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APPROXIMATION OF THE FINITE PREDICTION FOR A WEAKLY STATIONARY PROCESS

BY AKIO ARIMOTO

Musashi Institute of Technology

Let w be the spectral density function of a weakly stationary stochastic process with $w = |h|^2$, h being an outer function in the upper half plane, and let $\rho^*(a) = \mathrm{dist}(e^{ita}h/\bar{h}, H^\infty)$, where H^∞ is the space of boundary functions on R for bounded analytic functions in the upper half plane. It is shown that the standard deviation of the difference between the infinite predictor and the finite predictor from the past of length T does not exceed $\rho^*(T)/(1 - \rho^*(T))$ times the prediction error of the infinite predictor. Some other estimates relating to the difference between the infinite predictor and the finite predictor are also discussed.

1. Introduction and preliminaries. Let $X(t)$ be a weakly stationary stochastic process of $t \in R$ (the real line) and let $w(x)$ denote the spectral density function of $X(t)$, i.e., $w \in L^1(R, dx)$ and $w \geq 0$. It will be assumed that $(\log w)/(1 + x^2) \in L^1(R, dx)$. Thus $w = |h|^2$, where h is an outer function in H^2, the Hardy space for the upper half plane. Let $Z = L^2(R, w\,dx)$ and let $Z(a, b)$ denote the closed subspace in Z generated by $(e_t\colon a \leq t \leq b\}$, $e_t \equiv e^{itx}$.

The statistical prediction problem is to find the best estimator of $X(s)$, $s > 0$, by means of the linear combinations of observed values $X(t_k)$, $-T \leq t_k \leq 0$, $T > 0$, or of their limits. In the language of functional analysis this problem is to find the orthogonal projection of e_s on $Z(-T, 0)$. In order to get this projection we shall adopt Hayashi's method [6]: Under mild assumptions, $Z(-T, 0) = Z(-T, \infty) \cap Z(-\infty, 0)$, so the desired projection may be approximated by "projecting back and forth" on $Z(-\infty, 0)$ and $Z(-T, \infty)$ repeatedly. (This idea originally goes back to von Neumann [10]). Projections onto these last subspaces are straightforward. We need additional notation. Let the map S: $f \to hf$ be an isometry of Z onto $L^2(R, dx)$. Then S maps $Z(-\infty, 0)$ and $Z(-T, \infty)$ onto $(h/\bar{h})\bar{H}^2$ and $e_{-T}H^2$, respectively (where the bar denotes the complex conjugation). Let P_a and Q_b denote projections from $L^2(R, dx)$ onto e_aH^2 and $(e_bh/\bar{h})\bar{H}^2$, respectively. Then it is readily checked that $P_a\varphi = e_aPe_{-a}\varphi$ and $Q_b\varphi = e_bh/\bar{h}Qe_{-b}\bar{h}/h\varphi$, where P and Q denote the orthogonal projections of $L^2(R, dx)$ onto H^2 and $\bar{H}^2$, respectively. Let $M_T = e_{-T}H^2 \cap (h/\bar{h})\bar{H}^2$ and π_T be the projection onto M_T. Then $S^{-1}\pi_TS$ is the projection from Z onto $Z(-\infty, 0) \cap Z(-T, \infty)$.

The estimator of $X(s)$, $s > 0$, which uses the whole history of $X(t)$, $t \leq 0$, is called the infinite predictor and the estimator of $X(s)$ which uses the part of the history $X(t)$, $-T \leq t \leq 0$, is called the finite predictor. In this paper we study the variance of the difference between the infinite predictor and the finite

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predictor, and get a more precise order than the previous one obtained by the author in [1]. The order given below is related to $\rho^*(a) = \mathrm{dist}(e_a h/\bar{h}, H^\infty)$, where H^∞ is the space of boundary functions on R for bounded analytic functions in the upper half plane. Note that $\rho^*(a) \to 0$ as $a \to \infty$ if and only if $h/\bar{h} \in H^\infty + \mathrm{BUC}$, where BUC is the space of bounded uniformly continuous functions on R (Sarason [7]). Also we note that $\rho^*(a)$ is equal to the operator norm $\|Q_0 P_{-a}\|$ and is a nonincreasing function of a (Dym [4]).

2. Theorems. In view of the stationarity of our stochastic process, we have the following time-shifted modifications of the theorems of Dym and Hayashi.

THEOREM A (Dym [5]). *If $1/w$ is locally integrable, then*

$$Z(-\infty,0) \cap Z(-T,\infty) = \bigcap_{\varepsilon>0} Z(-T-\varepsilon, \varepsilon).$$

If also $\rho^*(c) < 1$ for some $c \geq 0$, then

$$Z(-T,0) = Z(-\infty,0) \cap Z(-T,\infty), \quad \text{for every } T > c.$$

THEOREM B (Hayashi [6]). *Let $1/w$ be locally integrable. Then the following are equivalent*:

(i) $\rho^*(T) < 1$.
(ii) $(Q_0 P_{-T})^n \to \pi_T$ *exponentially fast in the operator norm as $n \to \infty$, where π_T is the projection onto M_T.*

We shall henceforth tacitly assume that $1/w$ is locally integrable and $\rho^*(c) < 1$ for some $c \geq 0$. In terms of functional analysis the infinite predictor and the finite predictor are given by $S^{-1}Q_0 S e_s$ and $S^{-1}\pi_T S e_s = \lim_{n\to\infty} S^{-1}(Q_0 P_{-T})^n S e_s$, respectively.

