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ISOTROPIC STOCHASTIC FLOWS

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Dedicated to the memory of Mark Kac

We consider isotropic stochastic flows in a Euclidean space of d dimensions, $d \geq 2$. The tendency of two-point distances and of tangent vectors to shrink or expand is related to the dimension and the proportion of the flow that is solenoidal or potential. Tangent vectors from the same point tend to become aligned in the same or opposite directions. The purely potential flows are characterized by an analogue of the curl-free property. Liapounov exponents are treated briefly. The rate of increase or decrease of the length of an arc of small diameter is related to the shape of the arc. In the case $d = 2$ a sufficient condition is given under which the length of a short arc has a high probability of approaching 0.

1. Introduction. A *stochastic flow* is a family of random mappings X_{st}, $0 \leq s \leq t < \infty$, of a space M into itself, such that $X_{tu} \circ X_{st} = X_{su}$ if $s \leq t \leq u$, X_{ss} is the identity map, and $X_{s_1 t_1}, X_{s_2 t_2}, \dots$ are independent if $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots$. We treat the case $M = \mathbf{R}^d$, $d \geq 2$, with sufficient conditions imposed so that X has a version which for each $s \leq t$ is a diffeomorphism of $\mathbf{R}^d$ onto itself and such that $X_{st}(x)$, $X_{st}^{-1}(x)$, $DX_{st}(x) \equiv (\partial_j X_{st}^i(x))$, and $DX_{st}^{-1}(x)$ are jointly continuous in $0 \leq s \leq t < \infty$ and $x \in \mathbf{R}^d$. We put $X_t = X_{0t}$.

A number of authors have established conditions under which such flows exist; relevant references include [1], [4], [9], [10], [12], [13], [15], [16], [19], [20], [24]. For our purposes it is important to relate the flow to the correlation tensor $b(x) = (b^{pq}(x))$, $p, q = 1, 2, \dots, d$, of a certain $\mathbf{R}^d$-valued homogeneous random "generating" field $U(x) = U^p(x)$, $p = 1, 2, \dots, d$, where $EU^p(x) = 0$ and

$$\begin{aligned} b^{pq}(x) &= EU^p(y+x)U^q(y) \\ &= \lim_{t \downarrow 0} t^{-1} E\big(X_t^p(y+x) - y^p - x^p\big)\big(X_t^q(y) - y^q\big). \end{aligned} \tag{1.1}$$

This field enters more or less explicitly in several of the above references.

The tensor $b(x)$ determines the flow uniquely. (Inhomogeneous correlation tensors $b(x, y)$ have also been treated in some of the above references.) We shall assume that b is *isotropic* in the vector sense: If G is any real orthogonal matrix, proper or improper, then

$$b(x) = G^* b(Gx) G, \tag{1.2}$$

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where G^* is the transpose of G. If we think of U as Gaussian, which is appropriate, (1.2) implies that GU has the same law as $U \circ G$, and as we shall see later, that the process GX_t, $t \geq 0$ has the same law as $X_t \circ G$, $t \geq 0$; i.e., the flow is isotropic.

Isotropic flows have nice properties which simplify their computational treatment. Lengths of displacement and tangent vectors are Markovian processes under the flow; so are angles between pairs of tangent vectors. A decomposition of b into potential and solenoidal parts, available in the isotropic case, due to Itô [17], Yaglom [29], and Obukhov [26], is reflected in certain properties of the flow, which depend on how close it is to being potential or solenoidal. In particular if $d = 2$ or 3 and the flow is close to potential, displacement and tangent vectors tend to shrink.

After reviewing some properties of vector random fields in Section 2, we give an appropriate construction of our flow in Section 3 and discuss the motion of finite sets of points. The behavior of the distance between two points is important, and is related to the dimension as well as the proportions of potential or solenoidal components. Sections 4 and 5 treat the flow of tangent vectors associated with the flow X. This again is related to the dimension and the proportion of potential or solenoidal parts. Certain angle–length relations between pairs of tangent vectors are given; these appear later in the brief discussion of Liapounov exponents given in Section 7. In particular, tangent vectors from the same point tend to line up in the same or opposite directions. In Section 6 we characterize isotropic potential stochastic flows by an analogue of the curl-free property, although strictly speaking no curl exists.

Sections 8 and 9 treat the length of an arc under the flow. It is seen that the length-process of a short arc depends on the shape mainly through the integral in (8.11). Taking, e.g., a purely potential flow, we find that if $d = 2$, a short arc tends in a sense to shrink no matter what its shape, the shrinkage being most rapid for a straight arc and least for certain special shapes (see (8.13)). The main result here, for $d = 2$, is Theorem 9.4. If $d = 3$, with potential flow, a short straight arc tends to shrink but the special shapes tend to expand, and the situation is less clear.

Sections 10 and 11 have brief discussions of volumes and of the situation for homogeneous but nonisotropic flows.

NOTATION. $\langle M, N\rangle_t$ is the mutual variation process for continuous martingales M and N, $\langle M, N\rangle_t'$ is the time derivative, which always exists and is continuous for our processes; $\langle M\rangle_t = \langle M, M\rangle_t$. (ξ, η) is the Euclidean inner product, $|\xi| = (\xi, \xi)^{1/2}$, $e_1, e_2, \ldots, e_d$ are orthonormal basis vectors in $\mathbf{R}^d$. $L(\mathbf{R}^d)$ is the space of bounded linear operators $\mathbf{R}^d \to \mathbf{R}^d$. $C_b^r(\mathbf{R}^m, \mathbf{R}^n)$ is the set of mappings $\mathbf{R}^m \to \mathbf{R}^n$ which with their partial derivatives of order $\leq r$ are bounded and continuous; $C_b^r(\mathbf{R}^m, \mathbf{R}^1)$ may be written $C_b^r(\mathbf{R}^m)$ or C_b^r if m is understood. Omission of b means boundedness is not required. $C_b^r(\mathbf{R}^m, \mathbf{R}^n)$ is a separable Banach space under the norm

$$\| f \| = \sup\left\{ \left| \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m} f(x) \right| : 0 \leq |\alpha| = k \leq r,\ x \in \mathbf{R}^m \right\}.$$

Linear maps are identified with their matrix representations through the standard basis. (W_t) is a Wiener process with mean 0; the rate is 1 unless the contrary is stated. Processes W_t^α with different α's in the same formula are independent. A^* is the transpose of the matrix A.

ADDENDUM. While this paper was in the final stages of typing, the authors received Le Jan's preprint [23], treating essentially the same class of isotropic stochastic flows in $\mathbf{R}^d$. It appears that there is little overlap in the results beyond the analysis of the diffusion $\rho_t = |X_t(x) - X_t(y)|$ (cf. (3.16)) and the computation of the Liapounov spectrum (cf. (7.3)), where we have relied on [3]. Le Jan obtains detailed results on the effect of isotropic stochastic flows on volumes in $\mathbf{R}^d$, a topic we have treated only briefly in Section 10.

2. Homogeneous isotropic vector fields. Let $(U^i(x),\ i = 1, \dots, d,\ x \in \mathbf{R}^d)$ be an $\mathbf{R}^d$-valued homogeneous random field in $\mathbf{R}^d$ with mean 0 and finite second moments. We shall make direct use only of the correlation tensor

$$b^{pq}(x) = EU^p(y + x)U^q(y). \tag{2.1}$$

We denote the matrices $(b^{pq}(x))$ or (b^{pq}) by $b(x)$ or b.

CONDITIONS (2.2). $b(x)$ satisfies (1.2) and is not a constant matrix. The components $b^{pq}(x)$ are continuous and have continuous partial derivatives of order ≤ 4. (Boundedness is automatic.)

These conditions are assumed throughout. However, in some cases (e.g., [12]) homeomorphic flows exist if only continuous second derivatives are assumed.

We shall occasionally need the spectral formula $b^{pq}(x) = \int_{\mathbf{R}^d} e^{i(x,\lambda)} F^{pq}(d\lambda)$, where, using (2.2), $F^{pq}(B)$ for B a Borel subset of $\mathbf{R}^d$ is a real symmetric nonnegative definite matrix satisfying $F(B) = G^*F(GB)G$, in particular $F(B) = F(-B)$. From an analogue of (6.4.1) of [8] $\int |\lambda|^4 |F^{pq}(d\lambda)| < \infty$. Hence all first and third order partial derivatives of b^{pq} vanish at $x = 0$.

The rest of this section is mainly adapted from Yaglom [29]. Isotropic correlation depends on two scalar functions B_L and B_N, the *longitudinal* and *transverse* correlation functions:

$$B_L(r) = b^{pp}(re_p), \qquad r \ge 0, \tag{2.3}$$

$$B_N(r) = b^{pp}(re_q), \qquad r \ge 0,\ q \ne p. \tag{2.4}$$

It follows from (1.2) that B_L and B_R do not depend on the choice of p and q or of the basis vectors. We then have

$$b^{pq}(x) = (B_L(|x|) - B_N(|x|))x^p x^q/|x|^2 + B_N(|x|)\delta^{pq}, \qquad x \ne 0, \tag{2.5}$$

$$b^{pq}(0) = \delta^{pq}B_L(0) = \delta^{pq}B_N(0), \tag{2.6}$$

where δ^{pq} is the Kronecker delta.

We shall henceforth normalize by taking $b^{pq}(0) = \delta^{pq}$, so that $B_L(0) = B_N(0) = 1$. B_L and B_N are bounded and C^4 on $[0, \infty)$ (right-hand derivatives at 0) with

$$(2.7)\qquad \begin{aligned} B_L'(0) &= B_N'(0) = 0,\\ -B_L''(0) &= -\partial_p\partial_p b^{pp}(0) = E\big(\partial_p U^p(x)\big)^2 \overset{\text{def}}{=} \beta_L,\\ -B_N''(0) &= -\partial_q\partial_q b^{pp}(0) = E\big(\partial_p U^q(x)\big)^2 \overset{\text{def}}{=} \beta_N, \end{aligned}$$

using any p, q with $p \neq q$. Moreover

$$(2.8)\qquad B_L(r) = 1 - \tfrac{1}{2}\beta_L r^2 + O(r^4), \qquad r \to 0,$$

$$(2.9)\qquad B_N(r) = 1 - \tfrac{1}{2}\beta_N r^2 + O(r^4), \qquad r \to 0.$$

Then

$$(2.10)\qquad \begin{aligned} b^{pq}(x) &= \tfrac{1}{2}(\beta_N - \beta_L)x^p x^q + \big(1 - \tfrac{1}{2}\beta_N|x|^2\big)\delta^{pq} + O(|x|^4), \qquad x \to 0,\\ -\partial_i\partial_j b^{pq}(0) &= \tfrac{1}{2}(\beta_L - \beta_N)(\delta^{ip}\delta^{jq} + \delta^{iq}\delta^{jp}) + \beta_N\delta^{ij}\delta^{pq}. \end{aligned}$$

Using quadratic-mean derivatives, $E\partial_i U^p(x)\partial_j U^q(y) = -\partial_i\partial_j b^{pq}(x-y)$, so that from the second line of (2.10)

$$(2.11)\qquad E\big(\partial_i U^p(x) - \partial_p U^i(x)\big)^2 = (1 - \delta^{ip})(3\beta_N - \beta_L),$$

$$(2.12)\qquad E\Big(\sum_p \partial_p U^p(x)\Big)^2 \equiv E(\operatorname{div} U(x))^2 = \tfrac{1}{2}d\big[(d+1)\beta_L - (d-1)\beta_N\big].$$

We call U *potential* (= irrotational = curl-free) if $3\beta_N - \beta_L = 0$ and *solenoidal* (= incompressible = divergence-free) if $(d+1)\beta_L - (d-1)\beta_N = 0$. We shall also apply these terms to the correlation tensor b. If U is Gaussian, so that continuous realizations for U and its first derivatives may be assumed, then in the potential case U is the gradient of a scalar field. However, additional conditions are needed to insure that the latter field is homogeneous and isotropic; see [29], Sections 4 and 5. Similarly, in the solenoidal case, for a Gaussian U, the realizations are divergence-free vector fields.

