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ABSENCE OF A STATIONARY DISTRIBUTION FOR THE EDGE PROCESS OF SUBCRITICAL ORIENTED PERCOLATION IN TWO DIMENSIONS

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We prove that the edge process of oriented percolation in two dimensions does not have a stationary distribution in the subcritical case.

In this note we answer a question raised by Durrett (1984). We prove that the edge process of oriented percolation in two dimensions does not have a stationary distribution in the subcritical case. We suppose that the reader is familiar with at least Sections 2, 3, 4, 7, and 8 of Durrett (1984) and also adopt the notation used there.

First we recall briefly the definition of the model and the basic terminology. Let

$$\mathscr{L} = \{(m,n) \in \mathbb{Z}^2 \colon m+n \text{ is even}, n \geq 0\}$$

and draw an oriented arc from each $(m,n) \in \mathscr{L}$ to $(m+1, n+1)$ and to $(m-1, n+1)$. Each arc is independently open with probability p and closed with probability $(1-p)$. Write $x \to y$ if there is a sequence $x_0 = x, x_1, \dots, x_m = y$ of points in $\mathscr{L}$ such that for each $k \leq m$, the arc from x_{k-1} to x_k is open. Now for $\eta \subset 2\mathbb{Z}$ define the random configuration

$$\xi_n^\eta = \{m \colon (m,n) \in \mathscr{L} \text{ and there is } k \in \eta \text{ s.t. } (k,0) \to (m,n)\}.$$

If μ is a probability distribution on the subsets of $2\mathbb{Z}$, write ξ_n^μ for ξ_n^η if η is chosen at random according to the distribution μ independently of the percolation structure. Let

$$r_n^\mu = \sup \xi_n^\mu, \qquad \sup \varnothing = -\infty.$$

If $\eta \subset \mathbb{Z}$ and $x \in \mathbb{Z}$ we use the notation $\eta - x = \{z \in \mathbb{Z} \colon z + x \in \eta\}$. Now on the event $\{|r_n^\mu| < \infty\}$ define

$$\tilde{\xi}_n^\mu = \xi_n^\mu - r_n^\mu.$$

If μ has support on $\tilde{E} = \{\eta \subset 2\mathbb{Z} \colon |\eta| = \infty, \sup \eta = 0\}$, where $|\eta|$ is the cardinality of η, then $|r_n^\mu|$ is almost surely finite for all n and $(\tilde{\xi}_n^\mu, n = 0, 1, \dots)$ is the process $(\xi_n^\mu, n = 0, 1, \dots)$ as viewed from the right edge.

Finally remember that the critical value

$$p_c = \inf\{p \in [0,1] \colon P(\xi_n^0 \neq \varnothing \ \forall\, n \geq 0) > 0\}$$

belongs to $(0,1)$.

Durrett (1984) proved that if $p \geq p_c$, there is a distribution μ concentrated on $\tilde{E}$ such that $(\tilde{\xi}_n^\mu, n = 0, 1, \dots)$ is a stationary process. The uniqueness of such a distribution was proven by Galves and Presutti (1987) for $p > p_c$. Here we prove

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Theorem 1. *If $p < p_c$, there is no distribution μ concentrated on the subsets of $\{\dots, -4, -2, 0\}$ such that $(\tilde{\xi}_n^\mu, n = 0, 1, 2, \dots)$ is a stationary process.*

Proof. We will suppose that such a distribution exists and show that this gives rise to a contradiction. μ has to be concentrated on the infinite subsets of $\{\dots, -4, -2, 0\}$ for $(\tilde{\xi}_n^\mu)$ to be well defined, but we will prove below that for any positive even integer d, $\mu(\eta\colon -d \in \eta) \le Ce^{-\gamma d}$, where C and γ are positive constants. Then, by a Borel–Cantelli argument, μ is concentrated on finite configurations, which is absurd. *Warning*: We adopt the convention that C and γ are positive numbers but may change from equality to equality.

Given $d \in 2\mathbb{Z} \cap [0, \infty)$, take $n = \sup\{z \in 2\mathbb{Z}\colon z \le d/3\}$. If $(\tilde{\xi}_n^\mu)$ were stationary, then

$$\mu(\eta\colon -d \in \eta) = P(-d \in \tilde{\xi}_n^\mu) = \sum_{r=-\infty}^{+\infty} P(\sup \xi_n^\mu = r,\ r - d \in \xi_n^\mu)$$

$$\le \sum_{r=-\infty}^{+\infty} P(\sup \xi_n^\mu = r,\ r - d \in \xi_n^{2\mathbf{Z}}).$$

But the events $[r - d \in \xi_n^{2\mathbf{Z}}]$ and $[\sup \xi_n^\mu = r]$ are independent, since the former depends only on the percolation structure in the region $[r - d - n, r - d + n] \times [0, n]$ and the latter depends on the percolation structure in $[r - n, r + n] \times [0, n]$ and on the initial distribution μ. Therefore, by translation invariance,

$$\mu(\eta\colon -d \in \eta) \le \sum_{r=-\infty}^{+\infty} P(\sup \xi_n^\mu = r) P(r - d \in \xi_n^{2\mathbf{Z}})$$

$$= P(0 \in \xi_n^{2\mathbf{Z}}) \sum_{r=-\infty}^{+\infty} P(\sup \xi_n^\mu = r)$$

$$= P(0 \in \xi_n^{2\mathbf{Z}}).$$

Finally by self-duality [(2) in Section 8 of Durrett (1984)] and the exponential estimate (1) in Section 7 of the same article,

$$P(0 \in \xi_n^{2\mathbf{Z}}) = P(\xi_n^0 \ne \varnothing) \le Ce^{-\gamma n} \le Ce^{-\gamma d} \qquad \square$$

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