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LOWER TAIL PROBABILITY ESTIMATES FOR SUBORDINATORS AND NONDECREASING RANDOM WALKS[1]

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Let $X_1, X_2, \ldots$ be nonnegative i.i.d. random variables and $S_n = X_1 + \cdots + X_n$; $EX_1 = \mu \leq \infty$ and a is the infimum of the support of the distribution of X_1. For $a < x_n < \mu$ we obtain the asymptotic behavior of $\log P\{S_n \leq nx_n\}$ as $n \to \infty$. Under the additional assumption of stochastic compactness a stronger result is obtained which gives the asymptotic behavior of $P\{S_n \leq nx_n\}$ itself. Analogues of these results are given for subordinators when $t \to \infty$ or $t \to 0$.

1. Introduction. Let $\{X_k\}$ be a sequence of independent, identically distributed, nonnegative nondegenerate random variables with a common distribution function F. Let

$$S_n = X_1 + \cdots + X_n, \qquad n \geq 1; \quad S_0 = 0.$$

Our objective here is to find large deviation estimates for the lower tail of S_n under no additional assumptions. These estimates are in the literature for $P\{S_n \leq nx\}$, x fixed, in which case the probability approaches zero exponentially fast. Our method will allow us to estimate this probability even when x increases so that the probability goes to zero much more slowly—for example like n^{-1} or $(\log n)^{-1}$. This is, of course, what is needed in many applications.

Such estimates are particularly important since nonnegative i.i.d. random variables often arise as times between events for quite general processes. For example, X_n could be the time between the nth and the $(n+1)$st return to a state x by a recurrent random walk on the integer lattice; in this context we have been able to use these estimates in [10] to get quite detailed and precise information for the growth of the local time, and its maximal increments, for a recurrent random walk. Results on the growth of the maximal increments of the local time of a random walk were known only for a random walk with zero mean and finite second moment [3]; the estimates of the present paper have allowed us to go beyond the domain of attraction case and to give more complete results even for the simple random walk via a unified and essentially simpler approach to the problem.

To describe the results we need some notation. For $u > 0$, let

$$(1.1) \quad \varphi(u) = Ee^{-uX_1}, \qquad g(u) = -\frac{\varphi'(u)}{\varphi(u)}, \qquad R(u) = -\log\varphi(u) - ug(u).$$

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The functions φ, g, and R are continuous on $(0, \infty)$ and other relevant properties may be found in Lemma 2.1 in the next section. We define

$$a = \inf\{x > 0 \colon P\{X_1 \le x\} > 0\}, \qquad \mu = EX_1, \tag{1.2}$$

where $\mu \le \infty$. For $a < x_n < \mu$ define λ_n by $g(\lambda_n) = x_n$ (g is strictly decreasing). We will show in the next section that $P\{S_n \le nx_n\} \to 0$ iff $nR(\lambda_n) \to \infty$ as $n \to \infty$. If $nR(\lambda_n) \to \infty$, then we determine the rate at which $P\{S_n \le nx_n\} \to 0$. A weak result (given in Section 2) is the following: If X_1 has an atom at a or for some $\delta > 0$, $a + \delta \le x_n < \mu$, then

$$-\log P\{S_n \le nx_n\} \sim nR(\lambda_n), \qquad n \to \infty, \tag{1.3}$$

provided $nR(\lambda_n) \to \infty$. If X_1 has no atom at a and $x_n \to a$, then a corollary of (1.3) gives

$$\lim_{n\to\infty} \frac{1}{n} \log P\{S_n \le nx_n\} = -\infty; \tag{1.4}$$

this should be quite adequate for most applications, but we will show in Example 2.1 that (1.3) is not necessarily valid if $x_n \to a$ and X_1 has no atom at a.

To apply these results in the domain of attraction case it is helpful to know the behavior of R and g in terms of α, the exponent of stability, and F; this is done in Proposition 2.1.

We refer to (1.3) as the weak result since it only gives the asymptotic behavior of the log of the probability. If we assume further that X_1 satisfies Feller's condition for stochastic compactness (this is still quite general; the definition is in Section 3) then we obtain a strong result, namely we define a sequence $\{\rho_n\}$ in terms of φ and show that

$$P\{S_n \le nx_n\} \sim \rho_n e^{-nR(\lambda_n)},$$

provided that for some $\delta > 0$, $a + \delta \le x_n < \mu$. Thus we have the correct asymptotic behavior of $P\{S_n \le nx_n\}$ whenever it goes to zero except in the extreme case when x_n can approach a. This strong result is in Section 4. In order to prove it we need a local limit theorem for triangular arrays. This is proved in Section 3; it should also be of independent interest.

In Section 5 we give the analogues of these results in the case of subordinators (increasing Lévy processes) when $t \to \infty$ or $t \to 0$.

We would like to mention here a paper of Bahadur and Ranga Rao [1] where an asymptotic expansion for the above probability is obtained when $x_n \equiv x < \mu$. Their result includes our strong result in this special case of $x_n \equiv x$. Also, in the very special case that $\varphi(-\varepsilon) < \infty$ for some $\varepsilon > 0$, our results are included in Höglund's [9].

Before closing this section we mention some notation that will be used.

For real sequences $\{a_n\}$ and $\{b_n\}$ we will write $a_n \sim b_n$ to mean $a_n b_n^{-1} \to 1$ as $n \to \infty$, and $a_n \approx b_n$ to mean that there exist positive constants c_1 and c_2 such that $c_1 a_n \le b_n \le c_2 a_n$ for $n \ge$ some n_0. The same notation will be used for functions $a(u)$ and $b(u)$ except that in this case u may tend to 0 as well. We write $a(u) \approx b(u)$ for $u > 0$ to mean $c_1 a(u) \le b(u) \le c_2 a(u)$ for all $u > 0$.

For a random variable X we write $\operatorname{Var} X = EX^2 - (EX)^2$ if $EX^2 < \infty$. Positive constants will be denoted by c, C with or without suffixes. Their meaning may change from one context to another.

Weak convergence of a sequence of random variables $\{Y_n\}$ (or the corresponding distributions) to a probability distribution will simply mean the weak convergence of the corresponding probability measures (no mass escaping).

2. The weak result. In this section we will prove the general asymptotic result for the log of the probability that S_n is unusually small. The behavior of R in the domain of attraction case is given in Proposition 2.1. Following that we will give an example to show that the result may fail if x_n is allowed to approach a.

We start by giving useful properties of the functions g and R, defined in (1.1), in the following lemma.

LEMMA 2.1. *The functions g and R defined in* (1.1) *are continuous on* $(0, \infty)$, *g is strictly decreasing, and R is strictly increasing. In fact,*

$$g'(u) = -V(u), \qquad R'(u) = uV(u), \tag{2.1}$$

where

$$V(u) = \frac{\varphi(u)\varphi''(u) - (\varphi'(u))^2}{(\varphi(u))^2}. \tag{2.2}$$

The limiting behavior of g and R is given by

$$g(0) = \mu, \qquad g(\infty) = a, \qquad R(0) = 0, \qquad R(\infty) = -\log q, \tag{2.3}$$

where

$$q = P\{X_1 = a\} \tag{2.4}$$

and a is defined in (1.2). *If $q = 0$, then $R(\infty) = \infty$.*

REMARK 2.1. It is useful to recognize $V(u)$ as the variance of the random variable $Z_1(u)$ whose distribution is given by

$$H_u(x) = P\{Z_1(u) \le x\} = \frac{1}{\varphi(u)} \int_{[0,x]} e^{-uy}\, dF(y). \tag{2.5}$$

The dependence of H and Z_1 on u will be suppressed when there is no danger of confusion.

PROOF. It follows from Schwarz's inequality that $V > 0$ (since X_1 is nondegenerate) so that g is strictly decreasing and R is strictly increasing by (2.1). The limits in (2.3) are easy to check except that $R(\infty)$ can be a little tricky when $q = 0$; for this,

$$R(u) = \int_0^u (g(v) - g(u))\, dv \uparrow \int_0^\infty (g(v) - a)\, dv$$

by monotone convergence and

$$\int_0^u (g(v) - a)\, dv = -\log \varphi(u) - au = -\log(\varphi(u)e^{au}) \to -\log q. \qquad \square$$

Next we need to know the asymptotic behavior of $R(u)$ and an estimate for $V(u)$ as $u \to 0$. This is given in

LEMMA 2.2. *If $EX_1^2 < \infty$, then as $u \to 0$*

$$R(u) \sim \tfrac{1}{2}u^2 \operatorname{Var} X_1 \quad \textit{and} \quad V(u) \sim \operatorname{Var} X_1.$$

If $EX_1^2 = \infty$, then $R(u) \sim R_1(u)$ as $u \to 0$, where

$$R_1(u) = 1 - \varphi(u) + u\varphi'(u)$$

and

$$u^2 V(u) = O(R(u)), \qquad u \to 0.$$

If $Z_1(u)$ has distribution given in (2.5), *then*

$$V(u) \sim EZ_1^2(u), \qquad u \to 0, \quad \textit{when } EX_1^2 = \infty$$

and for $u \le C < \infty$,

$$V(u) \approx EZ_1^2(u), \qquad u \to 0, \quad \textit{even when } EX_1^2 < \infty.$$

PROOF. The case when $EX_1^2 < \infty$ is a straightforward expansion of R out to terms of order u^2. Thus assume $EX_1^2 = \infty$. We use the inequalities

$$-\log \varphi = \log \frac{1}{\varphi} \le \frac{1}{\varphi} - 1, \qquad -\log \varphi \ge -(\varphi - 1) = 1 - \varphi$$

to obtain

$$\begin{aligned} R(u) &\le \frac{1 - \varphi(u) + u\varphi'(u)}{\varphi(u)} = \frac{R_1(u)}{\varphi(u)}, \\ R(u) &\ge 1 - \varphi(u) + u\varphi'(u) - (1 - \varphi(u))ug(u) \\ &= R_1(u) - (1 - \varphi(u))ug(u). \end{aligned} \tag{2.6}$$

