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A ONE-DIMENSIONAL DIFFUSION PROCESS IN A WIENER MEDIUM

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We consider a one-dimensional diffusion process in a random medium, which is itself generated by a Wiener process. The asymptotic behaviour is investigated, using the self-similarity of the medium process. The results are analogous to the lattice case.

1. Introduction. In [4] the behaviour of a one-dimensional random walk in a random medium with independent increments was investigated and it was shown, among other things, that $x(n)$ is of the order of $(\log n)^2$, i.e., the "average speed" of $x(n)$ is surprisingly slow. In this paper we will treat the same problem for a one-dimensional diffusion process in a continuous random medium and show that things will, then, become more transparent. We will take as a medium a sample path of a Wiener process. An advantage of the continuous framework is the proper self-similarity of the basic processes. Using this, we can reformulate the problem to be dealt with in a more familiar form. We remark that the importance of self-similarity was already conjectured in [2].

In order to avoid unnecessary difficulties (say, explosions) we restrict the set of possible realizations of media by an extra condition, which is satisfied by almost all Wiener paths. Let $\mathscr{W}$ be the set of all continuous functions W: $\mathbb{R} \mapsto \mathbb{R}$ satisfying $W(0) = 0$ and

$$\left| \int_0^x 1_{[c,\infty)}(W(z))\,dz \right| \to \infty \quad \text{as } |x| \to \infty$$

for each $c \in \mathbb{R}$. We denote by $(\mathscr{W}, \mathscr{A}, \nu)$ the Wiener space, where $\mathscr{A}$ is the σ-field generated by all Borel cylinder sets and ν is the Wiener measure on $(\mathscr{W}, \mathscr{A})$, i.e., the coordinate process $\{W(x);\ x \in \mathbb{R}\}$ has independent homogeneous (Gaussian) increments with respect to ν. The Wiener space $(\mathscr{W}, \mathscr{A}, \nu)$ serves as a model of a random medium, i.e., a sample path $W \in \mathscr{W}$ is considered as a random potential. A well known property of such a medium is self-similarity with parameter $\frac{1}{2}$: For each $\alpha > 0$,

$$\{\alpha^{-1}W(\alpha^2 x);\ x \in \mathbb{R}\} =_d \{W(x);\ x \in \mathbb{R}\},$$

where $=_d$ denotes equality in law. In other words, ν is invariant under the transformation

$$\mathscr{W} \ni W \mapsto W^\alpha \in \mathscr{W},$$

where W^α is defined by

$$W^\alpha(x) = \alpha^{-1}W(\alpha^2 x).$$

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We consider a diffusion process $X(t)$ moving in a random medium given by a potential $W \in \mathscr{W}$, i.e., we consider a formal stochastic differential equation

$$dX(t) = d\beta - \tfrac{1}{2}W'(X(t))\,dt, \qquad X(0) = 0,$$

where W' denotes the formal derivative of W (white noise) and β is a one-dimensional Brownian motion. Rigorously speaking we are considering a Feller-diffusion process $X(t)$ on $\mathbb{R}$ with the generator of Feller's canonical form

$$\frac{1}{2\exp\{-W(x)\}}\frac{d}{dx}\left(\frac{1}{\exp\{W(x)\}}\frac{d}{dx}\right).$$

Let $B = (B(t))_{t \geq 0}$ be a Brownian motion starting from the origin defined on a probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. Then, thanks to Itô–McKean's construction of Feller-diffusion processes from a Brownian motion via scale-transformation and time-change, the diffusion process $X(t)$ can be explicitly given by

$$X(t) = S^{-1}(B(T^{-1}(t))), \qquad t \geq 0, \tag{1.1}$$

where S is the scale function and T the time-change function defined by

$$S(x) = \int_0^x e^{W(z)}\,dz, \qquad x \in \mathbb{R},$$

$$T(t) = \int_0^t \exp\{-2W(S^{-1}(B(u)))\}\,du, \qquad t \geq 0.$$

Here S^{-1} (respectively, T^{-1}) denotes the inverse of S (respectively, T) (cf. [1]). The definition of $\mathscr{W}$ ensures that S is a homeomorphism of $\mathbb{R}$ for each $W \in \mathscr{W}$. For a fixed $W \in \mathscr{W}$ we may consider $X(t) = X(W, t)$ as a nonadapted functional of B. Our main result is an analogue of [4].

THEOREM 1.2. *There exists a nontrivial measurable function* $m_1\colon \mathscr{W} \mapsto \mathbb{R}$ *such that for any* $\delta > 0$,

$$\lim_{\alpha \to \infty} \int P\big(|\alpha^{-2}X(W, e^{\alpha}) - m_1(W^{\alpha})| < \delta\big)\nu(dW) = 1.$$

Using the self-similarity of the medium and the Brownian motion, we can reformulate the problem. Since

$$B^{\alpha}(t) = \alpha^{-2}B(\alpha^4 t) =_d B(t),$$

we obtain, by formula (1.1), the following lemma.

