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NONUNIFORM ESTIMATES IN THE CONDITIONAL CENTRAL LIMIT THEOREM

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Let X_n, $n \in \mathbb{N}$, be i.i.d. with mean 0, variance 1, and $E(|X_1|^r) < \infty$ for some $r > 3$. Let B be a measurable set such that its distances from the σ fields $\sigma(X_1, \dots, X_n)$ are of order $O(n^{-1/2}(\log n)^{-r/2})$. We prove that for such B the conditional probabilities $P(n^{-1/2}\sum_{i=1}^n X_i \le t|B)$ can be approximated by the standard normal distribution $\Phi(t)$ up to the classical nonuniform bound $(1 + |t|^r)^{-1}n^{-1/2}$. An example shows that this is not true any more if the distances of B from $\sigma(X_1, \dots, X_n)$ are only of order $O(n^{-1/2}(\log n)^{-r/2+\varepsilon})$ for some $\varepsilon > 0$. For the case $r = 3$ one can obtain the corresponding assertion only under a strengthened assumption.

1. Introduction and notation. Let X_n, $n \in \mathbb{N}$, be a sequence of i.i.d. real valued random variables with mean 0 and variance 1. Put $S_n = \sum_{i=1}^n X_i$ and $S_n^* = n^{-1/2}\sum_{i=1}^n X_i$. Let $B \in \sigma(X_n: n \in \mathbb{N})$ with $P(B) > 0$. The conditional probabilities $P(S_n^* \le t|B)$ play an important role in several fields of application and have been investigated in a lot of papers (see e.g., [7], [2], [3], [4]). The classical conditional central limit theorem of Rényi (1958) states that for each B

$$P(S_n^* \le t|B) - \Phi(t) \underset{n \in \mathbb{N}}{\rightarrow} 0, \tag{1.1}$$

where Φ is the standard normal distribution.

The convergence order in (1.1), however, depends critically on the special set B: By suitable B you can make the convergence order in (1.1) as bad as you want (see Example 1 of [1]). In [2] and [4] approximation orders and second order expansions for the conditional probabilities $P(S_n^* \le t|B)$ are given for special sets B. It turns out that the distances

$$d(B, \sigma(X_1, \dots, X_n)) = \inf\{P(B \triangle A): A \in \sigma(X_1, \dots, X_n)\}$$

essentially determine the approximation results for the conditional probabilities. If, for instance, $d(B, \sigma(X_1, \dots, X_n)) = O(n^{-1/2}(\log n)^{-\beta})$ for some $\beta > \frac{3}{2}$, then

$$\sup_{t \in \mathbf{R}} |P(S_n^* \le t|B) - \Phi(t)| = O(n^{-1/2}), \tag{1.2}$$

a result which fails if we replace $\beta > \frac{3}{2}$ by $\beta = \frac{3}{2}$ (see Corollary 3 and Example 5 of [2]).

In this paper we give for a large class of sets B a nonuniform estimate for the conditional probabilities of the form

$$|P(S_n^* \le t|B) - \Phi(t)| \le c(1 \wedge |t|^{-r})n^{-1/2}. \tag{1.3}$$

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Such nonuniform estimates have wider applicability than approximation results of type (1.2), e.g., for obtaining inequalities for $\| \ \|_r$ norms and for the theory of moderate deviations. Until now nonuniform bounds of type (1.3) were only known for $B = \Omega$ (see, e.g., Petrov (1975), Theorem 13, page 125).

If $E(|X_1|^r) < \infty$ for some $r \geq 3$, we prove that relation (1.3) holds for all B satisfying

$$d(B, \sigma(X_1, \dots, X_n)) = O\big(n^{-1/2}(\log n)^{-\beta(r)}\big), \tag{1.4}$$

where $\beta(r) = r/2$ for $r > 3$ and $\beta(3) > \frac{3}{2}$. We show by an example that assumption (1.4) cannot be weakened, even if X_1 is standard normally distributed.

