The Annals of Probability
1987, Vol. 15, No. 1, 388–393

A RENEWAL THEOREM IN THE INFINITE MEAN CASE

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Let $F(\cdot)$ be a c.d.f. on $(0, \infty)$ such that $1 - F(x)$ is regularly varying with exponent $-\alpha$, $\frac{1}{2} < \alpha \le 1$. Let $Q(\cdot)$: $\mathscr{R}^+ \to \mathscr{R}^+$ be nonincreasing and regularly varying with exponent $-\beta$, $0 \le \beta < 1$. Then, as $t \to \infty$, $(U * Q)(t) \equiv \int_{[0,t]} Q(t-u)U(du)$ is asymptotic to $c(\alpha, \beta)(\int_0^t Q(u)\,du)(\int_0^t (1 - F(u))\,du)^{-1}$, where $U(\cdot)$ is the renewal function associated with $F(\cdot)$ and $c(\alpha, \beta)$ is a suitable constant. This is an improved version of a theorem due to Teugels, whose proof appears to be incomplete. Applications of the result to the second order behavior of $U(t)$ in some special cases are also given.

1. Introduction. Let $F(\cdot)$ be a c.d.f. on $[0,\infty)$ with $F(0) = 0$. Let $X_1, X_2, \ldots,$ be i.i.d. random variables with c.d.f. $F(\cdot)$ and let

$$U(t) \equiv \sum_{n=0}^{\infty} F^n(t) \equiv \sum_{n=0}^{\infty} P(S_n \le t),$$

where $S_0 = 0$ and $S_n = X_1 + X_2 + \cdots + X_n$ for $n \ge 1$, be the renewal function. The so-called key renewal theorems are results about the asymptotic behavior of $(U * Q)(t) \equiv \int_{[0,t]} Q(t-u)U(du)$ as $t \to \infty$ under suitable hypotheses on $Q(\cdot)$ and $F(\cdot)$. For example, Feller (1971) has a version which states: If $F(\cdot)$ is nonarithmetic with a finite mean μ and $Q(\cdot)$ is directly Riemann integrable on $(0, \infty)$, then

$$\lim_{t \to \infty} (U * Q)(t) = \mu^{-1} \int_0^{\infty} Q(u)\,du.$$

This improved the original result of Smith (1954).

There have been attempts in the literature to extend Smith's and Feller's versions to the case when $\mu = \infty$. Erickson (1970) gives the following: If $Q(\cdot)$ is directly Riemann integrable and satisfies $Q(t) = O(t^{-1})$, then $(U * Q)(t) \sim$ const.$(\int_0^{\infty} Q(u)\,du)(\int_0^t (1 - F(u))\,du)^{-1}$ as $t \to \infty$. Teugels (1968) proposed a version for functions $Q(\cdot)$ that do not satisfy Erickson's hypotheses. This version is particularly useful in studying the second order behavior of the renewal function $U(t)$. Unfortunately, Teugels' proof does not seem to be complete, as far as we understand it. In particular, his Lemmas 8 and 9 do not appear valid as they stand and we have not found any easy way to correct them. Erickson (1970) questions the necessity of the additional hypothesis on $U(\cdot)$ imposed by Teugels and the lack of use of the connection to renewal theory. Meanwhile, Mohan (1977, 1981) used Teugels' theorem to give second order estimates for $U(t)$ for some particular $F(\cdot)$ in the infinite mean case.

Received March 1985.

AMS 1980 *subject classification*. Primary 60K05.

Key words and phrases. Renewal function, regular variation, key renewal theorem, second order behavior.

Inspired by Erickson's comments and some of Mohan's techniques, we are able to give a correct proof of an improved version of Teugels' result when $F(\cdot)$ is nonarithmetic and satisfies

$$1 - F(x) = x^{-\alpha}L(x), \qquad \tfrac{1}{2} < \alpha \leq 1, \tag{1}$$

where $L(\cdot)$ is slowly varying, dropping Teugels' additional hypothesis on the renewal function.

