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STOPPING TIMES OF BESSEL PROCESSES[1]

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Let X_α^x be a Bessel process with parameter α, starting at $x \geq 0$. Gordon [3] obtained L^p inequalities which relate stopping times to stopping places for the case $\alpha = 1$, $x = 0$ and $p > \frac{1}{2}$. Rosenkrantz and Sawyer [5] extended them to $\alpha > 0$, $x = 0$ and $p \geq 1$. Burkholder [1] obtained results for α a positive integer, $x \geq 0$ and $p > 0$. Here we consider arbitrary starting points x, $\alpha > 0$ and $p > 0$. The L^p inequalities are valid for $\alpha \geq 2$ with $p > 0$, and also for $0 < \alpha < 2$ with $p > (2 - \alpha)/2$. Examples are constructed to show that for $0 < \alpha < 2$ with $p \leq (2 - \alpha)/2$, the L^p inequalities cannot hold.

0. Introduction. Let X_α^x be the Bessel process with index $\alpha > 0$, where $x \geq 0$ and $X_\alpha^x(0) = x$; i.e., $X_\alpha^x(\cdot)$ is that diffusion governed by the differential operator L_α on $[0, \infty)$ defined by

$$L_\alpha f = \frac{1}{2}\left[f''(x) + \frac{\alpha - 1}{x} f'(x)\right],$$

with domain

$$\mathscr{D}(L_\alpha) = \{f \in C_b^2([0,\infty)) \colon \text{for some } 0 < a_1 < a_2, \; f(x) = f(0) \text{ for } x \in [0, a_1] \text{ and } f(x) = 0 \text{ if } x \geq a_2\}$$

(see Ikeda and Watanabe [4], Example 8.3, pages 223–225).

In Gordon [3], it was shown for $\alpha = 1$ and starting point 0,

$$c_p E\tau^p \leq EX_1^0(\tau)^{2p} \leq C_p E\tau^p, \tag{0.1}$$

for any stopping time τ of $X_1^0(\cdot)$ with $E\tau^p < \infty$, $p > \frac{1}{2}$. He also pointed out that the right-hand inequality is true for *any* stopping time τ of $X_1^0(\cdot)$ and $p > 0$. Nothing was said about $p \leq \frac{1}{2}$ for the left-hand inequality and starting points other than 0 were not considered. Burkholder [1] allowed other starting points and showed that for $\alpha = 1, 2, 3 \ldots$

$$c_{p,n} E(\tau + |x|^2)^p \leq E[X_\alpha^x(\tau)^*]^{2p} \leq C_{p,n} E(\tau + |x|^2)^p, \tag{0.2}$$

for any stopping time τ of X_α^x and $p > 0$ where

$$X_\alpha^x(\tau)^* = \sup_{0 \leq t < \infty} X_\alpha^x(t \wedge \tau).$$

Next, Rosenkrantz and Sawyer [5] obtained (0.1) for general $\alpha > 0$, provided τ is bounded and $p \geq 1$. They did not consider other starting points or $0 < p < 1$. It is the purpose of this paper to discuss these results for all starting points x, powers $p > 0$, and indices $\alpha > 0$.

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1. Main results.

THEOREM 1.1. *There are positive constants* $c_{p,\alpha}$ *and* $C_{p,\alpha}$ *depending only on* α *and* p *such that*:

(i) *For* $\alpha > 0$, $p > 0$, *any stopping time* τ *of* $X_\alpha^x(\cdot)$,

$$c_{p,\alpha}E[\tau + x^2]^p \le E[X_\alpha^x(\tau)^*]^{2p} \le C_{p,\alpha}E[\tau + x^2]^p. \tag{1.1}$$

(ii) *For* $\alpha \ge 2$ *and* $p > 0$,

$$c_{p,\alpha}E[\tau + x^2]^p \le E[X_\alpha^x(\tau)]^{2p}, \tag{1.2}$$

provided either $P(\tau < \infty) = 1$ *for* $\alpha > 2$ *or* $E \log \tau < \infty$ *for* $\alpha = 2$.

(iii) *For* $0 < \alpha < 2$, $p > (2 - \alpha)/2$ *and* $E\tau^p < \infty$,

$$c_{p,\alpha}E[\tau + x^2]^p \le E[X_\alpha^x(\tau)]^{2p}. \tag{1.3}$$

It still remains to consider the case when $0 < \alpha < 2$ and $p \le (2 - \alpha)/2$. The next result shows (1.3) [or (1.2)] cannot hold for these values of p and α.