2. The results. For a bounded random variable Y define

$$d_1(Y, \sigma(X_1, \dots, X_n)) = \inf\{E(|Y - Z|) \colon Z \text{ is } \sigma(X_1, \dots, X_n)\text{-measurable}\}.$$

Observe that

$$d_1(1_B, \sigma(X_1, \dots, X_n)) \leq d(B, \sigma(X_1, \dots, X_n)) \leq 2 d_1(1_B, \sigma(X_1, \dots, X_n))$$

and that $d_1(Y, \sigma(X_1, \dots, X_n)) \to_{n \in \mathbb{N}} 0$ for each $\sigma(X_n\colon n \in \mathbb{N})$-measurable and bounded Y.

In the following we write $E(S_n^* \leq t, Y)$ instead of $E(Y \cdot 1_{\{S_n^* \leq t\}})$.

THEOREM. *Let $r \geq 3$ and X_n, $n \in \mathbb{N}$, be i.i.d. random variables with mean 0, variance 1, and $E(|X_1|^r) < \infty$. Let Y be a bounded random variable and assume that*

$$d_1(Y, \sigma(X_1, \dots, X_n)) = O\big(n^{-1/2}(\log n)^{-\beta(r)}\big),$$

where $\beta(r) = r/2$ for $r > 3$ and $\beta(3) > \frac{3}{2}$. Then there exists a constant c such that for all $t \in \mathbb{R}$, $n \in \mathbb{N}$

$$|E(S_n^* \leq t, Y) - \Phi(t)E(Y)| \leq c(1 \wedge |t|^{-r})n^{-1/2}.$$

PROOF. According to Theorem 4 of [3] we may assume that $|t| \geq 1$. It is well known that there exists a $\sigma(X_1, \dots, X_k)$-measurable random variable Y_k with

$$E(|Y - Y_k|) = d_1(Y, \sigma(X_1, \dots, X_k)). \tag{2.1}$$

Put $\mathbb{N}_1 = \{2^k\colon k \in \mathbb{N}\}$ and $N_n = \{\nu \in \mathbb{N}_1\colon \nu \leq n/4\}$. Let $Z_2 = Y_2$ and $Z_\nu = Y_\nu - Y_{\nu/2}$ for $4 \leq \nu \in \mathbb{N}_1$. In the following c_i denote constants depending only on Y, r, and the distribution of X_1. Using (2.1) and our assumption we have

$$E(|Z_\nu|) \leq c_1 \nu^{-1/2}(\log \nu)^{-\beta(r)}, \quad \nu \in \mathbb{N}_1. \tag{2.2}$$

Let $j(n) = \max N_n$. Since $Y = Y - Y_{j(n)} + \sum_{\nu \in N_n} Z_\nu$, we have

$$\begin{aligned} |E(S_n^* \leq t, Y) - \Phi(t)E(Y)| \leq{}& |E(S_n^* \leq t, Y - Y_{j(n)}) - \Phi(t)E(Y - Y_{j(n)})| \\ &+ \sum_{\nu \in N_n} |E(S_n^* \leq t, Z_\nu) - \Phi(t)E(Z_\nu)|. \end{aligned}$$

Hence it suffices to prove

(2.3) $\left|E\left(S_n^* \le t, Y - Y_{j(n)}\right) - \Phi(t)E\left(Y - Y_{j(n)}\right)\right| \le c_2(1 \wedge |t|^{-r})n^{-1/2}$,

(2.4) $$\sum_{\nu \in N_n} |E(S_n^* \le t, Z_\nu) - \Phi(t)E(Z_\nu)| \le c_3(1 \wedge |t|^{-r})n^{-1/2}.$$

For the property (2.3) let $t < 0$. Then

$$\left|\Phi(t)E\left(Y - Y_{j(n)}\right)\right| \le c_4(1 \wedge |t|^{-r})E\left(\left|Y - Y_{j(n)}\right|\right) \le_{(2.1)} c_5(1 \wedge |t|^{-r})n^{-1/2}.$$

Hence (2.3) is shown, if we prove

(2.5) $$\left|E\left(S_n^* \le t, Y - Y_{j(n)}\right)\right| \le c_6(1 \wedge |t|^{-r})n^{-1/2}.$$

Let $-(2\log n)^{1/2} \le t < 0$. Then

$$\begin{aligned}\left|E\left(S_n^* \le t, Y - Y_{j(n)}\right)\right| &\le E\left(\left|Y - Y_{j(n)}\right|\right) \le_{(2.1)} c_6\left(1 \wedge (\log n)^{-\beta(r)}\right)n^{-1/2}\\ &\le c_7(1 \wedge |t|^{-r})n^{-1/2}.\end{aligned}$$