LEMMA 1.3. *For each* $\alpha > 0$ *and* $W \in \mathscr{W}$,

$$\{X(\alpha W^{\alpha}, t);\, t \geq 0\} =_d \{\alpha^{-2}X(W, \alpha^4 t);\, t \geq 0\}.$$

The proof is tedious but can be carried out straightforwardly. If we define $x^{\alpha}(t)$ to be the r.h.s. of Lemma 1.3, then heuristically Lemma 1.3 can be seen as follows:

$$\begin{aligned} dx^{\alpha} &= d\beta^{\alpha} - \tfrac{1}{2}\alpha^2 W'(\alpha^2 x^{\alpha})\,dt \\ &= d\beta - \tfrac{1}{2}\alpha W^{\alpha\prime}(x^{\alpha})\,dt \end{aligned}$$

because of $W^{\alpha\prime}(x) = \alpha W'(\alpha^2 x)$ and $\beta^{\alpha} =_d \beta$.

Since the measure ν is invariant under the transformation $W \mapsto W^\alpha$ we can view the problem in another way: For $W \in \mathscr{W}$ and $\alpha > 0$ consider the process X^α defined by

$$X^\alpha(W, t) = X(\alpha W, t), \qquad t \ge 0.$$

Then, by the formula (1.1),

$$X^\alpha(W, t) = S_\alpha^{-1}\big(B\big(T_\alpha^{-1}(t)\big)\big),$$

where

$$S_\alpha(x) = S(\alpha W, x) = \int_0^x e^{\alpha W(z)}\, dz,$$

$$T_\alpha(t) = T(\alpha W, t) = \int_0^t \exp\big\{-2\alpha W\big(S_\alpha^{-1}(B(u))\big)\big\}\, du.$$

THEOREM 1.4. *Let $\alpha \mapsto h(\alpha)$, $\alpha > 0$, be a real function such that $h(\alpha) \to 1$ as $\alpha \to \infty$. Then for ν-a.a. $W \in \mathscr{W}$,*

$$\lim_{\alpha \to \infty} P\big(\big|X^\alpha(W, e^{\alpha h(\alpha)}) - m_1(W)\big| < \delta\big) = 1$$

for each $\delta > 0$.

Replacing t by $\alpha^{-4}e^\alpha$ in Lemma 1.3, we obtain Theorem 1.2 as an immediate corollary of Theorem 1.4, Lemma 1.3, and the invariance of ν. Theorem 1.4 will be proved in Sections 3 and 4 after some preparations on media in Section 2. With a little bit more work we can also obtain the precise analogue of [4]. For $\alpha > 0$ and m, n natural numbers let

$$p_{\alpha, m}(W) = P\big(\big|\alpha^{-2}X(W, e^\alpha) - m_1(W^\alpha)\big| < 1/m\big),$$

$$p_{n, m}(W) = \inf_{\alpha \ge n} P\big(\big|X^\alpha(W, \alpha^{-4}e^\alpha) - m_1(W)\big| < 1/m\big).$$

Then, by Theorem 1.4, $p_{n,m} \to 1$ ν-a.s for each m. Because of Egoroff's theorem, for any $\eta > 0$ there exists $C^{(m)} \subset \mathscr{W}$ with measure greater than $1 - \eta 2^{-m}$ such that $p_{n,m}$ converges uniformly on $C^{(m)}$. Let C be the intersection of all $C^{(m)}$, $m \ge 1$, and let C_α be the set of all W such that $W^{1/\alpha} \in C$. By the invariance of ν, C_α has measure greater than $1 - \eta$. Because of Lemma 1.3, and by interchanging the infima,

$$\inf_C p_{n, m} = \inf_{\alpha \ge n} \inf_{C_\alpha} p_{\alpha, m}.$$

Thus, for each m, the r.h.s. of the last equation tends to 1 as $n \to \infty$.

In the preceding discussions we restricted the process to start from the origin. In order to start the process from an arbitrary point $x \in \mathbb{R}$ we may shift the medium, i.e., we consider $W_x \in \mathscr{W}$ defined by

$$W_x(z) = W(x + z) - W(x).$$

Then we define the process X_x by

$$X_x(W, t) = x + X(W_x, t), \qquad t \ge 0,$$

and X_x^α accordingly. For fixed $W \in \mathscr{W}$ and arbitrary $x \in \mathbb{R}$ consider the measure $Q_x = P \circ X_x^{-1}$ on $C([0, \infty), \mathbb{R})$ endowed with the usual Borel filtration. It is well known that $\{Q_x;\ x \in \mathbb{R}\}$ is a strong Markovian family (cf. [1]).