THEOREM 1.2. *Let* $0 < \alpha < 2$ *and* $x \ge 0$.

(i) *There is a stopping time* τ *of* $X_\alpha^x(\cdot)$ *with* $0 < E\tau^p < \infty$ *for* $p < (2 - \alpha)/2$, $E\tau^p = \infty$ *for*

$$p \ge \frac{2 - \alpha}{2}, \quad \textit{and} \quad E[X_\alpha^x(\tau)]^{2p} = 0.$$

(ii) *There is a sequence* τ_n *of stopping times of* $X_\alpha^x(\cdot)$ *with* $E[\tau_n]^{(2-\alpha)/2} < \infty$ *and*

$$\frac{E[\tau_n]^{(2-\alpha)/2}}{E[X_\alpha^x(\tau_n)]^{2-\alpha}} \to \infty \quad \textit{as } n \to \infty.$$

REMARK. Note that (i) in Theorem 1.2 also shows the condition $E\tau^p < \infty$ in Theorem 1.1(iii) cannot be dropped. Also, (i) is well known for the case $\alpha = 1$.

2. Proofs of the main results. We use the martingale generating function approach of Gordon [3] and Rosenkrantz and Sawyer [5]. Our method differs in that the key to it all is the following representation of $X_\alpha^x(\cdot)$ (Ikeda and Watanabe [4], pages 223–225) which enables us to handle the cases left open by these authors: Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. Suppose $\{\mathscr{F}_t: t \ge 0\}$ is an increasing family of complete σ subalgebras of $\mathscr{F}$. Let $B(t)$ be a one dimensional $\{\mathscr{F}_t\}$ Brownian motion. Then we may represent

$$X_\alpha^x(t) = [Y_\alpha^{x^2}(t)]^{1/2}, \tag{2.1}$$

where $Y_\alpha^{x^2}$ is the unique (nonnegative) solution of

$$\begin{aligned} dY_t &= 2[Y_t \vee 0]^{1/2}\, dB_t + \alpha\, dt, \\ Y_0 &= x^2. \end{aligned} \tag{2.2}$$

Notice that the coefficients $\tilde{\sigma}(\omega, t) := 2[Y_t(\omega) \vee 0]^{1/2}$ and $\tilde{b}(\omega, t) := \alpha$ satisfy

$$\sup_{t,\,\omega} |\tilde{b}(\omega, t)| \vee |[\tilde{\sigma}(\omega, t)]^2 / Y_t(\omega)| \le \alpha \vee 4 < \infty.$$

Hence by Remark 1.3(ii) of DeBlassie [2], for $p > 0$ there are positive constants $C_{p,\alpha}$ and $c_{p,\alpha}$ such that for any stopping time τ of $X_\alpha^x(t)$,

$$c_{p,\alpha} E[\tau + x^2]^p \le E[X_\alpha^x(\tau)^*]^{2p} \le C_{p,\alpha} E[\tau + x^2]^p. \tag{2.3}$$

LEMMA 2.1. *Let $\alpha \ge 2$, $x \ge 0$ and τ be a stopping time of $X_\alpha^x(\cdot)$ satisfying $P(\tau < \infty) = 1$ for $\alpha > 2$ or $E \log \tau < \infty$ if $\alpha = 2$. Then for any $p > 0$*

$$E[X_\alpha^x(\tau)^*]^p \le C_{p,\alpha} E[X_\alpha^x(\tau)]^{2p},$$

where $C_{p,\alpha}$ is independent of τ and x.

