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CONVOLUTION OF THE IFRA SCALED-MINS CLASS

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The class of nonnegative random vectors $\mathbf{T} = (T_1, \ldots, T_n)$ for which $\min_{1 \le i \le n} a_i T_i$ is IFRA for all $0 < a_i \le \infty$, $i = 1, \ldots, n$, is closed under convolution.

1. Introduction and statement of main result. In recent years various multivariate extensions of the univariate classes of life distributions that are important in reliability theory have been proposed. A survey of many of these classes may be found in Block and Savits (1981). Central to the study of these classes is the determination of whether or not they are closed under the operation of convolution. In reliability theory, convolution corresponds to the natural operation of standby redundancy, i.e., replacing failed components with new ones.

In this paper we focus on an important extension of the IFRA (increasing failure rate average) class due to Esary and Marshall (1979): A nonnegative random vector $\mathbf{T} = (T_1, \ldots, T_n)$ is said to satisfy condition (F) if $\min_{1 \le i \le n} a_i T_i$ is IFRA for all choices $0 < a_i \le \infty$, $i = 1, \ldots, n$. (Recall that a nonnegative random variable T is called IFRA if $\bar{F}(\alpha t) \ge \bar{F}^{\alpha}(t)$ for all $t \ge 0$, $0 < \alpha < 1$, where $\bar{F}(t) = P(T > t)$ is the survival probability.) Here we interpret $\infty \cdot 0 = \infty$. We will denote the class of all nonnegative random vectors satisfying condition (F) by $\mathscr{I}$ and call it the IFRA scaled-mins class.

Since the life length of a series system corresponds to the minimum of its component life lengths, any life class that is closed with respect to minimums is natural and important in the reliability context. The mechanism of considering minimums and scaled minimums has also been considered by several authors: Marshall and Olkin (1967a, b) and Esary and Marshall (1974).

Although Esary and Marshall (1979) considered some closure properties of the class $\mathscr{I}$, they did not study closure with respect to the operation of convolution. Recently, El-Neweihi (1984) showed that the class is closed under convolution provided one of the two vectors has independent components; however, the general problem was not resolved. The purpose of this paper is to prove the general result as stated below.

THEOREM 1.1. *The IFRA scaled-mins class $\mathscr{I}$ is closed under convolution.*

The proof of this result is contained in Section 2. In Section 3 we consider a further characterization of the class $\mathscr{I}$. As usual R_+^n denotes the nonnegative orthant.

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(1.2) REMARK. The standard Gumbel exponential distribution (cf. Johnson and Kotz (1972), page 250) $\bar{F}(x, y) = \exp\{-x - y - \delta xy\}$, $0 \le \delta \le 1$, can be easily shown to belong to the IFRA scaled-mins class. Consequently our result shows that the convolution of any two such distributions belongs to the IFRA scaled-mins class. Block and Savits (1980) proposed another multivariate IFRA class called MIFRA. This class has all the properties that one would expect for a multivariate IFRA class including closure under convolution. It is not known, however, whether the Gumbel distribution belongs to the MIFRA class.

2. Proof of the main result. We shall make use of the following characterization of the class $\mathscr{I}$ due to El-Neweihi (1984).

THEOREM 2.1 (El-Neweihi). *A nonnegative random vector* $\mathbf{T} = (T_1, \dots, T_n)$ *belongs to* $\mathscr{I}$ *if and only if*

$$E\left[\prod_{i=1}^{n} h_i(T_i)\right] \le E^{1/\alpha}\left[\prod_{i=1}^{n} h_i^{\alpha}(T_i/\alpha)\right] \tag{2.1}$$

for all $0 < \alpha < 1$ *and all nonnegative nondecreasing functions* h_i *defined on* $[0, \infty)$, $i = 1, \dots, n$.

Thus to show that $\mathscr{I}$ is closed under convolution, we need only show that if $\mathbf{S} = (S_1, \dots, S_n)$ and $\mathbf{T} = (T_1, \dots, T_n)$ are independent vectors in $\mathscr{I}$, then

$$E\left[\prod_{i=1}^{n} h_i(S_i + T_i)\right] \le E^{1/\alpha}\left[\prod_{i=1}^{n} h_i^{\alpha}((S_i + T_i)/\alpha)\right]$$

for all $0 < \alpha < 1$ and all nonnegative nondecreasing h_i on $[0, \infty)$, $i = 1, \dots, n$. As usual we may assume without loss of generality that each h_i is continuous and bounded (see, e.g., Block and Savits (1980)).

First we prove a lemma.