REMARK. As the transformation formulas show, the continuity of the medium is not needed to construct the related diffusion process. Furthermore, the connection between Theorems 1.2 and 1.4 via Lemma 1.3 relies only on the self-similarity of the medium and of Brownian motion. A very interesting case is that of a stable symmetric medium process of exponent κ, $0 < \kappa < 2$. If Theorem 1.4 remains true in this case, we obtain $X(e^\alpha) \sim \alpha^\kappa$. On the other hand, a stationary ergodic medium corresponds formally to self-similarity with parameter 0. Assuming for example boundedness, it is not hard to show that in this case the law of $\alpha^{-1}X(\alpha^2 t)$ converges weakly for W-a.a. to an unnormalized Brownian motion with diffusion coefficient given by

$$\langle e^{W(0)} \rangle^{-1} \langle e^{-W(0)} \rangle^{-1}.$$

For a more detailed treatment of these and related problems, cf. [3].

2. Depressions of the medium. In this section we introduce the basic notion of a depression. The aim is to ensure the existence of suitable depressions for almost all realizations of the medium. It should be mentioned that the definition here is more restrictive than in [4], cf. the Remark at the end of this section.

For $W \in \mathscr{W}$ and two points $x, z \in \mathbb{R}$ we define

$$H_{x,z} = \max_{0 \le \lambda \le \xi \le 1} [W(x + \xi v) - W(x + \lambda v)],$$

where $v = z - x$. Roughly speaking, $H_{x,z}$ is the maximal ascent which one must surmount if one wanders along the path W from x to z. Note that in general $H_{x,z}$ is not equal to $H_{z,x}$. A triple of real numbers

$$\Delta = (a, m, b), \qquad a < m < b,$$

is called a depression for $W \in \mathscr{W}$ if

$$W(a) > W(x) > W(m) \quad \text{for } x \in (a, m),$$
$$W(m) < W(x) < W(b) \quad \text{for } x \in (m, b),$$

and

$$H_{a,m} < H_{m,b}, \qquad H_{b,m} < H_{m,a}.$$

The depth of a depression $\Delta = (a, m, b)$ is defined by

$$D = D(\Delta) = H_{m,a} \wedge H_{m,b},$$

and the inner directed ascent is defined by

$$A = A(\Delta) = H_{a,m} \vee H_{b,m}.$$

As an immediate consequence of the definitions we obtain for a depression

$$A(\Delta) < D(\Delta).$$

Note that W must have a strict local minimum at site m of depth at least D. Especially if $\Delta^{(1)}, \Delta^{(2)}$ are two depressions of W such that $m^{(1)} < m^{(2)}$, then we obtain for $i = 1.2$:

$$\max\{W(x) - W(m^{(i)}): m^{(1)} \leq x \leq m^{(2)}\} \geq D^{(1)} \wedge D^{(2)}. \tag{2.1}$$

The problem is now whether we have good chances to find, for given $r > 0$, a depression Δ of a sample W such that $a < 0$, $b > 0$ and

$$A < r < D.$$

Given $W \in \mathscr{W}$ and $r > 0$ we define the subset $M_r = M_r(W)$ of $\mathbb{R}$ by

$$M_r = \{m\text{: there exists a depression } \Delta = (a, m, b) \text{ of } W \text{ such that } D(\Delta) \geq r\}.$$

Since W is continuous and because of (2.1), M_r consists of isolated points without finite cluster point. We need the following facts about Wiener measure.

LEMMA 2.2. *Let $\tilde{\mathscr{W}}$ be the subset of all $W \in \mathscr{W}$ satisfying*

(i) $\limsup W(x) = \infty$ *as* $x \to \infty$, $\limsup W(x) = \infty$ *as* $x \to -\infty$;
(ii) *for any open interval* $G \subset \mathbb{R}$,

$$\operatorname{card}\left\{x \in G: W(x) = \sup_G W\right\} \leq 1,$$

$$\operatorname{card}\left\{x \in G: W(x) = \inf_G W\right\} \leq 1;$$

(iii) *W has no local maximum at site* 0. *Then* $\nu(\tilde{\mathscr{W}}) = 1$.

The proof is routine, and is omitted. If $W \in \tilde{\mathscr{W}}$, M_r is not empty because of (i) and (ii). Furthermore, (ii) ensures that there exists a unique maximum of W between two neighbours of M_r, i.e., two points of M_r such that there is no other point of M_r between them. Hence for each $W \in \tilde{\mathscr{W}}$ there exists exactly one point $m_r \in M_r(W)$ such that $a_r < 0$ and $b_r > 0$, where a_r (respectively b_r) is uniquely determined as the site of the maximum of W between m_r and its left (respectively right) neighbour in $M_r(W)$. Furthermore,

$$\Delta_r(W) = (a_r, m_r, b_r)$$

is a depression satisfying

$$A_r = A(\Delta_r) < r, \qquad D_r = D(\Delta_r) \geq r.$$

To check the latter note that (2.1) ensures $D_r \geq r$. If we assume $H(a_r, m_r) \geq r$, we can find by (ii) a depression contained in $[a_r, m_r)$ with depth at least r, but this contradicts the construction. The same holds for b_r, accordingly.