Let $t < -(2\log n)^{1/2}$; then $\Phi(t) \le c_8(1 \wedge |t|^{-r})n^{-1/2}$. Hence, by Petrov ((1975), Theorem 13, page 125),

$$\begin{aligned}\left|E\left(S_n^* \le t, Y - Y_{j(n)}\right)\right| &\le c_9 P(S_n^* \le t) \le c_9|P(S_n^* \le t) - \Phi(t)| + c_9\Phi(t)\\ &\le c_{10}(1 \wedge |t|^{-r})n^{-1/2} + c_9\Phi(t) \le c_{11}(1 \wedge |t|^{-r})n^{-1/2}.\end{aligned}$$

For property (2.4) let F_k be the distribution function of S_k^*, $\mathscr{A}_n := \sigma(X_1, \dots, X_n)$ and $t_{n,\nu}(\omega) = (tn^{1/2} - S_\nu(\omega))(n-\nu)^{-1/2}$. With $S_{n-\nu}^{\#} = \sum_{i=\nu+1}^{n} X_i$ we have $S_{n-\nu}^{\#} =_d S_{n-\nu}$, $S_{n-\nu}^{\#}$ is independent of $\mathscr{A}_\nu$, and hence,

$$\begin{aligned}E(S_n^* \le t, Z_\nu) &= E\left(E^{\mathscr{A}_\nu}1_{S_n^* \le t}Z_\nu\right)\\ &= E\left(Z_\nu P^{\mathscr{A}_\nu}(S_n^* \le t)\right) = E\left(Z_\nu P^{\mathscr{A}_\nu}(S_\nu + S_{n-\nu}^{\#} \le tn^{1/2})\right)\\ &= E\left(Z_\nu P^{\mathscr{A}_\nu}\left((n-\nu)^{-1/2}S_{n-\nu}^{\#} \le \left(tn^{1/2} - S_\nu\right)(n-\nu)^{-1/2}\right)\right)\\ &= E(Z_\nu F_{n-\nu}(t_{n,\nu})).\end{aligned}$$

Thus for $\nu \in N_n$,

$$\begin{aligned}E(S_n^* \le t, Z_\nu) - \Phi(t)E(Z_\nu) &= E(Z_\nu[F_{n-\nu}(t_{n,\nu}) - \Phi(t_{n,\nu})])\\ &\quad + E(Z_\nu[\Phi(t_{n,\nu}) - \Phi(t)]).\end{aligned}$$

Since furthermore (see Petrov (1975), Theorem 13, page 125)

$$|F_{n-\nu}(t_{n,\nu}) - \Phi(t_{n,\nu})| \le c_{12}\left(1 \wedge |t_{n,\nu}|^{-r}\right)n^{-1/2}, \qquad \nu \in N_n,$$

relation (2.4) is shown if we prove

(2.4a) $$a_n(t) = \sum_{\nu \in N_n} E\left(|Z_\nu|\left(1 \wedge |t_{n,\nu}|^{-r}\right)\right) \le c_{13}(1 \wedge |t|^{-r}),$$

(2.4b) $$\sum_{\nu \in N_n} E\left(|Z_\nu||\Phi(t_{n,\nu}) - \Phi(t)|\right) \le c_{14}(1 \wedge |t|^{-r})n^{-1/2}.$$

For property (2.4a) use

$$1 \wedge |t_{n,\nu}|^{-r} \le c_{15}\left(1 + |S_\nu n^{-1/2}|^r\right)(1 \wedge |t|^{-r});$$

hence, by (2.2),

$$\begin{aligned} a_n(t) &\le c_{16}(1 \wedge |t|^{-r}) \sum_{\nu \in N_n} \left[E(|Z_\nu|) + E\left(|S_\nu n^{-1/2}|^r\right)\right] \\ &\le c_{17}(1 \wedge |t|^{-r}) + c_{18}(1 \wedge |t|^{-r}) n^{-r/2} \sum_{\nu \in N_n} \nu^{r/2} \\ &\le c_{19}(1 \wedge |t|^{-r}). \end{aligned}$$

