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A SUFFICIENT CONDITION FOR ASSOCIATION OF A RENEWAL PROCESS

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Let $\{N(t)\colon t \geq 0\}$ be a renewal counting process with lifetime density $f(t)$. For each bounded Borel set A contained in $[0, \infty)$, denote the number of renewals in A by $N(A)$. The renewal process is called associated if the corresponding family of random variables, $N(A)$, is associated. The result of this note is that the renewal process is associated whenever $\log(f)$ is a convex function (which implies a decreasing failure rate).

Let $N = \{N(t) = N[0, t)\colon t \geq 0\}$ be a renewal counting process with lifetime density $f(t)$ and reliability function given by $F^c(t) = \int_t^\infty f(u)\,du$. We make the tacit assumption that the first renewal occurs at 0, so that the lifetime density is applicable from $t = 0$, without further specific mention of this point. The lifetime distribution has decreasing failure rate (DFR), if the failure rate function $f(t)/F^c(t)$ is decreasing. Let $g(t) = \log f(t)$. If g is convex then it is well known that the lifetime distribution has DFR; see Barlow and Proschan (1975), where, in addition, examples of such processes from engineering and biology are also described.

A nonempty, but possibly infinite, family $\mathscr{F}$ of random variables is called associated if, whenever $X_1, \dots, X_n \in \mathscr{F}$, $\operatorname{Cov}\{g(X_1, \dots, X_n), h(X_1, \dots, X_n)\} \geq 0$, for any (coordinatewise) nondecreasing real-valued functions g and h for which the indicated covariance exists. The literature in probability theory is rapidly becoming rich with results and applications for associated random variables. The most important recent result is the central limit theorem for associated random variables originally discovered by Newman (1980). Earlier results and applications of the notion of association in the context of reliability theory can be found in Barlow and Proschan (1975).

We extend the definition of N to any bounded Borel set A of nonnegative real numbers by setting $N(A)$ equal to the number of renewals which occur in A. The renewal process is then defined to be associated if the corresponding family of counting variables, $N(A)$, is associated.

Define Bernoulli random variables $Y(J)$, indexed by subintervals J of $[0, \infty)$, by $Y(J) = 0$ if $N(J) > 1$, and by $Y(J) = 1$ if $N(J) = 0$. It is not hard to check that if N is associated then the family of variables $Y(J)$ is associated; because $1 - Y(J) = \min\{1, N(J)\}$ is a nondecreasing function of the $N(J)$'s. Observe

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that for $J_1 = [0, \delta)$ and $J_2 = [\delta, 2\delta)$,

$$\begin{aligned} &\operatorname{Cov}\{Y(J_1), Y(J_2)\} \\ &\quad = P(N(J_1) = N(J_2) = 0) - P(N(J_1) = 0)P(N(J_2) = 0) \\ &\quad = F^c(2\delta) - F^c(\delta)P(N(J_2) = 0) \\ &\quad = F^c(2\delta) - F^c(\delta)\int_0^\delta \{F^c(2\delta - t)/F^c(\delta - t)\}H(dt), \end{aligned} \tag{0}$$

where the ratio $F^c(2\delta - t)/F^c(\delta - t)$ is the conditional probability that there are no renewals in J_2 given that the last renewal before δ is at time t, and $H(dt)$ represents the probability of the last occurrence before δ within dt of t. If the lifetime distribution has an increasing failure rate, equivalently, $\log F^c$ is concave (or f/F^c increasing in the case of a density), then, for $\log F^c$ strictly concave in the neighborhood $[0, 2\delta)$ and having no atoms, one can check that the covariance in (0) is strictly less than zero. In particular, such a process is not associated. The strict concavity rules out the case of the Poisson renewals which has increasing failure rate, though degenerately, and is nonetheless associated; see Burton and Waymire (1985). Our interest in the present note is to get a positive result in the case of DFR. In particular, we establish the following theorem.

THEOREM 1. *Let N be a renewal process whose lifetime density is log-convex. Then N is associated.*

The proof will be given as a consequence of two lemmas. First, though, we require some special notation for the description of the sample realizations of the process.