From (2.11) and (2.12)

$$(2.13)\qquad 0 \le \left(\frac{d-1}{d+1}\right)\beta_N \le \beta_L \le 3\beta_N.$$

From (2.13) if either β_L or β_N is 0, so is the other and then from (2.10) $b^{pp}(x) = 1 + O(|x|^4)$. Since b^{pp} is the characteristic function of the distribution F^{pp}, this implies $b^{pp}(x) = 1$, whence $E(U^p(x) - U^p(0))^2 = 0$, implying $b^{pq}(x) = \delta^{pq}$, excluded by (2.2). Hence β_L and β_N are > 0. Note that U is potential or solenoidal iff $\beta_L/\beta_N = 3$ or $(d-1)/(d+1)$, respectively.

For isotropic random fields we have the decomposition

$$(2.14)\qquad \begin{aligned} b^{pq} &= \mu_0\delta^{pq} + \mu_1 b_P^{pq} + \mu_2 b_S^{pq}, \qquad \mu_i \ge 0, \Sigma\mu_i = 1, \mu_0 < 1,\\ b_P^{pq}(0) &= b_S^{pq}(0) = \delta^{pq}, \qquad b_P^{pq}(\infty) = b_S^{pq}(\infty) = 0, \end{aligned}$$

where b_P is an isotropic potential covariance tensor and b_S is isotropic solenoidal. The representation is unique, with trivial exceptions if μ_1 or μ_2 is 0.

If U is a homogeneous vector field, isotropic or not, there is a decomposition $U = U_P + U_S$ into potential and solenoidal parts (see [17]), but without isotropy the parts are in general correlated and we do not have a corresponding decomposition (2.14).

Expressing b_P and b_S in terms of B_{PL}, B_{PN}, B_{SL} and B_{SN} as in (2.5), we get

$$\begin{aligned} B_L &= \mu_0 + \mu_1 B_{PL} + \mu_2 B_{SL}, \\ B_N &= \mu_0 + \mu_1 B_{PN} + \mu_2 B_{SN}. \end{aligned} \tag{2.15}$$

The four functions B_{PL}, etc., depend on two finite positive measures M_P and M_S supported on $(0, \infty)$. (In (4.37) of [29] these measures may have some mass at 0, here absorbed in μ_0.) Since these four functions vanish at ∞, we have $B_L(\infty) = B_N(\infty) = \mu_0$. Here are the representations:

$$\begin{aligned} B_{PL}(r) &= 2^{(d-2)/2}\Gamma\left(\tfrac{1}{2}d\right)\int_{(0,\infty)}\left[\frac{J_{d/2}(rs)}{(rs)^{d/2}} - \frac{J_{(d+2)/2}(rs)}{(rs)^{(d-2)/2}}\right]M_P(ds), \\ B_{PN}(r) &= 2^{(d-2)/2}\Gamma\left(\tfrac{1}{2}d\right)\int_{(0,\infty)}\frac{J_{d/2}(rs)}{(rs)^{d/2}}M_P(ds), \\ B_{SL}(r) &= 2^{(d-2)/2}\Gamma\left(\tfrac{1}{2}d\right)(d-1)\int_{(0,\infty)}\frac{J_{d/2}(rs)}{(rs)^{d/2}}M_S(ds), \\ B_{SN}(r) &= 2^{(d-2)/2}\Gamma\left(\tfrac{1}{2}d\right)\int_{(0,\infty)}\left[\frac{J_{(d-2)/2}(rs)}{(rs)^{(d-2)/2}} - \frac{J_{d/2}(rs)}{(rs)^{d/2}}\right]M_S(ds), \\ & M_P(0,\infty) = d,\ M_S(0,\infty) = \frac{d}{d-1}, \end{aligned} \tag{2.16}$$

the last line holding because each function must be 1 when $r = 0$. Because of (2.2) M_P and M_S have finite fourth moments, and we have

$$\begin{aligned} \beta_L &= \frac{3\mu_1}{d(d+2)}\int s^2 M_P(ds) + \frac{(d-1)\mu_2}{d(d+2)}\int s^2 M_S(ds), \\ \beta_N &= \frac{\mu_1}{d(d+2)}\int s^2 M_P(ds) + \frac{(d+1)\mu_2}{d(d+2)}\int s^2 M_S(ds). \end{aligned} \tag{2.17}$$

M_P and M_S are uniquely determined by b, except trivially when μ_1 or $\mu_2 = 0$; any M_P, M_S, μ_0, μ_1, μ_2 subject to the conditions given above determine an isotropic covariance tensor b.

REMARK (2.18). Since $B_L(r) = \int_{\mathbf{R}^d} e^{ir\lambda^1}\,dF^{11}(\lambda)$ is a characteristic function, $|B_L(r)| = 1$ for some $r > 0$ would imply $\limsup_{r\to\infty} B_L(r) = 1$, which is impossible because $\mu_0 < 1$ in (2.14). Hence, $|B_L(r)| < 1$ if $r \neq 0$, and similarly for B_N.

Le Jan [21] has treated an interesting class of flows based on covariance tensors of the form

$$b^{pq}(x) = \delta^{pq}C(x) = \delta^{pq}\int_{[0,\infty)} M(dr)\int_{S^{d-1}} \cos(r\theta\cdot x)\,d\theta,$$

where M is a positive measure on $[0,\infty)$ with finite moments. Taking $M\{0\} = 0$ for simplicity, we obtain these tensors among the general isotropic ones by taking $\mu_1 = 1/d$, $\mu_2 = (d-1)/d$ in (2.14), and $M_S = (d-1)^{-1}M_P = M$ in (2.16). Using some Bessel identities we find $B_L(r) = B_N(r)$ and, hence, from (2.5)

$$\begin{aligned} b^{pq}(x) &= \delta^{pq}B_N(|x|) \\ &= \delta^{pq}d^{-1}(d-1)2^{(d-2)/2}\Gamma\left(\tfrac{1}{2}d\right)\int_{(0,\infty)} \frac{J_{(d-2)/2}(|x|s)M(ds)}{(|x|s)^{(d-2)/2}}. \end{aligned}$$

Since

$$\int_{S^{d-1}} \cos(r\theta\cdot x)\,d\theta = \frac{2^{(d-2)/2}\Gamma\left(\frac{1}{2}d\right)J_{(d-2)/2}(r|x|)}{(r|x|)^{(d-2)/2}},$$

we have Le Jan's form.

3. The flow and its finite-point diffusions. Several authors have shown that under quite general conditions there is a unique diffeomorphic flow associated with a given correlation tensor. See for example Le Jan [20], Le Jan and Watanabe [24] and Baxendale [1]. Here we indicate briefly how the flow may be realized as the solution of a system of stochastic differential equations.

Let b^{pq} be a correlation tensor satisfying (2.2). Writing

$$b^{pq}(x,y) = b^{pq}(x-y), \tag{3.1}$$

we obtain a reproducing kernel $b^{pq}(\cdot,\cdot)$ and its real separable reproducing kernel Hilbert space $\mathbf{H}$ consisting of vector fields on $\mathbf{R}^d$. $\mathbf{H}$ may be realized by way of its identification with the L_2 space generated by the $U^p(x)$, $p = 1,\dots,d$, $x\in\mathbf{R}^d$. For ξ in this space we obtain the vector field f_ξ given by $f_\xi^p(x) = [\xi, U^p(x)] = E\xi U^p(x)$. Then $b^{p\cdot}(x,\cdot)\in\mathbf{H}$, corresponding to $\xi = U^p(x)$, and we have the reproducing property $[b^{p\cdot}(x,\cdot), f] = f^p(x)$. From (2.2) $b^{pq}(x,y)$ is C_b^2 in x and y separately; the bounds on b and its partial derivatives come from the positive definite property. Hence, there is a continuous inclusion of $\mathbf{H}$ in $C_b^2(\mathbf{R}^d,\mathbf{R}^d)$ (see [1]; in the present case this can also be seen from the finiteness of the fourth spectral moments.) In particular $\|f\|_{C_b^2} \le K\|f\|_{\mathbf{H}}$.

Let $\{V_\alpha\}_{\alpha\ge 1}$ be a complete orthonormal set in $\mathbf{H}$. From the definition of $\mathbf{H}$ in terms of b^{pq},

$$\sum_\alpha V_\alpha^p(x)V_\alpha^q(y) = b^{pq}(x,y) = b^{pq}(x-y). \tag{3.2}$$

In general $\mathbf{H}$ will be infinite dimensional so that $\{V_\alpha\}$ will be a countable family of bounded twice differentiable vector fields in $\mathbf{R}^d$ and (3.2) will involve an infinite series, but the continuous inclusion property insures the absolute convergence of (3.2) uniformly in x and y, as well as of

$$\sum_\alpha \partial_i V_\alpha^p(x)\partial_j V_\alpha^q(y) = \frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j} b^{pq}(x, y) = -\partial_i\partial_j b^{pq}(x - y). \tag{3.3}$$

In particular

$$\sum_\alpha (V_\alpha^p(x))^2 = b^{pp}(0) = 1, \tag{3.4}$$

$$\sum_\alpha (\partial_i V_\alpha^p(x))^2 = -\partial_i\partial_i b^{pp}(0) < \infty. \tag{3.5}$$

Note that if $\xi^1, \xi^2, \dots$ are independent standard Gaussian random variables, then $U(x) = \Sigma \xi^\alpha V_\alpha(x)$ is a homogeneous mean 0 Gaussian field with the correlation tensor b.

Consider, for each x, the Itô stochastic differential equation in $\mathbf{R}^d$

$$dX_t(x) = \sum_\alpha V_\alpha(X_t(x))\,dW_t^\alpha, \qquad X_0(x) = x, \tag{3.6}$$

where $W_t^1, W_t^2, \dots$ are independent standard Brownian motions. Kunita [19] considered such systems, in general not isotropic, for the case of finitely many W^α, showing that they determine a flow of diffeomorphisms; Fujiwara and Kunita consider the infinite case in [10], indicating that the reasoning is similar. In our case, in order to maintain isotropy we take the drift to be zero. This applies to both Stratonovich and Itô drift, since the correction term is

$$\frac{1}{2}\sum_\alpha \sum_{q=1}^d \frac{\partial V_\alpha^p(x)}{\partial x^q} V_\alpha^q(x) = \frac{1}{2}\sum_q \partial_q b^{pq}(0) = 0. \tag{3.7}$$

(See (2.10)). Following the reasoning of [19], we can then assert the existence of a flow of diffeomorphisms as follows.