Since $\varphi(u) \to 1$ as $u \to 0$, the upper bound is adequate. Next observe that

$$\begin{aligned} 1 - \varphi(u) &= E(1 - e^{-uX_1}) \le E(uX_1 \wedge 1), \\ ug(u) &\sim -u\varphi'(u) = EuX_1 e^{-uX_1} \le E(uX_1 \wedge 1), \end{aligned}$$

so that for small u

$$(1 - \varphi(u))ug(u) \le 2\{E(uX_1 \wedge 1)\}^2. \tag{2.7}$$

Since $v^{-2}(1 - e^{-v}(1 + v))\downarrow$ we have

$$(1 - 2e^{-1})(v^2 \wedge 1) \le 1 - e^{-v}(1 + v) \le v^2 \wedge 1, \quad v \ge 0,$$

so that

$$R_1(u) \approx E(u^2 X_1^2 \wedge 1). \tag{2.8}$$

Now our assumption that $EX_1^2 = \infty$ implies that

$$\{E(uX_1 1\{uX_1 \le 1\})\}^2 = o\{E(u^2X_1^2 1\{uX_1 \le 1\})\}, \qquad u \to 0 \tag{2.9}$$

(see page 11 of [11]), which together with (2.6), (2.7), and (2.8) completes the proof that $R(u) \sim R_1(u)$. Since

$$\begin{aligned} u^2V(u) &\le \frac{1}{\varphi(u)} Eu^2X_1^2e^{-uX_1} \sim Eu^2X_1^2e^{-uX_1} \\ &\le E(u^2X_1^2 \wedge 1) \approx R_1(u) \sim R(u), \end{aligned} \tag{2.10}$$

we have $u^2V(u) = O(R(u))$ as $u \to 0$. Finally, when $EX_1^2 = \infty$ we have $V(u) \sim EZ_1^2(u)$ as a consequence of (2.9); if $EX_1^2 < \infty$, then $V(u) \to \operatorname{Var} X_1$ as $u \to 0$, so that $V(u)$ has a positive minimum on $[0, C]$, whereas $EZ_1^2(u)$ has a finite maximum so the last assertion of the lemma follows. □

The following lemma contains the key estimates for the proof of Theorem 2.1.

LEMMA 2.3. *Let $a < x_n < \mu$ and $g(\lambda_n) = x_n$. Then*

$$P\{S_n \le nx_n\} \le \exp(-nR(\lambda_n)). \tag{i}$$

Furthermore, given $0 < \eta < R(\infty)$, there exists $c = c(\eta) > 0$ such that if for some $\varepsilon > 0$ and n

$$(1+\varepsilon)R(\lambda_n) \le \eta < R(\infty), \tag{2.11}$$

then for such ε and n we have

$$P\{S_n \le nx_n\} \ge \left(1 - \frac{(1+\varepsilon)c}{\varepsilon^2 nR(\lambda_n)}\right)\exp\{-(1+2\varepsilon)nR(\lambda_n)\}. \tag{ii}$$

PROOF. The upper bound is easy. By Chebyshev's inequality

$$P\{S_n \le nx_n\} \le E\exp(-\lambda_n S_n + n\lambda_n x_n) = \left(\varphi(\lambda_n)e^{\lambda_n x_n}\right)^n = e^{-nR(\lambda_n)}.$$

For the lower bound, let $\varepsilon > 0$ and n be fixed for which (2.11) holds. There exists $\gamma_n > \lambda_n$ such that

$$R(\gamma_n) = (1+\varepsilon)R(\lambda_n). \tag{2.12}$$

We use the transformed distributions introduced by Cramér [2]. Let $\{Z_k\}$ be an i.i.d. sequence of random variables with distribution $H_{\gamma_n} = H$ given by (2.5). We have suppressed the dependence of H on n which will cause no confusion. We let $z_n = g(\gamma_n)$ and note that

$$EZ_1 = z_n, \qquad \operatorname{Var} Z_1 = V(\gamma_n). \tag{2.13}$$

Denoting

$$\mathbb{R}_+^n = \{(y_1, \dots, y_n) \colon y_i \ge 0,\ i = 1, \dots, n\}$$

and

$$B_n = \{y \in \mathbb{R}_+^n \colon y_1 + \cdots + y_n \le nx_n\},$$

we have

$$\begin{aligned} P\{S_n \le nx_n\} &= \int_{B_n} dF(y_1) \cdots dF(y_n) \\ &= (\varphi(\gamma_n))^n \int_{B_n} e^{\gamma_n(y_1 + \cdots + y_n)} dH(y_1) \cdots dH(y_n) \\ &= e^{-nR(\gamma_n)} E\{e^{\gamma_n(Z_1 + \cdots + Z_n - nz_n)} 1\{Z_1 + \cdots + Z_n \le nx_n\}\}. \end{aligned} \tag{2.14}$$

We need a lower bound for the expectation. We let

$$w_n = z_n - \varepsilon\gamma_n^{-1}R(\lambda_n), \qquad I_n = [nw_n, nx_n] \tag{2.15}$$

and get

$$\begin{aligned} P\{S_n \le nx_n\} &\ge e^{-nR(\gamma_n)} e^{n\gamma_n(w_n - z_n)} P\{Z_1 + \cdots + Z_n \in I_n\} \\ &= e^{-nR(\lambda_n)(1+2\varepsilon)} P\{Z_1 + \cdots + Z_n \in I_n\}. \end{aligned} \tag{2.16}$$

By the generalized mean value theorem,

$$\begin{aligned} x_n - z_n = g(\lambda_n) - g(\gamma_n) &= \frac{g(\lambda_n) - g(\gamma_n)}{R(\gamma_n) - R(\lambda_n)} \varepsilon R(\lambda_n) \\ &= \frac{V(\xi_n)}{\xi_n V(\xi_n)} \varepsilon R(\lambda_n) \ge \frac{\varepsilon R(\lambda_n)}{\gamma_n}, \end{aligned} \tag{2.17}$$

where $\lambda_n < \xi_n < \gamma_n$. Recalling (2.15) and (2.17) we have

$$P\{Z_1 + \cdots + Z_n \notin I_n\} \le P\left\{ |Z_1 + \cdots + Z_n - nz_n| > \frac{\varepsilon n R(\lambda_n)}{\gamma_n} \right\},$$

so by Chebyshev's inequality (recall (2.13))

$$P\{Z_1 + \cdots + Z_n \notin I_n\} \le \frac{nV(\gamma_n)}{n^2\varepsilon^2 R^2(\lambda_n)\gamma_n^{-2}} = \frac{\gamma_n^2 V(\gamma_n)}{\varepsilon^2 n R^2(\lambda_n)}. \tag{2.18}$$

Let u_1 satisfy $R(u_1) = \eta$. By Lemma 2.2 there exists $c > 0$ such that

$$u^2 V(u) \le cR(u) \quad \text{for } u \le u_1,$$

which by (2.11) means that

$$\gamma_n^2 V(\gamma_n) \le cR(\gamma_n) = c(1 + \varepsilon)R(\lambda_n),$$

and using this in (2.18) we get

$$P\{Z_1 + \cdots + Z_n \notin I_n\} \le \frac{c(1 + \varepsilon)}{\varepsilon^2 n R(\lambda_n)},$$

which together with (2.16) gives the desired lower bound. □

We will now prove the main result of this section.

THEOREM 2.1. *Let $a < x_n < \mu$ and let λ_n be the unique solution of $g(\lambda_n) = x_n$. Then*

$$P\{S_n \le nx_n\} \to 0 \quad \textit{iff} \quad nR(\lambda_n) \to \infty.$$

Furthermore, if $nR(\lambda_n) \to \infty$ and either for some $\delta > 0$, $a + \delta \le x_n < \mu$ for all n or $q = P\{X = a\} > 0$, then

$$-\log P\{S_n \le nx_n\} \sim nR(\lambda_n). \tag{2.19}$$

REMARK 2.2. When $x_n \to \mu$, $\lambda_n \to 0$, so R may be replaced by R_1 in (2.19) when $EX_1^2 = \infty$ by Lemma 2.2.

PROOF. If $nR(\lambda_n) \to \infty$, then by the upper bound in Lemma 2.3 we have $P\{S_n \le nx_n\} \to 0$. Now assume that $nR(\lambda_n) \le C < \infty$ along a subsequence, in which case we will show that $P\{S_n \le nx_n\} \ge c > 0$ for large n in that subsequence. First observe that there is no loss of generality if we assume

$$nR(\lambda_n) = C \tag{2.20}$$

along the subsequence in question because we can always pick λ_n larger to satisfy (2.20) and the corresponding x_n will be smaller than before; if we can show $P\{S_n \le nx_n\} \ge c > 0$ with the smaller x_n, that will certainly suffice. Thus we assume (2.20) along a subsequence. Since $R(\lambda_n) \to 0$ along this subsequence, we can apply the lower bound in Lemma 2.3 with ε large so $c(1 + \varepsilon)/\varepsilon^2 C < \frac{1}{2}$ to conclude that

$$P\{S_n \le nx_n\} \ge \tfrac{1}{2}\exp(-(1 + 2\varepsilon)C)$$

for n sufficiently large. This proves the first assertion.

Now assume that $nR(\lambda_n) \to \infty$ and $x_n \ge a + \delta$. Then λ_n must be bounded above and $\sup_n R(\lambda_n) < R(\infty)$. Thus there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the condition (2.11) holds for some η and all n. Thus by Lemma 2.3,

$$nR(\lambda_n) \le -\log P\{S_n \le nx_n\} \le (1 + 2\varepsilon)nR(\lambda_n) - \log\left(1 - \frac{c(1+\varepsilon)}{\varepsilon^2 nR(\lambda_n)}\right),$$

which proves (2.19) in this case. If $q > 0$ and λ_n remains bounded along a subsequence, then $x_n \ge a + \delta$ for some $\delta > 0$ along that subsequence and this case has already been considered. So we need to consider the case $q > 0$ and $\lambda_n \to \infty$ along a subsequence and show that along that subsequence (2.19) holds. We have

$$P\{S_n \le nx_n\} \ge P\{X_j = a, 1 \le j \le n\} = q^n$$

so $-\log P\{S_n \le nx_n\} \le -n \log q \sim nR(\lambda_n)$, and since the lower bound is always valid by Lemma 2.3 this proves the theorem. □

If $x_n \to a$ and $q = 0$, then the following corollary of Theorem 2.1 should be adequate for most applications. However, Example 2.1 shows that (2.19) may fail if x_n can approach a and $q = 0$.