2. A renewal theorem when $\mu = \infty$. Our version of Teugels' result is the following.

THEOREM 1. *Let $F(\cdot)$ be nonarithmetic and satisfy* (1). *Let $Q(\cdot)$: $\mathscr{R}^+ \to \mathscr{R}^+$ be a nonincreasing function such that $Q(0) < \infty$ and*

$$Q(x) = x^{-\beta}L_\beta(x), \qquad 0 \leq \beta < 1,$$

where $L_\beta(\cdot)$ is slowly varying. Then,

$$(U * Q)(t) \sim C(\alpha, \beta)\left(\int_0^t Q(u)\, du\right)\left(\int_0^t (1 - F(u))\, du\right)^{-1} \quad \text{as } t \to \infty, \tag{2}$$

where $(C(\alpha, \beta))^{-1} = (2 - \beta)B(\alpha - \beta + 1, 2 - \alpha)$.

PROOF. The proof is modelled after the proof of a similar theorem by Mohan (1976) for the finite mean case. Given $\varepsilon > 0$, choose $\delta \in (\frac{1}{2}, 1)$ such that

$$\alpha B(\alpha, 1 - \beta) - \alpha\int_0^\delta (1 - u)^{-\beta}u^{\alpha - 1}\, du < \varepsilon \tag{3}$$

and $(1 - \delta)^{1-\beta} < \varepsilon$. We write

$$(U * Q)(t) = \left(\int_0^{[\delta t]} + \int_{[\delta t]}^{[t]} + \int_{[t]}^t\right)Q(t - x)U(dx) \equiv I(t) + J(t) + K(t).$$

We give some useful results and examine the integrals I, J and K, in a series of lemmas.

LEMMA 1. *Let $m(t) = \int_0^t(1 - F(u))\, du$. Under the assumptions of the theorem*:

(i) $U(t) \sim (C_\alpha/\alpha)t/m(t)$ *as* $t \to \infty$,
(ii) $m(t)$ *is regularly varying with exponent* $1 - \alpha$,
(iii) $\lim_{t\to\infty} m(t)(U(t + h) - U(t)) = C_\alpha h$ *and*
(iv) $\int_0^t Q(y)\, dy \sim tQ(t)/(1 - \beta)$ *as* $t \to \infty$, *where* $C_\alpha = [\Gamma(\alpha)\Gamma(2 - \alpha)]^{-1}$.

PROOF. (i), (ii) and (iii) can be found in Erickson (1970) (Theorems 5 and 2). A proof of (iv) can be found in Feller (1971), page 281. □

LEMMA 2.

$$\lim_{t\to\infty} I(t)(Q(t)U(t))^{-1} = \alpha\int_0^\delta (1 - u)^{-\beta}u^{\alpha - 1}\, du.$$

PROOF.

$$
\begin{aligned}
I(t)(Q(t)U(t))^{-1} &= \int_0^{[\delta t]} (Q(t-x)/Q(t))U(dx)/U(t)\\
&= \int_0^{\delta} \chi_{A_t}(u)(Q(t(1-u))/Q(t))U(t\,du)/U(t)\\
&\equiv \int_0^{\delta} f_t(u)\mu_t(du), \quad \text{where } A_t = \{u\colon u \le [\delta t]/t\}.
\end{aligned}
$$

The measures $\mu_t(du)$ converge weakly to $\mu(du) = \alpha u^{\alpha-1}\,du$ and the functions $f_t(u)$ are dominated by $Q(t(1-u))/Q(t)$, which converges uniformly to $(1-u)^{-\beta}$ on $u \in (0, \delta)$. Since

$$
\lim_{t\to\infty} \int_0^{\delta} (Q(t(1-u))/Q(t))\mu_t(du) = \alpha \int_0^{\delta} (1-u)^{-\beta} u^{\alpha-1}\,du
$$

and $f_t(u)$ converges pointwise to $(1-u)^{-\beta}$ on $u \in (0, \delta)$, a dominated convergence theorem (for example Royden (1968), Proposition 18, page 232) gives the result. □

LEMMA 3.

$$
\limsup_{t\to\infty} J(t)m(t) \Big/ \int_0^t Q(y)\,dy \le c(1-\delta)^{1-\beta},
$$

where c is a constant independent of δ.