For property (2.4b) let $t > 0$ and put $\varphi(t) = \Phi'(t)$. As

$$\begin{aligned} &\sum_{\nu \in N_n} E\left(|Z_\nu|\left|\Phi\left(tn^{1/2}(n-\nu)^{-1/2}\right) - \Phi(t)\right|\right) \\ &\quad \le \sum_{\nu \in N_n} t\left(n^{1/2}(n-\nu)^{-1/2} - 1\right)\varphi(t) E(|Z_\nu|) \\ &\quad \le c_{20}(1 \wedge |t|^{-r}) \sum_{\nu \in N_n} E(|Z_\nu|)\nu/n \\ &\quad \le c_{21}(1 \wedge |t|^{-r}) n^{-1} \sum_{\nu \in N_n} \nu^{1/2}(\log \nu)^{-\beta(r)} \\ &\quad \le c_{22}(1 \wedge |t|^{-r}) n^{-1/2}, \end{aligned}$$

it suffices to prove

$$\sum_{\nu \in N_n} E\left(|Z_\nu|\left|\Phi(t_{n,\nu}) - \Phi\left(tn^{1/2}(n-\nu)^{-1/2}\right)\right|\right) \le c_{23}(1 \wedge |t|^{-r}) n^{-1/2}. \tag{2.6}$$

Let at first $t \ge ((r+2)\log n)^{1/2}$. Then $\varphi(t/2) \le c_{24}(1 \wedge |t|^{-r}) n^{-1/2}$. Hence,

$$\begin{aligned} &\sum_{\nu \in N_n} E\left(|S_\nu| \le \tfrac{1}{2}tn^{1/2}, |Z_\nu|\left|\Phi(t_{n,\nu}) - \Phi\left(tn^{1/2}(n-\nu)^{-1/2}\right)\right|\right) \\ &\quad \le c_{25} \sum_{\nu \in N_n} n^{-1/2}\varphi(t/2) E(|S_\nu|) \\ &\quad \le c_{26}(1 \wedge |t|^{-r}) n^{-1} \sum_{\nu \in N_n} E(|S_\nu|) \\ &\quad \le c_{27}(1 \wedge |t|^{-r}) n^{-1/2}. \end{aligned} \tag{2.7}$$

Furthermore (use, e.g., Theorem 2 of Michel (1976) for (+))

$$\begin{aligned} &\sum_{\nu \in N_n} E\left(|S_\nu| > \tfrac{1}{2}tn^{1/2}, |Z_\nu|\left|\Phi(t_{n,\nu}) - \Phi\left(tn^{1/2}(n-\nu)^{-1/2}\right)\right|\right) \\ &\quad \le c_{28} \sum_{\nu \in N_n} P\{|S_\nu^*| > tn^{1/2}/(2\nu^{1/2})\} \\ &\quad \underset{(+)}{\le} c_{29} \sum_{\nu \in N_n} \left(t(n/\nu)^{1/2}\right)^{-r} \nu^{-(r-2)/2} \\ &\quad \le c_{30}(1 \wedge |t|^{-r}) n^{-(r-2)/2}. \end{aligned}$$

Together with (2.7) this yields (2.6) for $t \geq ((r+2)\log n)^{1/2}$. Let finally $1 < t < ((r+2)\log n)^{1/2}$. Put $a = r^{1/2}[E(|X_1|^3)]^{1/3}$ and $A_\nu = \{|S_\nu| > a(\nu \log \nu)^{1/2}\}$. Then we have (as in the proof of formula 15, page 233 of [2]) that

$$\sum_{\nu \in \mathbb{N}_1} E\left(|S_\nu| 1_{A_\nu}\right) < \infty. \tag{2.8}$$

Let

$$M_n = M_n(t) = \left\{\nu \in N_n: a(\nu \log \nu)^{1/2} \leq t n^{1/2}/2\right\}.$$