LEMMA 2.3. *For each $W \in \tilde{\mathscr{W}}$ the map*

$$r \mapsto \Delta_r(W) = (a_r, m_r, b_r) \in \mathbb{R}^3, \qquad r > 0,$$

has the following properties:

(i) *it is left continuous and changes only by finitely many jumps in each bounded strictly positive interval*;

(ii) *it is monotone in the sense that*

$$(a_r, b_r) \subseteq (a_u, b_u) \quad \text{if } 0 < r < u,$$

and if in addition m_r *and* m_u *have the same sign, then* $|m_r| \le |m_u|$;

(iii) $A_r < r < D_r$ *if* r *is not a jump point, and* r *is a jump point if and only if* $A_{r+} = r$.

PROOF. We have already shown that M_r is locally finite for each $r > 0$. Obviously $r \to M_r$ is decreasing, i.e., $M_r \supseteq M_u$ if $r < u$. But then the set of intermediate maxima is also locally finite for each $r > 0$ and decreasing in r. Furthermore, for each $r > 0$, M_r and the set of intermediate maxima generate a proper partition of W in depressions with depth at least r, Δ_r is the one who contains the origin. From these facts we can easily conclude (i), (ii), and the first part of (iii). If r is a jump point, then A_{r+} is greater than or equal to the depth of a depression just swallowed by Δ_{r+}, hence $A_{r+} \ge r$. But $A_r < r$ for any $r > 0$. This completes the proof. □

It should be noted that for a fixed $W \in \tilde{\mathscr{W}}$ the range of Δ_r consists of countably many different depressions which are, by construction, maximal in the following sense: If (a, m, b) is a depression of W such that $a < 0$ and $b > 0$, then, for an appropriate $r > 0$, $m = m_r$ and (a, b) is contained in (a_r, b_r). Conversely, $\{\Delta_r\}$ is the smallest system satisfying this condition.

By a simple geometric consideration we obtain for each $W \in \tilde{\mathscr{W}}$,

$$\Delta_r(W^\alpha) = \alpha^{-2}\Delta_{\alpha r}(W), \qquad r > 0, \alpha > 0.$$

If we set $\Delta_r \equiv 0$ on $\mathscr{W} \setminus \tilde{\mathscr{W}}$, we may consider $(\Delta_r)_{r>0}$ as a stochastic process defined on $(\mathscr{W}, \mathscr{A}, \nu)$. Then, since ν is invariant, we obtain for each $\alpha > 0$,

$$\{\alpha^{-2}\Delta_{\alpha r};\ r > 0\} =_d \{\Delta_r;\ r > 0\}.$$

LEMMA 2.4. *For each fixed* $r > 0$,

$$\nu(A_r < r < D_r) = 1.$$

PROOF. It is sufficient to prove, by Lemma 2.3, that almost surely r is not a jump point. By the scaling property of (Δ_r),

$$p = \nu(r \text{ is a jump point})$$

does not depend on r. Because of Lemma 2.3, the jump points are locally finite in each strict positive interval with probability 1. Hence we conclude $p = 0$. □

REMARK. The definition of a depression in [4] permits $A > D$, but then Lemma 3.1 of the next section would fail.

3. Exit times from depressions. In this section we evaluate the time that X^α needs to leave a depression. Suppose we have for some $W \in \mathscr{W}$ a depression

$\Delta = (a, m, b)$ with depth D. For a starting point $x \in (a, b)$ let

$$\tau_{x,\alpha} = \inf\{t \geq 0\colon X_x^\alpha(W, t) \notin (a, b)\},$$

i.e., $\tau_{x,\alpha}$ is the first exit time from the depression.

Lemma 3.1. *For each closed interval $I \subset (a, b)$ and any $\delta > 0$,*

$$\lim_{\alpha \to \infty} \inf_{x \in I} P\Big(\big|\alpha^{-1}\log\{\tau_{x,\alpha}\} - D\big| < \delta\Big) = 1.$$

Proof. To simplify the notation we omit W in the argument of X_x^α, etc. First we investigate how exit times are expressible in terms of the Brownian motion B. Let $\{L(t, y);\ t \geq 0,\ y \in \mathbb{R}\}$ be the local time of B. For $y_1 < 0,\ y_2 > 0$ we set

$$L(y_1, y_2, y) = L(\tau(y_1, y_2), y), \qquad y \in \mathbb{R},$$

where $\tau(y_1, y_2)$ is the first exit time of B from the interval (y_1, y_2). From self-similarity of Brownian motion we obtain for each $\lambda > 0$,

$$\{\lambda L(y_1, y_2, y);\ y \in \mathbb{R}\} =_d \{L(\lambda y_1, \lambda y_2, \lambda y);\ y \in \mathbb{R}\}. \tag{3.2}$$