Hence, using the isometry of S and Theorem B(ii), we can evaluate the variance of the difference of these predictors by $\sigma_T^2(s) = \|Q_0 h e_s - \pi_T h e_s\|^2 = \lim_{n\to\infty} \|Q_0 h e_s - (Q_0 P_{-T})^n h e_s\|^2$, where the norm $\| \ \|$ is in $L^2(R, dx)$. The following theorems and corollaries hold. The theorems are proved in Sections 3 and 4.

THEOREM 1. *For some $a \geq 0$, $e_a j \in H^\infty$, i.e., $\rho^*(a) = 0$ if and only if*

$$\sigma_a(s) = 0, \quad \text{for all } s > 0,$$

where $j = h/\bar{h}$. In this case, the finite predictor from the past of length a equals the infinite predictor.

COROLLARY 1. *If $\rho^*(a) = 0$ for some $a \geq 0$, then for every $T \geq a$ and every $s > 0$, we have*

$$\sigma_T(s) = 0.$$

This is an immediate consequence of Theorem 1 and the fact that $e_a j \in H^\infty$ implies $e_T j \in H^\infty$ for $T \geq a$.

COROLLARY 2. *If $e_{a/2}h$ agrees on R with the reciprocal of an entire function of exponential type $\leq a/2$, then*

$$\sigma_T(s) = 0, \quad \text{for } T \geq a.$$

This is also immediate from the fact that under the condition of Corollary 2, $e_a j$ agrees a.e. on R with an inner function (Dym [3]).

THEOREM 2. *Let $\rho^*(c) < 1$ for some $c \geq 0$. Then we have for $T \geq c$,*

$$\sigma_T(s) \leq \frac{\rho^*(T)}{1-\rho^*(T)} \left\{ \int_0^s |\hat{h}(t)|^2 \, dt \right\}^{1/2},$$

where $\rho^*(T) = \operatorname{dist}(e_T h/\bar{h}, H^\infty)$ and $\hat{h}$ is the L^2-Fourier transform of h.

COROLLARY 3. *Let $j = h/\bar{h}$. If $\bar{j}$ is an inner function, then for each $s > 0$, $\sigma_T(s)$ decreases to zero exponentially fast as $T \to \infty$.*

This is clear from Theorem 2 and the fact that $\rho^*(T) = \inf_{k \in H^\infty} \|e_T j - k\|_\infty$ decreases to zero exponentially fast as $T \to \infty$ when $\bar{j}$ is an inner function (Dym [2]).

THEOREM 3. *Let $\rho^*(c) < 1$ for some $c \geq 0$. Then we have for $j = h/\bar{h}$ and $T \geq c$,*

$$\sigma_T(s) \leq \frac{\varepsilon_T(j)}{1-\rho^*(T)} \left\{ \int_0^s |\hat{h}(t)|^2 \, dt \right\}^{1/2},$$

where $\varepsilon_T(\varphi)$ is the error of the best approximation of φ by entire functions of exponential type not greater than T.

COROLLARY 4. *If j is uniformly continuous, then for each $s > 0$,*

$$\sigma_T(s) = O(\omega(j, 1/T)), \quad \text{as } T \to \infty,$$

where ω is the modulus of continuity in L^∞. In particular, if $j \in \operatorname{Lip} \alpha$, then for each $s > 0$,

$$\sigma_T(s) = O(1/T^\alpha), \quad \text{as } T \to \infty.$$

This is a simple consequence of the well-known theorem which states $\varepsilon_T(\varphi) = O(\omega(\varphi, 1/T))$ and $= O(T^{-\alpha})$ for $\varphi \in \operatorname{Lip} \alpha$ (Timan [9]).

3. Proof of Theorem 1. Let $j = h/\bar{h}$ and P be the orthogonal projection from $L^2(R, dx)$ onto H^2, and define the operator V on $L^2(R, dx)$ by $V = jP\bar{j}$.

Then we may write $Q_0 = I - V = jQ\bar{j}$, where $Q = I - P$. Since for $T > 0$, $P_{-T} = e_{-T}Pe_T$ and $he_s \in H^2$ imply $P_{-T}he_s = he_s$, it follows that $Q_0P_{-T}he_s = Q_0he_s$ is the infinite predictor and that $Q_0P_{-T}he_s - (Q_0P_{-T})^2he_s = Q_0P_{-T}Vhe_s$. We will make frequent use of this last equality in what follows.