COROLLARY 2.1. *If $x_n \to a$ and $q = 0$, then we have*

$$\lim_{n\to\infty} \frac{1}{n}\log P\{S_n \le nx_n\} = -\infty.$$

PROOF. For $\delta > 0$ and $x_n \le a + \delta$ we get by Theorem 2.1,

$$-\frac{1}{n}\log P\{S_n \le nx_n\} \ge -\frac{1}{n}\log P\{S_n \le n(a+\delta)\} \sim R(\lambda),$$

where $g(\lambda) = a + \delta$. When $q = 0$, $R(\lambda) \to \infty$ as $\delta \to 0$ by Lemma 2.1 and the corollary is proved. □

To apply the theorem when X_1 is in the domain of attraction of a stable law H of index α, $0 < \alpha < 1$, supported on $[0, \infty)$, we need to know the behavior of R and g near 0; note that $\mu = \infty$ in this case and in most applications $x_n \to \infty$ ($\lambda_n \to 0$) is the important case. If X_1 is as above, then there exists a slowly varying function L (near infinity) such that ($G = 1 - F$)

$$G(x) \sim \frac{x^{-\alpha}L(x)}{\Gamma(1-\alpha)} \tag{2.21}$$

(see [5], page 448). The next proposition gives the behavior of R and g in this situation.

PROPOSITION 2.1. *If X_1 is in the domain of attraction of a stable law H of index α, $0 < \alpha < 1$, then H is supported on $[0, \infty)$, and with L as in (2.21) we have*

$$R(\lambda) \sim (1-\alpha)\lambda^{\alpha}L(1/\lambda) \sim \Gamma(2-\alpha)G(1/\lambda), \qquad \lambda \to 0,$$

and

$$g(\lambda) \sim \alpha\lambda^{\alpha-1}L(1/\lambda) \sim \alpha\Gamma(1-\alpha)G(1/\lambda)\lambda^{-1}, \qquad \lambda \to 0.$$

PROOF. Clearly $EX_1^2 = \infty$, so by Lemma 2.2

$$R(\lambda) \sim R_1(\lambda) = 1 - \varphi(\lambda) + \lambda\varphi'(\lambda), \qquad \lambda \to 0.$$

We have

$$1 - \varphi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\, dF(x) = \lambda \int_0^\infty e^{-\lambda x} G(x)\, dx = \lambda \int_0^\infty e^{-\lambda x}\, dU_1(x)$$

and

$$-\lambda\varphi'(\lambda) = \int_0^\infty \lambda x e^{-\lambda x}\, dF(x) = \lambda \int_0^\infty e^{-\lambda x}\, dU_2(x),$$

where by (2.21) and Theorem 1 ([5], page 281) as $t \to \infty$,

$$U_1(t) = \int_0^t G(x)\, dx \sim \frac{t^{1-\alpha}L(t)}{\Gamma(2-\alpha)},$$

$$U_2(t) = \int_0^t x\, dF(x) = -tG(t) + \int_0^t G(x)\, dx \sim \frac{\alpha t^{1-\alpha}L(t)}{\Gamma(2-\alpha)}.$$

Now by Theorem 2 ([5], page 445) we get as $\lambda \to 0$,

$$1 - \varphi(\lambda) \sim \lambda^\alpha L(1/\lambda)$$

and

$$\lambda g(\lambda) \sim -\lambda\varphi'(\lambda) \sim \alpha\lambda^\alpha L(1/\lambda),$$

from which the conclusions of the proposition follow immediately. □

EXAMPLE 2.1. Let X_1 have mass

$$p_k = \exp(-\exp(k^2)) \quad \text{at} \quad b_k = \exp(-\exp(k)),\ k = 1, 2, \dots,$$

with the remaining mass at 1. We consider

$$\lambda_n = b_n^{-1}(\log(p_n/p_{n+1}) - n).$$

In estimating $\varphi(\lambda_n)$ it is easy to see that the main contributions will come from $k = n$ and $n + 1$. The following estimates are all straightforward:

$$\begin{aligned}
\varphi(\lambda_n) &= p_n e^{-\lambda_n b_n} + p_{n+1}e^{-\lambda_n b_{n+1}} + o(p_{n+1}) = p_{n+1}(e^n + 1 + o(1)),\\
-\varphi'(\lambda_n) &= p_{n+1}e^n b_n + p_{n+1}b_{n+1} + o(p_{n+1}b_n) = b_n p_{n+1}(e^n + o(1)),\\
g(\lambda_n) &= b_n(1 - e^{-n}(1 + o(1))),\\
R(\lambda_n) &= -\log p_{n+1} - n - (\log(p_n/p_{n+1}) - n)(1 - e^{-n}(1 + o(1))) + o(1)\\
&= -\log p_n + e^{-n}\log(p_n/p_{n+1})(1 + o(1)) + o(1)\\
&= e^{n^2} + e^{-n}(e^{(n+1)^2} - e^{n^2})(1 + o(1)) + o(1)\\
&\sim e^{-n}e^{(n+1)^2}.
\end{aligned}$$

The critical thing is that $x_n = g(\lambda_n) < b_n$. Since $b_{n-1} > nb_n$, $S_n \le nx_n$ implies that all summands are at most b_n and at least one is smaller. Thus

$$P\{S_n \le nx_n\} \le Cnp_{n+1}(p_n)^{n-1},$$

and so

$$\begin{aligned}
-\log P\{S_n \le nx_n\} &\ge -(n-1)\log p_n - \log p_{n+1} - \log n - \log C\\
&= (n-1)e^{n^2} + e^{(n+1)^2} - \log n - \log C\\
&\sim e^{(n+1)^2}.
\end{aligned}$$

Thus

$$-\log P\{S_n \le nx_n\}(nR(\lambda_n))^{-1} \to \infty.$$

3. The local limit theorem. In proving the strong result we will need a local limit theorem for triangular arrays. We have not found one in the literature which allows the distribution to change from row to row. Although we will only use this result in the case of normal convergence, the more general result requires no more work.

We start by describing the stochastic compactness condition of Feller. For any random variable X define

$$
\begin{aligned}
(3.1)\qquad & G(x) = P\{|X| > x\}, \qquad K(x) = x^{-2}E\big(X^2 1\{|X| \le x\}\big),\\
& Q(x) = G(x) + K(x) = E\big\{(x^{-1}X)^2 \wedge 1\big\},
\end{aligned}
$$

for $x > 0$. Q is continuous and strictly decreasing once the support of $|X|$ is reached. The analytic form of the stochastic compactness assumption is

$$
(3.2)\qquad \limsup_{x\to\infty} \frac{G(x)}{K(x)} < \infty.
$$

Since $G(x)/K(x)$ actually converges to a finite limit when X is in the domain of attraction of any stable law, this condition is more general.

We will also need some facts about symmetrization. Consider a sequence of distributions F_n with associated functions G_n, K_n, Q_n as defined in (3.1). The associated functions for the symmetrized distributions will be denoted by F_n^s, G_n^s, K_n^s, and Q_n^s. The following lemma uses the ideas of [7] where analogous facts are proved for a single distribution.

LEMMA 3.1. *Suppose that $F_n \to F_0$ weakly with F_n, F_0 nondegenerate and the F_n are uniformly stochastically compact, i.e.,*

$$
(3.3)\qquad \limsup_{x\to\infty} \sup_n \frac{G_n(x)}{K_n(x)} < \infty.
$$

Then the F_n^s are also uniformly stochastically compact and there exist positive constants C, c such that

$$
(3.4)\qquad cQ_n^s(x) \le Q_n(x) \le CQ_n^s(x) \quad \text{for all } n, x.
$$

PROOF. With the obvious notation,

$$
G_n^s(x) = P\{|X_{n1} - X_{n2}| > x\} \le P\{|X_{n1}| > \tfrac{1}{2}x\} + P\{|X_{n2}| > \tfrac{1}{2}x\} = 2G_n(\tfrac{1}{2}x)
$$

and

$$
G_n^s(x) \ge P\{|X_{n1}| \le x\}P\{|X_{n2}| > 2x\} = (1 - G_n(x))G_n(2x).
$$

Because of the convergence we can find x_0 such that $G_n(x) \le \frac{1}{2}$ for all n and all $x \ge x_0$. Thus we have

$$
\tfrac{1}{2}G_n(2x) \le G_n^s(x) \le 2G_n(\tfrac{1}{2}x) \quad \text{for all } x \ge x_0.
$$

Next

$$
\begin{aligned}
x^2Q_n^s(x) &= \int_0^x 2yG_n^s(y)\,dy \le 2\int_0^x 2yG_n(\tfrac{1}{2}y)\,dy\\
&= 8\int_0^{x/2} 2zG_n(z)\,dz \le 8x^2Q_n(x),
\end{aligned}
$$

so we have the lower bound in (3.4). Similarly, for $x \geq x_0$,

$$4x^2Q_n(2x) = \int_0^{2x} 2yG_n(y)\,dy = 4\int_0^x 2zG_n(2z)\,dz \leq 4x_0^2 + 8\int_{x_0}^x 2zG_n^s(z)\,dz$$

$$\leq 4x_0^2 + 8x^2Q_n^s(x) \leq \left(Q_n^s(x_0)\right)^{-1}4x^2Q_n^s(x) + 8x^2Q_n^s(x),$$

where we have used the fact that $x^2Q_n^s(x)\uparrow$. Again because of the convergence, the sequence $(Q_n^s(x_0))^{-1}$ is bounded so that this gives the upper bound in (3.4) for $x \geq x_0$. For $x < x_0$, it is trivial:

$$Q_n(x) \leq 1 \leq \left(Q_n^s(x_0)\right)^{-1}Q_n^s(x).$$

By (3.3) we know there is an x_1 and C_1 such that

$$G_n(x) \leq C_1K_n(x) \quad \text{for all } x \geq x_1, \text{ all } n.$$

Then by Lemma 2.4 of [11] there is a $\lambda > 0$ such that $x^\lambda Q_n(x)\downarrow$ for $x \geq x_1$. Choose b small enough that $b^{\lambda/2} < cC^{-1}$, the constants in (3.4). Take $x > b^{-1}x_1$ and let

$$\alpha_n = \frac{G_n^s(x)}{K_n^s(x)},$$

where we have suppressed the dependence on x for simplicity. Now if $bx \leq y \leq x$,

$$G_n^s(y) \geq G_n^s(x) = \alpha_nK_n^s(x) \geq \alpha_nx^{-2}y^2K_n^s(y) \geq \alpha_nb^2K_n^s(y).$$

Then if $\delta_n = 2/(1 + \alpha_nb^2)$ we have $y^{\delta_n}Q_n^s(y)\uparrow$ on $[bx, x]$ by Lemma 2.4 of [11]. Therefore

$$b^{\delta_n}Q_n^s(bx) \leq Q_n^s(x) \leq c^{-1}Q_n(x) \leq c^{-1}b^\lambda Q_n(bx) \leq Cc^{-1}b^\lambda Q_n^s(bx),$$

and so

$$b^{\delta_n} \leq Cc^{-1}b^\lambda < b^{\lambda/2}$$

by our choice of b. Thus $\delta_n > \lambda/2$, or

$$\frac{G_n^s(x)}{K_n^s(x)} = \alpha_n < (4\lambda^{-1} - 1)b^{-2} \quad \text{for all } x > b^{-1}x_1.$$

This is the uniform stochastic compactness for F_n^s. □

Before we state the local limit theorem we should say a word about a lattice distribution and the associated period. By a lattice distribution we shall mean one that is discrete and assumes values in the integer lattice $\mathbb{Z}$. A lattice distribution F_0 is called full lattice (*aperiodic* in the terminology of Spitzer [14]) if the group generated by the atoms of F_0 is $\mathbb{Z}$. The assumption that a lattice distribution is full lattice is harmless because it amounts to a rescaling of the atoms of F_0. A lattice distribution is said to have period p if p is the (maximum) span of the distribution lattice. We also note that p is the period of F_0 iff $|f_0(\theta)| < 1$ for $0 < \theta < 2\pi/p$ and $|f_0(2\pi/p)| = 1$, where f_0 is the characteristic function of F_0; see [14] for details. We now state the local limit theorem.