Let T be an arbitrary positive real number and let $R = \{x_1, \dots, x_n\}$ be a nonempty finite subset of $[0, T]$, with $0 = x_0 < x_1 < x_2 < \cdots < x_n < T$. R corresponds in a natural way to a possible sample path of N on the interval $[0, T]$. For each $i = 1, \dots, n$, $[x_{i-1}, x_i]$ is called an *intermediate segment* of length $a_i = x_i - x_{i-1}$, and $[x_n, T]$ is a *terminal segment* of length $a_{n+1} = T - x_n$. We define the absolute product densities by,

$$r_T(R) = F^c(a_{n+1})\prod_1^n f(a_i). \tag{1}$$

In the case $R = \phi$, take $n = 0$ and assign the empty product in (1) the value one. For small enough Δx, $r_T(R)(\Delta x)^n$ is approximately the probability that there are renewals in each of the intervals of length Δx about $x_1, \dots, x_n$, and no other renewals in $[0, T]$.

A well-known sufficient condition for a family of random variables to be associated is the "convexity" condition of Fortuin, Kasteleyn, and Ginibre (1971). In the context of point processes this condition takes the form of (2) below; see also Burton and Waymire (1985). In particular, the renewal process will be associated provided that the following relation holds for all $T > 0$ and all finite

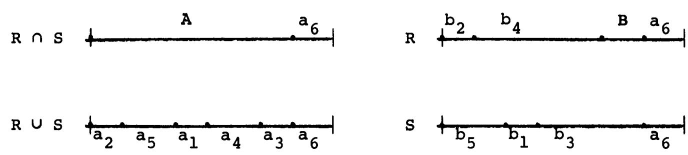

FIG. 1.

subsets R and S of $[0, T]$:

$$r_T(R \cup S) r_T(R \cap S) > r_T(R) r_T(S). \tag{2}$$

The inequality (2), and consequently Theorem 1, is established as a consequence of the following two lemmas.

LEMMA 2. *Suppose that $R \cup S$ consists of the points $0 < x_1 < \cdots < x_n < T$, and $R \cap S = \{x_n\}$. Then inequality* (2) *holds.*

PROOF. Label the lengths of the intermediate segments of $R \cup S$ in nondecreasing order as $a_1 \le a_2 \le \cdots \le a_n$, and let a_{n+1} be the length of the terminal segment of $R \cup S$ (as well as that of $R \cap S$). Set $A = \sum_1^n a_i$. It is clear by a marriage lemma argument [i.e., Philip Hall's theorem on distinct representatives; see Mirsky and Perfect (1966)], or by direct construction, that for each $i = 1, 2, \dots, n$, there is a (distinct) intermediate segment from R or S of length b_i that covers the intermediate segment corresponding to a_i. In particular $b_i \ge a_i$ for each $i = 1, \dots, n$. The length of the remaining segment is $B = 2A - \sum_1^n b_i$. Figure 1 illustrates a partial case.

It now follows that

$$r_T(R \cup S) r_T(R \cap S) = f(A) \prod_1^n f(a_i) F^c(a_{n+1})^2$$

and

$$r_T(R) r_T(S) = f(B) \prod_1^n f(b_i) F^c(a_{n+1})^2.$$

So it suffices to show that

$$f(A) \prod_1^n f(a_i) \ge f(B) \prod_1^n f(b_i). \tag{3}$$

Taking logarithms, we will establish the equivalent

$$g(A) + \sum_1^n g(a_i) \ge g(B) + \sum_1^n g(b_i). \tag{4}$$

First notice that because $a_1, \dots, a_k$ are the kth smallest of the a_i's, we have $a_1 + \cdots + a_k \le B + b_1 + \cdots + b_{k-1}$, or $a_k \le B + \sum_1^{k-1}(b_i - a_i)$. Since g is convex, whenever $0 < \Delta$ and $x < y$, we have $g(x + \Delta) - g(x) \le g(y + \Delta) - g(y)$. Applying this with $\Delta = b_k - a_k$, $x = a_k$, and $y = B + \sum_1^{k-1}(b_i - a_i)$, we get,

$$(5)\qquad g(b_k) - g(a_k) \le g\left(B + \sum_1^k (b_i - a_i)\right) - g\left(B + \sum_1^{k-1} (b_i - a_i)\right).$$

