The Annals of Probability
1987, Vol. 15, No. 1, 382–387

A NEW PROOF OF THE COMPLETE CONVERGENCE THEOREM FOR CONTACT PROCESSES IN SEVERAL DIMENSIONS WITH LARGE INFECTION PARAMETER

BY ROBERTO HENRIQUE SCHONMANN

Universidade de São Paulo

A new proof is given of the complete convergence theorem for the d-dimensional basic contact process provided that the infection parameter is larger than the critical value in the one-dimensional case. This proof is much more elementary than the known one since it does not depend on exponential estimates and does not use the subadditive ergodic theory in the extension from one to more dimensions.

In [2] Durrett and Griffeath proved a growth theorem for a class of systems, including the contact processes, in any dimension, provided that the infection parameter λ is larger than the critical value in the one-dimensional case, which we call λ_1. One of the main corollaries is the complete convergence theorem (Theorem 1 below).

Here we give a much more elementary proof of this theorem. Our main tools are the complete convergence theorem in the one-dimensional case, proved for $\lambda > \lambda_1$ in [1] and the lemma on page 383 of [3]. The idea is to find imbedded one-dimensional contact processes. We do not need exponential estimates, and the subadditive ergodic theory is used only in the proof of the theorem in the one-dimensional case [1] and not in the extension for $d > 1$. (One should observe that for $\lambda > \lambda_1^+$ = critical value of λ for the one-dimensional contact process with infection in only one direction [4], the proof of the analogue of Theorem 1 is then very elementary since, in this case, there are elementary proofs in the one-dimensional case given in [3] and [4].) On the other hand, we must observe that the present approach does not give a proof of the pointwise ergodic theorem and the law of large numbers for the number of infected individuals on $\mathbb{Z}^d$ (when one starts with one infected individual) as in [2].

Notation. $(\xi_d^\mu(t), t \geq 0)$ is the basic contact process in d dimensions, with initial configuration taken randomly with distribution μ and λ as the infection parameter. It is defined on $\mathscr{P}(\mathbb{Z}^d)$ through the rates

$$A \to A \cup \{x\} \quad \text{with rate } \lambda|A \cap \{y: |x - y| = 1\}| \quad \text{if } x \notin A,$$

$$A \to A \setminus \{x\} \quad \text{with rate } 1 \quad \text{if } x \in A,$$

Received April 1985; revised September 1985.
AMS 1980 *subject classification*. Primary 60K35.
Key words and phrases. Contact process, complete convergence theorem.

where $|\cdot|$ is the euclidean norm on $\mathbb{Z}^d$.

$$\tau_d^\mu = \inf\{t \geq 0: \xi_d^\mu(t) = \varnothing\}.$$

$$\nu_d \text{ is the weak limit of } \xi_d^{\mathbb{Z}^d}(t) \quad \text{as } t \to \infty.$$

$$\lambda_d = \sup\{\lambda \geq 0: \nu_d = \delta_\varnothing\}.$$

THEOREM 1. *For any $\lambda > \lambda_1$ and any μ*

$$\xi_d^\mu(t) \to P(\tau_d^\mu < \infty)\delta_\varnothing + P(\tau_d^\mu = \infty)\nu_d \tag{1}$$

weakly, as $t \to \infty$.

PROOF. We employ the lemma on page 383 of [3]. Since the d-dimensional basic contact process is self-dual we construct two independent versions of this process, which we represent, respectively, by $(\xi_d^A(t), t \geq 0)$ and $(\bar{\xi}_d^B(t), t \geq 0)$, where $A, B \subset \mathbb{Z}^d$ are the initial configurations. The above mentioned lemma states that (1) is equivalent to

$$\lim_{t\to\infty} P\big(\xi_d^A(t) \cap \bar{\xi}_d^B(t) = \varnothing, \tau_d^A = \infty, \bar{\tau}_d^B = \infty\big) = 0, \tag{2}$$

$$\forall A, B \subset \mathbb{Z}^d, \qquad 0 < |A| < \infty, \qquad 0 < |B| < \infty,$$

where

$$\tau_d^A = \inf\{t \geq 0: \xi_d^A(t) = \varnothing\},$$

$$\bar{\tau}_d^B = \inf\{t \geq 0: \bar{\xi}_d^B(t) = \varnothing\}.$$

To simplify the notation and ideas we consider first the case $d = 2$ and later we indicate the differences in the general case.

The reader is assumed to be familiar with the construction of the contact processes with percolation structures (see [4], page 5) and the corresponding terminology. This construction will be needed for the definition of the imbedded one dimensional processes (see Definition 4 below).

