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A CONVERSE TO A THEOREM OF P. LÉVY

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By a well-known theorem of P. Lévy, if (X_t) is a standard Brownian motion on $\mathbb{R}$ with $X_0 = 0$ and if $H_t = \min_{u \le t} X_u$, then $(Y_t) = (X_t - H_t)$ is Brownian motion with 0 as a reflecting lower boundary. More generally, if X is allowed to have nonzero drift or a reflecting lower boundary at $A < 0$, then the process $Y = X - H$ is still a diffusion process. We prove the converse result: If X is a diffusion on an interval $I \subset \mathbb{R}$ which contains 0 as an interior point, and if $(Y_t) = (X_t - H_t)$ is a time homogeneous strong Markov process (when $X_0 = 0$), then X must be a Brownian motion on I (with drift μ, variance parameter $\sigma^2 > 0$, killing rate $c \ge 0$ and reflection at $\inf I$ in case $\inf I > -\infty$).

1. Introduction and the main result. Let $X = (X_t\colon t \ge 0)$ be a Brownian motion on $\mathbb{R}$, started at 0, with drift μ and variance parameter σ^2. Let $H_t = \min_{s \le t} X_s$ and $Y_t = X_t - H_t$. In case $\mu = 0$, a theorem of Lévy (1948) states $Y = (Y_t\colon t \ge 0)$ is a Brownian motion, with 0 as a reflecting lower boundary. In fact, this result is true even if $\mu \ne 0$ [see Fristedt (1974)], and in either case $(-H_t\colon t \ge 0)$ is a multiple of local time at 0 for Y. The Brownian nature of Y persists if X is modified by placing a reflecting lower boundary at $A < 0$.

Our purpose in this paper is to prove a converse to Lévy's theorem (Theorem 1 below). To state this result, we need to introduce some notation. Let $(X_t\colon t \ge 0)$ now denote a conservative, regular diffusion on an interval $I \subset \mathbb{R}$. Being conservative, X can be realized as the coordinate process on the space Ω of continuous paths from $[0, +\infty[$ to I. On Ω are defined the usual coordinate maps $(X_t\colon t \ge 0)$, shift operators $(\theta_t\colon t \ge 0)$ and σ-algebras $\mathscr{F}_t^\circ = \sigma(X_s\colon 0 \le s \le t)$, $\mathscr{F}^\circ = \sigma(X_s\colon s \ge 0)$. The evolution of X is described by a family $(P^x\colon x \in I)$ of probabilities on $(\Omega, \mathscr{F}^\circ)$. We assume that, for each $F \in \mathscr{F}^\circ$, the mapping $x \to P^x(F)$ is Borel measurable. Our hypothesis that X is a regular diffusion is expressed as follows. Let $A = \inf I$, $B = \sup I$ and

$$T_x = \inf(t > 0\colon X_t = x), \qquad \inf \varnothing = +\infty.$$

Then

$$P^x(X_0 = x) = 1, \qquad x \in I, \tag{1}$$

$$P^x(T_y < +\infty) > 0, \qquad x \in I,\ y \in I^\circ =]A, B[, \tag{2}$$

(3) for any $(\mathscr{F}_{t+}^\circ)$-optional time T, the conditional distribution of $\theta_T = (X_{T+u}\colon u \ge 0)$, given $\mathscr{F}_{T+}^\circ$, is P^{X_T} on $(T < +\infty)$.