Then

$$N_n - M_n(t) \subset \left\{\nu \in N_n: \nu > nt^2/\left(4a^2 \log n\right)\right\}$$

and hence

$$\begin{aligned}
&\sum_{\nu \in N_n} E\left(|Z_\nu|\left|\Phi(t_{n,\nu}) - \Phi\left(tn^{1/2}(n-\nu)^{-1/2}\right)\right| 1_{\bar{A}_\nu}\right)\\
&\quad \leq \sum_{\nu \in M_n} \Big[\Phi\left(tn^{1/2}(n-\nu)^{-1/2}\right)\\
&\qquad\qquad - \Phi\left(tn^{1/2}(n-\nu)^{-1/2} - a(\nu \log \nu)^{1/2}(n-\nu)^{-1/2}\right)\Big] E\left(|Z_\nu|\right)\\
&\qquad + \sum_{\nu \in N_n - M_n} E\left(|Z_\nu|\right)\\
&\quad \leq a\sqrt{2}\, n^{-1/2} \varphi(t/2) \sum_{\nu \in M_n} (\nu \log \nu)^{1/2} E\left(|Z_\nu|\right)\\
&\qquad + \sum_{N_1 \ni \nu > nt^2(4a^2 \log n)^{-1}} \nu^{-1/2} (\log \nu)^{-\beta(r)}\\
&\quad \leq c_{31}(1 \wedge |t|^{-r}) n^{-1/2} \sum_{\nu \in \mathbb{N}_1} (\log \nu)^{-\beta(r)+1/2} + c_{32}(\log n/n)^{1/2} t^{-1} (\log n)^{-\beta(r)}\\
&\quad \leq c_{33}(1 \wedge |t|^{-r}) n^{-1/2},
\end{aligned}$$

where the last inequality follows from $-\beta(r) + \frac{1}{2} < -1$ and $1 \leq t < ((r+2)\log n)^{1/2}$. Consequently the proof of (2.6)—and hence the assertion—is shown if we prove

$$\begin{aligned}
b_n(t) &= \sum_{\nu \in N_n} E\left(|Z_\nu| 1_{A_\nu}\left|\Phi(t_{n,\nu}) - \Phi\left(tn^{1/2}(n-\nu)^{-1/2}\right)\right|\right)\\
&\leq c_{34}(1 \wedge |t|^{-r}) n^{-1/2}.
\end{aligned} \tag{2.9}$$

Let

$$M_n = M_n(t) = \left\{\nu \in N_n: tn^{1/2} > 2\left(\nu(r-1)\log \nu\right)^{1/2}\right\}.$$

Then

$$N_n - M_n(t) \subset \left\{\nu \in N_n: \nu \geq t^2 n\left(4(r-1)\log n\right)^{-1}\right\}$$

and we have

$$
\begin{aligned}
b_n(t) \le & \sum_{\nu \in N_n} E\Big(|Z_\nu| 1_{A_\nu} 1_{\{|S_\nu| \le (1/2) t n^{1/2}\}} \big|\Phi(t_{n,\nu}) - \Phi\big(t n^{1/2}(n-\nu)^{-1/2}\big)\big|\Big) \\
& + \sum_{\nu \in M_n} E\Big(|Z_\nu| 1_{\{|S_\nu| > (1/2) t n^{1/2}\}} \big|\Phi(t_{n,\nu}) - \Phi\big(t n^{1/2}(n-\nu)^{-1/2}\big)\big|\Big) \\
& + \sum_{\nu \in N_n - M_n} E\Big(|Z_\nu| 1_{A_\nu} \big|\Phi(t_{n,\nu}) - \Phi\big(t n^{1/2}(n-\nu)^{-1/2}\big)\big|\Big) \\
\le & \; c_{35} \sum_{\nu \in N_n} n^{-1/2} \varphi(t/2) E\big(|S_\nu| 1_{A_\nu}\big) + c_{36} \sum_{\nu \in M_n} P\{|S_\nu^*| > (1/2) t n^{1/2} \nu^{-1/2}\} \\
& + \sum_{\nu \in N_n - M_n} E(|Z_\nu|).
\end{aligned}
$$