LEMMA 2.2. *Let* $H(\mathbf{s}, \mathbf{t})$ *be bounded, nonnegative, and continuous on* $R_+^n \times R_+^n$. *Let* μ *and* ν *be two finite measures on* R_+^n. *For* $0 < \alpha < 1$, *define* $\|H(\cdot, \mathbf{t})\|_\alpha = \{\int H^\alpha(\mathbf{s}, \mathbf{t})\, d\nu(\mathbf{s})\}^{1/\alpha}$. *Then*

$$\begin{aligned} \int \|H(\cdot, \mathbf{t})\|_\alpha \, d\mu(\mathbf{t}) &\le \left\{\int \left[\int H(\mathbf{s}, \mathbf{t})\, d\mu(\mathbf{t})\right]^\alpha d\nu(\mathbf{s})\right\}^{1/\alpha} \\ &= \left\|\int H(\cdot, \mathbf{t})\, d\mu(\mathbf{t})\right\|_\alpha . \end{aligned} \tag{2.2}$$

PROOF. If $m > 0$ and $\mathbf{i} = (i_1, \dots, i_n)$, let $A_{\mathbf{i}}^m = [(i_1 - 1)/2^m, i_1/2^m) \times \cdots \times [(i_n - 1)/2^m, i_n/2^m)$ for $1 \le i_j \le m \cdot 2^m$, $j = 1, \dots, n$, $m = 1, 2, \dots$. Set $H_m(\mathbf{s}, \mathbf{t}) = H(\mathbf{s}, 2^{-m}\mathbf{i})$ for $t \in A_{\mathbf{i}}^m$ and zero otherwise. Since H is bounded and continuous, $H_m(\mathbf{s}, \mathbf{t}) \to H(\mathbf{s}, \mathbf{t})$ boundedly as $m \to \infty$. Hence $\|H_m(\cdot, \mathbf{t})\|_\alpha \to \|H(\cdot, \mathbf{t})\|_\alpha$ boundedly and

$$\int \|H_m(\cdot, \mathbf{t})\|_\alpha \, d\mu(\mathbf{t}) \to \int \|H(\cdot, \mathbf{t})\|_\alpha \, d\mu(\mathbf{t}) \quad \text{as } m \to \infty$$

by the bounded convergence theorem. But

$$\int \|H_m(\cdot,\mathbf{t})\|_\alpha \, d\mu(\mathbf{t}) = \sum_{\mathbf{i}} \|H_m(\cdot, 2^{-m}\mathbf{i})\|_\alpha \mu(A_{\mathbf{i}}^m)$$

$$= \sum_{\mathbf{i}} \|\mu(A_{\mathbf{i}}^m) H_m(\cdot, 2^{-m}\mathbf{i})\|_\alpha$$

$$\le \left\| \sum_{\mathbf{i}} \mu(A_{\mathbf{i}}^m) H_m(\cdot, 2^{-m}\mathbf{i}) \right\|_\alpha$$

$$= \left\| \int H_m(\cdot, \mathbf{t}) \, d\mu(\mathbf{t}) \right\|_\alpha .$$

The inequality follows from Minkowski's inequality for $0 < \alpha < 1$ (cf. Hewitt and Stromberg (1965), page 192). The desired result is obtained by passing to the limit as $m \to \infty$. □

(2.3) REMARK. The above lemma remains valid if we weaken the continuity assumption on H. It suffices that $H(\mathbf{s}, \mathbf{t})$ be measurable and right-continuous in $\mathbf{t}$ for each fixed $\mathbf{s}$. We can also replace right-continuity with left-continuity if we redefine $H_m(\mathbf{s}, \mathbf{t})$ as $H(\mathbf{s}, 2^{-m}(\mathbf{i} - \mathbf{1}))$ on $A_{\mathbf{i}}^m$ where $\mathbf{1} = (1, \ldots, 1)$.

We are now ready to prove the main result. Let $\mathbf{S} = (S_1, \ldots, S_n)$ and $\mathbf{T} = (T_1, \ldots, T_n)$ be independent vectors in $\mathscr{I}$ with corresponding distribution functions F and G, respectively. Fix $0 < \alpha < 1$ and let h_i be nonnegative, nondecreasing, continuous bounded functions on $[0, \infty)$. Then

$$E\left[\prod_{i=1}^n h_i(S_i + T_i)\right] = \iint \prod_{i=1}^n h_i(s_i + t_i) \, dF(\mathbf{s}) \, dG(\mathbf{t})$$