THEOREM 3.1. *For fixed n, $\{X_{nk}\}$, $k = 1, 2, \dots, n$, are to be i.i.d. with distribution F_n, characteristic function f_n, and associated functions G_n, K_n, Q_n as defined in* (3.1). *Define α_n by $Q_n(\alpha_n) = n^{-1}$. We assume that there exist probability distributions F_0, nondegenerate, and H such that*

$$X_{n1} \to F_0 \quad \textit{weakly}, \qquad \alpha_n^{-1} \sum_{k=1}^{n} X_{nk} - b_n \to H \quad \textit{weakly},$$

and that the F_n are uniformly stochastically compact, i.e., satisfy (3.3). *We also assume that if F_0 is lattice then it is full lattice and moreover, that F_0 and F_n have the same periodicity. Then H must have a density h. If $\alpha_n^{-1}x_n(k) - b_n \to x$ uniformly in k and F_0 is nonlattice, then for fixed $\eta > 0$ as $n \to \infty$,*

$$\alpha_n P\left\{ \sum_j X_{nj} \in (x_n(k) - \eta, x_n(k) + \eta] \right\} \to 2\eta h(x)$$

uniformly in k; in case F_0 is lattice and $x_n(k)$ is a possible value of $\sum_j X_{nj}$, then as $n \to \infty$,

$$\alpha_n P\left\{ \sum_j X_{nj} = x_n(k) \right\} \to p h(x)$$

uniformly in k, where p is the period of F_0. There is also a uniform upper bound is both cases:

$$\alpha_n P\left\{ \sum_j X_{nj} \in (y - \eta, y + \eta] \right\} \le C$$

for η fixed.

Before giving the proof, we need a lemma:

LEMMA 3.2. *Under the assumptions of Theorem* 3.1, *if F_0 is nonlattice, then given $\rho > 0$ there exist $C, c, \lambda > 0$ such that*

$$\left| f_n\left(v\alpha_n^{-1}\right)\right|^n \le C \exp\left(-c|v|^\lambda\right), \qquad |v| \le \rho\alpha_n,\ n \ge 1. \tag{3.5}$$

If F_0 is lattice then (3.5) *is valid for $\rho = \pi/p$.*

PROOF. By (3.3) we have x_0 and M such that

$$G_n(x) \le M K_n(x), \qquad x \ge x_0,\ n \ge 1.$$

Then by Lemma 2.4 of [11] there is a $\lambda > 0$ such that $x^\lambda Q_n(x)\downarrow$ for $x \ge x_0$. Now if ρ is fixed we can find c such that

$$x^\lambda Q_n(x) \ge c y^\lambda Q_n(y) \quad \text{for } \rho^{-1} \le x \le y;$$

to see this, if $\rho^{-1} \le x \le y \le x_0$ we have

$$x^\lambda Q_n(x) \ge x^\lambda Q_n(y) \ge (\rho x_0)^{-\lambda} y^\lambda Q_n(y),$$

so that $c = (\rho x_0)^{-\lambda}$ suffices. Then for $1 \le v \le \rho\alpha_n$,

$$nQ_n(\alpha_n v^{-1}) \ge cnv^{\lambda}Q_n(\alpha_n) = cv^{\lambda}. \tag{3.6}$$

Next we observe that by Lemma 3.1 for $|u| \le u_0$

$$\begin{aligned} 2(1 - |f_n(u)|) \ge 1 - |f_n(u)|^2 &= \int (1 - \cos ux)\, dF_n^s(x) \\ &\ge c_1 \int_{|ux| \le 1} u^2 x^2\, dF_n^s(x) = c_1 K_n^s(|u|^{-1}) \\ &\ge c_2 Q_n^s(|u|^{-1}) \ge c_3 Q_n(|u|^{-1}), \end{aligned}$$

so that

$$|f_n(u)| \le 1 - cQ_n(|u|^{-1}) \quad \text{for } |u| \le u_0. \tag{3.7}$$

In the nonlattice case, (3.7) may be extended to $|u| \le \rho$, with c depending on ρ, since $f_n \to f$, uniformly on compacts, $Q_n \to Q$ uniformly (since Q is continuous and decreasing), and since $|f(u)|$ is never 1. In the lattice case this is still valid for $\rho = \pi/p$ since $|f(u)| < 1$ for $0 < |u| < 2\pi/p$. To finish the proof, (3.5) is clearly valid for $|v| \le 1$, and for $1 \le |v| \le \rho\alpha_n$ we have by (3.7)

$$\begin{aligned} |f_n(v\alpha_n^{-1})|^n &\le (1 - cQ_n(\alpha_n|v|^{-1}))^n \le \exp(-cnQ_n(\alpha_n|v|^{-1})) \\ &\le \exp(-c_1|v|^{\lambda}), \end{aligned}$$

where we have used (3.6) at the final step. This proves the lemma. □

Proof of Theorem 3.1. We start with the lattice case. Suppose that y_n is a possible value of ΣX_{nk} and

$$|\alpha_n^{-1} y_n - b_n - x| < \varepsilon$$

for large n. Now $f_n^n(u)\exp(-iuy_n)$ is the characteristic function of $\Sigma X_{nk} - y_n$; since this random variable takes its values in $p\mathbb{Z}$, the characteristic function has period $2\pi/p$. Also note that $\alpha_n \to \infty$ since $Q_n \to Q$ uniformly. Thus

$$\begin{aligned} \alpha_n P\{\textstyle\sum X_{nk} = y_n\} &= \alpha_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iuy_n} f_n^n(u)\, du \\ &= \alpha_n \frac{p}{2\pi} \int_{-\pi/p}^{\pi/p} e^{-iuy_n} f_n^n(u)\, du \\ &= \frac{p}{2\pi} \int_{-\pi\alpha_n/p}^{\pi\alpha_n/p} e^{-iv\alpha_n^{-1}y_n} f_n^n(v\alpha_n^{-1})\, dv \\ &= \frac{p}{2\pi} \int_{-\pi\alpha_n/p}^{\pi\alpha_n/p} e^{-ivx} f_n^n(v\alpha_n^{-1}) e^{-ivb_n}\, dv \\ &\quad + \frac{p}{2\pi} \int_{-\pi\alpha_n/p}^{\pi\alpha_n/p} \left(e^{-iv(\alpha_n^{-1}y_n - b_n - x)} - 1\right) e^{-ivx} f_n^n(v\alpha_n^{-1}) e^{-ivb_n}\, dv. \end{aligned}$$

Now by Lemma 3.2 the first term converges to $ph(x)$ by dominated convergence;

note that the lemma also implies the existence of h. The second term is bounded by

$$\frac{p}{\pi}\int_0^\infty \varepsilon v C \exp(-cv^\lambda)\,dv,$$

which can be made small by choice of ε. The uniform upper bound comes from applying the bound in Lemma 3.2 to the next to last expression above. This completes the proof in the lattice case.

For the nonlattice case we introduce the functions

$$k(u) = \frac{\sin u}{u}, \qquad r(u) = (1-|u|)^+, \qquad s(x) = 2\frac{1-\cos x}{x^2}.$$

By the inversion formula,

$$\begin{aligned} J_n &= \frac{\rho}{2\eta}\int s(\rho(x_n - y))\alpha_n P\{|\textstyle\sum X_{nk} - y| \le \eta\}\,dy \\ &= \alpha_n \int \exp(-iux_n) r(u\rho^{-1}) k(u\eta) f_n^n(u)\,du \\ &= \int \exp(-iv(\alpha_n^{-1}x_n - b_n)) r(v\alpha_n^{-1}\rho^{-1}) k(v\alpha_n^{-1}\eta) f_n^n(v\alpha_n^{-1}) e^{-ivb_n}\,dv. \end{aligned}$$

Note that the integrand in the last expression vanishes for $|v| \ge \rho\alpha_n$ due to the r factor. Thus we may use Lemma 3.2 as above to see that this last expression converges to $2\pi h(x)$ uniformly in the sequence $\{x_n\}$ when ρ is fixed. Fix $\delta > 0$ and let $I = \{y: |y - x_n| \le \delta\eta\}$. Then we obtain

$$J_n \ge \frac{\rho}{2\eta}\alpha_n P\{|\textstyle\sum X_{nk} - x_n| \le (1-\delta)\eta\}\int_I s(\rho(x_n - y))\,dy$$

and

$$\rho\int_I s(\rho(x_n - y))\,dy = \int_{|z|\le\rho\delta\eta} s(z)\,dz = 2\pi - \int_{|z|>\rho\delta\eta} s(z)\,dz \ge 2\pi - \frac{8}{\rho\delta\eta}.$$

Now replace η by $(1-\delta)^{-1}\eta$ and we have for large n,

$$\alpha_n P\{|\textstyle\sum X_{nk} - x_n| \le \eta\} \le (1+\varepsilon)2\pi h(x)2\eta(1-\delta)^{-1}\left(2\pi - \dfrac{8(1-\delta)}{\rho\delta\eta}\right)^{-1}.$$

By taking δ small and then ρ large we obtain the desired upper bound. Note that since h is bounded a similar argument gives an upper bound

$$\alpha_n P\{|\textstyle\sum X_{nk} - y| \le \eta\} \le C,$$

which is uniform in y. This allows us to estimate J_n above:

$$\begin{aligned} J_n &\le \frac{\rho}{2\eta}\alpha_n P\{|\textstyle\sum X_{nk} - x_n| \\ &\qquad \le (1+\delta)\eta\}\int_I s(\rho(x_n - y))\,dy + \frac{\rho}{2\eta}C\int_{I^c} s(\rho(x_n - y))\,dy \\ &\le \frac{2\pi}{2\eta}\alpha_n P\{|\textstyle\sum X_{nk} - x_n| \le (1+\delta)\eta\} + C\frac{4}{\rho\delta\eta^2}. \end{aligned}$$

The argument is completed as above although this time the error term must be made small compared to $h(x)$; of course, a lower bound is not needed if $h(x) = 0$. This completes the proof. □

4. The strong result. The exact asymptotic behavior of $P\{S_n \le nx_n\}$ is now fairly easy to obtain in the stochastically compact case. This is the content of Theorem 4.1. First we need a lemma which will provide more information about V, $\{\alpha_n\}$, and $\{\lambda_n\}$.