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Now define, for $t \geq 0$ and $\omega \in \Omega$,

$$H_t(\omega) = \min_{u \leq t} X_u(\omega),$$
$$Y_t(\omega) = X_t(\omega) - H_t(\omega),$$
$$\mathscr{G}_t^\circ = \sigma(Y_u : 0 \leq u \leq t).$$

It is easy to check that the bivariate process $((Y_t, H_t); \mathscr{F}_{t+}^\circ; P^0)$ is a time homogeneous strong Markov process. However, $(Y_t; \mathscr{G}_{t+}^\circ; P^0)$ is strong Markov only under special circumstances. For example, if $A = 0$, then $H_t \equiv 0$ and $Y_t = X_t$. This trivial case aside, we have the following:

THEOREM 1. *Suppose that* $A < 0 < B$ *and that* $(Y_t; \mathscr{G}_{t+}^\circ; P^0)$ *is a time homogeneous strong Markov process. Then* $B = +\infty$ *and there are numbers* $\mu \in \mathbb{R}$, $\sigma > 0$ *such that for each* $x \in I = [A, +\infty[(= \mathbb{R}$ *if* $A = -\infty)$, $(X; P^x)$ *is a Brownian motion started at* x, *with drift* μ, *variance parameter* σ^2 *and reflection of* A *in case* $A > -\infty$.

This result complements the recent work of Rogers (1981) and Rogers and Pitman (1982) characterizing those diffusions X for which $2(\max_{s \leq t} X_s) - X_t$ is a diffusion.

The proof of Theorem 1 occupies the next two sections. *In these sections we work under the blanket hypotheses*: (i) $A < 0 < B$, (ii) $(Y_t; \mathscr{G}_{t+}^\circ; P^0)$ is a time homogeneous strong Markov process. The precise meaning of (ii) is this: Let T be a $(\mathscr{G}_{t+}^\circ)$-optional time and let Y^T denote the path $t \to Y_{T(\omega)+t}(\omega)$ in case $T(\omega) < +\infty$. Then there is a kernel $Q = Q^y(F)(y \geq 0,\ F \in \mathscr{F}^\circ)$ such that for any $(\mathscr{G}_{t+}^\circ)$-optional time T,

$$P^0\left(Y^T \in F \mid \mathscr{G}_{T+}^\circ\right) = Q^{Y_T}(F) \tag{4}$$

a.s. P^0 on $(T < +\infty)$.

In the final section we prove the analogue of Theorem 1 for the nonconservative case.

2. Preliminary results. Our aim in this section is to show that, as a consequence of the Markovian nature of $(Y_t; \mathscr{G}_{t+}^\circ; P^0)$, X enjoys a certain translation invariance property. This property will be used in the next section to determine the generator of X, thereby proving Theorem 1.

Let $(\mathscr{G}_t)$ and $(\mathscr{F}_t)$ denote the usual P^0-augmentations of $(\mathscr{G}_{t+}^\circ)$ and $(\mathscr{F}_{t+}^\circ)$, respectively. Clearly, $\mathscr{G}_t \subset \mathscr{F}_t$ for $t \geq 0$; a standard argument shows that (4) holds for any $(\mathscr{G}_t)$-optional time T.

Define a "random set" $M \subset \Omega \times [0, +\infty[$ by

$$M = \{(\omega, t) : t \geq 0,\ X_t(\omega) = H_t(\omega)\}$$
$$= \{(\omega, t) : t \geq 0,\ Y_t(\omega) = 0\}.$$

Evidently M is $(\mathscr{G}_t)$-optional and has sections $M(\omega)$ which are closed in $[0, +\infty[$. From the identity

$$H_{t+u}(\omega) = H_t(\omega) \wedge H_u(\theta_t \omega),$$

it follows that M is *intrinsically homogeneous*:

(5) $$M(\theta_t\omega) = \{u \geq 0: t + u \in M(\omega)\}, \qquad t \in M(\omega).$$