Let the standard Brownian motions W^α be defined on a complete probability space $(\Omega, \mathbf{F}, P)$, relative to a right-continuous filtration $(\mathbf{F}_t)$ such that $\mathbf{F}_0$ contains all the P-null sets of $\mathbf{F}$. Then for each x, $X_t(x)$ is $\mathbf{F}_t$-measurable and so is $X_t^{-1}(x)$. From the results of Kunita we see that $\exists\ \Omega_0 \in \mathbf{F}$, $P(\Omega_0) = 0$, such that $\omega \notin \Omega_0$ implies the following: X_t and X_t^{-1} are diffeomorphisms of $\mathbf{R}^d$ onto itself for each $t \geq 0$; $X_t(x)$ and $X_t^{-1}(x)$ and their first partial derivatives are jointly continuous in (t, x). Putting $X_{st} = X_t \circ X_s^{-1}$, $0 \leq s \leq t < \infty$, it can be shown that for $\omega \notin \Omega_0'$, where $P(\Omega_0') = 0$, we have $X_{tu} \circ X_{st} = X_{su}$ for all $0 \leq s \leq t \leq u < \infty$. Moreover, $X_{s_1 t_1}, X_{s_2 t_2}, \dots$ have the independence property of Section 1.

In coordinate form (3.6) is $dX_t^p(x) = \Sigma_\alpha V_\alpha^p(X_t(x)\,dW_t^\alpha$, whence $X_t^p(x)$ is a martingale. Moreover, if $x_1, \dots, x_k \in \mathbf{R}^d$, then $(X_t(x_1), \dots, X_t(x_k))$ is a diffusion whose mutual variation processes satisfy

$$\begin{aligned}\langle X^p(x_i), X^q(x_j)\rangle_t' &\equiv \frac{d}{dt}\langle X^p(x_i), X^q(x_j)\rangle_t \\ &= \sum_\alpha V_\alpha^p(X_t(x_i))V_\alpha^q(X_t(x_j)) = b^{pq}(X_t(x_i) - X_t(x_j)),\end{aligned} \tag{3.8}$$

which relates the flow to b It is known [1] that b and the flow determine one another uniquely.

From (1.2) and (3.8) we see that if G is any rigid motion of $\mathbf{R}^d$, the diffusion $(X_t(x_1),\dots,X_t(x_k))$ has the same law as $(G^{-1}X_t(Gx_1),\dots,G^{-1}X_t(Gx_k))$. More generally the stochastic flows $\{X_{st}: 0 \le s \le t < \infty\}$ and $\{G^{-1}\circ X_{st}\circ G: 0 \le s \le t < \infty\}$ have the same law, expressing the isotropy of the flow. Conversely any stochastic flow having the above isotropy properties arises from a covariance tensor satisfying (1.2). The normalization $b^{pq}(0) = \delta^{pq}$ insures that the one-point motion of the flow is standard Brownian motion in $\mathbf{R}^d$. The condition that b^{pq} is not constant merely removes the case where the flow consists of rigid translations by a single Brownian motion in $\mathbf{R}^d$.

LEMMA (3.9). *Fix $x, y \in \mathbf{R}^d$, $x \ne y$ and let $\tilde{V}_t = X_t(x) - X_t(y)$. Then $\tilde{V}_t$ is a diffusion with* 0 *drift and diffusion matrix*

$$\langle \tilde{V}^p, \tilde{V}^q\rangle_t' = 2(\delta^{pq} - b^{pq}(\tilde{V}_t)).$$

This follows readily from (3.8).

LEMMA (3.10). *Let $\rho_t = |\tilde{V}_t|$ (see* (3.9)*). Then ρ_t may be extended to a diffusion on $\mathbf{R}^1$ with operator* $\mathbf{A}$:

$$\text{(3.11)}\quad \mathbf{A}g(\rho) = (1 - B_L(\rho))g''(\rho) + (d-1)\left(\frac{1 - B_N(\rho)}{\rho}\right)g'(\rho), \qquad g \in C_b^2.$$

Here B_L and B_N are extended smoothly to $\mathbf{R}^1$ by reflection. From (2.18), $|B_L(\rho)|$ and $|B_N(\rho)|$ are < 1 if $\rho \ne 0$, so the drift and diffusion coefficients do not vanish except at 0. We can deduce (3.10) readily from (3.9) (for a special case see [12], Section 4).

LEMMA (3.12). *For the diffusion ρ_t of* (3.10) *the point* 0 *is absorbing and is inaccessible from any other point. If S is a natural scale function for ρ_t on* $(0,\infty)$ *then*: (a) *If* $d \ge 4$, $S(0) \equiv S(0+) = -\infty$; $S(\infty) < \infty$. (b) *If* $d = 3$, $S(0) > -\infty$ *iff* $\beta_L/\beta_N > 2$; $S(\infty) < \infty$. (c) *If* $d = 2$, $S(0) > -\infty$ *iff* $\beta_L/\beta_N > 1$; $S(\infty) = \infty$.

REMARK (3.13). If $d = 2$ or 3, $S(0)$ is finite in the potential case and $-\infty$ in the solenoidal case.

PROOF OF (3.12). Since ρ_t measures the distance between two points in a flow of diffeomorphisms, 0 is absorbing and inaccessible from any other point. (Alternatively one can use Feller's criterion with (3.14), (2.8), and (2.9).) Now let S be a natural scale function ([5], Chapter 16):

$$\text{(3.14)}\qquad S'(y) = \exp\left\{-(d-1)\int_1^y \frac{1 - B_N(s)}{s(1 - B_L(s))}\,ds\right\}.$$

From (2.8) and (2.9) we have $(1 - B_N(s))/(1 - B_L(s)) = \beta_N/\beta_L + O(s^2)$, $s \to 0$.

Hence, $S'(y) \sim Ky^{-(d-1)\beta_N/\beta_L}$ as $y \to 0$, where K is a positive constant, whence $S(0)$ is finite iff $\beta_N(d-1)/\beta_L < 1$. This justifies the statements about $S(0)$ (see (2.13)).

Since $\lim_{s\to\infty} B_L(s) = \lim_{s\to\infty} B_N(s) = \mu_0 < 1$ (see (2.14)), we may write

$$S'(y) = y^{-(d-1)}\exp\left\{-(d-1)\int_1^y \frac{\delta(s)}{s}\,ds\right\},$$

where $\delta(s) = (B_L(s) - B_N(s))/(1 - B_L(s)) \to 0$ as $s \to \infty$. This shows $S(\infty) < \infty$ if $d \geq 3$. If $d = 2$ it suffices to prove that $\int_1^y(\delta(s)/s)\,ds$ converges as $y \to \infty$, since then $S'(y) \sim Ly^{-1}$ as $y \to \infty$, where L is a constant > 0. From (2.16)

$$\text{(3.15)}\qquad \begin{aligned} B_{PL}(s) - B_{PN}(s) &= \int_{(0,\infty)} -J_2(s\lambda)M_P(d\lambda),\\ B_{SL}(s) - B_{SN}(s) &= \int_{(0,\infty)}\left[\frac{2J_1(s\lambda)}{s\lambda} - J_0(s\lambda)\right]M_S(d\lambda). \end{aligned}$$

Using (2.15) and noting that $\inf_{s\geq 1}(1 - B_L(s)) > 0$, it suffices to observe that

$$\int_{(0,\infty)} M_P(d\lambda)\int_0^\infty\left|\frac{J_2(s\lambda)}{s\lambda}\right|d(s\lambda) < \infty,$$

$$\int_{(0,\infty)} M_S(d\lambda)\int_0^\infty\left|\frac{2J_1(s\lambda)}{s^2\lambda^2} - \frac{J_0(s\lambda)}{s\lambda}\right|d(s\lambda) < \infty,$$

since $J_2(x)/x$ and $2J_1(x)/x^2 - J_0(x)/x$ are absolutely integrable on $(0,\infty)$. □

COROLLARY (3.16). (a) *If* $d \geq 4$, $P\{\rho_t \to \infty\} = 1$. (b) *If* $d = 3$, $P\{\rho_t \to \infty\} > 0$; $P\{\rho_t \to 0\} > 0$ *iff* $\beta_L/\beta_N > 2$; $P\{\rho_t \to \infty\} + P\{\rho_t \to 0\} = 1$. (c) *If* $d = 2$, $P\{\rho_t \to \infty\} = 0$; $P\{\rho_t \to 0\} = 1$ *if* $\beta_L/\beta_N > 1$ *and* $= 0$ *if* $\beta_L/\beta_N \leq 1$.

In particular ρ_t is transient except in the case $d = 2$ and $\beta_L/\beta_N \leq 1$.

4. The tangent flow. From (3.6) the derivative $DX_t(x)$ (in matrix form $(\partial_j X_t^i(x))$, i = row, j = column), considered as an element of $L(\mathbf{R}^d)$, satisfies

$$\text{(4.1)}\qquad dDX_t(x) = \sum_\alpha DV_\alpha(X_t(x))\,DX_t(x)\,dW_t^\alpha,\qquad DX_0(x) = I,$$

or in component form

$$\text{(4.2)}\qquad d\partial_q X_t^p(x) = \sum_\alpha\sum_{i=1}^d \partial_i V_\alpha^p(X_t(x))\,\partial_q X_t^i(x)\,dW_t^\alpha,\qquad \partial_q X_0^p(x) = \delta^{pq}.$$

Fixing x, put $v_t = DX_t(x)(v)$. Then $v_t^p = \Sigma_q \partial_q X_t^p(x)v^q$ satisfies

$$\text{(4.3)}\qquad dv_t^p = \sum_\alpha\sum_i \partial_i V_\alpha^p(X_t(x))\,v_t^i\,dW_t^\alpha,\qquad v_0^p = v^p.$$

Lemma (4.4). *For fixed x and fixed $v \in T_x = \mathbf{R}^d$, v_t is a diffusion in $\mathbf{R}^d$ with zero drift and diffusion matrix*

$$\langle v^p, v^q \rangle_t' = \beta_N |v_t|^2 \delta^{pq} + (\beta_L - \beta_N) v_t^p v_t^q.$$

Proof. (Inside the angular bracket v^p means the process (v_t^p).) From (4.3), (3.3), and (2.10), v_t^p is a martingale and

$$\begin{aligned}\langle v^p, v^q \rangle_t' &= \sum_\alpha \sum_i \partial_i V_\alpha^p(X_t(x)) v_t^i \sum_j \partial_j V_\alpha^q(X_t(x)) v_t^j \\ &= -\sum_{ij} v_t^i v_t^j \partial_i \partial_j b^{pq}(0) \\ &= \sum_{ij} v_t^i v_t^j \left[\tfrac{1}{2}(\beta_L - \beta_N)(\delta^{ip}\delta^{jq} + \delta^{iq}\delta^{jp}) + \beta_N \delta^{ij}\delta^{pq}\right],\end{aligned}$$

giving the asserted value. □

In (4.4), $v \in T_x$, the tangent space at x, and $v_t \in T_{X_t(x)}$; we may speak of v_t as a "tangent vector from T_x."