For $x_1 < 0,\ x_2 > 0$ let $\tau(x_1, x_2; X^\alpha)$ be the first exit time of X^α from (x_1, x_2). The exit time is then given by

$$\begin{aligned}\tau(x_1, x_2; X^\alpha) &= T_\alpha(\tau(S_\alpha(x_1), S_\alpha(x_2)))\\ &= \int_0^{\tau(S_\alpha(x_1), S_\alpha(x_2))} \exp\{-2\alpha W(S_\alpha^{-1}(B(u)))\}\, du.\end{aligned}$$

Using the local time, this is equal to

$$\int \exp\{-2\alpha W(S_\alpha^{-1}(y))\} L(S_\alpha(x_1), S_\alpha(x_2), y)\, dy.$$

Since $S'_\alpha = \exp\{\alpha W\}$, we finally obtain

$$\tau(x_1, x_2; X^\alpha) = \int_{x_1}^{x_2} \exp\{-\alpha W(z)\} L(S_\alpha(x_1), S_\alpha(x_2), S_2(z))\, dz. \tag{3.3}$$

In order to prove the lemma we define for a starting point $x \in (a, b)$ the interval Λ by

$$\Lambda = \begin{cases} (a, m) & \text{if } x \leq m,\\ (m, b) & \text{if } x > m.\end{cases}$$

Since I is a closed subset of (a, b),

$$\sup_{\Lambda \cap I} W < \sup_{\Lambda} W$$

for both possible intervals Λ. Let

$$\begin{aligned}\sigma_{x,\alpha} &= \inf\{t \geq 0\colon X_x^\alpha(t) \notin \Lambda\},\\ \sigma'_{x,\alpha} &= \inf\{t \geq 0\colon X_x^\alpha(t) = m\},\\ \tau'_{m,\alpha} &= \inf\{t \geq 0\colon X_x^\alpha(\sigma'_{x,\alpha} + t) \notin (a, b)\},\end{aligned}$$

where we set $\tau'_{m,\alpha} = \infty$ if $\sigma'_{x,\alpha} = \infty$. As an immediate consequence of these definitions,

$$\tau_{x,\alpha} = \sigma_{x,\alpha} + \tau'_{m,\alpha} 1\{\sigma_{x,\alpha} = \sigma'_{x,\alpha}\}.$$

For an arbitrary $x \in I$ set $\tilde{S}_\alpha(z) = S_\alpha(W_x, z - x)$. Then

$$\begin{aligned} P(\sigma_{x,\alpha} \neq \sigma'_{x,\alpha}) &= P(B \text{ hits } \tilde{S}_\alpha(a) \text{ or } \tilde{S}_\alpha(b) \text{ before } \tilde{S}_\alpha(m)) \\ &= \left| \int_x^m e^{\alpha W(z)}\, dz \right| \Big/ \left| \int_\Lambda e^{\alpha W(z)}\, dz \right| \\ &\le (b-a) \exp\Big\{ \alpha \sup_{I \cap \Lambda} W \Big\} \Big/ \left| \int_\Lambda e^{\alpha W(z)}\, dz \right|. \end{aligned}$$

Hence we obtain for any $x \in I$,

$$\alpha^{-1} \log P(\tau_{x,\alpha} \neq \sigma_{x,\alpha} + \tau'_{m,\alpha}) \le \sup_{I \cap \Lambda} \{W\} - \sup_{\Lambda} \{W\} + o(1),$$

where $o(1)$ tends to zero not depending on I, x.

By a simple calculation we get

$$\log\{\sigma_{x,\alpha} + \tau'_{m,\alpha}\} = [\log \sigma_{x,\alpha}] \vee [\log \tau'_{m,\alpha}] + O(1),$$

where $|O(1)| \le \log 2$. Then, since $A < D$ and because of the strong Markov property, the proof is complete if we can show the following:

(i) for each $x \in (a, b)$ and any $\delta > 0$,

$$P\big(\alpha^{-1} \log \sigma_{x,\alpha} \ge A + \delta + 2\alpha^{-1} \log(b-a)\big) \le p_{\alpha,\delta},$$

where $p_{\alpha,\delta}$ depends only on α, δ and $p_{\alpha,\delta} \to 0$ as $\alpha \to \infty$;

(ii) $$\alpha^{-1} \log \tau_{m,\alpha} \to D$$

in distribution, as $\alpha \to \infty$.

To prove (i) we can assume, by shifting and reflection, that $x = 0$ and $a < 0$, $m > 0$. Let be $n \in [0, m]$ such that

$$W(n) = \max_{[0,m]} W.$$

Then, by (3.3),

$$\sigma_{0,\alpha} = I_\alpha^{(1)} + I_\alpha^{(2)},$$

where

$$I_\alpha^{(1)} = \int_a^n e^{-\alpha W(z)} L(S_\alpha(a), S_\alpha(m), S_\alpha(z))\, dz,$$

$$I_\alpha^{(2)} = \int_n^m e^{-\alpha W(z)} L(S_\alpha(a), S_\alpha(m), S_\alpha(z))\, dz.$$