Combining (4) with (5) we see that $(M; \mathscr{G}_t; P^0)$ is a *regenerative set* [Hoffmann-Jørgensen (1969) and Maisonneuve (1974, 1983)]: If T is a $(\mathscr{G}_t)$-optional time with $T(\omega) \in M(\omega)$ a.s. P^0 on $(T < +\infty)$, then $M(\theta_T)$ is independent of $\mathscr{G}_T|_{(T<+\infty)}$ [on $(T < +\infty)$ under P^0] and has the same law as M under P^0. The regularity of X implies that for each $\varepsilon > 0$, the set $M(\omega) \cap [0, \varepsilon[$ is infinite for P^0-a.e. ω. By Proposition X.1 of Maisonneuve (1974), $M(\omega)$ is a perfect set for P^0-a.e. ω. Thus, there exists a local time process $(L_t: t \geq 0)$ for $(M; \mathscr{G}_t; P^0)$: (L_t) is path-continuous and increasing, $L_0 = 0$, (L_t) is adapted to $(\mathscr{G}_t)$ and the measure $dt \to L(\omega, dt) \equiv dL_t(\omega)$ has support $M(\omega)$ for P^0-a.e. ω. Moreover, (L_t) can be chosen to have the following homogeneity property: Let $\underline{M}(\omega)$ denote the minimal right closed set whose closure is $M(\omega)$. Then for $\omega \in \Omega$, $s \geq 0$,

(6) $$L_{t+s} = L_t(\omega) + L_s(\theta_t\omega), \qquad t \in \underline{M}(\omega).$$

See for example Maisonneuve (1974), Theorem X.2. A local time with the above properties is determined only up to a multiplicative constant. For definiteness we normalize (L_t) so that $P^0(\int_0^\infty e^{-t}\, dL_t) = 1$.

The following result is the key to later developments.

PROPOSITION 2. *There exists a continuous decreasing map* ψ: $[0, +\infty[\to [A, 0]$ *such that*

(7) $$H_t = \psi(L_t), \qquad \forall\, t \geq 0, \quad a.s.\ P^0.$$

PROOF. Let $\tau_t = \inf(u: L_u > t)$, $\hat{X}_t = X_{\tau_t}$, $\hat{\mathscr{F}}_t = \mathscr{F}_{\tau_t}$, where $t \geq 0$. By a standard argument $(\hat{X}_t; \hat{\mathscr{F}}_t; P^0)$ is a time homogeneous strong Markov process. Clearly, $\tau_t \in M$, $\forall\ t \geq 0$, a.s. P^0 and so $\hat{X}_t = H_{\tau_t}$, $\forall\ t \geq 0$, a.s. P^0. Put $K = \inf(t: \hat{X}_t = A$ or $\tau_t = +\infty)$ and note that $t \to \hat{X}_t$ is continuous and strictly decreasing on $[0, K[$. By a result of Çinlar (1979) [see also Itô and McKean (1965), pages 146–147], there exists a continuous, strictly decreasing map ϕ: $]A, 0] \to [0, +\infty[$ such that a.s. P^0,

(8) $$\hat{X}_t = \phi^{-1}(t), \qquad \forall\, t \in [0, K[,$$

where ϕ^{-1}: $[0, \phi(A+)[\to]A, 0]$ denotes the inverse of ϕ. But if $t = L_u$, then $\hat{X}_t = H_{\tau_t} = H_u$ and so (8) implies

(9) $$H_u = \phi^{-1}(L_u), \quad \text{whenever } 0 \leq L_u < K.$$

But $L_u \geq K$ if and only if $u \geq \rho \equiv \inf(t: X_t = \lim_{u\to+\infty} H_u)$; if $u \geq \rho$, then either (i) $\rho = T_A$ in which case $H_u = A$, or (ii) $\rho < T_A = +\infty$ in which case $H_u = H_\rho > A$. Thus, if we define ψ by $\psi = \phi^{-1}$ on $[0, \phi(A+)[$, $\psi = A$ on $[\phi(A+), +\infty[$, then the proposition follows. □

The following result is now obvious.

COROLLARY 3. *For* $t \geq 0$, $\mathscr{G}_t = \mathscr{F}_t$. *Consequently*, $(Y_t; \mathscr{F}_t; P^0)$ *is a time homogeneous strong Markov process.*

For $a \geq 0$ define

$$S_a = \inf(t > 0\colon Y_t = a)$$

and set $J = \{a \geq 0\colon P^0(S_a < +\infty) > 0\}$. Clearly J is the state space of Y [and, in fact, $(Y_t; \mathscr{F}_t; P^0)$ is a diffusion on J, although we shall not use this fact]. Recall that $A = \inf I$, $B = \sup I$.