LEMMA 4.1. *Assume that X_1 is stochastically compact, i.e., F satisfies (3.2). Let Z_n have distribution $H_{\lambda_n} = H$ defined in (2.5). Let G_n, K_n, Q_n (see (3.1)) correspond to Z_n. Then the Z_n are uniformly stochastically compact, i.e., satisfy (3.3), and for $0 < u \le C < \infty$ and $\lambda_n \le C < \infty$ we have the following*:

(a) $u^2V(u) \approx R(u) \approx R_1(u) \approx Q(u^{-1})$.

(b) *If* $nQ_n(\alpha_n) = 1$ *and* $nQ(a_n) = 1$, *then* $\lambda_n\alpha_n \to \infty$ *iff* $nR(\lambda_n) \to \infty$ *iff* $\lambda_n a_n \to \infty$.

(c) *If* $nR(\lambda_n) \to \infty$, *then* $\alpha_n^2 \approx nV(\lambda_n)$, *where* $nQ_n(\alpha_n) = 1$.

(d) *If* $nR(\lambda_n) \to \infty$, *then*

$$\lim_{n\to\infty} \frac{nE|Z_n - EZ_n|^3}{(n \operatorname{Var} Z_n)^{3/2}} = 0.$$

PROOF. For the uniform stochastic compactness, note that

$$\frac{G_n(x)}{K_n(x)} = \frac{E\left(e^{-\lambda_n X_1}1\{X_1 > x\}\right)}{x^{-2}E\left(X_1^2 e^{-\lambda_n X_1}1\{X_1 \le x\}\right)} \le \frac{e^{-\lambda_n x}G(x)}{e^{-\lambda_n x}K(x)} = \frac{G(x)}{K(x)}.$$

(a) Since these functions are all positive and continuous we only need to consider $u \to 0$. The behavior when $EX_1^2 < \infty$ is easy—see Lemma 2.2. We also showed $R(u) \sim R_1(u) \approx Q(u^{-1}) \ge cu^2V(u)$ when $EX_1^2 = \infty$ in Lemma 2.2—see (2.8) also. The remaining inequality depends on stochastic compactness:

$$\begin{aligned} u^2V(u) &= \frac{1}{\varphi(u)}\int u^2x^2e^{-ux}\,dF(x) - \left(\frac{1}{\varphi(u)}\int uxe^{-ux}\,dF(x)\right)^2 \\ &\ge e^{-1}K(u^{-1}) - C\left(\int (ux \wedge 1)\,dF(x)\right)^2 \\ &\sim e^{-1}K(u^{-1}) \ge cQ(u^{-1}) \end{aligned}$$

by (2.9).

(b) First note that

$$(4.1)\quad x^2Q_n(x) = E\{Z_n^2 \wedge x^2\} \le EZ_n^2 \approx V(\lambda_n) \approx \lambda_n^{-2}Q(\lambda_n^{-1}) \quad \text{for all } x,$$

where we used Lemma 2.2 at the third step and (a) at the last step; for $x \ge \lambda_n^{-1}$ we have

$$(4.2)\quad \begin{aligned} x^2Q_n(x) &\ge E\left(Z_n^2 1\{Z_n \le x\}\right) \\ &\ge E\left(Z_n^2 1\{Z_n \le \lambda_n^{-1}\}\right) \ge e^{-1}\lambda_n^{-2}K(\lambda_n^{-1}) \ge c\lambda_n^{-2}Q(\lambda_n^{-1}), \end{aligned}$$

using stochastic compactness of X_1 at the last step. In particular, evaluating at λ_n^{-1},

$$Q_n(\lambda_n^{-1}) \approx Q(\lambda_n^{-1}) \approx R(\lambda_n). \tag{4.3}$$

Next, since $x^2Q_n(x)\uparrow$ and $x^\lambda Q_n(x)\downarrow$ for large x as in the proof of Lemma 3.1, we have for large n

$$nQ_n(\varepsilon\alpha_n) \le \varepsilon^{-2}nQ_n(\alpha_n) = \varepsilon^{-2}, \qquad nQ_n(\varepsilon\alpha_n) \ge \varepsilon^{-\lambda}nQ_n(\alpha_n) = \varepsilon^{-\lambda}. \tag{4.4}$$

Now if $nR(\lambda_n) \to \infty$, then $nQ_n(\lambda_n^{-1}) \to \infty$ by (4.3) and so $\lambda_n^{-1} \le \varepsilon\alpha_n$ by (4.4). On the other hand, if $nR(\lambda_n)$ is bounded (for a subsequence) then so is $nQ_n(\lambda_n^{-1})$ and then $\lambda_n^{-1} \ge \varepsilon\alpha_n$ for small enough ε by (4.4). The proof for a_n is the same with Q_n replaced by Q.

(c) Since $nR(\lambda_n) \to \infty$, $\lambda_n\alpha_n \to \infty$ by (b) so $\alpha_n \ge \lambda_n^{-1}$. Then by (4.2),

$$\alpha_n^2 n^{-1} = \alpha_n^2 Q_n(\alpha_n) \ge c\lambda_n^{-2}Q(\lambda_n^{-1}) \approx V(\lambda_n).$$

Similarly, by (4.1)

$$\alpha_n^2 n^{-1} = \alpha_n^2 Q_n(\alpha_n) \le C\lambda_n^{-2}Q(\lambda_n^{-1}) \approx V(\lambda_n).$$

(d) First we have

$$\lambda_n EZ_n = \frac{1}{\varphi(\lambda_n)}\int \lambda_n z e^{-\lambda_n z}\,dF(z) \le \frac{1}{\varphi(\lambda_n)} \le C_1.$$

Then with $C_2 > C_1$,

$$\begin{aligned}
E|Z_n - EZ_n|^3 &= E\left\{|Z_n - EZ_n|^3 1\{|Z_n - EZ_n| \le C_2\lambda_n^{-1}\}\right\} \\
&\quad + E\left\{|Z_n - EZ_n|^3 1\{|Z_n - EZ_n| > C_2\lambda_n^{-1}\}\right\} \\
&\le C_2\lambda_n^{-1}V(\lambda_n) \\
&\quad + \frac{1}{\varphi(\lambda_n)}\lambda_n^{-3}\int_{|z-EZ_n|>C_2\lambda_n^{-1}} \lambda_n^3|z - EZ_n|^3 e^{-\lambda_n z}\,dF(z) \\
&\le C_2\lambda_n^{-1}V(\lambda_n) + C_4\lambda_n^{-3}G(C_3\lambda_n^{-1}) \approx \lambda_n^{-1}V(\lambda_n),
\end{aligned}$$

where the boundedness of $\lambda_n EZ_n$ is used to get the second inequality and the last step follows by (a). Thus

$$\frac{nE|Z_n - EZ_n|^3}{(nV(\lambda_n))^{3/2}} \le C_5\left(n\lambda_n^2 V(\lambda_n)\right)^{-1/2} \approx (nR(\lambda_n))^{-1/2} \to 0. \qquad \square$$

THEOREM 4.1. *Assume that X_1 is stochastically compact. Let λ_n be the unique solution of $g(\lambda_n) = x_n$ where for some $\delta > 0$, $a + \delta \le x_n < \mu$. If $nR(\lambda_n) \to \infty$ and X_1 is nonlattice, then*

$$P\{S_n \le nx_n\} \sim \frac{1}{\sqrt{2\pi}\,\lambda_n s_n} e^{-nR(\lambda_n)}, \tag{4.5}$$

where $s_n = (nV(\lambda_n))^{1/2}$. If X_1 is lattice, (4.5) is still valid if $x_n \to \mu$ but in *general there is an oscillating factor*:

$$P\{S_n \le nx_n\} \sim \frac{p}{\sqrt{2\pi}\, s_n} \frac{e^{\lambda_n d_n}}{(1 - e^{-\lambda_n p})} e^{-nR(\lambda_n)} \tag{4.6}$$

where d_n *is the maximum possible nonpositive value of* $S_n - nx_n$.

REMARK 4.1. (i) This result may fail if x_n is allowed to approach a even if $q = P\{X_1 = a\} > 0$. (Take $P\{X_1 = 0\} = P\{X_1 = 1\} = \frac{1}{2}$ and $nx_n = 1$.)

(ii) If only a $\approx$ estimate is needed in (4.5), then $\lambda_n s_n$ may be replaced by $(nR(\lambda_n))^{1/2}$ by Lemma 4.1.