It follows from (4.4) that the infinitesimal generator of v_t acting on $f(v) \in C_b^2(\mathbf{R}^d)$ is

$$\mathbf{A} f(v) = \tfrac{1}{2} \sum_{pq} \left(\beta_N |v|^2 \delta^{pq} + (\beta_L - \beta_N) v^p v^q\right) \partial_p \partial_q f(v).$$

Changing to spherical coordinates $r = |v|$, $u = v/|v| \in S^{d-1}$, noting that $v_0 \neq 0$ implies that a.s. $v_t \neq 0$ for all $t \geq 0$, we find that if $f(v) = g(r, u)$ where $g \in C_b^2$ on $(0, \infty) \times S^{d-1}$,

$$\mathbf{A} f = \frac{1}{2}\beta_L r^2 \frac{\partial^2 g}{\partial r^2} + \frac{1}{2}\beta_N (d-1) r \frac{\partial g}{\partial r} + \beta_N \mathbf{A}_0 g, \tag{4.5}$$

where $\mathbf{A}_0$, acting on u only, is half the Laplacian operator on S^{d-1}:

$$\mathbf{A}_0 g(r, u) = \frac{1}{2} \sum_{pq} (\delta^{pq} - u^p u^q) \frac{\partial}{\partial u^p} \frac{\partial}{\partial u^q} \tilde{g}(r, u) - \frac{1}{2}(d-1) \sum_p u^p \frac{\partial}{\partial u^p} \tilde{g}(r, u),$$

where $\tilde{g}$ is any C^2 extension of g to $(0, \infty) \times$ (neighborhood of S^{d-1}). It follows (see [18], 7.15 for more general related propositions) that *$|v_t|$ and u_t are independent, u_t having the law $B(\beta_N t)$, where B is Brownian motion on S^{d-1}, and $|v_t|$ having the law*

$$|v_t| = |v_0| \exp\left\{\sqrt{\beta_L}\, W_t + \tfrac{1}{2}\left[(d-1)\beta_N - \beta_L\right] t\right\}. \tag{4.6}$$

In particular $|v_t| \to 0$ a.s. as $t \to \infty$ iff $(d-1)\beta_N - \beta_L < 0$. This was also the condition for the finiteness of $S(0)$ (see proof of Lemma 3.12). It can hold only if $d = 2$ or 3 and then only for flows that are close to potential.

Now consider a displacement vector $\tilde{V}_t = X_t(x_2) - X_t(x_1)$ and the tangent vectors $v_{it} = DX_t(x_i)(v_i)$, $i = 1, 2$, where $v_1, v_2 \neq 0$.

LEMMA (4.7). *$(v_{1t}, v_{2t}, \tilde{V}_t)$ is a diffusion in $\mathbf{R}^{3d}$, and each component v_{it}^p or $\tilde{V}_t^q$ is a martingale with*

$$\begin{aligned}\langle v_1^p, v_2^q\rangle_t' &= \sum_{ij\alpha} \partial_i V_\alpha^p(X_t(x_1)) \partial_j V_\alpha^q(X_t(x_1) + \tilde{V}_t) v_{1t}^i v_{2t}^j \\ &= -\sum_{ij} \partial_i \partial_j b^{pq}(\tilde{V}_t) v_{1t}^i v_{2t}^j,\end{aligned} \tag{4.8}$$

$$\begin{aligned}\langle v_1^p, \tilde{V}^q\rangle_t' &= \sum_{i\alpha} \partial_i V_\alpha^p(X_t(x_1))\left[V_\alpha^q(X_t(x_1) + \tilde{V}_t) - V_\alpha^q(X_t(x_1))\right] v_{1t}^i \\ &= -\sum_i \partial_i b^{pq}(\tilde{V}_t) v_{1t}^i,.\end{aligned} \tag{4.9}$$

$$\begin{aligned}\langle v_2^p, \tilde{V}^q\rangle_t' &= \sum_{i\alpha} \partial_i V_\alpha^p(X_t(x_1) + \tilde{V}_t) \\ &\quad \times\left[V_\alpha^q(X_t(x_1) + \tilde{V}_t) - V_\alpha^q(X_t(x_1))\right] v_{2t}^i \\ &= -\sum_i \partial_i b^{pq}(\tilde{V}_t) v_{2t}^i.\end{aligned} \tag{4.10}$$

We can get $\langle v_i^p, v_i^q\rangle_t'$ from (4.4) and $\langle \tilde{V}^p, \tilde{V}^q\rangle_t'$ from (3.9).

PROOF. The diffusion property follows from standard results on the solution of the martingale problem [28] and the fact that the right sides of (4.8)–(4.10) depend only on v_{1t}, v_{2t}, and $\tilde{V}_t$. We obtain (4.8)–(4.10) from the following relations deduced from (3.6):

$$\begin{aligned} dv_{1t}^p &= \sum_\alpha \sum_i \partial_i V_\alpha^p(X_t(x_1)) v_{1t}^i \, dW_t^\alpha, \\ dv_{2t}^q &= \sum_\alpha \sum_i \partial_i V_\alpha^q(X_t(x_1) + \tilde{V}_t) v_{2t}^i \, dW_t^\alpha, \\ d\tilde{V}_t^p &= \sum_\alpha \left(V_\alpha^p(X_t(x_1) + \tilde{V}_t) - V_\alpha^p(X_t(x_1))\right) dW_t^\alpha. \qquad \square \end{aligned}$$

Note that $b^{pq}(-x) = b^{pq}(x)$ and $\partial_i b^{pq}(-x) = -\partial_i b^{pq}(x)$, so $\partial_i b^{pq}(0) = 0$. In case $\tilde{V}_0 = 0$, then $\tilde{V}_t = 0$ for all t and (4.8) gives

$$\begin{aligned}\langle v_1^p, v_2^q\rangle_t' &= -\sum_{ij} \partial_i \partial_j b^{pq}(0) v_{1t}^i v_{2t}^j \\ &= \beta_N(v_{1t}, v_{2t})\delta^{pq} + \tfrac{1}{2}(\beta_L - \beta_N)(v_{1t}^p v_{2t}^q + v_{2t}^p v_{1t}^q).\end{aligned} \tag{4.11}$$

For later use the mutual variation process for the martingale parts M_{1t} and M_{2t} of $|v_{1t}|$ and $|v_{2t}|$, $v_{it} = DX_t(x_i)(v_i)$, is

$$\begin{aligned}\langle M_1, M_2\rangle_t' &= |v_{1t}|^{-1}|v_{2t}|^{-1} \sum_{pq} v_{1t}^p v_{2t}^q \langle v_1^p, v_2^q\rangle_t' \\ &= -|v_{1t}|^{-1}|v_{2t}|^{-1} \sum_{ijpq} v_{1t}^i v_{2t}^j v_{1t}^p v_{2t}^q \partial_i \partial_j b^{pq}(\tilde{V}_t),\end{aligned} \tag{4.12}$$

where $\tilde{V}_t = X_t(x_2) - X_t(x_1)$. This is because $d|v_{it}| = |v_{it}|^{-1} \Sigma_p v_{it}^p \, dv_{it}^p$ + drift terms. In the special case $x_1 = x_2$, using (2.10), (4.12) becomes

$$\frac{1}{2}(\beta_L - \beta_N)|v_{1t}|\,|v_{2t}| + \frac{1}{2}(\beta_L + \beta_N)\frac{(v_{1t}, v_{2t})^2}{|v_{1t}|\,|v_{2t}|}. \tag{4.13}$$

In our version $|v_{it}|$ is never 0, so the right side of (4.13) is always defined.

5. Angle-length relations for tangent vectors.

LEMMA (5.1). *Let $u, v \in T_x$, x fixed, be noncollinear (implying neither is 0); let $u_t = DX_t(x)(u)$, $v_t = DX_t(x)(v)$. Let*

$$\gamma_t = \frac{(u_t, v_t)}{|u_t|\,|v_t|}, \qquad \theta_t = \cos^{-1}\gamma_t,$$

taking $0 < \theta_t < \pi$. Then γ_t is a diffusion on $(-1,1)$ with operator $\frac{1}{2}(\beta_L + \beta_N) \times (1-\gamma^2)^2\partial^2/\partial\gamma^2 - \frac{1}{2}(\beta_L + \beta_N)\gamma(1-\gamma^2)\partial/\partial\gamma$; θ_t is a diffusion on $(0,\pi)$ with operator $\frac{1}{2}(\beta_L + \beta_N)\sin^2\theta(\partial^2/\partial\theta^2)$.

This follows from the Itô calculus, using (4.11). Note that -1 and 1 (resp. 0 and π) are inaccessible for γ_t (resp. θ_t) from the interiors of their respective intervals.

Since θ_t is a bounded martingale and $\langle\theta\rangle'_t > 0$ for $0 < \theta_t < \pi$, $\lim \theta_t$ exists a.s. and by familiar reasoning is π with probability θ_0/π and otherwise 0. Hence, $\lim \sin\theta_t = 0$ a.s. This means that under the flow any two initial tangent vectors in the same tangent space tend to line up in the same or directly opposite directions. The alignment takes place exponentially fast, as is seen in the following result. (See the remark about Furstenberg [11] in paragraph 2 of Section 7.)

LEMMA (5.2). $\lim_{t\to\infty}(\log\sin\theta_t)/t = -\frac{1}{2}(\beta_L + \beta_N)$ *a.s.*

PROOF. θ_t satisfies the stochastic differential equation

$$d\theta_t = \sqrt{\beta_L + \beta_N}\,\sin\theta_t\,dW_t.$$

Using Itô's lemma

$$\log\sin\theta_t = \log\sin\theta_0 + \sqrt{\beta_L + \beta_N}\int_0^t \cos\theta_s\,dW_s - \tfrac{1}{2}(\beta_L + \beta_N)t$$

and the lemma follows. □

THEOREM (5.3). *Let $v_1, v_2 \in T_x$ be noncollinear. Let $v_{it} = DX_t(x)(v_i)$, $i = 1,2$; $S_t = \log(|v_{1t}|/|v_{2t}|) - \log(|v_{10}|/|v_{20}|)$; and let θ_t be the angle between v_{1t} and v_{2t}. Then S_t is a martingale; also $\langle S\rangle'_t = \langle\theta\rangle'_t = (\beta_L + \beta_N)\sin^2\theta_t$, whence $\lim S_t = S_\infty$ exists a.s. and in quadratic mean.*

PROOF. Since $\log|v_{1t}|$ and $\log|v_{2t}|$ have the same constant drift $\frac{1}{2}[(d-1)\beta_N - \beta_L]$ (see (4.6)), S_t is a martingale; also $dS_t = d|v_{1t}|/|v_{1t}| -$

$d|v_{2t}|/|v_{2t}|$. A routine calculation based on (4.13) gives the desired result for $\langle S\rangle_t'$. Then $ES_t^2 = E\langle S\rangle_t = E\langle\theta\rangle_t = E\theta_t^2 - \theta_0^2 \le \pi^2$, from which the rest of (5.3) follows. □

(5.3) will be used in Section 7 for some special results when $d = 2$. In particular we shall find the distribution of S_∞.

REMARK (5.4). If $d = 2$, let θ_{it} be the angle between v_{it} and the x^1 axis, $\theta_t = \theta_{1t} - \theta_{2t}$. Then $\theta_t, \theta_{1t}, \theta_{2t}$ are diffusions with 0 drift and

$$\langle\theta_1\rangle_t' = \langle\theta_2\rangle_t' = \beta_N, \qquad \langle\theta\rangle_t' = (\beta_L + \beta_N)\sin^2\theta_t,$$

$$\langle\theta_1, \theta_2\rangle_t' = \beta_N - \tfrac{1}{2}(\beta_L + \beta_N)\sin^2\theta_t.$$

Here θ_t is not necessarily confined to $(0, \pi)$ as it was in Lemma 5.1.