LEMMA 4. $B = +\infty$ *and* $J = [0, +\infty[$.

PROOF. The regularity of X leads easily to the inclusion $[0, B - A[\subset J$. If B were not $+\infty$, then, with positive probability, Y would hit $B - \max(A, -1)/2 > B$ before X hit $\max(A, -1)/2$; this would lead to the absurd conclusion that X hits points greater than B, and so $B = +\infty$. □

Here is the promised translation invariance property of X.

PROPOSITION 5. *Let* $a > 0$ *be fixed. Then the* P^{y+a} *distribution of* $(X_t - y;\ 0 \leq t < T_y)$ *does not depend on* $y \in]A, 0]$.

PROOF. Let $S_0' = \inf(t > 0\colon Y_{S_a + t} = 0)$ and consider the path fragment $Z = (Y_{S_a+t}\colon 0 \leq t < S_0')$. By the strong Markov property of X at S_a, the P^0 conditional distribution of Z, given $\mathscr{F}_{S_a}$, is the P^{y+a} distribution of $(X_t - y\colon 0 \leq t < T_y)$, where $y = H_{S_a}(\omega)$. But also, by Corollary 3, the P^0 conditional distribution of Z, given $\mathscr{F}_{S_a}$, coincides with the distribution of Y_t started at a and stopped upon hitting 0. This last distribution does not depend on $y = H_{S_a}(\omega)$. The proposition now follows since the P^0 distribution of $H_{S_a} = \psi(L_{S_a})$ has support $[A, 0] (=]-\infty, 0]$ if $A = -\infty)$. Indeed, $\psi([0, +\infty[) \supset]A, 0]$, and L_{S_a} follows the exponential distribution under P^0 [according to a result of Kesten, for which see page 112 of Maisonneuve (1974)]. □

3. The generator of *X*. In this section, we use Proposition 5 to determine explicitly the infinitesimal generator of X. Let s (resp. m) denote a scale function (resp. speed measure) for X. The scale s is a strictly increasing map from $]A, B[$ into $\mathbb{R}$, while m is a strictly positive Radon measure on $]A, B[$. Recall from Itô and McKean (1965) that the generator G of our conservative, regular diffusion X takes the form $G = (d/dm)/(d^+/ds^+)$; more precisely, for $f \in D(G)$ (the domain of G) and $a < b$ both in $]A, B[$,

$$\int_{]a, b]} Gf(y)m(dy) = f^+(b) - f^+(a), \tag{10}$$

where $f^+(x) \equiv (d^+f/ds^+)(x) = \lim_{y \downarrow x}(f(y) - f(x))/(s(y) - s(x))$.

The scale s is only determined modulo the family of transformations $s \to \alpha s + \beta$ ($\alpha > 0$, $\beta \in \mathbb{R}$), but once s is chosen, m is uniquely determined. We assume that $s(0) = 0$, determining s up to a positive multiple; we reserve the particular choice of s until later.

The generator of X determines its resolvent and so its distribution; expressed in terms of G, the conclusion of Theorem 1 is the statement that G takes the special form

$$G = (\sigma^2/2)D^2 + \mu D, \qquad D = d/dx, \tag{11}$$

on $]A, +\infty[$, with the appropriate boundary condition at A in the case $A > -\infty$. But (11) holds if and only if s and m can be written as

$$\begin{aligned} s(x) &= (\sigma^2/2\mu)(1 - e^{-2\mu x/\sigma^2}) \qquad (= x \text{ if } \mu = 0), \\ m(dx) &= (2/\sigma^2)e^{2\mu x/\sigma^2}\,dx. \end{aligned} \tag{12}$$