PROOF. We start with (2.14) with $\gamma_n = \lambda_n$:

$$P\{S_n \le nx_n\} = e^{-nR(\lambda_n)} E\{e^{\lambda_n(Z_1 + \cdots + Z_n - nx_n)} 1\{Z_1 + \cdots + Z_n \le nx_n\}\}.$$

We will use the local limit theorem to evaluate the expectation. Here the Z_i are i.i.d. with the distribution H_{λ_n} defined in (2.5). Since $x_n \ge a + \delta$, λ_n is bounded and by passing to a subsequence we may assume that $\lambda_n \to \lambda < \infty$ (possibly zero). (There is no harm in restricting to a subsequence since we could start with an arbitrary subsequence.) Thus with X_{n1} corresponding to Z_1, we have

$$P\{X_{n1} \le z\} = \frac{1}{\varphi(\lambda_n)} E\{e^{-\lambda_n X_1} 1\{X_1 \le z\}\} \to \frac{1}{\varphi(\lambda)} E\{e^{-\lambda X_1} 1\{X_1 \le z\}\}.$$

All the distributions have the same support so the lattice assumption is satisfied. By (d) of Lemma 4.1 we have Lyapounov's condition for the central limit theorem satisfied so that

$$s_n^{-1}\left(\sum X_{nk} - nx_n\right) \to N(0,1) \quad \text{weakly}.$$

By (c) of Lemma 4.1 we have $\alpha_n \approx s_n$, so by passing to a further subsequence we may assume that $s_n \sim c\alpha_n$ and then

$$\alpha_n^{-1}\left(\sum X_{nk} - nx_n\right) \to N(0, c^2) \quad \text{weakly}.$$

We have already checked in Lemma 4.1 that the $\{X_{n1}\}$ are uniformly stochastically compact. Thus we have the conditions of Theorem 3.1 satisfied. Let $\varepsilon > 0$ and

$$\Lambda_k = \{\lambda_n(Z_1 + \cdots + Z_n - nx_n) \in (-k\varepsilon, -(k-1)\varepsilon]\}.$$

For k and ε fixed we have $k\varepsilon\lambda_n^{-1}\alpha_n^{-1} \to 0$ by (b) of Lemma 4.1, so by Theorem 3.1 we have in the nonlattice case if $\lambda > 0$,

$$P(\Lambda_k) \sim \alpha_n^{-1}\varepsilon\lambda_n^{-1} \frac{1}{\sqrt{2\pi}\, c} \sim \varepsilon s_n^{-1}\lambda_n^{-1} \frac{1}{\sqrt{2\pi}}.$$

This is still valid if $\lambda_n \to 0$, even in the lattice case, since we estimate the probability that the sum is in a long interval of length $\varepsilon\lambda_n^{-1}$ by adding the estimates for intervals of fixed length and using the uniformity. Now

$$E\left(e^{\lambda_n(Z_1+\cdots+Z_n-nx_n)}1\{Z_1+\cdots+Z_n\le nx_n\}\right)\ge\sum_{k=1}^{k_0}e^{-k\varepsilon}P(\Lambda_k)$$

$$\sim\frac{e^{-\varepsilon}-e^{-(k_0+1)\varepsilon}}{1-e^{-\varepsilon}}P(\Lambda_1)$$

and we may choose ε small and k_0 large so that this is close to $\varepsilon^{-1}P(\Lambda_1)$, which is sufficient for the lower bound. The upper bound for these terms is similar. For the terms with $k\ge k_0$ we use the uniform upper bound from the local limit theorem to obtain a bound of

$$e^{-k_0\varepsilon}(1-e^{-\varepsilon})^{-1}C_5\varepsilon s_n^{-1}\lambda_n^{-1},$$

which will be relatively small if $k_0\varepsilon$ is large. The lattice case is similar with the nonpositive possible values of $Z_1+\cdots+Z_n-nx_n$ being d_n-kp, $k=0,1,\ldots,$ and the probability of each being $\sim p/\sqrt{2\pi}\,s_n$. This completes the proof. □

EXAMPLE 4.1. As an application of Theorem 4.1, we will obtain an exact asymptotic formula for the lower tail of a nonnegative stable random variable of index $\alpha<1$. More precise results are available in the literature (see Skorohod [13]) but they depend on contour integrations; the present method is more probabilistic. For simplicity we choose the scale factor so that $\varphi(\lambda)=e^{-\lambda^\alpha}$. Then

$$g(\lambda)=\alpha\lambda^{\alpha-1},\qquad R(\lambda)=(1-\alpha)\lambda^\alpha.$$

By the scaling property we have

$$P\{X_1\le x\}=P\{n^{1/\alpha}X_1\le n^{1/\alpha}x\}=P\{S_n\le nn^{-1+1/\alpha}x\},$$

where we will want $x\to0$. Since n is arbitrary we can take n large enough so that $x_n=n^{-1+1/\alpha}x\to\infty$. Then

$$\alpha\lambda_n^{\alpha-1}=n^{-1+1/\alpha}x\quad\text{so that}\quad\lambda_n=\left(\frac{\alpha}{x}\right)^{1/(1-\alpha)}n^{-1/\alpha}$$

and

$$nR(\lambda_n)=(1-\alpha)\left(\frac{\alpha}{x}\right)^{\alpha/(1-\alpha)}\to\infty$$

since $x\to0$. Finally $V(\lambda)=\alpha(1-\alpha)\lambda^{\alpha-2}$, so that

$$s_n\sim\{n\alpha(1-\alpha)\}^{1/2}\left(\frac{\alpha}{x}\right)^{(\alpha-2)/2(1-\alpha)}n^{-(\alpha-2)/2\alpha}$$

and so

$$\lambda_ns_n\sim\{\alpha(1-\alpha)\}^{1/2}\left(\frac{\alpha}{x}\right)^{\alpha/2(1-\alpha)}.$$

Thus

$$P\{X_1\le x\}\sim\{2\pi\alpha(1-\alpha)\}^{-1/2}\left(\frac{x}{\alpha}\right)^{\alpha/2(1-\alpha)}\exp\left(-(1-\alpha)(\alpha x^{-1})^{\alpha/(1-\alpha)}\right),$$

$$x\to0.$$

5. Subordinators. As usual, we will call a process with stationary, independent increments a Lévy process. In this section we adapt the results for sums of independent random variables to subordinators (nondecreasing Lévy processes), but now we consider both $t \to \infty$ and $t \to 0$. $\{X_t, t \ge 0\}$ will denote a subordinator with drift parameter $b \ge 0$ and Lévy measure ν. Then ν has all its mass on $(0, \infty)$ and $\int_0^\infty (x \wedge 1)\, d\nu(x) < \infty$; furthermore for $u > 0$,

$$Ee^{-uX_t} = e^{-t\psi(u)},$$

where $\psi(u) = bu + \int_0^\infty (1 - e^{-ux})\nu(dx)$. For u real, the characteristic function of X_t is given by

$$Ee^{iuX_t} = e^{t\psi_1(u)},$$

where $\psi_1(u) = ibu + \int_0^\infty (e^{iux} - 1)\nu(dx)$. We still use

$$\varphi(u) = Ee^{-uX_1} = e^{-\psi(u)}.$$

In this case

$$g(u) = -\varphi'(u)/\varphi(u) = \psi'(u) = b + \int_0^\infty xe^{-ux}\nu(dx)$$

and

$$R(u) = -\log \varphi(u) - ug(u) = \psi(u) - u\psi'(u) = \int_0^\infty (1 - e^{-ux}(1 + ux))\nu(dx).$$

We clearly have

$$g(0) = b + \int_0^\infty x\nu(dx), \qquad g(\infty) = b, \qquad R(0) = 0, \qquad R(\infty) = \nu(0, \infty).$$

As before, $g \downarrow$ and $R \uparrow$.

We now describe the transformation on the process X_t corresponding to the Cramér transformation. Let $b < y_t < g(0)$ and let γ_t be defined by $g(\gamma_t) = y_t$. The transformed process (corresponding to y_t) is denoted by Z_t, $t \ge 0$; the dependence of Z_t on y_t is suppressed. It is a nondecreasing process with independent increments whose infinitely divisible law has parameter bt and Lévy measure $te^{-\gamma_t x}\nu(dx)$. For $u \ge -\gamma_t$ we have

$$Ee^{-uZ_t} = \exp\left\{-t\left(bu + \int_0^\infty (1 - e^{-ux})e^{-\gamma_t x}\nu(dx)\right)\right\}. \tag{5.1}$$

With the help of (5.1) we can now prove the following relation corresponding to (2.14): For $x \ge 0$,

$$P\{X_t \le x\} = e^{-tR(\gamma_t)} \int_{[0, x]} e^{\gamma_t(y - ty_t)}\, dP(Z_t \le y). \tag{5.2}$$

This relation is the same as

$$P\{X_t \le x\} = e^{-t\psi(\gamma_t)} \int_{[0, x]} e^{\gamma_t y}\, dP(Z_t \le y),$$

and to verify this we simply take the Laplace transform of each side. We also define the variance function for the transformed process by

$$V(u) = -g'(u) = \int_0^\infty x^2 e^{-ux}\nu(dx). \tag{5.3}$$

It is easily seen that $EZ_t = t\gamma_t$ and $\mathrm{Var}(Z_t) = tV(\gamma_t)$. For $x > 0$, let

$$G_\nu(x) = \nu(x, \infty), \qquad K_\nu(x) = x^{-2}\int_{(0,x]} y^2\nu(dy), \qquad Q_\nu(x) = G_\nu(x) + K_\nu(x).$$

In the next lemma we summarize some useful relations between the functions R, V, and Q_ν.

LEMMA 5.1. *For $u > 0$ we have*

$$R(u) \approx Q_\nu(u^{-1}); \tag{5.4}$$

consequently,

$$u^2V(u) = O(R(u)). \tag{5.5}$$

Furthermore,

$$u^2V(u) \approx R(u), \quad \text{as } u \to 0 \ (\text{resp. } u \to \infty), \tag{5.6}$$

provided that, respectively,

$$\limsup_{x\to\infty} \frac{G_\nu(x)}{K_\nu(x)} < \infty, \qquad \limsup_{x\to 0} \frac{G_\nu(x)}{K_\nu(x)} < \infty. \tag{5.7}$$

In particular, if both conditions in (5.7) *hold, then*

$$u^2V(u) \approx R(u) \quad \text{for } u > 0. \tag{5.8}$$

REMARK 5.1. The first condition in (5.7) is equivalent to the stochastic compactness of X_1. To see this, $Q_\nu(x) \approx Q(x)$ by (5.4), (2.8), and Lemma 2.2 and then the proof of Lemma 2.5 of [7] does the rest. With the proper definition of K_ν this equivalence is true for a general Lévy process. (See the lemma in [12].)