PROOF. By Theorem 2.2 in Burkholder [1], page 189, the case $\alpha = 2$ is true. By the proof of that Theorem 2.2, for the case $\alpha > 2$ it suffices to show

$$P\big(|X_\alpha^x(t)| \le r \text{ for some } t \ge 0\big) = (r/x)^{\alpha - 2}, \qquad 0 < r < x. \tag{2.4}$$

But this is immediate since both sides solve the problem

$$L_\alpha f(x) = 0 \quad \text{for } x > r,$$
$$f(r) = 1,$$
$$f(\infty) = 0. \qquad \square$$

PROOF OF THEOREM 1.1. Part (i) follows immediately from (2.3). Part (ii) follows from (2.3) and Lemma 2.1. For part (iii), let $0 < \alpha < 2$, $p > (2 - \alpha)/2$ and τ be a stopping time of $X_\alpha^x(\cdot)$ with $E\tau^p < \infty$. Choose

$$q > 1 \vee (2p/[2(p-1) + \alpha]) \tag{2.5}$$

and note if $p > 1$, q can also satisfy

$$q < \frac{p}{p-1}. \tag{2.6}$$

Let q' be conjugate to q,

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Then

$$\frac{p}{q'} < 1. \tag{2.7}$$

For $\varepsilon > 0$ and $x \ge 0$, pick $u_\varepsilon(t, y) \in C^2(\mathbf{R} \times \mathbf{R})$ with

$$u_\varepsilon(t, y) = (t + \varepsilon + x^2)^{p/q}(y + \varepsilon)^{p/q'}, \qquad (t, y) \in [0, \infty) \times [0, \infty).$$

Notice on $[0, \infty) \times [0, \infty)$, by (2.7)

$$
\begin{aligned}
&2y\frac{\partial^2 u_\varepsilon}{\partial y^2} + \alpha\frac{\partial u_\varepsilon}{\partial y} + \frac{\partial u_\varepsilon}{\partial t}\\
&\quad \geq 2\frac{p}{q'}\left(\frac{p}{q'} - 1\right)(y+\varepsilon)^{p/q'-1}(t+\varepsilon+x^2)^{p/q}\\
&\qquad + \alpha\frac{p}{q'}(y+\varepsilon)^{p/q'-1}(t+\varepsilon+x^2)^{p/q} + \frac{p}{q}(t+\varepsilon+x^2)^{p/q-1}(y+\varepsilon)^{p/q'}\\
&\quad = (t+\varepsilon+x^2)^{p-1}\left\{\frac{p}{q'}\left(\frac{y+\varepsilon}{t+\varepsilon+x^2}\right)^{p/q'-1}\left[2\left(\frac{p}{q'}-1\right)+\alpha\right]\right.\\
&\qquad\qquad \left. + \frac{p}{q}\left(\frac{y+\varepsilon}{t+\varepsilon+x^2}\right)^{p/q'}\right\}.
\end{aligned}
$$

Since $p/q' - 1 < 0$ and $2(p/q' - 1) + \alpha > 0$ [by (2.5) and $p > (2-\alpha)/2$],

$$
\begin{aligned}
&\frac{p}{q'}s^{p/q'-1}\left(2\left[\frac{p}{q'}-1\right]+\alpha\right) + \frac{p}{q}s^{p/q'} \geq \inf_{s\leq 1}('') \wedge \inf_{s>1}('')\\
&\qquad \geq \left[\frac{p}{q'}\left(2\left[\frac{p}{q'}-1\right]+\alpha\right)\right] \wedge \frac{p}{q}\\
&\qquad =: C_1 > 0.
\end{aligned} \tag{2.8}
$$

Thus on $[0, \infty) \times [0, \infty)$

$$
2y\frac{\partial^2 u_\varepsilon}{\partial y^2} + \alpha\frac{\partial u_\varepsilon}{\partial y} + \frac{\partial u_\varepsilon}{\partial t} \geq C_1(t+\varepsilon+x^2)^{p-1}, \tag{2.9}
$$

where C_1 is independent of $\varepsilon > 0$.

Hence by Itô's formula and optional stopping,

$$
\begin{aligned}
&E[t\wedge\tau+\varepsilon+x^2]^{p/q}\left[Y_\alpha^{x^2}(t\wedge\tau)+\varepsilon\right]^{p/q'}\\
&\quad = Eu_\varepsilon\left(t\wedge\tau, Y_\alpha^{x^2}(t\wedge\tau)\right)\\
&\quad \geq (\varepsilon+x^2)^p + E\int_0^{t\wedge\tau} C_1(s+\varepsilon+x^2)^{p-1}\,ds\\
&\quad = \frac{C_1}{p}E\left[(\tau\wedge t+\varepsilon+x^2)^p - (x^2+\varepsilon)^p\right] + (\varepsilon+x^2)^p\\
&\quad \geq \frac{C_1}{p}E(\tau\wedge t+\varepsilon+x^2)^p \quad \left[\text{since } C_1 \leq \frac{p}{q} < p \text{ by (2.8)}\right],
\end{aligned}
$$

and using Hölder's inequality we end up with

$$
\frac{C_1}{p}E(\tau\wedge t+\varepsilon+x^2)^p \leq \left\{E[\tau\wedge t+\varepsilon+x^2]^p\right\}^{1/q}\left\{E\left[Y_\alpha^{x^2}(t\wedge\tau)+\varepsilon\right]^p\right\}^{1/q'}.
$$