6. Characterization of the potential (irrotational) case. Although the terms "potential" and "solenoidal" were picked as descriptive of the generating field U (see Section 2), we should expect them to be reflected in appropriate properties of the flow itself. Among homogeneous not necessarily isotropic flows, the solenoidal ones are characterized by the preservation of Lebesgue measure; this is known in various settings and is to be expected from the analogous relations between fluid flows and their velocity fields. Alternatively they are the ones whose finite-point diffusions are reversible for Lebesgue measure. The argument for both properties is like that given in [12] for flows in $\mathbf{R}^2$.

To characterize potential isotropic stochastic flows in a manner related to rotation, we recall that for smooth nonrandom flows in $\mathbf{R}^2$ the curl at a point x is the sum of the angular velocities of any two mutually perpendicular tangent vectors in T_x; see [27], 10.2 and 10.5 for this and an analogous property for $\mathbf{R}^3$. Recalling the definition of θ_{it} in (5.4), we shall see that among isotropic flows in $\mathbf{R}^2$ the potential ones are characterized by having $\langle\theta_1 + \theta_2\rangle_t' = 0$ whenever $\theta_{1t} - \theta_{2t}$ is an odd multiple of $\pi/2$. Here is the result for $\mathbf{R}^d$. It is not quite a straightforward analogue of the three-dimensional result given in [27] for the deterministic case.

THEOREM (6.1). *Let v_{1t}, v_{2t} be tangent vectors from T_x. We assume that the initial values v_1 and v_2 (and, hence, v_{1t} and v_{2t}) are noncollinear. Let*

$$\phi_t = \left(\frac{v_{1t}}{|v_{1t}|} + \frac{v_{2t}}{|v_{2t}|}\right)\Big/\left|\frac{v_{1t}}{|v_{1t}|} + \frac{v_{2t}}{|v_{2t}|}\right|$$

be the unit angle bisector of v_{1t} and v_{2t} and let

$$\psi_t = \left(\frac{v_{1t}}{|v_{1t}|} - \frac{v_{2t}}{|v_{2t}|}\right)\Big/\left|\frac{v_{1t}}{|v_{1t}|} - \frac{v_{2t}}{|v_{2t}|}\right|$$

be a unit vector in the span of v_{1t} and v_{2t}, orthogonal to ϕ_t. Then $M_t = \int_0^t(d\phi_s, \psi_s)$, which measures the rotation of ϕ_t in the plane spanned by v_{1t} and v_{2t}, is a *martingale with*

$$\langle M\rangle_t' = \frac{1}{4}\left(3\beta_N - \beta_L + (\beta_L + \beta_N)\frac{(v_{1t}, v_{2t})^2}{|v_{1t}|^2|v_{2t}|^2}\right). \tag{6.2}$$

In particular, only the potential flows, among the isotropic ones, have the property that $\langle M\rangle_t' = 0$ whenever v_{1t} and v_{2t} are orthogonal vectors.

PROOF. Let $u_t = v_{1t}$, $v_t = v_{2t}$, $\tilde{u}_t = u_t/|u_t|$, $\tilde{v}_t = v_t/|v_t|$. If Y is a vector quasimartingale (in [15], Chapter 3, a quasimartingale is defined as a continuous semimartingale) with $|Y_t| \neq 0$, we have from Itô calculus

$$\begin{aligned} d\left(\frac{Y}{|Y|}\right) &= \frac{dY}{|Y|} - \frac{(Y, dY)Y}{|Y|^3} \\ &\quad + \frac{1}{2}\left\{-\frac{1}{|Y|^3}[2(Y, dY)\,dY + Y(dY, dY)] + \frac{3Y(Y, dY)^2}{|Y|^5}\right\} \\ &= \frac{dY}{|Y|} - \frac{(Y, dY)\,dY}{|Y|^3} + \text{terms in the direction of } Y. \end{aligned} \tag{6.3}$$

Here, as usual, $dY^i dZ^j$ means $d\langle M^i, N^j\rangle$, where M^i and N^j are the martingale parts of Y^i and Z^j, respectively. Applying (6.3) to $\phi = Y/|Y|$, where $Y = \tilde{u} + \tilde{v}$, and noting that ϕ and ψ are orthogonal, we have, for M_t as defined in the statement of the theorem,

$$dM = \left(\frac{d(\tilde{u} + \tilde{v})}{|\tilde{u} + \tilde{v}|} - \frac{(\tilde{u} + \tilde{v}, d\tilde{u} + d\tilde{v})d(\tilde{u} + \tilde{v})}{|\tilde{u} + \tilde{v}|^3}, \frac{\tilde{u} - \tilde{v}}{|\tilde{u} - \tilde{v}|}\right). \tag{6.4}$$

The martingale part M^* of $\tilde{u} + \tilde{v}$ satisfies

$$dM^* = \frac{du}{|u|} + \frac{dv}{|v|} - \left[\frac{(u, du)}{|u|^3}u + \frac{(v, dv)}{|v|^3}v\right].$$

This contributes a martingale part N to the right side of (6.4) satisfying

$$\begin{aligned} dN &= \frac{-(du, F) + (dv, G)}{|\tilde{u} + \tilde{v}|\,|\tilde{u} - \tilde{v}|\,|u|\,|v|}, \qquad F = v - \frac{(u, v)u}{|u|^2}, \\ G &= u - \frac{(u, v)v}{|v|^2}, \qquad (F, u) = (G, v) = 0. \end{aligned} \tag{6.5}$$

After some calculations, using (4.11), we find that $\langle N\rangle_t'$ has the value given on the right side of (6.2). To complete the proof notice that the drift term in dM is antisymmetric in $\tilde{u}$ and $\tilde{v}$ and is also invariant under the replacement of $\tilde{u}, \tilde{v}$ by $G\tilde{u}, G\tilde{v}$ for any real orthogonal G. This follows directly from the isotropy of the flow. Choosing G so that $G\tilde{u} = \tilde{v}$, $G\tilde{v} = \tilde{u}$, we see that dM has zero drift, as required. □

7. Liapounov characteristic numbers (exponents). For fixed $x \in \mathbf{R}^d$ the $L(\mathbf{R}^d)$-valued process $DX_t(x)$, whose law does not depend on x, satisfies

$$DX_{t+s}(x) \overset{\text{(law)}}{=} DX'_s(x)\, DX_t(x), \qquad s, t \geq 0, \tag{7.1}$$

where X' is an independent copy of X; this follows from the independence of the "increments" of the flow. Hence, we should expect the behavior of $M_t = DX_t(x)$ to resemble that of a product of independent random matrices. Since M_t is nonsingular, because X_t is a diffeomorphism, we have $M_t = (M_t N_t^{-1})N_t$, where $N_t = (M_t^* M_t)^{1/2}$ is symmetric strictly positive definite and $M_t N_t^{-1}$ is unitary. Thus the semiaxes of the ellipsoid $M_t S^{d-1}$ are $(\lambda_{it})^{1/2}$, $1 \leq i \leq d$, where $\lambda_{1t} \geq \lambda_{2t} \geq \cdots \geq \lambda_{dt} > 0$ are the characteristic values of N_t^2. The Liapounov exponents μ_i may be defined as the limits

$$\mu_i = \lim_{t \to \infty} \frac{\log \lambda_{it}}{2t}, \tag{7.2}$$

which can be shown to exist and to be constants a.s.; actually $\mu_1 > \mu_2 > \cdots > \mu_d$ for isotropic flows.

Carverhill [6], [7] has discussed the exponents for stochastic flows on compact manifolds, and has treated their relation to asymptotic properties of the flow. (In this case (7.1) does not hold, and a multiplicative ergodic theorem is used.) Many of his results remain valid for isotropic flows in $\mathbf{R}^d$. Baxendale [3] has treated flows on manifolds by different methods and has evaluated the μ_i for isotropic flows in $\mathbf{R}^d$. Le Jan [21] and Newman [25] have evaluated the μ_i for certain operators related to $DX_t(x)$ and Le Jan [22] has done it for isotropic flows. Furstenberg [11] discusses the asymptotic behavior of products of independent identically distributed random matrices. In particular his Theorem 8.3 shows (in a slightly different situation from ours) the exponentially fast alignment of columns in the random product. See the remark above (5.2).

It has been shown in [3] that for isotropic flows

$$\mu_i = \tfrac{1}{2}\left[(d-i)\beta_N - i\beta_L\right], \qquad i = 1, \ldots, d. \tag{7.3}$$

Here we note only that the value of μ_1 follows readily from our earlier calculations, although we shall shortly also obtain μ_2 in the case $d = 2$. In fact, we have

$$M_t = \begin{pmatrix} v_{1t}^1 & \cdots & v_{dt}^1 \\ \vdots & & \\ v_{1t}^d & \cdots & v_{dt}^d \end{pmatrix},$$

where $v_{it} = DX_t(x)(e_i)$. Then the trace of $M_t^* M_t$ is $\Sigma_i |v_{it}|^2$ and, hence,

$$\begin{aligned} \mu_1 &= \lim_{t \to \infty} \frac{\log \lambda_{1t}}{2t} = \lim \frac{\log \Sigma \lambda_{it}}{2t} \\ &= \lim \frac{\log \Sigma |v_{it}|^2}{2t} = \frac{1}{2}\left[(d-1)\beta_N - \beta_L\right] \end{aligned} \tag{7.4}$$

from (4.6). Compare Theorem 2 of [7]. From (2.13) we see that *if* $d \leq 3$ *and the flow is close to potential, then* $\mu_1 < 0$ (*Liapounov stability.*) Conversely if $d > 4$, or if $d = 4$ and the flow is not potential, then $\mu_1 > 0$.

For the case $d = 2$, *assumed in the rest of this section*, we can use the results of Section 5 to get additional information about the shape of $M_t S^1$, and at the same time we show how to find the distribution of $\lim_{t\to\infty}|v_{2t}|/|v_{1t}|$ for two tangent vectors from T_x. We shall put $S^1 = C$, $DX_t(x)\, C = C_t$.

Let e_1, e_2 be an orthonormal basis in T_x and represent C as $\{v_\alpha, 0 \leq \alpha < 2\pi\}$, where $v_\alpha = \cos\alpha e_1 + \sin\alpha e_2$. Let $v_{it} = DX_t(x)(e_i)$ and let α_t be the value of α in $[0, \pi)$ for which $|DX_t(x)(v_\alpha)|$ is maximum. There are minima at $\alpha_t \pm \pi/2$, as we see from the polar form of M_t. We shall see that a.s. α_t has a unique determination, continuous in t, for sufficiently large t.

Let A_t and a_t be the lengths of the semimajor and semiminor axes of C_t. Let θ_t be the angle between v_{1t} and v_{2t}. Let $R = \lim_{t\to\infty}|v_{2t}|/|v_{1t}|$, $0 < R < \infty$ (see (5.3)).