Using (2.8) and Theorem 2 of Michel (1976) we obtain

$$
\begin{aligned}
b_n(t) \le & \; c_{37}(1 \wedge |t|^{-r}) n^{-1/2} + c_{38} \sum_{\nu \in M_n} \nu^{r/2} t^{-r} n^{-r/2} \nu^{-(r-2)/2} \\
& + \sum_{N_n \ni \nu \ge t^2 n (4(r-1)\log n)^{-1}} \nu^{-1/2} (\log \nu)^{-\beta(r)} \\
\le & \; c_{37}(1 \wedge |t|^{-r}) n^{-1/2} + c_{39}(1 \wedge |t|^{-r}) n^{-r/2} \sum_{\nu \in M_n} \nu \\
& + c_{40} (\log n)^{1/2} t^{-1} n^{-1/2} (\log n)^{-\beta(r)} \\
\le & \; c_{41}(1 \wedge |t|^{-r}) n^{-1/2}.
\end{aligned}
$$

This proves (2.9) and hence the proof is finished. □

Example 5 of [2] shows that for $r = 3$ we cannot obtain the assertion of the preceding theorem any more if we replace in the assumption $\beta(3) > \frac{3}{2}$ by $\beta(3) = \frac{3}{2}$. The following example shows that for $r > 3$ we cannot weaken the assumption from $\beta(r) = r/2$ to $\beta(r) < r/2$. The examples work with indicator functions $g = 1_B$.

3. Example. Let X_n, $n \in \mathbb{N}$, be i.i.d. standard normally distributed random variables. Let $r > 3$ and $\beta < r/2$. Put $\hat{\mathbb{N}} = \{2^{2^k}: k \in \mathbb{N}\}$. Using Lemma 3 of [2] and the theorem of Liapounov for nonatomic measures it is easy to see that there exist constants $c_1, c_2 > 0$ and disjoint sets B_ν, $\nu \in \hat{\mathbb{N}}$, with

$$B_\nu \in \sigma(X_1, \dots, X_\nu), \tag{3.1}$$

$$B_\nu \subset \{S_\nu^* \le -c_1 (\log \nu)^{1/2}\}, \tag{3.2}$$

$$P(B_\nu) = c_2 \nu^{-1/2} (\log \nu)^{-\beta}. \tag{3.3}$$

Put $B = \sum_{\nu \in \hat{\mathbb{N}}} B_\nu$. By (3.1) and (3.3) we have

$$d(B, \sigma(X_1, \dots, X_n)) = O\big(n^{-1/2} (\log n)^{-\beta}\big). \tag{3.4}$$

Furthermore, we have for all $n \in \hat{\mathbb{N}}$ and $t_n = -c_1(\log n)^{1/2}$,

$$\begin{aligned} P(S_n^* \le t_n, B) - \Phi(t_n)P(B) &\\ \ge \sum_{\hat{\mathbb{N}} \ni \nu < n} \left(P(S_n^* \le t_n, B_\nu) - \Phi(t_n)P(B_\nu)\right) &\\ + P(S_n^* \le t_n, B_n) - \Phi(t_n)P(B_n) - \sum_{\hat{\mathbb{N}} \ni \nu > n} P(B_\nu). \end{aligned}$$

Since

$$P(S_n^* \le t_n, B_\nu) - \Phi(t_n)P(B_\nu) \ge 0 \quad \text{for } \nu < n,\ \nu \in \hat{\mathbb{N}},$$

$$P(S_n^* \le t_n, B_n) - \Phi(t_n)P(B_n) = (1 - \Phi(t_n))P(B_n) \ge \tfrac{1}{2}P(B_n), \qquad n \in \hat{\mathbb{N}},$$

and

$$\sum_{\hat{\mathbb{N}} \ni \nu > n} P(B_\nu) = o(P(B_n)), \qquad n \in \hat{\mathbb{N}},$$

we obtain for all sufficiently large $n \in \hat{\mathbb{N}}$,

$$\begin{aligned} P(S_n^* \le t_n, B) - \Phi(t_n)P(B) \ge \tfrac{1}{4}P(B_n) &= \tfrac{1}{4}c_2 n^{-1/2}(\log n)^{-\beta} \\ &\ge c_3\left(1 \wedge |t_n|^{-r}\right)n^{-1/2}|t_n|^{r-2\beta}. \end{aligned}$$

Since $r - 2\beta > 0$, $|t_n| \to \infty$, relation (3.4) with $\beta < r/2$ does not imply the assertion of our theorem.

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