}(\mathbf{x},\mathbf{y})$$
$$-\left(n!^2(n-k)^{n-k}\frac{(n-m)^{n-m}}{\{n^{2n}(n-k)!(n-m)!\}}\right)$$
$$\left.\times\left(1+\frac{x_k+k}{n-k}\right)^{n-k}\left(1+\frac{y_1+m}{n-m}\right)^{n-m}1_{B_n}(\mathbf{x},\mathbf{y})\right|d(\mathbf{x},\mathbf{y}),$$

where 1_A denotes the indicator function of a set A.

After the preceding standardization, our proof will be analogous to the proof of the asymptotic expansion of $Q^n * (n(Z_{n-k+1:n} - 1))$ established in Theorem 2.6 of Reiss (1981). For some details we will therefore refer to this article.

Define now

$$C_n := \left\{\mathbf{u}\in(-n,0)^k: u_1 > \cdots > u_k, |u_k+k| < n^{1/2}\right\}$$
$$\times\left\{\mathbf{v}\in(-n,0)^m: v_1 < \cdots < v_m, |v_1+m| < n^{1/2}\right\}.$$

Then $C_n \subset A_n$ ($\subset B_n$) if n is large and thus, by (13),

$$\Delta_Q(n,k,m) = 2^{-1}\int_{C_n}\left|\left(n!\frac{(n-m-k)^{n-m-k}}{\{n^n(n-m-k)!\}}\right)\left(1+\frac{x_k+y_1+m+k}{n-m-k}\right)^{n-m-k}\right.$$
$$-\left(n!^2(n-k)^{n-k}\frac{(n-m)^{n-m}}{\{n^{2n}(n-k)!(n-m)!\}}\right)$$
$$\left.\times\left(1+\frac{x_k+k}{n-k}\right)^{n-k}\left(1+\frac{y_1+m}{n-m}\right)^{n-m}\right|d(\mathbf{x},\mathbf{y}) + R_n$$

$$(14)\quad = 2^{-1}\int_{-k-n^{1/2}}^{0\wedge(-k+n^{1/2})}\int_{-m-n^{1/2}}^{0\wedge(-m+n^{1/2})}p_k(x)p_m(y)$$
$$\times\left|\left(n!\frac{(n-m-k)^{n-m-k}}{\{n^n(n-m-k)!\}}\right)\right.$$
$$\times\exp(-x-y)\left(1+\frac{x+y+m+k}{n-m-k}\right)^{n-m-k}$$
$$-\left(n!^2(n-k)^{n-k}\frac{(n-m)^{n-m}}{\{n^{2n}(n-k)!(n-m)!\}}\right)$$
$$\left.\times\exp(-x-y)\left(1+\frac{x+k}{n-k}\right)^{n-k}\left(1+\frac{y+m}{n-m}\right)^{n-m}\right|dx\,dy + R_n.$$

Now, it is easy to see that

$$R_n = O\left(\frac{(k+m)^2}{n^2}\right) \tag{15}$$

(use, for example, Lemma 1 in Wellner (1977)).

Furthermore, we will utilize the following inequalities (see formula (21.10) of Krickeberg and Ziezold (1977)):

$$\begin{aligned}&(2\pi)^{1/2} n^{n+1/2} \exp(-n) \exp\left(\frac{1}{12n+1}\right)\\ &\quad< n! < (2\pi)^{1/2} n^{n+1/2} \exp(-n)\exp\left(\frac{1}{12n}\right), \qquad n \in \mathbb{N},\end{aligned} \tag{16}$$

leading to

$$\begin{aligned}&2^{-1}\log(2\pi) + \left(n+\tfrac{1}{2}\right)\log(n) - n + (12n)^{-1} - (12n(12n+1))^{-1}\\ &\quad< \log(n!) < 2^{-1}\log(2\pi) + \left(n+\tfrac{1}{2}\right)\log(n) - n + (12n)^{-1},\end{aligned} \tag{17}$$

and the expansions

$$\begin{aligned}\log\left(1+\frac{x+y+m+k}{n-m-k}\right) &= \frac{x+y+m+k}{n-m-k}\\ &\quad - \frac{(x+y+m+k)^2}{2(n-m-k)^2} + O\left(\frac{(x+y+m+k)^3}{n^3}\right),\\ &\qquad |x+k| < n^{1/2},\ |y+m| < n^{1/2},\end{aligned} \tag{18}$$

$$\begin{aligned}\log\left(1+\frac{x+k}{n-k}\right) = \frac{x+k}{n-k} - \frac{(x+k)^2}{2(n-k)^2} + O\left(\frac{(x+k)^3}{n^3}\right),\\ |x+k| < n^{1/2}.\end{aligned} \tag{19}$$

Employing (17)–(19) in (13) we derive

$$\begin{aligned}\Delta_Q(n,k,m) &= 2^{-1}\int_{-k-n^{1/2}}^{0\wedge(-k+n^{1/2})}\int_{-m-n^{1/2}}^{0\wedge(-m+n^{1/2})} p_k(x)p_m(y)\\ &\quad\times\Bigg|\exp\Bigg\{\frac{m+k}{2n} + (12n)^{-1} - (12(n-m-k))^{-1}\\ &\quad\quad - \frac{(x+y+m+k)^2}{2(n-m-k)} + O\left(\frac{(x+y+m+k)^3}{n^2} + \frac{(m+k)^2}{n^2}\right)\Bigg\}\\ &\quad - \exp\Bigg\{\frac{k+m}{2n} + (6n)^{-1} - (12(n-k))^{-1}\end{aligned} \tag{20}$$

$$- (12(n-m))^{-1} - \frac{(x+k)^2}{2(n-k)} - \frac{(y+m)^2}{2(n-m-k)}$$

$$+ O\left(\frac{(x+k)^3 + (y+m)^3}{n^2} + \frac{(k+m)^2}{n^2}\right)\Bigg\}\Bigg|\, dx\, dy$$

$$+ O\left(\frac{(k+m)^2}{n^2}\right).$$

Using the expansion $\exp(x) = 1 + x + O(x^2)$, $|x| \le C$, and the bound $\int |x + k|^i p_k(x)\, dx \le i! k^{i/2}$ (see formula (2.3) in Reiss (1981)), it is now easy to see that (20) yields

$$\Delta_Q(n,k,m) = 2^{-1} \int_{-k-n^{1/2}}^{0 \wedge (-k+n^{1/2})} \int_{-m-n^{1/2}}^{0 \wedge (-m+n^{1/2})} p_k(x) p_m(y)$$

$$\times \left| \frac{(x+y+m+k)^2}{2(n-m-k)} - \frac{(x+k)^2}{2(n-k)} - \frac{(y+m)^2}{2(n-m)} \right| dx\, dy$$

$$(21) \qquad + O\left(\frac{(k+m)^2}{n^2}\right)$$

$$= 2^{-1} \int_{-k-n^{1/2}}^{0 \wedge (-k+n^{1/2})} \int_{-m-n^{1/2}}^{0 \wedge (-m+n^{1/2})} p_k(x) p_m(y) |x+k|$$

$$\times \frac{|y+m|}{n-m-k}\, dx\, dy + O\left(\frac{(k+m)^2}{n^2}\right),$$

which completes the proof. □

Acknowledgment. The authors would like to thank R.-D. Reiss for stimulating this research work.

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