THEOREM (7.5). *With the above definitions*:

(a)
$$A_t \sim \left(|v_{1t}|^2 + |v_{2t}|^2\right)^{1/2} \sim |v_{1t}|(1 + R^2)^{1/2}, \qquad t \to \infty,$$
$$a_t \sim \frac{|v_{1t}|\,|v_{2t}|\sin\theta_t}{\left(|v_{1t}|^2 + |v_{2t}|^2\right)^{1/2}} \sim \frac{|v_{1t}|R\sin\theta_t}{(1 + R^2)^{1/2}}, \qquad t \to \infty,$$
$$\frac{a_t}{A_t} \sim \frac{R\sin\theta_t}{1 + R^2}, \qquad t \to \infty.$$

(b) *Almost surely there exists a random* t_0 *such that for* $t > t_0$ *there is a continuous choice of* $\alpha_t \in (0, \pi/2) \cup (\pi/2, \pi)$, *such that* $\lim \alpha_t = \alpha^*$ *and*

(i) *if* $\theta_t \to 0$, *then* $\alpha^* \in (0, \pi/2)$ *and* $\tan\alpha^* = R$;

(ii) *if* $\theta_t \to \pi$ *then* $\alpha^* \in (\pi/2, \pi)$ *and* $\tan\alpha^* = -R$.

(*From* (5.1), $\lim\theta_t = 0$ *or* π *with probability* $\frac{1}{2}$ *each.*)

(c) *If* $v^* = v_{\alpha^*}$ *and* v *is any vector in* C *making an acute* (*obtuse*) *angle with* v^*, *then the angle between* $DX_t(x)(v)$ *and* $DX_t(x)(v^*) \to 0(\pi)$ *as* $t \to \infty$.

Note that v^* is orthogonal to Carverhill's $V^{(1)}_{(x,w)}$ in Theorem 2.1 of [6]. Note also that (b) and (c) can be obtained from Liapounov theory once we know that $\mu_1 > \mu_2$. Conversely from (a) we obtain $\mu_2 - \mu_1 = \lim(\log(a_t/A_t))/t = \lim(\log\sin\theta_t)/t = -\frac{1}{2}(\beta_L + \beta_N) < 0$, verifying (7.3) when $d = 2$.

We are indebted to R. Arratia for the following corollary and its proof.

COROLLARY (7.6). *The distribution function of* R *has the density* $(2/\pi)(1 + r^2)^{-1}$, $r > 0$. (*From this one can deduce the density of* $\lim|v_{2t}|/|v_{1t}|$ *when* v_{10} *and* v_{20} *are arbitrary nonzero vectors in* T_x, *using the linearity of* $DX_t(x)$.)

PROOF OF (7.5). (a) λ_{1t} and λ_{2t} are given by
$$\frac{|v_{1t}|^2 + |v_{2t}|^2 \pm \left[\left(|v_{1t}|^2 + |v_{2t}|^2\right)^2 - 4|v_{1t}|^2|v_{2t}|^2\sin^2\theta_t\right]^{1/2}}{2}$$

with the + and − sign, respectively. Since A_t and a_t are $\sqrt{\lambda_{1t}}$ and $\sqrt{\lambda_{2t}}$, (a) follows from (5.2) and (5.3).

(b) Note that since $\theta_t \in (0, \pi)$ and $\lim \theta_t = 0$ or π, (v_{1t}, v_{2t}) ultimately stays > 0 or < 0 for $t >$ some random t_0, henceforth supposed. Putting $F_t(\alpha) = |DX_t(x)v_\alpha|$, we have

$$F_t(\alpha) = \tfrac{1}{2}\big(|v_{1t}|^2 + |v_{2t}|^2\big) + \tfrac{1}{2}(\cos 2\alpha)\big(|v_{1t}|^2 - |v_{2t}|^2\big) + (\sin 2\alpha)(v_{1t}, v_{2t}), \qquad 0 \le \alpha < \pi.$$

From this it follows that if $(v_{1t}, v_{2t}) > 0$, the smallest critical value of α in $(0, \pi)$ is at $\alpha_t \in (0, \pi/2)$ where $\cot 2\alpha_t = \frac{1}{2}(1 - R_t^2)/(R_t \cos\theta_t)$, $R_t = |v_{2t}|/|v_{1t}|$. This determines $2\alpha_t \in (0, \pi)$ and $\alpha_t \in (0, \pi/2)$ uniquely with $F_t''(\alpha_t) < 0$, so that $F_t(\alpha_t)$ is maximum. Then $\lim \cot 2\alpha_t = \cot 2\alpha^* = \frac{1}{2}(1 - R^2)/R$, which determines α^* uniquely by $\alpha^* \in (0, \pi/2)$, $\tan\alpha^* = R$. If $(v_{1t}, v_{2t}) < 0$, $F_t(\alpha)$ has a minimum at $\tilde{\alpha}_t \in (0, \pi/2)$ and a maximum at $\alpha_t = \tilde{\alpha}_t + \pi/2 \in (\pi/2, \pi)$, satisfying $\cot 2\alpha_t = \cot 2\tilde{\alpha}_t = \frac{1}{2}(1 - R_t^2)/(R_t \cos\theta_t)$. In this case $\lim \cot 2\alpha_t = \cot 2\alpha^* = -\frac{1}{2}(1 - R^2)/R$ and α^* is the unique angle in $(\pi/2, \pi)$ satisfying $\tan\alpha^* = -R$.

(c) Put $v_t = DX_t(x)(v)$ where $v = \cos\phi e_1 + \sin\phi e_2 \in T_x$, $v^* = \cos\alpha^* e_1 + \sin\alpha^* e_2$ where α^* is as in (b), $v_t^* = DX_t(x)(v^*)$. Then $v_t = \cos\phi v_{1t} + \sin\phi v_{2t}$, $v_t^* = \cos\alpha^* v_{1t} + \sin\alpha^* v_{2t}$. If $\theta_t \to 0$, we have $\sin\alpha^* = R/\sqrt{1 + R^2}$, $\cos\alpha^* = 1/\sqrt{1 + R^2}$, $(v_{1t}, v_{2t}) \sim |v_{1t}|\,|v_{2t}|$, and we find

$$\lim_{t\to\infty} \frac{(v_t, v_t^*)}{|v_t|\,|v_t^*|} = \frac{\cos(\phi - \alpha^*)}{|\cos(\phi - \alpha^*)|}.$$

If $\theta_t \to \pi$ we have $\sin\alpha^* = R/\sqrt{1 + R^2}$, $\cos\alpha^* = -1/\sqrt{1 + R^2}$, $(v_{1t}, v_{2t}) \sim -|v_{1t}|\,|v_{2t}|$, and we find that the limit is the same, proving (c). (*Note.* We shall see in a moment that α^* has an absolutely continuous distribution, so the above limit is a.s. well defined.) □

PROOF OF (7.6). Although α^* was defined with respect to a particular coordinate system, one can see from isotropy that its distribution is the same for any orthonormal basis e_1, e_2. It can be seen from this that α^* must have an absolutely continuous distribution and in fact must be uniform on $(0, \pi)$. Since $R = \tan\alpha^*$ if $\alpha^* \in (0, \pi/2)$ and $-\tan\alpha^*$ if $\alpha^* \in (\pi/2, \pi)$, (7.6) follows. □

8. Arc lengths. Let $\gamma(u)$, $a \le u \le b$ be a piecewise-C^1 continuous curve in $\mathbf{R}^d$ (having bounded continuous first derivatives on each piece), with $|\gamma'(u)| \ne 0$ at each C^1 point. Put $\gamma_t(u) = X_t(\gamma(u))$, $L_t = \int_a^b |\gamma_t'(u)|\,du$ = length of γ_t. Because of the diffeomorphic property of the flow γ_t is also piecewise-C^1 and $|\gamma_t'(u)| \ne 0$ at each C^1 point.

Since $\gamma_t'(u)$ is a tangent vector, we have in law for fixed u (see (4.6))

$$|\gamma_t'(u)| = |\gamma'(u)| \exp\Big\{\sqrt{\beta_L}\, W_t + \tfrac{1}{2}\big[(d-1)\beta_N - \beta_L\big]t\Big\}, \tag{8.1}$$

$$d|\gamma_t'(u)| = |\gamma_t'(u)|\Big\{\sqrt{\beta_L}\, dW_t + \tfrac{1}{2}(d-1)\beta_N\, dt\Big\}. \tag{8.2}$$

To exhibit the dependence of the r.h.s. of (8.2) on u write

$$|\gamma_t'(u)| = |\gamma'(u)| + M_t(u) + \tfrac{1}{2}(d-1)\beta_N \int_0^t |\gamma_s'(u)|\, ds, \tag{8.3}$$

where, using (4.13) with $v_{1t} = v_{2t} = \gamma_t'(u)$,

$$\langle M(u)\rangle_t' = \beta_L |\gamma_t'(u)|^2. \tag{8.4}$$

Since $|\gamma'(u)|$ is integrable and $DX_t(x)$ is continuous in (t, x), we may integrate (8.3) on u to get

$$L_t = L_0 + N_t + \tfrac{1}{2}(d-1)\beta_N \int_0^t L_s\, ds, \qquad N_t = \int_a^b M_t(u)\, du. \tag{8.5}$$

Using (8.1) and (8.4), one can get estimates insuring that N_t is an L_2 martingale. It has continuous sample functions. One can check that

$$\langle N\rangle_t = \int_a^b du_1 \int_a^b du_2 \langle M(u_1), M(u_2)\rangle_t. \tag{8.6}$$

In differential form (8.5) is

$$dL_t = dN_t + \tfrac{1}{2}(d-1)\beta_N L_t\, dt, \tag{8.7}$$

whence L_t is a submartingale with $EL_t = L_0 \exp\{\frac{1}{2}(d-1)\beta_N t\} \to \infty$ as $t \to \infty$.

If $u_1, u_2 \in [a, b]$, denote $\gamma_t(u_2) - \gamma_t(u_1) = X_t(\gamma(u_2)) - X_t(\gamma(u_1))$ by $\tilde{V}_t(u_1, u_2)$ or by $\tilde{V}_t$ if u_1 and u_2 are understood. Fixing u_1 and u_2 for the moment, let v_{1t} and v_{2t} denote $\gamma_t'(u_1)$ and $\gamma_t'(u_2)$, respectively. Since $M_t(u_\iota)$ is the martingale part of $|v_{it}|$, $i = 1, 2$ we have from (4.12)

$$\langle M(u_1), M(u_2)\rangle_t' = -|v_{1t}|^{-1}|v_{2t}|^{-1} \sum_{ijpq} v_{1t}^{\iota} v_{2t}^{j} v_{1t}^{p} v_{2t}^{q} \partial_\iota \partial_j b^{pq}(\tilde{V}_t). \tag{8.8}$$

Referring to the remarks below (2.2) we see that $\partial_i \partial_j b^{pq}(\tilde{V}_t) = \partial_\iota \partial_j b^{pq}(0) + \Theta K(1 \wedge \tilde{V}_t^2)$ where $K \in (0, \infty)$ depends only on the correlation tensor b; here $|\Theta| \le 1$. From (2.10)

$$\begin{aligned} \langle M(u_1), M(u_2)\rangle_t' &= \tfrac{1}{2}(\beta_L - \beta_N)|v_{1t}|\,|v_{2t}| + \tfrac{1}{2}(\beta_L + \beta_N)\frac{(v_{1t}, v_{2t})^2}{|v_{1t}|\,|_{2t}|} \\ &\quad + \Theta K(1 \wedge \tilde{V}_t^2)|v_{1t}|\,|v_{2t}|. \end{aligned} \tag{8.9}$$

From (8.6) and (8.9)

$$\begin{aligned} \langle N\rangle_t' &= \frac{1}{2}(\beta_L - \beta_N)\int_a^b du_1 \int_a^b du_2 |\gamma_t'(u_1)|\,|\gamma_t'(u_2)| \\ &\quad + \frac{1}{2}(\beta_L + \beta_N)\int_a^b du_1 \int_a^b du_2 \frac{(\gamma_t'(u_1), \gamma_t'(u_2))^2}{|\gamma_t'(u_1)|\,|\gamma_t'(u_2)|} \\ &\quad + \Theta K(1 \wedge \Delta_t^2)\int_a^b du_1 \int_a^b du_2 |\gamma_t'(u_1)|\,|\gamma_t'(u_2)| \\ &= \frac{1}{2}(\beta_L - \beta_N)L_t^2 + \frac{1}{2}(\beta_L + \beta_N)\int_a^b du_1 \int_a^b du_2 \frac{(\gamma_t'(u_1), \gamma_t'(u_2))^2}{|\gamma_t'(u_1)|\,|\gamma_t'(u_2)|} \\ &\quad + \Theta K(1 \wedge \Delta_t^2)L_t^2, \end{aligned} \tag{8.10}$$

where $\Delta_t \le L_t$ is the diameter of γ_t.