Letting $\varepsilon \to 0$,

$$
E(\tau\wedge t+|x|^2)^p \leq C_{\alpha,p}E\left[Y_\alpha^{x^2}(t\wedge\tau)\right]^p.
$$

Since $E\tau^p < \infty$, (2.3) gives $E[Y_\alpha^{x^2}(\tau)^*]^p < \infty$. So by dominated convergence and the fact that $\tau \wedge t \uparrow \tau$ as $t \to \infty$,

$$E(\tau + x^2)^p \le C_{\alpha,p} E\left[Y_\alpha^{x^2}(\tau)\right]^p = C_{\alpha,p} E\left[X_\alpha^x(\tau)\right]^{2p},$$

as desired. □

PROOF OF THEOREM 1.2(i). Let $0 < \alpha < 2$ and consider any $x > 0$. Define $\tau_x := \inf\{t > 0: X_\alpha^x(t) = 0\}$. Below we show that for some finite positive $C_{p,\alpha}$

$$E\tau_x^p = \begin{cases} C_{p,\alpha}\Gamma\left(1 - p - \dfrac{\alpha}{2}\right)x^{2p}, & \text{for } 0 < p < \dfrac{2-\alpha}{2}, \\ \infty, & \text{for } p \ge \dfrac{2-\alpha}{2}. \end{cases} \tag{2.10}$$

Then we have $0 < E\tau_x^p < \infty$ for $p < (2-\alpha)/2$, $E\tau_x^p = \infty$ for $p \ge (2-\alpha)/2$ and $E[X_\alpha^x(\tau_x)]^{2p} = 0$ as desired.

If $x = 0$, let

$$\sigma = \inf\{t > 0: X_\alpha^0(t) = 1\},$$
$$\tau = \inf\{t > \sigma: X_\alpha^0(t) = 0\}.$$

Then by the strong Markov property and the case $x > 0$ (above), we have for $p < (2-\alpha)/2$

$$E\tau^p > E(\tau - \sigma)^p = E(\tau_1)^p > 0,$$

and

$$E(\tau - \sigma)^p = E\tau_1^p < \infty.$$

Since $E[X_\alpha^0(\sigma)^*]^{2p} = 1$, (2.3) gives that $E\sigma^p < \infty$. Thus

$$\begin{aligned} E\tau^p &= E[(\tau - \sigma) + \sigma]^p \\ &\le C_p E(\tau - \sigma)^p + C_p E\sigma^p \\ &< \infty. \end{aligned}$$

By (2.10), for $p \ge (2-\alpha)/2$, $E\tau^p > E\tau_1^p = \infty$. Finally since $EX_\alpha^0(\tau)^{2p} = 0$, this case is complete. □

PROOF THAT $E\tau_x^p = C_{p,\alpha}\Gamma(1 - p - \alpha/2)x^{2p}$ FOR $x > 0$. Define

$$\beta^{-1} = \int_0^\infty u^{(\alpha-4)/2} e^{-1/2u}\, du < \infty \quad (\text{since } 0 < \alpha < 2)$$

and

$$u(t, y) = 1 - \beta \int_0^{t/y} u^{(\alpha-4)/2} e^{-1/2u}\, du.$$

Then

$$\begin{aligned} &2yu_{yy} + \alpha u_y - u_t = 0 \qquad \text{for } y \text{ and } t > 0, \\ &u \in C^\infty((0,\infty) \times (0,\infty)). \end{aligned} \tag{2.11}$$

Noting that also $\tau_x = \inf\{t > 0\colon Y_\alpha^{x^2}(t) = 0\}$, we see that $u(t, x^2)$ should be $P(\tau_x > t)$. However, the differential equation in (2.11) is degenerate and the boundary data is not continuous, so the equality $u(t, x^2) = P(\tau_x > t)$ is not quite trivial.