Sum (5) over $k = 1, 2, \dots, n$ to get,

$$\begin{aligned}\sum_1^n \{g(b_k) - g(a_k)\} &\le \sum_1^n \left[g\left(B + \sum_1^k \{b_i - a_i\}\right) - g\left(B + \sum_1^{k-1} \{b_i - a_i\}\right)\right] \\ &= g\left(B + \sum_1^n \{b_i - a_i\}\right) - g(B) \\ &= g(A) - g(B),\end{aligned}$$

which is a rearrangement of (4). □

REMARK. It is important for the proof of the next lemma to notice that the terminal segment probability cancels in the calculation resulting in inequality (3). This particular inequality is merely a property of log-convex functions for suitable orderings of the sets of real numbers.

LEMMA 3. *Suppose that $R \cap S = \phi$ in Lemma 2. Then inequality (2) holds.*

PROOF. Let $a_1, \dots, a_n$ be the lengths of the intermediate segments in $R \cup S$, and let a_{n+1} be the length of the terminal segment in $R \cup S$. Set $A = \sum_1^{n+1} a_i$. Let $b_1, \dots, b_n$ be the lengths of the intermediate segments in R and in S, respectively. One of R or S has a terminal segment of length a_{n+1}, let B denote the length of the other terminal segment. By calculations as in Lemma 2, see the remark following the proof of Lemma 2,

$$(6)\qquad f(A) \prod_1^{n+1} f(a_i) \ge f(B) f(a_{n+1}) \prod_1^n f(b_i).$$

Since $g = \log(f)$ is convex, the failure rate f/F^c is decreasing. In particular, $f(A)/F^c(A) \le f(B)/F^c(B)$. Substituting this into (6) we get,

$$(7)\qquad F^c(A) \prod_1^n f(a_i) \ge F^c(B) \prod_1^n f(b_i),$$

which is (2). □

PROOF OF THEOREM 1. Let $A_1, \dots, A_k$ be the lengths of the intermediate segments in $R \cap S$, and let A_{k+1} be the length of the terminal segment of $R \cap S$.

For $i = 1, \dots, k+1$, let $a_1^i, \dots, a_{j_i}^i$ be the segments corresponding to A_i. Also let $b_1^i, \dots, b_{1_i}^i$ be the lengths of the segments in R and S contained in segments corresponding to A_i. Then by Lemma 2 for $i = 1, 2, \dots, k$, and by Lemma 3 for

$i = k + 1$, we have for each $i = 1, 2, \ldots, k + 1$,

$$f(A_i)\prod_1^{j_i} f(a_m^i) \geq \prod_1^{l_i} f(b_m^i). \tag{8}$$

Therefore, taking the product over $i = 1, \ldots, k$ in (8), one obtains the inequality (2). □

NOTE. The authors recently learned from Thomas Liggett (private communication) that a version of the result presented here can also be established with a very interesting argument using the machinery of interacting particle systems. Specifically, log-convexity of the (positive) lifetime density $f(t)$, for integer lattice lifetimes, makes the nearest particle system, with constant unit death rate and with birth rates of the form, $\beta(m, n) = f(m)f(n)/f(m + n)$, attractive [see Liggett (1983)]. Under a finite first-moment assumption on f, the distribution of the renewal process with lifetime density f is the unique time-reversible equilibrium state for the system concentrated on configurations with infinitely many occupied sites to the left and right of the origin [see Spitzer (1977)]. Moreover the system started with all sites occupied will converge time-asymptotically in distribution to the renewal measure corresponding to f [see Liggett (1983)]. A version of our result now follows from Harris' inequality after a suitable finite system approximation [see Harris (1977) and Cox (1984)]. The case of noninteger lattice lifetimes is then obtainable by approximations; see Burton and Waymire (1985) for similar approximations.

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