If (12) holds, then Feller's classification of boundaries [Itô and McKean (1965), page 130] tells us that $B = +\infty$ is neither an exit nor an entrance boundary point. The same is true of A if $A = -\infty$. If $A > -\infty$, then A is an entrance–exit (i.e., regular) boundary point. Now an argument of Vervaat (1979) shows that $P^0(t \in M) = P^0(Y_t = 0) = 0$ for $t > 0$. Thus, if $A > -\infty$, and if $t > 0$ is arbitrary, using (4) with $T = T_A$,

$$\begin{aligned} 0 &= P^0\big(Y_{T_A+t} = 0;\ T_A < +\infty\big) \\ &= P^0\big(X_{T_A+t} = A;\ T_A < +\infty\big) \\ &= P^0(T_A < +\infty)P^A(X_t = A), \end{aligned}$$

since $X_{T_A+t} - A = Y_{T_A+t}$ if $T_A < +\infty$. But

$$P^0(T_A < +\infty) > 0, \text{ so } P^A(X_t = A) = 0 \text{ for } t > 0.$$

Thus, in case $A > -\infty$, A is an instantaneously reflecting lower boundary for X, as claimed in Theorem 1. In view of the preceding discussion, to finish the proof of Theorem 1, it suffices to show that (12) holds.

Before proceeding to the verification of (12) let us rephrase Proposition 5 in terms of G. We shall say that a function f is in the *local domain of G at* $x \in]A, B[$ provided there is an interval $]a, b[\subset]A, B[$ containing x and a function $g \in D(G)$ such that $f = g$ on $]a, b[$. In this event we write $f \in D(x)$ and $(Gf)(x) = (Gg)(x)$.

PROPOSITION 6. *For each* $y \in]A, 0[$ *and* $x \in]y, +\infty[$, $f \in D(x - y)$ *iff* $f_y \equiv f(\cdot - y) \in D(x)$. *If* $f \in D(x - y)$, *then* $Gf(x - y) = Gf_y(x)$.

PROOF. This is an immediate consequence of Proposition 5, once we observe that the generator of $(X_t\colon 0 \le t < T_y;\ P^x,\ x > y)$ may be identified with the restriction of G to $\{f \in D(G)\colon f \equiv 0 \text{ on } [A, y]\}$. □

In verifying (12) we first consider the scale s. It is easy to check that if $z \in]A, B[$, then $s \in D(z)$ and $Gs(z) = 0$. By Proposition 6,

$$Gs_y(x) = Gs(x - y) = 0, \qquad A < y < x \wedge 0. \tag{13}$$

Combining (13) with (10) we obtain

$$s_y^+(z) = s_y^+(y), \qquad A < y < z \wedge 0, \tag{14}$$

where $s_y^+ = (s_y)^+ = d^+s_y/ds^+$. Integrating (14) over $]y, x]$ with respect to $ds(z)$ we obtain

$$s_y(x) - s_y(y) = s(x-y) = (s(x) - s(y))s_y^+(y), \qquad A < y < x \wedge 0, \tag{15}$$

where the first equality results from $s(0) = 0$. It follows quickly from (15) that the map ϕ: $y \to s_y^+(y)$ satisfies $\phi(y_1 + y_2) = \phi(y_1)\phi(y_2)$ if y_1, y_2 and $y_1 + y_2$ all lie in $]A, 0[$. Thus, $s_y^+(y) = e^{ay}$, $y \in]A, 0[$ for some $a \in \mathbb{R}$. Substituting this expression for $s_y^+(y)$ back into (15) and differentiating in x, we obtain

$$s(dx - y) = e^{ay}s(dx), \qquad A < y < x \wedge 0.$$

From this last relation we easily deduce that $s(dx) = a'e^{-ax}\,dx$, $A < x$, for some $a' > 0$. We can (and do) choose $a' = 1$; since $s(0) = 0$, we must have

$$\begin{aligned} s(x) &= a^{-1}(1 - e^{-ax}), \qquad x > A, \quad \text{if } a \neq 0, \\ &= x, \qquad\qquad\qquad\quad\; x > A, \quad \text{if } a = 0. \end{aligned} \tag{16}$$

Note that (16) implies that $f_y^+(x)$ exists if and only if $f^+(x-y)$ exists, and then $f_y^+(x) = e^{ay}f^+(x-y)$.