Let $T > \varepsilon > 0$. Let $w(t, y) \in C^\infty(\mathbf{R} \times \mathbf{R})$ such that

$$w(t, y) = u(T - t - \varepsilon, y + \varepsilon) \quad \text{for } \frac{-\varepsilon}{2} < y \quad \text{and} \quad -\varepsilon < t \le T - 2\varepsilon.$$

See that for $y > -\varepsilon/2$ and $-\varepsilon < t \le T - 2\varepsilon$

$$2yw_{yy} + \alpha w_y + u_t = 0.$$

Then by Itô's formula, optional stopping and (2.11),

$$Ew\big((T - 2\varepsilon) \wedge \tau_x, Y_\alpha^{x^2}((T - 2\varepsilon) \wedge \tau_x)\big) - w(0, x^2) = 0,$$

i.e.,

$$\begin{aligned} 0 &= Ew\big(\tau_x, Y_\alpha^{x^2}(\tau_x)\big)I(\tau_x \le T - 2\varepsilon) \\ &\quad + Ew\big(T - 2\varepsilon, Y_\alpha^{x^2}(T - 2\varepsilon)\big)I(\tau_x > T - 2\varepsilon) - w(0, x^2) \\ &= Eu(T - \tau_x - \varepsilon, \varepsilon)I(\tau_x \le T - 2\varepsilon) \\ &\quad + Eu\big(\varepsilon, Y_\alpha^{x^2}(T - 2\varepsilon) + \varepsilon\big)I(\tau_x > T - 2\varepsilon) - u(T - \varepsilon, x^2 + \varepsilon) \\ &= \text{①} + \text{②} - \text{③}, \quad \text{say}. \end{aligned}$$

Note u is bounded, so ① $\to 0$ as $\varepsilon \to 0$. Since u is continuous on $(0, \infty) \times (0, \infty)$, ③ $\to u(T, x^2)$ as $\varepsilon \to 0$. Now

$$\begin{aligned} \text{②} &= Eu\big(\varepsilon, Y_\alpha^{x^2}(T - 2\varepsilon) + \varepsilon\big)I(\tau_x > T - 2\varepsilon)I\big(Y_\alpha^{x^2}(T) \ne 0\big) \\ &\qquad \Big[\text{since } P\big(Y_\alpha^{x^2}(T) = 0\big) = 0 \text{ and } u \text{ is bounded}\Big] \\ &\to P(\tau_x \ge T) \quad \text{as } \varepsilon \to 0 \end{aligned}$$

since $u(\varepsilon, Y_\alpha^{x^2}(T - 2\varepsilon) + \varepsilon) \to 1$ on $\{Y_\alpha^{x^2}(T) \ne 0\}$. Thus

$$u(T, x^2) = P(\tau_x \ge T).$$

An easy calculation shows

$$E(\tau_x)^p = 2^{(4-\alpha)/2-p-1}\beta x^{2p}\Gamma\left(1 - \frac{\alpha}{2} - p\right). \qquad \square$$

Proof of Theorem 1.2(ii). Let $0 < \alpha < 2$ and $x > 0$. For convenience write $p = (2 - \alpha)/2$ and define

$$\sigma_{n,x} = \inf\{t > 0\colon X_\alpha^x(t) \le n^{-1}\}, \qquad n \ge 1. \tag{2.12}$$

Then as $n \uparrow \infty$, $E(\sigma_{n,x})^p \uparrow E\tau_x^p$ where τ_x is as in the proof of Theorem 1.2(i). But by (2.10), $E\tau_x^p = \infty$, so as $n \uparrow \infty$,

$$E(\sigma_{n,x})^p \uparrow \infty.$$

Thus for each fixed x and M, there is an $n = N(x, M)$ which is greater than $1/x$ and such that for each $n \geq N$ we have

$$E[\sigma_{n,x}]^p > (M+1)|x|^{2p}. \tag{2.13}$$

Then choose $t_n := t_n(M, x) > 0$ such that

$$E[t_n \wedge \sigma_{n,x}]^p > M|x|^{2p}, \qquad n \geq N(x, M). \tag{2.14}$$

Suppose $u_n \in C^\infty(\mathbf{R})$ satisfies $u_n \geq 0$, $u_n(y) = y^p$ for $y > 1/2n^2$ and $u_n(y) \leq (2n^2)^{-p}$ for $y \leq 1/2n^2$. Then by Itô's formula and optional stopping, for $n \geq N(x, M)$