PROOF. Since $1 - e^{-u}(1 + u) \approx u^2 \wedge 1$ for all $u > 0$, we have

$$R(u) \approx \int (u^2x^2 \wedge 1)\nu(dx) = Q_\nu(u^{-1})$$

for $u > 0$; also

$$u^2V(u) = \int_0^\infty u^2x^2e^{-ux}\nu(dx) \le \int_0^\infty (u^2x^2 \wedge 1)\nu(dx),$$

so (5.4) and (5.5) are proved. Furthermore, for $u > 0$

$$u^2V(u) \ge \int_{(0,u^{-1}]} u^2x^2e^{-1}\nu(dx) = e^{-1}K_\nu(u^{-1}).$$

This proves (5.6) via (5.4) and the corresponding assumption in (5.7). Since both $u^2V(u)$ and $R(u)$ are continuous and positive on $(0, \infty)$, (5.8) follows. □

LEMMA 5.2. *If $g(\infty) < x_t < g(0)$ and λ_t is determined by $g(\lambda_t) = x_t$, then*

(i) $$P\{X_t \le tx_t\} \le \exp(-tR(\lambda_t)).$$

Furthermore, there exists $c > 0$ *such that for all* $\varepsilon > 0$ *and* t,

$$\text{(ii)} \qquad P\{X_t \le tx_t\} \ge \left(1 - \frac{(1+\varepsilon)c}{\varepsilon^2 tR(\lambda_t)}\right)\exp\{-(1+2\varepsilon)tR(\lambda_t)\}.$$

PROOF. We get (i) via Chebyshev's inequality as in the proof of Lemma 2.3. To prove the lower bound if $(1+\varepsilon)R(\lambda_t) < R(\infty)$, we proceed as in the proof of Lemma 2.3 and get the obvious analogue of inequality (2.16); the only difference here is that (5.2) must replace the equality (2.14). The analogue of (2.18) is the inequality

$$P\{Z_t \notin I_t\} \le \frac{\gamma_t^2 V(\gamma_t)}{\varepsilon^2 tR^2(\lambda_t)}, \tag{5.9}$$

where $R(\gamma_t) = (1+\varepsilon)R(\lambda_t)$, Z_t is the transformed process corresponding to $y_t = g(\gamma_t)$, $w_t = y_t - \varepsilon\gamma_t^{-1}R(\lambda_t)$ and $I_t = [tw_t, tx_t]$. Now c is picked to satisfy

$$u^2V(u) \le cR(u), \qquad u > 0,$$

by (5.5). Then (5.9) gives

$$P\{Z_t \notin I_t\} \le \frac{c(1+\varepsilon)R(\lambda_t)}{\varepsilon^2 tR^2(\lambda_t)} = \frac{(1+\varepsilon)c}{\varepsilon^2 tR(\lambda_t)},$$

and the rest of the argument is completed as before. If $(1+\varepsilon)R(\lambda_t) \ge R(\infty)$ then we have

$$P\{X_t \le tx_t\} \ge P\{X_t = tb\} = e^{-t\nu(0,\infty)} \ge \exp\{-(1+\varepsilon)tR(\lambda_t)\}. \qquad \square$$

The next theorem is the analogue of Theorem 2.1. The proof is an immediate consequence of Lemma 5.2.

THEOREM 5.1. *Let* $g(\infty) < x_t < g(0)$ *and* λ_t *be determined by* $g(\lambda_t) = x_t$. *Then as* $t \to 0$ $(t \to \infty)$,

$$P\{X_t \le tx_t\} \to 0 \quad \textit{iff} \quad tR(\lambda_t) \to \infty, \textit{ as } t \to 0\ (t \to \infty).$$

Furthermore, if $tR(\lambda_t) \to \infty$ *as* $t \to 0$ $(t \to \infty)$, *then*

$$-\log P\{X_t \le tx_t\} \sim tR(\lambda_t), \quad \textit{as } t \to 0\ (t \to \infty).$$

We now proceed to establish the strong result which gives the asymptotic behavior of $P\{X_t \le tx_t\}$ itself. Let $g(\infty) < x_t < g(0)$ and λ_t be determined by $g(\lambda_t) = x_t$. Let Z_t be the transformed process corresponding to x_t. By (5.2) we have

$$P\{X_t \le tx_t\} = e^{-tR(\lambda_t)}\int_{[0,\, tx_t]} e^{\lambda_t(y - tx_t)}\, dP\{Z_t \le y\}, \tag{5.10}$$

and what we need is the behavior of the integral in (5.10) as $t \to 0$ $(t \to \infty)$. For this we prove a local limit theorem for Z_t. The next proposition gives the central limit theorem.

PROPOSITION 5.1. *Let*

$$U_t = \frac{Z_t - \mu_t}{\sigma_t},$$

where Z_t is the transformed process corresponding to x_t as in (5.1), $\mu_t = tx_t = EZ_t$, *and* $\sigma_t^2 = tV(\lambda_t) = \mathrm{Var}(Z_t)$; *here V is the function given in* (5.3). *Whether we are dealing with $t \to 0$ or $t \to \infty$, assume* (5.7) *if* $\liminf \lambda_t = 0$ (*resp.* $\limsup \lambda_t = \infty$) *and that $tR(\lambda_t) \to \infty$. Then $U_t \to N(0,1)$ weakly in either case.*

REMARK 5.2. If $t \to 0$ we can only have $\lambda_t \to \infty$, whereas if $t \to \infty$ the possibility that $\liminf \lambda_t = 0$ cannot be ruled out.

PROOF. As usual, we will use Lévy's continuity theorem for characteristic functions. For real u

$$E(e^{iuU_t}) = \exp\left\{t\int_0^\infty \left(e^{iuy/\sigma_t} - 1 - \frac{iuy}{\sigma_t}\right)e^{-\lambda_t y}\nu(dy)\right\}.$$

If $|y| \le M\lambda_t^{-1}$, then by (5.6) we have $\lambda_t^2 V(\lambda_t) \approx R(\lambda_t)$, so

$$\left|\frac{uy}{\sigma_t}\right| \le \frac{|u|M}{\lambda_t\sigma_t} = \frac{|u|M}{(t\lambda_t^2 V(\lambda_t))^{1/2}} \approx \frac{|u|M}{(tR(\lambda_t))^{1/2}} \to 0;$$

therefore

$$\begin{aligned} &t\int_{(0,M\lambda_t^{-1}]}\left(e^{iuy/\sigma_t} - 1 - \frac{iuy}{\sigma_t}\right)e^{-\lambda_t y}\nu(dy) \\ (5.11)\qquad &\sim -\frac{tu^2}{2\sigma_t^2}\int_{(0,M\lambda_t^{-1}]} y^2 e^{-\lambda_t y}\nu(dy) \\ &= -\frac{u^2}{2} + \frac{u^2}{2V(\lambda_t)}\int_{(M\lambda_t^{-1},\infty)} y^2 e^{-\lambda_t y}\nu(dy). \end{aligned}$$

Also,

$$\begin{aligned} &\left|t\int_{(M\lambda_t^{-1},\infty)}\left(e^{iuy/\sigma_t} - 1 - \frac{iuy}{\sigma_t}\right)e^{-\lambda_t y}\nu(dy)\right| \\ (5.12)\qquad &\le \frac{tu^2}{2\sigma_t^2}\int_{(M\lambda_t^{-1},\infty)} y^2 e^{-\lambda_t y}\nu(dy); \end{aligned}$$

thus (5.11) and (5.12) give

$$\begin{aligned} &\left|t\int_0^\infty\left(e^{iuy/\sigma_t} - 1 - \frac{iuy}{\sigma_t}\right)e^{-\lambda_t y}\nu(dy) + \frac{u^2}{2}\right| \\ (5.13)\qquad &= O\left\{\frac{1}{V(\lambda_t)}\int_{(M\lambda_t^{-1},\infty)} y^2 e^{-\lambda_t y}\nu(dy)\right\}, \end{aligned}$$

where O depends on u which is fixed, but does not depend on t. We have for $M > 2$

$$\int_{(M\lambda_t^{-1},\infty)} y^2 e^{-\lambda_t y}\nu(dy) = \lambda_t^{-2}\int_{(M\lambda_t^{-1},\infty)} \lambda_t^2 y^2 e^{-\lambda_t y}\nu(dy)$$
$$\le \lambda_t^{-2}M^2 e^{-M}\nu(\lambda_t^{-1},\infty)$$
$$\le \lambda_t^{-2}M^2e^{-M}Q_\nu(\lambda_t^{-1}) \approx \lambda_t^{-2}M^2e^{-M}R(\lambda_t),$$

where Lemma 5.1 is used at the last step, and in view of (5.7) we get the last quantity $\approx M^2e^{-M}V(\lambda_t)$ by Lemma 5.1 again. Therefore, under our assumptions, whether $t \to 0$ or ∞,

$$\frac{1}{V(\lambda_t)}\int_{(M\lambda_t^{-1},\infty)} y^2e^{-\lambda_t y}\nu(dy) = O(M^2e^{-M})$$

uniformly in t, so the proposition follows from (5.13). □

Next we give the local limit theorem for Z_t.

THEOREM 5.2. *Let Z_t be the transformed process corresponding to x_t and $\mu_t = EZ_t$, $\sigma_t^2 = \mathrm{Var}(Z_t)$. Assume* (5.7) *if* $\liminf \lambda_t = 0$ *(resp.* $\limsup \lambda_t = \infty$) *and that $tR(\lambda_t) \to \infty$ where either $t \to 0$ or $t \to \infty$ is allowed. Then the following hold:*

(i) *There exists $C > 0$ such that for any $\eta > 0$ and x real*

$$P\{Z_t \in (x - \eta\lambda_t^{-1}, x + \eta\lambda_t^{-1}]\} \le \frac{C\eta}{\sigma_t\lambda_t}.$$

(ii) *If ν is nonlattice, then for $\eta > 0$*

$$P\{Z_t \in (x - \eta\lambda_t^{-1}, x + \eta\lambda_t^{-1}]\} \sim \frac{2\eta}{(2\pi)^{1/2}\sigma_t\lambda_t}$$

uniformly in x provided $\sigma_t^{-1}(x - \mu_t) \to 0$ uniformly in x.

(iii) *If ν is lattice with support in $S = \{p, 2p, 3p, \dots\}$ for some $p > 0$, where p is the maximal real with this property, then for $x \in S$*

$$P\{Z_t = bt + x\} \sim \frac{p}{(2\pi)^{1/2}\sigma_t}$$

uniformly in x provided $\sigma_t^{-1}(x + bt - \mu_t) \to 0$ uniformly in x.

REMARK 5.3. (i) If ν is lattice then $\nu(0,\infty) < \infty$ so that

$$t\nu(0,\infty) \ge tR(\lambda_t) \to \infty$$

and t must go to infinity for lattice ν.

(ii) If ν is lattice then $K_\nu(x) = 0$ for $x < p$ so the second condition in (5.7) must fail. Thus we are excluding the possibility that $\limsup \lambda_t = \infty$ in the lattice case. It is easy to check that the result is not valid if $x_t = b + o(t^{-1})$ when ν is lattice.

Before giving the proof, we need the analogue of Lemma 3.2.