PROOF. Monotonicity of $Q(\cdot)$ and $m(\cdot)$ gives

$$
\begin{aligned}
J(t)m(t) &\le m(t) \sum_{k=[\delta t]}^{[t]-1} Q(t-k-1)(U(k+1)-U(k))\\
&\le (m(t)/m([\delta t])) \sum_{k=[\delta t]}^{[t]-1} Q(t-k-1)m(k)(U(k+1)-U(k)).
\end{aligned}
$$

Applying Lemma 1(iii), it follows that, for large t,

$$
\begin{aligned}
&\le (m(t)/m([\delta t]))(C_\alpha + 1) \sum_{k=[\delta t]}^{[t]-1} Q(t-k-1)\\
&\le (m(t)/m([\delta t]))(C_\alpha + 1)\left(\int_{t-[t]}^{t-[\delta t]-1} Q(y)\,dy + Q(t-[t])\right)\\
&\le (m(t)/m([\delta t]))(C_\alpha + 1)\left(\int_0^{t(1-\delta)} Q(y)\,dy + Q(0)\right).
\end{aligned}
$$

Using Lemma 1, parts (ii) and (iv), to pass to the limit, we get

$$
\limsup_{t\to\infty} J(t)m(t) \Big/ \int_0^t Q(y)\,dy \le \delta^{\alpha-1}(C_\alpha + 1)(1-\delta)^{1-\beta},
$$

from which the result follows. □

LEMMA 4.

$$\limsup_{t\to\infty} K(t)m(t)\Big/\int_0^t Q(y)\,dy = 0.$$

PROOF.

$$K(t)m(t)\Big/\int_0^t Q(y)\,dy \le Q(0)(U(t) - U([t]))m(t)\Big/\int_0^t Q(y)\,dy$$
$$\le Q(0)(U(t) - U(t-1))m(t)\Big/\int_0^t Q(y)\,dy.$$

The conclusion follows from Lemma 1(iii) and (iv). □

We now complete the proof of the theorem.

$$\left|(U*Q)(t)(Q(t)U(t))^{-1} - \alpha B(\alpha, 1-\beta)\right|$$
$$\le \left|I(t)(Q(t)U(t))^{-1} - \alpha\int_0^\delta (1-u)^{-\beta}u^{\alpha-1}\,du\right| + \left|J(t)(Q(t)U(t))^{-1}\right|$$
$$+ \left|K(t)(Q(t)U(t))^{-1}\right| + \left|\alpha B(\alpha, 1-\beta) - \alpha\int_0^\delta (1-u)^{-\beta}u^{\alpha-1}\,du\right|.$$

Applying the lemmas and (3) gives

$$\limsup_{t\to\infty}\left|(U*Q)(t)(Q(t)U(t))^{-1} - \alpha B(\alpha, 1-\beta)\right| \le (c+1)\varepsilon.$$

Therefore, by letting $\varepsilon \to 0^+$,

$$(U*Q)(t) \sim \alpha B(\alpha, 1-\beta)U(t)Q(t) \quad \text{as } t\to\infty,$$

which is equivalent to (2) by Lemma 1(i) and (iv). □

REMARK. The restriction of α in Theorem 1 is due to the unavailability of Lemma 1(iii) for $0 < \alpha \le \frac{1}{2}$. Erickson (1970) gives only a limit inferior result for the restricted interval. Of course, should Lemma 1(iii) hold for a particular renewal function with $0 < \alpha \le \frac{1}{2}$, then (2) would follow. In general, however, Theorem 1 is the best that can be given.

3. Second order estimation of the renewal function. Viewing Lemma 1(i) as a first order estimate of the renewal function, Mohan (1977, 1981) tried to prove some second order estimation results for a special class of c.d.f.'s using Teugels' version of the key renewal theorem. We restate valid versions of Mohan's results which follow from Theorem 1.

In what follows, $F(x)$ is the c.d.f. of a random variable X of the form

$$X = Z_\alpha Y^{1/\alpha}, \tag{4}$$

where Z_α is a nonnegative random variable with Lebesgue–Stieltjes transform (LST) $\exp\{-s^\alpha\}$, $s \ge 0$, $\frac{1}{2} < \alpha < 1$, and independent of Y, a nonnegative random variable with finite mean μ and LST $f_0(s)$, $s \ge 0$.

With this formulation, the LST of $F(\cdot)$, $f(s)$, satisfies

$$1 - f(s) = 1 - f_0(s^\alpha) \sim \mu s^\alpha \quad \text{as } s \to 0^+.$$

By Karamata's Tauberian theorem (see Feller (1971), page 447), this is equivalent to

$$1 - F(x) \sim x^{-\alpha}\mu/\Gamma(1-\alpha) \quad \text{as } x \to \infty,$$

from which

$$U(t) \sim t^\alpha(\mu\Gamma(1+\alpha))^{-1} \quad \text{as } t \to \infty$$

follows (Feller (1971), page 471).