To consider the speed measure m, we define a function h on $]A, +\infty[$ by

$$\begin{aligned} h(x) &= \int_0^x m]0, y]s(dy), \qquad x \geq 0, \\ &= \int_x^0 m]y, 0]s(dy), \qquad A < x < 0. \end{aligned} \tag{17}$$

Then $h(0) = 0 = h^+(0)$, and for $x > A$, $h \in D(x)$ with $Gh(x) = 1$. By Proposition 6,

$$Gh_y(x) = 1, \qquad A < y < x \wedge 0. \tag{18}$$

Using (10) and the fact that $h_y^+(y) = e^{ay}h^+(0) = 0$, we obtain from (18)

$$h_y^+(x) = m]y, x], \qquad A < y < x \wedge 0.$$

But by (17), $h_y^+(x) = e^{ay}h^+(y-x) = e^{ay}m]0, x-y]$ if $x > y > A$. Thus,

$$m]y, x] = e^{ay}m]0, x-y], \qquad A < y < x \wedge 0. \tag{19}$$

It follows from (19) that there is a constant $b > 0$ such that $m(dx) = be^{ax}\,dx$, $x > A$. Setting $\sigma^2 = 2/b$, $\mu = a/b$ and recalling (16) we see that (12) holds. The proof of Theorem 1 is complete. □

4. The nonconservative case. Let $(\overline{X}_t\colon t \geq 0)$ be a Brownian motion on $\mathbb{R}$, started at 0, with drift μ and variance parameter σ^2. Let $\overline{Y}_t = \overline{X}_t - \overline{H}_t$, where $\overline{H}_t = \min_{u \leq t}\overline{X}_u$. Then $(\overline{Y}_t\colon t \geq 0)$ is a Brownian motion on $[0, +\infty[$ with 0 as a reflecting lower boundary. Let X denote the c-subprocess of $\overline{X}$: X is formed by killing $\overline{X}$ at an independent exponential time ζ of mean $1/c$. The process X is a diffusion and if we consider $Y_t = X_t - H_t$ ($H_t = \min_{u \leq t}X_u$), then clearly (Y_t:

$0 \le t < \zeta$) is the c-subprocess of $\bar{Y}$. Thus, (Y_t) is a strong Markov process. Note that the generator of X is given by

$$Gf(x) = (\sigma^2/2)f''(x) + \mu f'(x) - cf(x). \tag{20}$$

The Markovian nature of Y persists if the original process $\bar{X}$ is modified by placing a reflecting lower boundary at $A < 0$, since as above, Y is the c-subprocess of the Markov process $\bar{Y}$.

In this section we shall prove the converse statement, which is the analogue of Theorem 1 when X is allowed to have a positive killing measure (and so a finite lifetime).

We will adhere to the notation set in earlier sections with one essential change. Since X will no longer be assumed conservative, we must introduce a cemetery to allow for a finite lifetime. Let $\Delta \notin I$ be this cemetery point; Ω now denotes the space of paths ω: $[0, +\infty[\to I \cup \{\Delta\}$ which are continuous on $[0, \zeta(\omega)[$ and which are absorbed in Δ at time $\zeta(\omega)$. Thus, $\zeta(\omega) = \inf(t\colon \omega(t) = \Delta)$ and $\omega(t) = \Delta$ if $t \ge \zeta(\omega)$. The objects X_t, $\mathscr{F}_t$, T_x are defined as before so that $X = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, P^x)$ is a regular diffusion on I with Δ as cemetery. Set $H_t = \min_{u \le t} X_u$ if $t < \zeta$ and $= H_{\zeta-}$ if $t \ge \zeta$. Define Y as before by $Y_t = X_t - H_t$ (with the convention $Y_t = \Delta$ if $t \ge \zeta$).