Definitions and remarks. Until the last paragraph, where we discuss the case of general d, we are always considering $d = 2$.

DEFINITION 1. $Q: \mathbb{Z}^2 \to \mathbb{Z}$, $Q(x, y) = x$.

DEFINITION 2. Given $n \in N$,

$$K_n = \{(C, D) \subset \mathbb{Z}^2 \times \mathbb{Z}^2: \exists\{c_1, c_2, \dots, c_n\} \subset C, \quad \exists\{d_1, d_2, \dots, d_n\} \subset D,$$
$$\text{s.t. } Q(c_n) < Q(c_{n-1}) < \cdots < Q(c_1) < Q(d_1) < \cdots < Q(d_n)\}.$$

DEFINITION 3. Given $c_1, c_2, \dots, c_n, d_1, \dots, d_n$ which satisfy the conditions of Definition 2 and $l \in \mathbb{N}$ such that for each i, $c_i \in [-l, +l]^2$, $d_i \in [-l, +l]^2$, consider for $i = 1, 2, \ldots, n$ the one-dimensional paths

$$\begin{aligned} H_i = \{(x, y) \in \mathbb{R}^2 &: (x = Q(c_i) \text{ and } y \le l + i) \text{ or} \\ &(Q(c_i) \le x \le Q(d_i) \text{ and } y = l + i) \text{ or} \\ &(x = Q(d_i) \text{ and } y \le l + i)\}, \\ L_i = H_i \cap \mathbb{Z}^2&. \end{aligned}$$

REMARK 1. $c_i, d_i \in L_i$, $i = 1, 2, \ldots, n$.

REMARK 2. $L_i \cap L_j = \varnothing$, if $i \neq j$.

DEFINITION 4. The percolation structure defining $(\xi_d^A(t))$ defines on each L_i a percolation structure corresponding to a one-dimensional contact process. It is enough to consider the arrows of the percolation structure only when they join sites on L_i. We use the notation $(\xi_{1,i}(t))$ for these processes.

Analogously the percolation structure defining $(\bar{\xi}_d^B(t))$ defines the processes $(\bar{\xi}_{1,i}(t))$, on each L_i, $i = 1, \ldots, n$.

REMARK 3. Remark 2 implies that the $2n$ processes

$$(\xi_{1,1}(t)), \ldots, (\xi_{1,n}(t)), (\bar{\xi}_{1,1}(t)), \ldots, (\bar{\xi}_{1,n}(t))$$

are mutually independent.

DEFINITION 5. Given $n \in \mathbb{N}$ we define the stopping time

$$\Theta_n = \inf\{t \ge 0: (\xi_d^A(t), \bar{\xi}_d^B(t)) \in K_n \text{ or } (\bar{\xi}_d^B(t), \xi_d^A(t)) \in K_n\}.$$

DEFINITION 6. Given $l \in \mathbb{N}$ and $t \in \mathbb{R}_+$ we define the event

$$E_{l,t} = [\xi_d^A(s) \subset [-l, l]^2, \bar{\xi}_d^B(s) \subset [-l, l]^2, \forall s \le t].$$

DEFINITION 7. Given $n \in \mathbb{N}$, $l \in \mathbb{N}$, and $t \in \mathbb{R}_+$ we define the event

$$F_{n,l,t} = [\Theta_n \le t] \cap E_{l,t}.$$

REMARK 4. We will use the notation $\xi_d^x(t)$ instead of $\xi_d^{\{x\}}(t)$ for $x \in \mathbb{Z}^d$.

Now we prove (2). For any $n \in \mathbb{N}$, $l \in \mathbb{N}$, and $t_0 \in \mathbb{R}_+$,

$$\begin{aligned} &P(\tau_d^A = \infty, \bar{\tau}_d^B = \infty, \xi_d^A(t) \cap \xi_d^B(t) = \varnothing) \\ &\quad \le P(\tau_d^A = \infty, \bar{\tau}_d^B = \infty, (F_{n,l,t_0})^c) + P(F_{n,l,t_0}, \xi_d^A(t) \cap \bar{\xi}_d^B(t) = \varnothing) \\ &\quad \le P(\tau_d^A = \infty, \bar{\tau}_d^B = \infty, \Theta_n > t_0) + P((E_{l,t_0})^c) \\ &\qquad + P(F_{n,l,t_0}, \xi_d^A(t) \cap \bar{\xi}_d^B(t) = \varnothing). \end{aligned} \tag{3}$$