The second order behavior can be examined using Theorem 1 by finding a $Q(t)$ such that

$$U(t) - t^\alpha(\mu\Gamma(1+\alpha))^{-1} = (U * Q)(t). \tag{5}$$

Let $G(x)$ be the c.d.f. of $X_0 = Z_\alpha(Y')^{1/\alpha}$, where Y' has c.d.f. $\mu^{-1}\int_0^x P(Y > u)\,du$ and is independent of Z_α as defined above. Since Y' has as its LST $(1 - f_0(s))(\mu s)^{-1}$, the LST of $G(x)$ is $(1 - f_0(s^\alpha))(\mu s^\alpha)^{-1}$. With $S_n' = X_0 + S_n$ and $H(t) = U(t) - 1$, the delayed renewal function $H'(t) = \sum_{n=0}^\infty P(S_n' \le t)$ satisfies the renewal equation

$$H'(t) = G(t) + (H * G)(t), \qquad t \ge 0.$$

The LST of $H'(t)$, $h'(s)$, satisfies

$$\begin{aligned} h'(s) &= (1 - f_0(s^\alpha))(\mu s^\alpha)^{-1}(1 + h(s)) \\ &= (\mu s^\alpha)^{-1}, \end{aligned}$$

which means $H'(t) = t^\alpha(\mu\Gamma(1+\alpha))^{-1}$, $t \ge 0$. Therefore,

$$\begin{aligned} U(t) - t^\alpha(\mu\Gamma(1+\alpha))^{-1} &= H(t) - H'(t) + 1 \\ &= H(t) - (H * G)(t) + 1 - G(t) \\ &= (U * (1 - G))(t) \end{aligned}$$

and $Q(t) = 1 - G(t)$ is the solution to (5).

The succeeding theorems follow easily using repeated applications of Karamata's Tauberian theorem (to show the regular variation of $1 - G(t)$) and our Theorem 1. (See Mohan (1977, 1981) for details.)

THEOREM 2. *If Y has mean μ and finite variance σ^2, then*

$$U(t) - t^\alpha(\mu\Gamma(1+\alpha))^{-1} \sim \frac{\sigma^2 + \mu^2}{2\mu^2} \quad \text{as } t \to \infty.$$

THEOREM 3. *If Y is in the domain of attraction of a stable law of order 2, then*

$$U(t) - t^\alpha(\mu\Gamma(1+\alpha))^{-1} \sim \mu^{-1}\int_0^{t^\alpha} P(Y' > u)\,du \quad \text{as } t \to \infty.$$

THEOREM 4. *If Y is in the domain of attraction of a stable law of order β with* $1 < \beta < 2$, *then*

$$U(t) - t^{\alpha}(\mu\Gamma(1+\alpha))^{-1} \sim t^{\alpha(2-\beta)}L_{\beta}(t^{\alpha})D(\alpha,\beta)/\mu^2 \quad \text{as } t \to \infty,$$

where $P(Y > t) = t^{-\beta}L_{\beta}(t)$, $L_{\beta}(\cdot)$ *is slowly varying, and*

$$D(\alpha,\beta) = \alpha B(\alpha, 1 - \alpha(\beta-1))\Gamma(2-\beta)(\Gamma(1+\alpha)\Gamma(1-\alpha(\beta-1))(\beta-1))^{-1}.$$

These results show quite a range of possible second order behaviors for the renewal function within a relatively small class of c.d.f.'s $F(x)$ in (1) with $\frac{1}{2} < \alpha < 1$. This is in contrast to the one particular behavior the second order estimate takes when $F(\cdot)$ satisfies (1) with $1 < \alpha < 2$ (see Mohan (1976), Theorem 2.2, which improves Theorem 4 of Teugels (1968)).

These results would follow for $0 < \alpha \leq \frac{1}{2}$ if one could be certain that the strong renewal theorem of Lemma 1(iii) holds. For example, if, in (4), the random variable Y has an exponential distribution with mean μ, then

$$U(t) - t^{\alpha}(\mu\Gamma(1+\alpha))^{-1} = 1 \quad \text{for } t \geq 0$$

and

$$\lim_{t\to\infty} m(t)(U(t+h) - U(t)) = hC_{\alpha} \quad \text{for } 0 < \alpha < 1.$$

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