Since X is no longer assumed to be conservative, its generator takes the more general form

$$Gf(x)m(dx) = df^+(x) - k(dx)f(x), \qquad A < x < B, \tag{21}$$

where k is a positive Radon measure on $]A, B[$; k is the killing measure of X. As before, s (resp. m) is the scale (resp. speed measure) of X; in (21) $f^+ = d^+f/ds^+$ as before. In addition to (21) the elements of $D(G)$ are constrained by appropriate boundary conditions at the regular endpoints of I.

Here is our general version of Theorem 1.

THEOREM 7. *Suppose that $A < 0 < B$ and that $(Y_t; \mathscr{G}^\circ_{t+}; P^0)$ is a time homogeneous strong Markov process. Then $B = +\infty$ and the generator of X takes the form*

$$Gf(x) = (\sigma^2/2)f''(x) + \mu f'(x) - cf(x), \qquad A < x, \tag{22}$$

where $\sigma^2 > 0$, $\mu \in \mathbb{R}$, $c \ge 0$. B is a natural (no exit–no entrance) boundary, as is A in case $A = -\infty$. If $A > -\infty$, then A is an instantaneously reflecting boundary for X.

REMARK. In terms of s, m and k, (22) amounts to the statement that

$$\begin{aligned} s(x) &= (\sigma^2/2\mu)(1 - e^{-2\mu x/\sigma^2}),\\ m(dx) &= (2/\sigma^2)e^{2\mu x/\sigma^2}\,dx,\\ k(dx) &= cm(dx). \end{aligned} \tag{23}$$

In other words, if $(Y_t; \mathscr{G}^\circ_{t+}, P^0)$ is strongly Markovian, then X must be the c-subprocess of a Brownian motion with drift μ, variance σ^2 (and reflection at A if $A > -\infty$) for some choice of c, μ, σ^2, A.

Before proceeding with the proof of Theorem 7, let us note that in previous sections the conservation hypothesis on X was used only in Section 3. Thus, the results of Section 2 remain valid under the present circumstances (that is, under the hypotheses of Theorem 7). In particular, $B = +\infty$ (Lemma 4).

PROOF OF THEOREM 7. Define a function r on $]A, +\infty[$ by

$$r(x) = P^x(T_0 < +\infty), \qquad x \geq 0,$$
$$= \left[P^0(T_x < +\infty)\right]^{-1}, \qquad A < x < 0.$$

Clearly, $r(0) = 1$, r is decreasing and, since X is regular, strictly positive and finite. Arguing as in Section 4.6 of Itô and McKean (1965), one checks that for each $x > A$, $r \in D(x)$ (the local domain of G at x) and $Gr(x) = 0$. Moreover, Proposition 6 implies that $P^{y+a}(T_y < +\infty) = P^a(T_0 < +\infty)$ if $A < y \leq 0 < a$. Thus, $r(y + a) = r(y)r(a)(A < y \leq 0 < a)$. Since r is decreasing, we must have

$$r(x) = e^{-\gamma x}, \qquad x > A, \tag{24}$$

for some $\gamma \geq 0$. A second application of Proposition 6, using (24), shows that the law of the *conditioned* process

$$\left(X_t - y: 0 \leq t < T_y;\ P^{y+a}(\cdot | T_y < +\infty),\ a > 0\right) \tag{25}$$

does not depend on $y \in]A, 0]$. Because the process

$$\left(X_t: 0 \leq t < T_y;\ P^x(\cdot | T_y < +\infty),\ x > y\right)$$

cannot die while in $]y, +\infty[$, its generator G^y has the form (10). Moreover, G^y is related to G by $G^y f(x) = (r(x))^{-1} G(fr)(x)$, $x > y$. See Itô and McKean (1965), Section 4.3. The *argument* of Section 4 now shows that for some $\hat{\sigma}^2 > 0$, $\hat{\mu} \in \mathbb{R}$,

$$G^y = r^{-1}Gr = (\hat{\sigma}^2/2)D^2 + \hat{\mu}D, \quad \text{on }]y, +\infty[. \tag{26}$$