Setting $s_\alpha(z) = S_\alpha(z)/S_\alpha(m)$, we obtain by (3.2),

$$\begin{aligned} I_\alpha^{(1)} &=_d S_\alpha(m)\int_a^n e^{-\alpha W(z)} L(s_\alpha(a), 1, s_\alpha(z))\,dz \\ &\le S_\alpha(m)\exp\Big\{-\alpha \min_{[a,n]} W\Big\}\int_a^n L(-\infty, 1, s_\alpha(z))\,dz \\ &\le m\exp\Big\{\alpha W(n) - \alpha\min_{[a,n]} W\Big\}(n-a)\sup_{y\le 1} L(-\infty,1,y) \\ &\le (b-a)^2 e^{\alpha A}\sup_{y\le 1} L(-\infty, 1, y). \end{aligned}$$

In a similar way,

$$\begin{aligned} I_\alpha^{(2)} &=_d S_\alpha(m)\int_n^m e^{-\alpha W(z)} L(s_\alpha(a), 1, s_\alpha(z))\,dz \\ &\le S_\alpha(m)\int_n^m e^{-\alpha W(z)} L(-\infty, 1, 1 - \tilde{s}_\alpha(z))\,dz, \end{aligned}$$

where

$$\tilde{s}_\alpha(z) = 1 - s_\alpha(z) = \int_z^m e^{\alpha W(x)}\,dx / S_\alpha(m).$$

Let $R(\cdot)$ be a two-dimensional Bessel process starting from the origin (cf. [1], Section 2.8, problem 6, page 75). Then the last expression of the estimate is equal in distribution to

$$S_\alpha(m)\int_n^m e^{-\alpha W(z)}\tfrac{1}{2}R^2(\tilde{s}_\alpha(z))\,dz.$$

Using the well known transformation time $\to 1/\text{time}$ of Brownian motion, the latter expression is equal in distribution to

$$S_\alpha(m)\tfrac{1}{2}\int_n^m \{e^{-\alpha W(z)}\tilde{s}_\alpha(z)\}\tilde{s}_\alpha(z)R^2(1/s_\alpha(z))\,dz.$$

This expression is smaller than or equal to

$$\begin{aligned} &\Big\{\max_{n\le z\le m}\Big[e^{-\alpha W(z)}\int_z^m e^{\alpha W(x)}\,dx\Big]\Big\}\tfrac{1}{2}\int_n^m \tilde{s}_\alpha(z)R^2(1/\tilde{s}_\alpha(z))\,dz \\ &\quad\le (b-a)\exp\Big\{\alpha\max_{n\le z\le m}\Big[-W(z) + \max_{[z,m]} W\Big]\Big\} \\ &\qquad\times\tfrac{1}{2}\int_n^m \tilde{s}_\alpha(z)R^2(1/\tilde{s}_\alpha(z))\,dz \\ &\quad\le (b-a)^2 e^{\alpha A}\tfrac{1}{2}\int_n^m \tilde{s}_\alpha(z)R^2(1/\tilde{s}_\alpha(z))\,dz/(m-n). \end{aligned}$$

Denote the last integral by I_α, then

$$E\big[(I_\alpha)^2\big] \le \int_n^m E\big[\tilde{s}_\alpha^2(z)R^4(1/\tilde{s}_\alpha(z))\big]\,dz/(m-n) = 12.$$

Putting this all together and using Chebyshev's inequality,

$$P\left(\alpha^{-1}\log \sigma_{0,\alpha} \geq A + \delta + 2\alpha^{-1}\log(b-a)\right)$$
$$\leq P\left(\sup_{y\leq 1} L(-\infty, 1, y) \geq \tfrac{1}{2}e^{\alpha\delta}\right) + 12e^{-2\alpha\delta}.$$

To prove (ii) we can assume, by shifting, that $m = 0$, i.e., we consider a depression $(a, 0, b)$ with depth D. Moreover, by reflection, we can assume $W(b) \geq W(a)$, especially D is equal to $W(b) - W(0)$. Setting now $s_\alpha(z) = S_\alpha(z)/S_\alpha(b)$, we obtain by (3.2) and (3.3),

$$\tau_{0,\alpha} =_d S_\alpha(b)\int_a^b e^{-\alpha W(z)} L(s_\alpha(a), 1, s_\alpha(z))\, dz$$
$$\leq S_\alpha(b)\exp\left\{-\alpha \min_{[a,b]} W\right\}\int_a^b L(-\infty, 1, s_\alpha(z))\, dz$$
$$\leq (b-a)^2 \exp\{\alpha[W(b) - W(0)]\}\sup\{L(-\infty, 1, y): y \leq 1\}.$$

Hence we obtain for each $\delta > 0$,

$$P\left(\alpha^{-1}\log \tau_{0,\alpha} \leq D + \delta\right) \to 1.$$

For $0 < \eta < 1$ let

$$a' = \sup\{x \leq 0: W(x) - W(0) = \eta D\},$$
$$b' = \inf\{x \geq 0: W(x) - W(0) = \eta D\}.$$