We will show that given $\varepsilon > 0$ it is possible to choose n, l, and t_0 in such a way that the three terms on the right-hand side of (3) are smaller than $\varepsilon/3$ for large t. The first term is controlled in the following way: For fixed n (its value will be specified later) define the events

$$G_i = \left[\tau_d^A > i,\ \bar{\tau}_d^B > i,\ \Theta_n > i\right], \qquad i \in \mathbb{N},$$

and the number

$$\alpha = P\big(\xi_d^0(1) \supset \{(-n,0),(-n+1,0),\dots,(n,0)\} \text{ and } \bar{\xi}_d^0(1) \supset \{(-n,0),(-n+1,0),\dots,(n,0)\}\big).$$

Then $\alpha > 0$ and by translation invariance, additivity, and the fact that $A \neq \varnothing$, $B \neq \varnothing$,

$$P(G_1) \leq 1 - \alpha, \qquad P(G_i | G_{i-1}) \leq 1 - \alpha, \qquad i = 2,3,\dots .$$

Then for $i = 2,3,\dots,$

$$P(G_i) = P(G_i \cap G_{i-1}) = P(G_i|G_{i-1})P(G_{i-1}) \leq P(G_{i-1})(1-\alpha)$$

and

$$P(G_i) \leq (1-\alpha)^i.$$

So

$$P\big(\tau_d^A = \infty,\ \bar{\tau}_d^B = \infty,\ \Theta_n > t_0\big) \leq P\big(G_{[t_0]}\big) \leq (1-\alpha)^{[t_0]},$$

where $[t_0]$ is the integer part of t_0. Then we can take t_0 such that

$$P\big(\tau_d^A = \infty,\ \bar{\tau}_d^B = \infty,\ \Theta_n > t_0\big) < \varepsilon/3. \tag{4}$$

Now, for fixed $t_0 \in \mathbb{R}_+$, almost surely $|\xi_d^A(t)| < \infty$, and $|\bar{\xi}_d^B(t)| < \infty$ for $0 \leq t \leq t_0$. So there is $l \in \mathbb{N}$ such that

$$P\big((E_{l,t_0})^c\big) < \varepsilon/3. \tag{5}$$

To control the last term in (3) we employ the imbedded one-dimensional contact processes. As (1) is in force in the case $d = 1$ [1], it follows that (2) is valid also in this case. In particular for any $x, y \in \mathbb{Z}$

$$\lim_{t\to\infty} P\big(\xi_1^x(t) \cap \bar{\xi}_1^y(t) = \varnothing,\ \tau_1^x = \infty,\ \bar{\tau}_1^y = \infty\big) = 0.$$

This convergence is not uniform in $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, but it is uniform in $(x, y) \in M = \{(i, j) \in \mathbb{Z} \times \mathbb{Z}\colon |i - j| \leq 6l + 2n\}$, by reduction to a finite (except for translations) number of cases. Then there exists $t_1 > 0$ such that if $t \geq t_1$,

$$P\big(\xi_1^x(t) \cap \bar{\xi}_1^y(t) = \varnothing,\ \tau_1^x = \infty,\ \bar{\tau}_1^y = \infty\big) \leq \rho^2/2$$

for any $(x, y) \in M$, where

$$\rho := P\big(\xi_1^0(t) \neq \varnothing,\ \forall t \geq 0\big),$$

and $\rho > 0$ since $\lambda > \lambda_1$. Then, if $t \geq t_1$,

$$
\begin{aligned}
P\big(\xi_1^x(t) \cap \bar{\xi}_1^y(t) \neq \varnothing\big) &\geq P\big(\xi_1^x(t) \cap \bar{\xi}_1^y(t) \neq \varnothing,\, \tau_1^x = \infty,\, \bar{\tau}_1^y = \infty\big) \\
&= P\big(\tau_1^x = \infty,\, \bar{\tau}_1^y = \infty\big) \\
&\quad - P\big(\xi_1^x(t) \cap \bar{\xi}_1^y(t) = \varnothing,\, \tau_1^x = \infty,\, \bar{\tau}_1^y = \infty\big) \\
&\geq \rho^2 - \rho^2/2 = \rho^2/2.
\end{aligned} \tag{6}
$$

Now we use (6) and the processes in Definition 4 to control the last term in (3). We use a notation which corresponds to the case $(\xi_d^A(\Theta_n), \bar{\xi}_d^B(\Theta_n)) \in K_n$. In this case take $c_1, \dots, c_n, d_1, \dots, d_n \in \mathbb{Z}^d$ as in Definition 2 and such that $\{c_1, \dots, c_n\} \subset \xi_d^A(\Theta_n)$ and $\{d_1, \dots, d_n\} \subset \bar{\xi}_d^B(\Theta_n)$. The case $(\bar{\xi}_d^B(\Theta_n), \xi_d^A(\Theta_n)) \in K_n$ is analogous; just interchange the roles of $\{c_1, \dots, c_n\}$ and $\{d_1, \dots, d_n\}$ in what follows.