Since $y \in]A, 0]$ was arbitrary, we conclude that $r^{-1}Gr$ has the form given by the right-hand term in (26) on all of $]A, +\infty[$. From Section 4.3 of Itô and McKean we know that the generator $r^{-1}Gr$ has scale $\hat{s}$ and speed measure $\hat{m}$ given in terms of s and m by

$$\begin{aligned} \hat{s}(dx) &= (r(x))^{-2} s(dx), \\ \hat{m}(dx) &= (r(x))^2 m(dx). \end{aligned} \tag{27}$$

On the other hand, (26) implies directly that $\hat{s}(dx) = e^{-2\hat{\mu}x/\hat{\sigma}^2}\,dx$, $\hat{m}(dx) = (2/\hat{\sigma}^2)e^{2\hat{\mu}x/\hat{\sigma}^2}\,dx$. Comparison of (26) and (27) thus yields, since $r(x) = e^{-\gamma x}$,

$$s(dx) = e^{-2(\gamma+\lambda)x}\,dx,$$
$$m(dx) = (2/\hat{\sigma}^2)e^{2(\gamma+\lambda)x}\,dx,$$

where $\lambda = \hat{\mu}/\hat{\sigma}^2$. Since $Gr(x) = 0$, if $x > A$, we have, using (25), $dr^+(x) = r(x)k(dx)$; thus $k(dx) = dr^+(x)/r(x) = cm(dx)$, where $c = \gamma\hat{\sigma}^2/2 \geq 0$. Setting $\mu = (\gamma + \lambda)\hat{\sigma}^2$, $\sigma^2 = \hat{\sigma}^2$, we see that (23) holds so that G has the form (22) as claimed in Theorem 7.

As in the conservative case, the boundaries $B = +\infty$ and A (in the case $A = -\infty$) are neither exit nor entrance boundaries. If $A > -\infty$, then A is both an entrance and an exit boundary. Thus, to complete the proof of Theorem 7, it suffices to check that, in the case $A > -\infty$, A is an instantaneously reflecting lower boundary for X. If $A > -\infty$, then any $f \in D(G)$ is subject to the boundary condition

$$Gf(A)m(A) = f^+(A) - k(A)f(A),$$

where $m(A) \geq 0$ represents the "stickiness" of A, while $k(A) \geq 0$ is the killing rate at A. We claim that $m(A) = k(A) = 0$ so that A is instantaneously reflecting for X. To see that $m(A) = 0$, note that as in the conservative case, $P^0(Y_t = 0) = 0$ if $t > 0$. But the law of $(Y_{T_A+t}\colon t \geq 0)$ under $P^0(\cdot \mid T_A < +\infty)$ is the same as that of both $(Y_t\colon t \geq 0)$ under P^0 and $(X_t - A\colon t \geq 0)$ under P^A. Thus, for $t > 0$, $P^A(X_t = A) = P^0(Y_{T_A+t} = 0 \mid T_A < +\infty) = P^0(Y_t = 0) = 0$ and $m(A) = 0$ as claimed.

That $k(A) = 0$ as well is intuitively obvious. Roughly speaking, if $k(A)$ were > 0, then we would have

$$P^0(Y_{\zeta-} = 0;\, T_A < \zeta) > 0. \tag{28}$$

But clearly

$$P^0(Y_{\zeta-} = 0;\, T_A \geq \zeta) = 0, \tag{29}$$

since the probability on the left is zero when $k(A) = 0$ and does not depend on $k(A)$. Since Y starts afresh in state 0 at time T_A, (28) and (29) are contradictory. Thus, $k(A) = 0$ as claimed. The interested reader is invited to supply the details of this argument.

The proof of Theorem 7 is complete. □

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