Obviously, $a' > a$ and $b' < b$. Let c_α be the minimum of $|S_\alpha(a)|$, $S_\alpha(b)$. Then, setting now $\tilde{s}_\alpha(z) = S_\alpha(z)/c_\alpha$,

$$\tau_{0,\alpha} =_d c_\alpha \int_a^b e^{-\alpha W(z)} L(\tilde{s}_\alpha(a), \tilde{s}_\alpha(b), \tilde{s}_\alpha(z))\, dz$$
$$\geq c_\alpha \int_{a'}^{b'} e^{-\alpha W(z)} L(-1, 1, \tilde{s}_\alpha(z))\, dz$$
$$\geq c_\alpha \exp\{-\alpha[W(0) + \eta D]\}(b' - a')$$
$$\times \inf\{L(-1, 1, y): \tilde{s}_\alpha(a') \leq y \leq \tilde{s}_\alpha(b')\}.$$

Then, by Laplace's method,

$$\alpha^{-1}\log c_\alpha \to W(b),$$
$$\alpha^{-1}\log|\tilde{s}_\alpha(a')| \to -(1-\eta)D,$$

and the same holds for b'. Since local time is continuous and because $L(-1, 1, 0) > 0$ with probability 1,

$$P\left(\alpha^{-1}\log \tau_{0,\alpha} \geq (1-\eta)D - \delta\right) \to 1$$

for each $\delta > 0$, $\eta > 0$. □

4. Localization. In this section we investigate the distribution of X^α for times of the order of $e^{\alpha r}$, $r > 0$. Suppose we have for some $W \in \mathscr{W}$ a depression $\Delta = (a, m, b)$ such that $a < 0$, $b > 0$.

PROPOSITION 4.1. *Let r_1, r_2 be real numbers such that*

$$A(\Delta) < r_1 < r_2 < D(\Delta),$$

and let K be the closed interval $[r_1, r_2]$. Then for each $\delta > 0$,

$$\lim_{\alpha \to \infty} \inf_{r \in K} P\big(|X^\alpha(W, e^{\alpha r}) - m| < \delta\big) = 1.$$

PROOF. To simplify the notation we omit W in the argument of X^α. First we construct a stationary Markov process Z^α on the state space $[a, b]$ which solves

$$\begin{cases} dZ^\alpha = d\beta - \frac{1}{2}\alpha W'(Z^\alpha)\, dt & \text{in } (a, b), \\ \text{reflection at } a, b. \end{cases}$$

The invariant measure should then be given by

$$\mu_\alpha(dx) = e^{-\alpha W(x)}\, dx \Big/ \int_a^b e^{-\alpha W(z)}\, dz.$$

For an arbitrary starting point $x \in [a, b]$ define the process Z_x^α by

$$Z_x^\alpha(t) = S_\alpha^{-1}\big(\hat{B}_{\alpha, x}(T_\alpha^{-1}(t))\big), \qquad t \geq 0,$$

where $\hat{B}_{\alpha, x}$ is a one-dimensional Brownian motion starting from $S_\alpha(x)$ and reflected at $S_\alpha(a)$, $S_\alpha(b)$. Here T_α is defined accordingly replacing B by $\hat{B}_{\alpha, x}$. Let $\hat{Q}_x^\alpha$ the probability measure on $C([0, \infty), [a, b])$ induced by Z_x^α; then we may realise Z^α as the coordinate process with respect to the probability measure

$$\int_a^b \mu_\alpha(dx) \hat{Q}_x^\alpha.$$

It is easy to see that μ_α is invariant because Z_x^α was obtained by a scale transformation from a diffusion process governed by the generator

$$\begin{cases} \exp\{2\alpha W(y)\}\, d^2/dy^2 & \text{in } (S_\alpha(a), S_\alpha(b)), \\ \text{reflection at } S_\alpha(a), S_\alpha(b). \end{cases}$$

Note that for each neighbourhood U of m,

$$\mu_\alpha(U) \to 1.$$

We know that $\{\hat{Q}_x^\alpha;\ x \in [a, b]\}$, as well as $\{Q_x^\alpha;\ x \in \mathbb{R}\}$, is a strong Markovian family (cf. [1]). Thus, on the interval $[a, b]$, we can couple the processes X^α, Z^α: They move independently up to the first collision; then they move together up to the next exit from (a, b); here we stop the coupled process. The coupling can be realised on a product space with factors according X^α, Z^α. The product measure, as well as other matters related to the coupling, is indicated by a bar. As we shall see later on,

$$\bar{\sigma}_\alpha = \inf\{t \geq 0:\ X^\alpha(t) = Z^\alpha(t)\}$$

is at most of the order of $e^{\alpha A}$ with high probability, as $\alpha \to \infty$. Let

$$\tau_\alpha = \inf\{t \geq 0\colon Z^\alpha(t) \notin (a, b)\},$$
$$\bar{\tau}_\alpha = \inf\{t \geq \bar{\sigma}_\alpha\colon Z^\alpha(t) \notin (a, b)\}.$$