Lemma 5.3. *Under the assumptions of Theorem* 5.2, *if* ν *is nonlattice, then given* $\rho > 0$ *there exists* $c > 0$ *such that for t sufficiently large or small*

$$t\int\left(1-\cos\left(v\sigma_t^{-1}y\right)\right)e^{-\lambda_t y}\nu(dy) \geq cv^2, \qquad |v| \leq \rho\sigma_t\lambda_t. \tag{5.14}$$

Furthermore, there exists $\delta > 0$ *such that for t sufficiently large or small*

$$t\int\left(1-\cos\left(v\sigma_t^{-1}y\right)\right)e^{-\lambda_t y}\nu(dy) \geq c|v|^{\delta}, \qquad 1 \leq |v| \leq \rho\sigma_t. \tag{5.15}$$

If ν *is lattice,* (5.15) *is valid for* $\rho = \pi/p$.

Proof. We let

$$h(t,v) = t\int\left(1-\cos\left(v\sigma_t^{-1}y\right)\right)e^{-\lambda_t y}\nu(dy).$$

Observe that

$$h(t,v) \geq c_1 tv^2\sigma_t^{-2}\int_{(0,\lambda_t^{-1}]} y^2\nu(dy) = c_1 tv^2\sigma_t^{-2}\lambda_t^{-2}K_\nu\left(\lambda_t^{-1}\right) \quad \text{if } |v| \leq \sigma_t\lambda_t, \tag{5.16}$$

and

$$h(t,v) \geq c_1 tv^2\sigma_t^{-2}\int_{(0,\sigma_t|v|^{-1}]} y^2\nu(dy) = c_1 tK_\nu\left(\sigma_t|v|^{-1}\right), \quad \text{if } |v| \geq \sigma_t\lambda_t. \tag{5.17}$$

Furthermore, by Lemma 5.1

$$\sigma_t^2\lambda_t^2 = t\lambda_t^2 V(\lambda_t) \approx tR(\lambda_t), \tag{5.18}$$

since we have assumed as much of (5.7) as is required for this. The first case we consider is when $\liminf \lambda_t = 0, \limsup \lambda_t = \infty$. Then $K_\nu(x) \approx Q_\nu(x)$ for all x since $Q_\nu(x)\downarrow$, $x^2K_\nu(x)\uparrow$ and we are assuming both conditions in (5.7). Thus K_ν may be replaced by Q_ν in (5.16), (5.17) and we have, using Lemma 5.1 and (5.18) for fixed $\rho > 1$,

$$h(t,v) \geq c_2 tQ_\nu\left(\lambda_t^{-1}\right)\sigma_t^{-2}\lambda_t^{-2}v^2 \geq c_3 v^2, \quad \text{if } |v| \leq \sigma_t\lambda_t, \tag{5.19}$$

$$\begin{aligned} h(t,v) &\geq c_2 tQ_\nu\left(\sigma_t|v|^{-1}\right) \geq c_2 tQ_\nu\left(\lambda_t^{-1}\right) \\ &\geq c_3\sigma_t^2\lambda_t^2 \geq c_4 v^2, \quad \text{if } \sigma_t\lambda_t \leq |v| \leq \rho\sigma_t\lambda_t, \end{aligned} \tag{5.20}$$

$$\begin{aligned} h(t,v) &\geq c_2 tQ_\nu\left(\sigma_t|v|^{-1}\right) \geq c_2 t\left(\frac{|v|}{\sigma_t\lambda_t}\right)^{\delta} Q_\nu\left(\lambda_t^{-1}\right) \\ &\geq c_3|v|^{\delta}\left(tR(\lambda_t)\right)^{1-\delta/2} \geq c_4|v|^{\delta}, \quad \text{if } \sigma_t\lambda_t \leq |v|, \end{aligned} \tag{5.21}$$

where in (5.21) we have used $tR(\lambda_t) \to \infty$ and the fact that $x^{\delta}Q_\nu(x)\downarrow$ on

$[\sigma_t|v|^{-1}, \lambda_t^{-1}]$ for some $\delta > 0$ which follows from $K_\nu \approx Q_\nu$ on this interval by Lemma 2.4 of [11]. Thus we have completed the proof in this case. Next assume $\liminf \lambda_t > 0$, $\limsup \lambda_t = \infty$. Then by (5.7) we have

$$G_\nu(x) \le CK_\nu(x), \qquad x \le x_0.$$

Next take $\varepsilon < \lambda_t$ and observe that

$$G_\nu(x) \le G_\nu(x_0) \le CK_\nu(x_0) \le Cx_0^{-2}x^2K_\nu(x) \le C_1K_\nu(x), \qquad x_0 \le x \le \varepsilon^{-1}.$$

Since (5.19)–(5.21) only required this inequality for $x \le \lambda_t^{-1}$ they also apply in this case. If $\liminf \lambda_t = 0$, $\limsup \lambda_t < \infty$, then we only know

$$G_\nu(x) \le CK_\nu(x), \qquad x \ge x_1,$$

where x_1 is large. Thus (5.19) still applies if $\lambda_t \le x_1^{-1}$, and (5.20) and (5.21) do if $|v| \le x_1^{-1}\sigma_t$. To complete this case take $M > \lambda_t$ and observe that

$$\int (1 - \cos uy)e^{-\lambda_t y}\nu(dy) \ge \int (1 - \cos uy)e^{-My}\nu(dy) \ge c_1u^2, \quad \text{if } |u| \le \rho M,$$

since this is clear for small u by integrating out to u^{-1} and then it holds for larger u since the integral does not vanish. (In the lattice case, we only need this for $|u| \le \rho$, the integral being positive on $(0, 2\rho)$.) Thus we have

$$h(t,v) \ge c_1t\sigma_t^{-2}v^2 \quad \text{if } |v| \le \rho M\sigma_t,\ \lambda_t \le M,$$

and so by (5.18)

$$(5.22) \quad h(t,v) \ge c_2\frac{\lambda_t^2}{R(\lambda_t)}v^2 \ge c_3v^2, \quad \text{if } \rho^{-1}x_1^{-1} \le \lambda_t \le M,\ |v| \le \rho M\sigma_t.$$

This proves (5.14) in this case. To finish the proof of (5.15), it only remains to consider $\lambda_t \le \rho^{-1}x_1^{-1}$, $x_1^{-1}\sigma_t \le |v| \le \rho\sigma_t$ since (5.19)–(5.22) take care of the other possibilities. Then since $x^\delta Q_\nu(x)\downarrow$ on $[x_1, \infty)$ we have, by Lemma 5.1 and (5.18),

$$x_1^\delta Q_\nu(x_1) \ge \lambda_t^{-\delta}Q_\nu(\lambda_t^{-1}) \ge c_1\lambda_t^{-\delta}R(\lambda_t) \ge c_2t^{-1}\sigma_t^2\lambda_t^{2-\delta} \ge t^{-1}\sigma_t^\delta,$$

so that

$$h(t,v) \ge c_1t\sigma_t^{-2}v^2 \ge c_2\sigma_t^{\delta-2}v^2 \ge c_3|v|^\delta.$$

Finally, if $\liminf \lambda_t > 0$, $\limsup \lambda_t < \infty$, then (5.22) applies and this is sufficient for both (5.14) and (5.15). □

PROOF OF THEOREM 5.2. In the nonlattice case we let

$$k(u) = \frac{\sin u}{u}, \qquad r(u) = (1 - |u|)^+, \qquad s(x) = 2\frac{1 - \cos x}{x^2},$$

and by the inversion formula

$$J_t = \frac{\rho\sigma_t\lambda_t^2}{2\eta}\int_{-\infty}^{\infty} s(\rho\lambda_t(x - y))P\{|Z_t - y| \le \eta\lambda_t^{-1}\}\,dy$$
$$= \int_{-\infty}^{\infty}\exp\{-iv\sigma_t^{-1}(x - \mu_t)\}r(v\rho^{-1}\sigma_t^{-1}\lambda_t^{-1})k(v\eta\sigma_t^{-1}\lambda_t^{-1})\exp\{t\hat{\psi}_t(v)\}\,dv,$$

where

$$\hat{\psi}_t(v) = \int_0^\infty \left(e^{ivy/\sigma_t} - 1 - ivy\sigma_t^{-1}\right)e^{-\lambda_t y}\nu(dy).$$

Since $r(u) = 0$ for $|u| \geq 1$ and

$$\operatorname{Re} t\hat{\psi}_t(v) = -t\int \left(1 - \cos\left(v\sigma_t^{-1}y\right)\right)e^{-\lambda_t y}\nu(dy),$$

(5.14) gives the necessary bound for the integrand in the second expression for J_t. Thus J_t is bounded and by Proposition 5.1 and (5.18) the integrand approaches $e^{-v^2/2}$ pointwise uniformly for x such that $\sigma_t^{-1}(x - \mu_t) \to 0$ uniformly so that by dominated convergence $J_t \to (2\pi)^{1/2}$ uniformly in x. Now the first expression for J_t is bounded as in the proof of Theorem 3.1 to complete the proof of (i) and (ii). For the lattice case we have for $x \in S$,

$$\sigma_t P\{Z_t = bt + x\} = \frac{p}{2\pi}\int_{-\pi\sigma_t/p}^{\pi\sigma_t/p} e^{-iv(x+bt-\mu_t)/\sigma_t}e^{t\hat{\psi}_t(v)}\,dv,$$

and (5.15) and Proposition 5.1 complete the proof. □

The strong theorem for subordinators now follows easily from the local limit theorem for Z_t via (5.10).

THEOREM 5.3. *Let $b < x_t < \mu = EX_1$, and λ_t be the unique solution of $g(\lambda_t) = x_t$. Assume* (5.7) *if* $\liminf \lambda_t = 0$ (*resp.* $\limsup \lambda_t = \infty$) *and that $tR(\lambda_t) \to \infty$ where t may tend to zero or infinity. Then if ν is nonlattice*

$$P\{X_t \leq tx_t\} \sim \frac{1}{\sqrt{2\pi}\,\sigma_t\lambda_t}e^{-tR(\lambda_t)}, \tag{5.23}$$

where $\sigma_t^2 = tV(\lambda_t)$. If ν is lattice, (5.23) *is still valid if $x_t \to \mu$ but in general there is an oscillating factor*:

$$P\{X_t \leq tx_t\} \sim \frac{p}{\sqrt{2\pi}\,\sigma_t}\frac{e^{\lambda_t d_t}}{\left(1 - e^{-\lambda_t p}\right)}e^{-tR(\lambda_t)}, \tag{5.24}$$

where d_t is the largest nonpositive possible value of $X_t - tx_t$.

REMARK 5.4. (i) If only a $\approx$ estimate is needed in (5.23), note that by (5.18) $\sigma_t\lambda_t$ may be replaced by $(tR(\lambda_t))^{1/2}$. (ii) As with the local limit theorem (5.24) fails when ν is lattice and $x_t = b + o(t^{-1})$.

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