$$\le \int \left[\int \prod_{i=1}^n h_i^\alpha\left(\frac{s_i}{\alpha} + t_i\right) dF(\mathbf{s})\right]^{1/\alpha} dG(\mathbf{t}) \quad (\text{since } \mathbf{S} \in \mathscr{I})$$

$$\le \left\{\int \left[\int \prod_{i=1}^n h_i\left(\frac{s_i}{\alpha} + t_i\right) dG(\mathbf{t})\right]^\alpha dF(\mathbf{s})\right\}^{1/\alpha} \quad (\text{by Lemma 2.2})$$

$$\le \left\{\int \left[\left(\int \prod_{i=1}^n h_i^\alpha\left(\frac{s_i}{\alpha} + \frac{t_i}{\alpha}\right) dG(\mathbf{t})\right)^{1/\alpha}\right]^\alpha dF(\mathbf{s})\right\}^{1/\alpha} \quad (\text{since } \mathbf{T} \in \mathscr{I})$$

$$= E^{1/\alpha}\left[\prod_{i=1}^n h_i^\alpha\left(\frac{S_i + T_i}{\alpha}\right)\right].$$

(2.4) REMARK. Suppose $\mathscr{H}$ is any class of nonnegative functions and we define $\mathbf{T}$ to be $\mathscr{H}$IFRA if $E[h(\mathbf{T})] \le E^{1/\alpha}[h^\alpha(\mathbf{T}/\alpha)]$ for all $h \in \mathscr{H}$, $0 < \alpha < 1$. Then this same argument shows that such a class is closed under convolution provided whenever $h \in \mathscr{H}$, it follows that $h(\mathbf{s} + \mathbf{t})$ belongs to $\mathscr{H}$ for fixed $\mathbf{s}$ and for fixed $\mathbf{t}$.

3. Another characterization of $\mathscr{I}$. As was mentioned in Section 2, El-Neweihi characterized the class $\mathscr{I}$ by the requirement that

$$E[H(\mathbf{T})] \leq E^{1/\alpha}[H^{\alpha}(\mathbf{T}/\alpha)] \tag{3.1}$$

for all $0 < \alpha < 1$ and $H(\mathbf{t})$ of the form $\prod_{i=1}^{n} h_i(t_i)$, where each h_i is nonnegative and nondecreasing on $[0, \infty)$. With the help of Lemma 2.2 we can extend the inequality (3.1) to a larger class.

Let $\mathscr{H}$ denote the class of all nonnegative distribution functions on R_+^n; i.e., $H \in \mathscr{H}$ if and only if there exists a measure μ on R_+^n such that $H(\mathbf{t}) = \mu([\mathbf{0}, \mathbf{t}])$. We denote this unique measure μ by dH.

THEOREM 3.2. *$\mathbf{T} \in \mathscr{I}$ if and only if*

$$E[H(\mathbf{T})] \leq E^{1/\alpha}[H^{\alpha}(\mathbf{T}/\alpha)]$$

for all $0 < \alpha < 1$ and all $H \in \mathscr{H}$.

PROOF. The sufficiency is clear since $h(\mathbf{t}) = \prod_{i=1}^{n} h_i(t_i) \in \mathscr{H}$ whenever each h_i is nonnegative, nondecreasing, and right-continuous. Now suppose $\mathbf{T} \in \mathscr{I}$ and $H \in \mathscr{H}$. Let F be the distribution of $\mathbf{T}$. Then

$$\begin{aligned}
E[H(\mathbf{T})] &= \int H(\mathbf{t})\, dF(\mathbf{t}) = \iint I_{[\mathbf{0},\mathbf{t}]}(\mathbf{s})\, dH(\mathbf{s})\, dF(\mathbf{t}) \\
&= \int \left[\int I_{[\mathbf{s},\infty)}(\mathbf{t})\, dF(\mathbf{t}) \right] dH(\mathbf{s}) \\
&\leq \int \left[\int I_{[\mathbf{s},\infty)}(\mathbf{t}/\alpha)\, dF(\mathbf{t}) \right]^{1/\alpha} dH(\mathbf{s}) \quad (\text{since } \mathbf{T} \in \mathscr{I}) \\
&\leq \left\{ \int \left[\int I_{[\mathbf{0},\mathbf{t}/\alpha]}(\mathbf{s})\, dH(\mathbf{s}) \right]^{\alpha} dF(\mathbf{t}) \right\}^{1/\alpha} \quad (\text{Remark } (2.3)) \\
&= E^{1/\alpha}[H^{\alpha}(\mathbf{T}/\alpha)].
\end{aligned}$$

(3.3) REMARK. The characterization of the NBU class considered by El-Neweihi (1984) also extends to this class $\mathscr{H}$.

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