Obviously, $\tau_\alpha \leq \bar{\tau}_\alpha$. Because of Lemma 3.1, for each $\delta > 0$,

$$\operatorname{prob}\big(|\alpha^{-1}\log \tau_\alpha - D| < \delta\big) = \int_a^b \mu_\alpha(dx) P\big(|\alpha^{-1}\log \tau_{x,\alpha} - D| < \delta\big) \to 1.$$

Therefore

$$\bar{p}_\alpha \equiv \bar{P}\big(\bar{\sigma}_\alpha \leq e^{\alpha r_1}, e^{\alpha r_2} \leq \bar{\tau}_\alpha\big) \to 1.$$

Hence we obtain for any $r \in K$ and each neighbourhood U of m,

$$P\big(X(e^{\alpha r}) \in U\big) \geq \bar{P}\big(\bar{\sigma}_\alpha \leq e^{\alpha r_1}, Z^\alpha(e^{\alpha r}) \in U, e^{\alpha r_2} \leq \bar{\tau}_\alpha\big)$$
$$\geq \bar{p}_\alpha + \mu_\alpha(U) - 1 \to 1.$$

Since the latter estimate does not depend on r, we obtain the desired result.

It remains to show that the first collision time of independent processes X^α, Z^α is at most of the order of $e^{\alpha A}$. The idea is that we let X^α cross properly over m. First we consider the case $m \neq 0$, i.e., the starting point of X^α, by definition the origin, is located in (a, m) or (m, b). Then, by reflection, we can assume that $a < 0$, $m > 0$. For $\eta > 0$, $\eta < D - A$ let

$$b' = \inf\{x \geq m\colon W(x) - W(m) = A + \eta\}.$$

Obviously $b' < b$, $W(a) > W(b')$, and $H(b', m)$ is at most equal to A. Hence (a, m, b') is a depression with depth $A + \eta$. Similar to the proof of Lemma 3.1 let σ_α be the first exit time of X^α from (a, b'), and let σ'_α be the first entrance time of X^α to the point b'. By a natural scale argument as in the proof of Lemma 3.1, and because of Lemma 3.1,

$$\lim_\alpha \alpha^{-1}\log P(\sigma_\alpha \neq \sigma'_\alpha) < 0,$$
$$\alpha^{-1}\log \sigma_\alpha \to_d A + \eta.$$

Since X^α, Z^α are independent,

$$Z^\alpha(\sigma'_\alpha) =_d \mu_\alpha,$$

and therefore

$$\bar{P}(\bar{\sigma}_\alpha \leq \sigma'_\alpha) \geq \bar{P}\big(Z^\alpha(0) \in [0, b], Z^\alpha(\sigma'_\alpha) \in [a, b']\big)$$
$$\geq \mu_\alpha[0, b] + \mu_\alpha[a, b'] - 1 \to 1.$$

Thus, in the case $m \neq 0$, we obtain for each $\eta > 0$,

$$\bar{P}\big(\alpha^{-1}\log \bar{\sigma}_\alpha < A + 2\eta\big) \to 0.$$

For the case $m = 0$ consider

$$a'' = \sup\{x \leq 0\colon W(x) - W(0) = A/2\},$$
$$b'' = \inf\{x \geq 0\colon W(x) - W(0) = A/2\}.$$

Then $a'' > a$, $b'' < b$, and $(a'', 0, b'')$ is a depression with depth $A/2$. Because of

Lemma 3.1,

$$P\left(\alpha^{-1}\log \sigma_\alpha'' < A\right) \to 1,$$

where σ_α'' is the first exit time of X^α from (a'', b''). Since the conditional distribution of $Z^\alpha(\sigma_\alpha'')$ given $X^\alpha(\sigma_\alpha'')$ is equal to μ_α, and because of the strong Markov property, we can now use the result for the case $m \neq 0$. Hence we obtain for arbitrary small $\eta > 0$,

$$\lim_{\alpha \to \infty} \overline{P}\left(\alpha^{-1}\log \bar{\sigma}_\alpha < A + 3\eta\right) = 1$$

whenever $a < 0$, $b > 0$. □

PROOF OF THEOREM 1.4. Because of Lemma 2.4, $\Delta_1(W)$ is a depression satisfying $A_1 < 1$, $D_1 > 1$ for ν-a.a. W. Since we assumed $h(\alpha) \to 1$, we can apply Proposition 4.1. □

REMARK. As an immediate consequence of Lemma 2.2, Lemma 2.3, and Proposition 4.1, we obtain for ν-a.a. $W \in \mathscr{W}$,

$$X^\alpha(W, e^{\alpha r}) \to_d m_r(W) \quad \text{as } \alpha \to \infty$$

for any $r > 0$ up to jump points of $\Delta_r(W)$. But the sample paths oscillate more and more rapidly in the according depressions as $\alpha \to \infty$. Hence we cannot expect more than convergence of the finite-dimensional distributions.

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