From (8.7) and (8.10) the behavior of L_t when it is small depends on the shape of γ_t (approximately) only through the integral

$$(8.11) \qquad I_t = \int_a^b \int_a^b \frac{(\gamma_t'(u_1), \gamma_t'(u_2))^2}{|\gamma_t'(u_1)|\,|\gamma_t'(u_2)|}\, du_1\, du_2.$$

LEMMA (8.12). *If γ is an arc satisfying the conditions at the beginning of this section, put*

$$I(\gamma) = \int_a^b du_1 \int_a^b du_2 \frac{(\gamma'(u_1), \gamma'(u_2))^2}{|\gamma'(u_1)|\,|\gamma'(u_2)|}$$

and let $L(\gamma)$ be the length of γ. Then $(L(\gamma))^2/d \le I(\gamma) \le (L(\gamma))^2$, and both bounds are attained, the maximum when and only when γ is (geometrically) a straight line segment.

NOTE (8.13). There are many minimizing shapes. One example occurs when γ consists of d pairwise perpendicular line segments of equal length. If $d = 2$, circles, semicircles, and all regular polygons are minimizing shapes.

PROOF OF (8.12). The statement about the maximizing shape is obvious. For the lower bound, since $I(\gamma)$ is invariant under a change of the arc parameter, we may as well assume $|\gamma'(u)| = 1$. If $\gamma'(u)$ has components $\gamma'^i(u)$, let $h^{ij} = \int_a^b \gamma'^i(u)\gamma'^j(u)\, du$. Then $I(\gamma) = \sum_{i,j}(h^{ij})^2 \ge \sum_i (h^{ii})^2 \ge (1/d)(\sum_i h^{ii})^2 = (1/d)(b-a)^2 = (L(\gamma))^2/d$. One may easily check that the first example mentioned in (8.13) attains this minimum, and the values for the others can readily be calculated. □

Putting together (8.7), (8.10), and (8.12) applied to γ_t, we have the following result.

THEOREM (8.14). *With the initial arc γ described at the beginning of this section*

$$dL_t = dN_t + \tfrac{1}{2}(d-1)\beta_N L_t\, dt,$$

where for the martingale N we have

$$(8.15) \qquad \frac{(d+1)\beta_L - (d-1)\beta_N}{2d} L_t^2 - K(1 \wedge \Delta_t^2)L_t^2 \le \langle N \rangle_t' \le \beta_L L_t^2 + K(1 \wedge \Delta_t^2)L_t^2,$$

the upper bound corresponding to a geometrically straight γ_t, the lower to any of the shapes in (8.13).

From (2.13), $(d+1)\beta_L - (d-1)\beta_N \ge 0$, with equality only in the solenoidal case.

We shall be interested in conditions under which $L_t \to 0$. A logarithmic transformation is suggestive. From (8.14)

$$(8.16)\quad d\log\left(\frac{L_t}{L_0}\right) = \frac{dN_t}{L_t} + A'(t)\,dt, \qquad A'(t) = \frac{1}{2}(d-1)\beta_N - \frac{1}{2}\frac{\langle N\rangle_t'}{L_t^2},$$

and from (8.15)

$$\frac{1}{2}(d-1)\beta_N - \frac{1}{2}\beta_L - \frac{1}{2}K(1\wedge\Delta_t^2) \le A'(t)$$

$$(8.17)\qquad \le \left(\frac{2d^2-d-1}{4d}\right)\beta_N - \left(\frac{d+1}{4d}\right)\beta_L + \frac{1}{2}K(1\wedge\Delta_t^2),$$

$$(8.18)\quad \frac{(d+1)\beta_L-(d-1)\beta_N}{2d} - K(1\wedge\Delta_t^2) \le \left\langle \int\frac{dN}{L}\right\rangle_t' \le \beta_L + K(1\wedge\Delta_t^2).$$

If $\Delta_t \le L_t$ is small then $\langle \int dN/L\rangle_t' = \langle N\rangle_t'/L_t^2$ is bounded between positive constants (the lower one is strictly positive except in the solenoidal case). If $d=2$, and if the flow is potential or not too far from potential as measured by β_L/β_N, the upper bound in (8.17), which corresponds to one of the shapes of (8.13), is negative when L_t is small. Since L_t, no matter what the shape, thus has a negative logarithmic drift rate when small, it should have a chance of approaching 0, as will be seen to be the case in the next section.

For $d=3$ the situation is interesting because a short straight arc has a negative logarithmic drift in potential or near potential cases (the lower bound in (8.17)), but a short arc of the type in (8.13) has a positive logarithmic drift (the upper bound in (8.17)). We have not settled what happens in this case. One might surmise from the results of Section 7 that a short arc tends to straighten itself out, in which case the lower bound in (8.17) might be relevant.

For $d \ge 4$ (3.16)(a) shows that a fortiori $L_t \to \infty$ a.s.

9. Decrease of arc lengths. Rather than the logarithmic transformation of Section 8, it seems simpler to use the following proposition, stated here for *any* quasimartingale $L_t > 0$ and continuous martingale N_t subject to the indicated conditions, and adapted to a common filtration. (See the definition of "quasimartingale" above (6.3).)

PROPOSITION (9.1). *Suppose* $dL_t = dN_t + \frac{1}{2}\lambda L_t\,dt$ (*cf.* (8.14)), $L_t > 0$, $\langle N\rangle_t' \ge \delta L_t^2 - K_1 L_t^3$, *where* $0 < \lambda < \delta$ *and* $K_1 > 0$ *are constants. Then there exist constants* $\bar{L} > 0$, $B > 0$, *and* $0 < A < 1$ *such that*

$$(9.2)\qquad P\{L_t < e^{-Bt}\bar{L}\ \forall\ t \ge 0\} \ge 1 - (L_0/\bar{L})^A$$

whenever $L_0 \le \bar{L}$. *In fact we may take any* $A \in (0,1)$, $B > 0$, $\bar{L} > 0$ *such that* (i) $K_1\bar{L} < \delta - \lambda$ *and* (ii) $A(\delta - K_1\bar{L}) + 2B \le \delta - \lambda - K_1\bar{L}$.

PROOF. Let A, B, $\bar{L}$ satisfy (i) and (ii) and put $f_t = (e^{Bt}L_t)^A$. Then $f_0 = L_0^A \le \bar{L}^A$ and

$$df_t = Ae^{ABt}L_t^{A-1}\,dN_t + \frac{1}{2}Ae^{ABt}L_t^A \cdot \left\{2B + \lambda - \frac{(1-A)}{L_t^2}\langle N\rangle_t'\right\}dt.$$

Let $\tau = \inf\{t \ge 0\colon f_t = \bar{L}^A\}$. Then $df_t = Ae^{ABt}L_t^{A-1}\,dN_t + C_t'\,dt$ where, for $t \le \tau$,

$$\begin{aligned} C_t' &\le \tfrac{1}{2}Ae^{ABt}L_t^A\{2B + \lambda - (1-A)(\delta - K_1 L_t)\} \\ &\le \tfrac{1}{2}Ae^{ABt}L_t^A\{A(\delta - K_1\bar{L}) + 2B - (\delta - \lambda - K_1\bar{L})\} \le 0 \end{aligned}$$

by (i) and (ii). It follows that $f_{t\wedge\tau}$ is a positive supermartingale with values in $[0, \bar{L}^A]$ and

$$(9.3)\quad P\{\tau = \infty\} = 1 - P\{\tau < \infty\} = 1 - \lim_{t\to\infty} P\{\tau \le t\} \ge 1 - (L_0/\bar{L})^A,$$

i.e., $P\{L_t < e^{-Bt}\bar{L}\ \forall\ t \ge 0\} \ge 1 - (L_0/\bar{L})^A$ if $L_0 \le \bar{L}$. □

Considering (9.1) in relation to (8.14) and (8.15), and noting $K(1 \wedge \Delta_t^2) \le K\Delta_t \le KL_t$, we may take $K = K_1$, $\delta = [(d+1)\beta_L - (d-1)\beta_N]/2d$, $\lambda = (d-1)\beta_N$. Then $\delta - \lambda = [(d+1)\beta_L - (d-1)(1+2d)\beta_N]/2d$. Using (2.13) we see that $\delta - \lambda > 0$ iff $d = 2$ and $\frac{5}{3} < \beta_L/\beta_N \le 3$. We can then choose A and B so that (ii) holds. Combining the above results:

THEOREM (9.4). *Suppose $d = 2$ and $\frac{5}{3} < \beta_L/\beta_N \le 3$. Let K be as determined below* (8.8). *Pick $\bar{L} > 0$, $A \in (0,1)$, $B > 0$ so that* (i) *and* (ii) *of* (9.1) *hold, where $\delta = (3\beta_L - \beta_N)/4$, $\lambda = \beta_N$, $K_1 = K$. Let γ be an arc as in Section* 8 *with initial length $L_0 \le \bar{L}$. Then $P\{L_t < e^{-Bt}\bar{L}\ \forall\ t \ge 0\} \ge 1 - (L_0/\bar{L})^A$.*

EXAMPLE. $d = 2$, $\beta_L = 3\beta_N$ (potential case). Take $K\bar{L} = \mu\beta_N$, $B = \nu\beta_N$, where $\mu, \nu > 0$, $\mu + 2\nu < 1$; $A = (1 - \mu - 2\nu)/(2 - \mu)$. Note that $A < \frac{1}{2}$ but we can make $\frac{1}{2} - A$ arbitrarily small.

We can strengthen (9.4) as follows. The general idea is that if the diameter Δ_0 of γ is small, then $P(L_t \to 0)$ is close to 1 even if L_0 is large.