We employ the processes $(\xi_{1,i}(t))$ starting at time Θ_n with configuration $\{c_i\}$. For them we use the notation $({}_{\Theta_n}\xi_{1,i}^{c_i}(t), t \geq \Theta_n)$. Analogously we define $({}_{\Theta_n}\bar{\xi}_{1,i}^{d_i}(t), t \geq \Theta_n)$.

By additivity, for $t > t_0$,

$$
\big[\xi_d^A(t) \cap \bar{\xi}_d^B(t) = \varnothing\big] \cap F_{n,l,t_0} \subset \bigcap_{i=1}^{n} \big[{}_{\Theta_n}\xi_{1,i}^{c_i}(t) \cap {}_{\Theta_n}\bar{\xi}_{1,i}^{d_i}(t) = \varnothing\big] \cap F_{n,l,t_0}. \tag{7}
$$

On the event F_{n,l,t_0} we have: (a) the distance between c_i and d_i along H_i is bounded above by $6l + 2n$. (b) $\Theta_n \leq t_0$. Using these facts plus Remark 3, (6), and the strong Markov property we get, for $t > t_0 + t_1$,

$$
P\left(\bigcap_{i=1}^{n} \big[{}_{\Theta_n}\xi_{1,i}^{c_i}(t) \cap {}_{\Theta_n}\bar{\xi}_{1,i}^{d_i}(t) = \varnothing\big] \cap F_{n,l,t_0}\right) \leq (1 - \rho^2/2)^n. \tag{8}
$$

So it is enough to fix n such that $(1 - \rho^2/2)^n < \varepsilon/3$. Then (7) and (8) imply

$$
P\big(F_{n,l,t_0}, \xi_d^A(t) \cap \bar{\xi}_d^B(t) = \varnothing\big) < \varepsilon/3. \tag{9}
$$

(4), (5), and (9) applied to (3) complete the proof (in the case $d = 2$).

In dimensions larger than 2 we just need to modify the definition of L_i. This must be done preserving the properties of Remarks 1 and 2 and the statements in Definition 4. A solution is to construct H_i in the following way: Keep the definition of K_n, where now $Q(x_1, \dots, x_n) = x_1$. Take points $c_i = (c_{i1}, \dots, c_{id})$, $d_i = (d_{i1}, \dots, d_{id})$ as in Definition 2. Trace a half straight line through c_i along which only the last coordinate varies and which ends at the point $(c_{i1}, c_{i2}, \dots, c_{in-1}, l)$. Now join this last point to $(c_{i1}, c_{i2}, \dots, c_{in-2}, l, l)$ using a straight segment. Join this to $(c_{i1}, c_{i2}, \dots, c_{in-3}, l, l, l)$ and so on until we reach $(c_{i1}, c_{i2}, l, l, \dots, l)$; then to $(c_{i1}, l + i, l, l, \dots, l)$ and from this to $(d_{i1}, l + i, l, l, \dots, l)$, then to $(d_{i1}, d_{i2}, l, l, \dots, l)$, and now invert the rule above to reach $(d_{i1}, d_{i2}, \dots, d_{in-1}, l)$. Finally, trace a half straight line starting at this point and passing through d_i. The definition of $E_{l,t}$ must be changed in an obvious way and the bound $6l + 2n$ for the distance between c_i and d_i along H_i may be changed to $4ld + 2n$. □

Acknowledgments. I thank Enrique Andjel for helpful conversations. I thank also M. Eulalia Vares and two anonymous referees for carefully reading the manuscript and helping me to make this paper more readable.

REFERENCES

[1] DURRETT, R. (1980). On the growth of the one dimensional contact processes. *Ann. Probab.* **8** 890–907.

[2] DURRETT, R. and GRIFFEATH, D. (1982). Contact processes in several dimensions. *Z. Wahrsch. verw. Gebiete* **59** 535–552.

[3] GRIFFEATH, D. (1978). Limit theorems for nonergodic set-valued Markov processes. *Ann. Probab.* **6** 379–387.

[4] GRIFFEATH, D. (1979). *Additive and Cancellative Interacting Particle Systems. Lecture Notes in Math.* **724.** Springer, New York.

DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853