$$Eu_n\big(Y_\alpha^{x^2}(t_n \wedge \sigma_{n,x})\big) - u_n(x^2) = 0,$$

since $2yu_n''(y) + \alpha u_n'(y) = 0$ on $y > 1/2n^2$ and $Y_\alpha^{x^2}(s) \geq 1/n^2$ for $0 \leq s < \sigma_{n,x}$. But this is none other than

$$E[X_\alpha^x(t_n \wedge \sigma_{n,x})]^{2p} = x^{2p} \quad [\text{since } n \geq N(x, M) > 1/x]. \tag{2.15}$$

Then if $\tau_M = t_{N(x,M)} \wedge \sigma_{N(x,M),x}$, $M = 1, 2, 3, \dots$ we see that

$$E\tau_M^p < \infty$$

and by (2.14)–(2.15)

$$\frac{E(\tau_M)^p}{EX_\alpha^x(\tau_M)^{2p}} > \frac{Mx^{2p}}{x^{2p}} = M \to \infty \quad \text{as } M \to \infty.$$

For $x = 0$ let

$$\sigma = \inf\{t > 0\colon X_\alpha^0(t) = 1\}, \qquad \sigma_n = \inf\{t > \sigma\colon X_\alpha^0(t) \leq n^{-1}\}.$$

Then

$$\begin{aligned} E\sigma_n^p &\geq E[\sigma_n - \sigma]^p \\ &= E[\sigma_{n,1}]^p \quad [\sigma_{n,1} \text{ as in (2.12)}] \\ &\geq M + 1 \quad \text{for } n \geq N(1, M) \ [\text{by (2.13)}]. \end{aligned}$$

Since $t \wedge \sigma_n \geq t \wedge (\sigma_n - \sigma)$ and $\text{Law}(t \wedge (\sigma_n - \sigma)) = \text{Law}(t \wedge \sigma_{n,1})$, by (2.14) there are t_n with

$$E[t_n \wedge \sigma_n]^p > M, \qquad n \geq N(1, M). \tag{2.16}$$

For u_n as before, Itô's formula and optional stopping give for $m \geq n \geq N(1, M)$

$$\begin{aligned} &Eu_m\big(Y_\alpha^0(t_n \wedge \sigma_n)\big) - Eu_m\big(Y_\alpha^0(t_n \wedge \sigma)\big) \\ &= E\int_{t_n \wedge \sigma}^{t_n \wedge \sigma_n} [2yu_m''(y) + au_m'(y)]\Big|_{y = Y_\alpha^0(s)} ds \\ &= 0 \quad \text{since } Y_\alpha^0(s) \geq \frac{1}{n^2} \geq \frac{1}{m^2} \text{ for } s \in [\sigma, \sigma_n]. \end{aligned}$$

Thus

$$\begin{aligned} 1 &\geq Eu_m\big(Y_\alpha^0(t_n \wedge \sigma)\big) \\ &= Eu_m\big(Y_\alpha^0(t_n \wedge \sigma_n)\big) \\ &= E\big[Y_\alpha^0(t_n \wedge \sigma_n)\big]^p I\big(Y_\alpha^0(t_n \wedge \sigma_n) > (2m^2)^{-1}\big) \\ &\quad + Eu_m\big(Y_\alpha^0(t_n \wedge \sigma_n)\big) I\big(Y_\alpha^0(t_n \wedge \sigma_n) \leq (2m^2)^{-1}\big). \end{aligned}$$

Recalling that $u_m \leq (2m^2)^{-p}$ on $[0, (2m^2)^{-1}]$ we see that as $m \to \infty$,

$$1 \geq EY_\alpha^0(t_n \wedge \sigma_n)^p. \tag{2.17}$$

Let $\tau_M = t_{N(1,M)} \wedge \sigma_{N(1,M)}$ and see

$$E\tau_M < \infty$$

and by (2.16)–(2.17)

$$\frac{E\tau_M}{EX_\alpha^0(\tau_M)^{2p}} \geq \frac{M}{1} \to \infty \quad \text{as } M \to \infty$$

as desired. □

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