THEOREM (9.5) (Conditions and notation of (9.4), but allowing any $L_0 < \infty$). (a) *If $\Delta_0 \le \bar{L}/5$, then $P(L_t \to 0) \ge 1 - (5\Delta_0/\bar{L})^A$.* (b) *If Δ_t = diameter of $\gamma_t = X_t(\gamma)$ then $P(L_t \to 0 | \Delta_t \to 0) = 1$, whatever the values of Δ_0 and L_0, provided $P(\Delta_t \to 0) > 0$.*

PROOF. Let γ be as in (9.4) with $\Delta_0 \le \bar{L}/5$. By a simple geometrical construction we see that there is a C^1 simple closed curve γ' of length $< 5\Delta_0$ having γ in its interior. Let L_t' (resp. Δ_t') be the length (resp. diameter) of $\gamma_t' = X_t(\gamma')$. From (9.4) we have, using $\Delta_t < \Delta_t' < \frac{1}{2}L_t'$,

$$(9.6)\quad \begin{aligned} &P\left(L_t' \le e^{-Bt}\bar{L}\ \forall\ t \ge 0\right) \ge 1 - (5\Delta_0/\bar{L})^A, \\ &P\left(\Delta_t \le \tfrac{1}{2}e^{-Bt}\bar{L}\ \forall\ t \ge 0\right) \ge 1 - (5\Delta_0/\bar{L})^A, \end{aligned}$$

the second line of (9.6) following from the first. Part (a) will now follow as soon as we have proved part (b).

From (8.15) we have

$$\langle N\rangle_t' \geq \delta L_t^2 - K\Delta_t L_t^2. \tag{9.7}$$

Defining f_t and C_t' as in the proof of (9.1), we have, using (9.7),

$$C_t' \leq \tfrac{1}{2}Ae^{ABt}L_t^A\left[2B + \lambda - (1-A)\delta + (1-A)K\Delta_t\right], \tag{9.8}$$

and $C_0' < 0$ from (9.2)(ii). Hence if $\Delta_t \to 0$ then eventually $C_t' < 0$. Let $\sigma_n = \inf\{t \geq n\colon C_t' \geq 0\}$, $\sigma_n = \infty$ if $C_t' < 0\ \forall\ t \geq n$. Since $\{f_{t\wedge\sigma_n}\colon t \geq n\}$ is a positive supermartingale, $\lim_{t\to\infty} f_{t\wedge\sigma_n}$ exists and is finite, whence the same is true of $\lim f_t$ on $\{\sigma_n = \infty\}$. Hence $L_t \to 0$ on $\{\sigma_n = \infty\}$, $n = 1, 2, \ldots$. Then

$$P(\Delta_t \to 0) \leq P\Big(\bigcup_n (\sigma_n = \infty)\Big) = \lim P(\sigma_n = \infty) \leq P(L_t \to 0). \tag{9.9}$$

This completes the proof, since $L_t \geq \Delta_t$. □

REMARKS (9.10). (1) Under the conditions of (9.4), if $x \in \gamma$ there is a.s. a subarc of random length containing x whose length $\to 0$ (use the Borel–Cantelli lemma). (2) Under the conditions of (9.4), if B_r is a ball of radius r, then $\lim_{r\to 0} P\{\lim_{t\to\infty}\operatorname{diam}(X_t(B_r)) = 0\} = 1$. For a general result for balls in compact manifolds of dimension $d \geq 2$ in cases of Liapounov stability see Carverhill [6], 2.3.3. (3) Lemma (3.12) (c) implies a fortiori that if $d = 2$ and $\beta_L/\beta_N \leq 1$ then $P(L_t \to 0) = 0$. We cannot say what happens if $1 < \beta_L/\beta_N \leq 5/3$. (4) We cannot say what happens to an arc of large diameter under the conditions of (9.4).

10. Volumes. We have already mentioned that results on volumes in compact manifolds have been obtained by Carverhill [6]. Here we confine ourselves to a few remarks that are readily established from what has gone before, using also the following fact: If $\{X_t,\ t \geq 0\}$ is an isotropic diffeomorphic flow in $\mathbf{R}^d$ of the type studied above, then for each *fixed* t the transformation X_t^{-1} has the same law as X_t. This can be established by an argument similar to that in [2], Section 6 dealing with flows on spheres.

Let $J_t(x) \equiv J(X_t, x) = \det(DX_t(x))$. $J_t(x) > 0$ since X_t is diffeomorphic and X_0 is the identity. From (7.1), extended to several factors, and the continuity in t of $J_t(x)$, we see that for fixed x

$$\begin{aligned} J_t(x) &\overset{\text{(law)}}{=} \exp\{aW_t(x) + bt\}, \\ EJ_t(x) &= \exp\left\{\tfrac{1}{2}a^2 t + bt\right\}. \end{aligned} \tag{10.1}$$

On the other hand, arguing as in [12], Section 3, if $\phi \in C_0^\infty(\mathbf{R}^d, \mathbf{R}^1)$,

$$\begin{aligned} \int_{\mathbf{R}^d}\phi(x)\,dx &= E\int_{\mathbf{R}^d}\phi(X_t(x))\,dx \\ &= E\int_{\mathbf{R}^d}\phi(y)J(X_t^{-1}, y)\,dy \\ &= e^{(a^2/2)t+bt}\int_{\mathbf{R}^d}\phi(y)\,dy, \end{aligned}$$

whence $b = -\frac{1}{2}a^2$ and $J_t(x)$ is a martingale with finite moments of order $1, 2, \dots$.

Using (7.2) and (7.3) we have

$$\lim_{t\to\infty} \frac{\log J_t(x)}{t} = \lim \frac{\Sigma \log \lambda_{it}}{2t} = \Sigma\mu_i = \frac{d}{4}\left[(d-1)\beta_N - (d+1)\beta_L\right].$$

But $\lim \log J_t(x)/t = -\frac{1}{2}a^2$, whence $a^2 = (d/2)[(d+1)\beta_L - (d-1)\beta_N]$. From (2.12) this is $E(\operatorname{div} U(x))^2$.

If B is a bounded Borel set in $\mathbf{R}^d$, $|\cdot|$ denotes Lebesgue measure, and $B_t = X_t(B)$, then $|B_t| = \int_B J_t(x)\,dx$ is a positive martingale with finite moments. Hence $\lim_{t\to\infty}|B_t|$ exists and is finite a.s.

The quadratic variation of the martingale can be evaluated, but is not given here. We have not settled in general what the nature of $\lim|B_t|$ is, although under the conditions of Theorem 9.4, taking γ as a simple closed curve and B as the interior of γ, we can get a lower bound for $P(|B_t| \to 0)$.

See also Le Jan [23] and the Addendum at the end of Section 1.

11. Homogeneous nonisotropic flows. In this section we indicate briefly the extent to which our results are valid when the isotropy condition (1.2) is dropped. We still assume homogeneity. Consider the vector field $V(x)$ on $\mathbf{R}^d$ given by

$$V^p(x) = \lim_{t\downarrow 0} t^{-1}E\left(X_t^p(x) - x^p\right). \tag{11.1}$$

$V(x)$ is the (Itô) drift of the flow. By homogeneity $V(x) \equiv V_0$, a constant vector field. The flow determined by the correlation tensor b and the drift V_0 now satisfies

$$dX_t(x) = \sum_{\alpha\geq 1} V_\alpha(X_t(x))\,dW_t^\alpha + V_0\,dt, \qquad X_0(x) = x, \tag{11.2}$$

where the V_α and the W_t^α are as in Section 3. It can be seen that the effect of V_0 is merely to superpose a constant drift in the direction V_0 on a flow with zero drift. The correction term for Stratonovich drift, again $\frac{1}{2}\Sigma_q \partial_q b^{pq}(0)$ as in the isotropic case, is now not zero in general, so we may take either the Itô or the Stratonovich drift to be 0, but not both.

The displacement $\tilde{V}_t = X_t(x) - X_t(y)$ is still a diffusion in $\mathbf{R}^d$. It has diffusion matrix $\langle \tilde{V}^p, \tilde{V}^q\rangle_t' = 2(b^{pq}(0) - b^{pq}(\tilde{V}_t))$. The tangent flow $DX_t(x)$ satisfies (4.1). For fixed x and v, $v_t = DX_t(x)(v)$ is a diffusion in $\mathbf{R}^d$ with zero drift and diffusion matrix

$$\langle v^p, v^q\rangle_t' = -\sum_{i,j} v_t^i v_t^j \partial_i \partial_j b^{pq}(0). \tag{11.3}$$

Since v_t satisfies a linear stochastic differential equation we may follow the method of Has'minskii [14] to determine the asymptotic behavior of $|v_t|$. Putting $\tilde{v}_t = v_t/|v_t|$, $\tilde{v}_t$ is a diffusion on S^{d-1} whose generator may be computed in terms of $\partial_i\partial_j b^{pq}(0)$. In particular, under the nondegeneracy condition that

$$-\sum_{i,j,p,q} v^i v^j u^p u^q \partial_i \partial_j b^{pq}(0) > 0$$

for all $u, v \neq 0$ (notice that ≥ 0 is automatic), the process $\tilde{v}_t$ will be a nondegenerate diffusion with a (unique) smooth invariant probability measure ν on S^{d-1}. $\sigma_t = \log|v_t|$ can be written in the form

$$d\sigma = f(\tilde{v}_t)\,dW_t + g(\tilde{v}_t)\,dt, \tag{11.4}$$

where the functions f, g: $S^{d-1} \to \mathbf{R}$ are given by

$$\left(f(\tilde{v}_t)\right)^2 = -\sum_{i,j,p,q} \tilde{v}_t^p \tilde{v}_t^q \tilde{v}_t^i \tilde{v}_t^j \partial_i \partial_j b^{pq}(0), \tag{11.5}$$

$$g(\tilde{v}_t) = -\tfrac{1}{2} \sum_{i,j,p,q} \left(\delta^{pq} - 2\tilde{v}_t^p \tilde{v}_t^q\right) \tilde{v}_t^i \tilde{v}_t^j \partial_i \partial_j b^{pq}(0). \tag{11.6}$$

We see that $\sigma_t = \log|v_t|$ is not a diffusion; its rate of change depends upon the direction of $\tilde{v}_t = v_t/|v_t|$. In particular Has'minskii shows that the asymptotic behavior of $|v_t|$ is determined by the average value $\int_{S^{d-1}} g(\tilde{v})\,d\nu(\tilde{v})$.

The Liapounov exponents $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_d$ exist as before. Assuming that $\mu_1 > \mu_2$ it will again follow that tangent vectors $v_{it} = DX_t(x)(v_i)$, $i = 1, 2$ will align themselves exponentially fast, for any noncollinear $v_1, v_2 \in T_x$. Also the ratio $|v_{2t}|/|v_{1t}|$ will converge. However in general the angle θ_t between v_{1t} and v_{2t}, and $\log(|v_{2t}|/|v_{1t}|)$, will not be martingales and θ_t will not be a diffusion.

The result on the shrinking of arcs depends upon a comparison of the sizes of the martingale and drift parts of L_t. In the nonisotropic case both of these terms will depend strongly not only on the shape of the curve but also on its orientation inside $\mathbf{R}^d$, so that our techniques do not apply.

Finally, our result in Section 10 that the Jacobian $J_t(x)$ is of the form $J_t(x) = \exp(aW_t - \frac{1}{2}a^2 t)$ depends only on homogeneity together with the fact that X_t and X_t^{-1} have the same law. By the comment earlier in this section we may assume without loss of generality that the flow has zero Stratonovich drift. Our result now follows. The constant a is given by

$$a^2 = \sum_{\alpha \geq 1} \left(\operatorname{div} V_\alpha(x)\right)^2 = -\sum_{i,j} \partial_i \partial_j b^{ij}(0).$$

The argument that $|B_t|$ is also a martingale follows as before.

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