If $e_aj \in H^\infty$, then we have $P_{-a}Vhe_s = e_{-a}Pe_ajP(\overline{he}_s) = jP(\overline{he}_s)$. On the other hand, $Q_0jP(\overline{he}_s) = jQP(\overline{he}_s) = 0$. Consequently, we have $(Q_0P_{-a})Vhe_s = 0$, from which $Q_0P_{-a}he_s - (Q_0P_{-a})^2he_s = 0$. At the same time, we have $Q_0P_{-a}he_s - (Q_0P_{-a})^nhe_s = 0$, or $\sigma_a(s) = 0$.

Conversely, we assume that $\sigma_a(s) = 0$ for all $s > 0$. Then $Q_0P_{-a}he_s = (Q_0P_{-a})^2he_s$. Indeed, if we take $n \to \infty$ in the inequality

$$\|Q_0P_{-a}he_s - (Q_0P_{-a})^2he_s\| \le \|Q_0P_{-a}he_s - (Q_0P_{-a})^nhe_s\| + \|Q_0P_{-a}\|\,\|(Q_0P_{-a})^{n-1}he_s - (Q_0P_{-a})he_s\|,$$

the right-hand side becomes zero. Since $P_{-a}he_s = he_s$, we have $Q_0P_{-a}(I - Q_0P_{-a})he_s = Q_0P_{-a}Vhe_s = Q_0P_{-a}jb_s = 0$, where we denoted $P\overline{he}_s \in H^2$ by b_s. Thus we have $Q_0P_{-a}jb_s = jQ\bar{j}e_{-a}Pe_ajb_s = 0$, and $j \neq 0$ implies that $Q\bar{j}e_{-a}Pe_ajb_s = Q\bar{j}e_{-a}Qe_ajb_s = 0$. To complete the proof of Theorem 1, it is enough to show that the last equality implies $Qe_ajb_s = 0$ because this is just Seghier's condition ([8], page 396). By just the same arguments Seghier used, it can be shown that the linear subspace spanned by $\{b_s,\ s \ge 0\}$ equals H^2 and that e_aj is the inner function which characterizes the invariant subspace e_ajH^2 and is uniquely determined by the subspace e_ajH^2, save for multiplication by a complex constant of modulus 1. Thus we only have to prove the next lemma to complete the proof of Theorem 1.

LEMMA 1. *If* $Q\bar{j}e_{-a}Qe_ajb_s = 0$, *then* $Qe_ajb_s = 0$.

PROOF. Under the condition, we can write

$$\bar{h}Qe_ajb_s = e_ahf, \tag{$*$}$$

for some $f \in H^2$. Now let $\theta = Qe_ajb_s \in \overline{H}^2$. Then the L^1-Fourier transform of $\bar{h}\theta$, $(\bar{h}\theta)\hat{}(u)$ is zero for $u \ge 0$, because $\bar{h}\theta \in \overline{H}^1$ (the L^1-Hardy class in the lower half plane). On the other hand, the L^1-Fourier transform of e_ahf is zero for $u \le a$, since $hf \in H^1$. Therefore we see that the both sides of the equality $(*)$ are zero everywhere because their L^1-Fourier transforms are zero everywhere, and conclude that $Qe_ajb_s = 0$ since h is an outer function so that $h \neq 0$ a.e. □

4. Proofs of Theorems 2 and 3. We first prove the following lemma.

LEMMA 2.

$$\sigma_T(s) \le \frac{1}{1 - \rho^*(T)}\|(Q_0P_{-T})Vhe_s\|.$$

Proof. We saw in Section 3 that $Q_0P_{-T}he_s = Q_0he_s$ is the infinite predictor and that $Q_0P_{-T}he_s - (Q_0P_{-T})^2he_s = Q_0P_{-T}Vhe_s$. Hence we can write

$$\begin{aligned}\sigma_T(s) &= \lim_{n\to\infty} \|Q_0P_{-T}he_s - (Q_0P_{-T})^n he_s\| \\ &\le \lim_{n\to\infty} \|\{I + (Q_0P_{-T}) + (Q_0P_{-T})^2 + \cdots + (Q_0P_{-T})^n\}\| \\ &\quad \times \|\{Q_0P_{-T} - (Q_0P_{-T})^2\}he_s\| \\ &\le \frac{1}{1-\rho^*(T)}\|(Q_0P_{-T})Vhe_s\|,\end{aligned}$$

where the last inequality follows from $\|Q_0P_{-T}\| = \rho^*(T) < 1$ (Dym [3], page 37). □

Proof of Theorem 2. We have $\|Q_0P_{-T}Vhe_s\| \le \|Q_0P_{-T}\|\,\|Vhe_s\| \le \rho^*(T)\|P\bar{h}e_s\|$ and $\|P\bar{h}e_s\|^2 = \int_0^s|\hat{h}(u)|^2\,du$, $\hat{h}(u)$ being the Fourier transform of h, which follows by Parseval's identity. Now the proof of Theorem 2 is completed by using Lemma 2. □

In order to prove Theorem 3, the following lemma is needed.

Lemma 3. *If k is an entire function of exponential type a which is bounded on R, then $e_ak \in H^\infty$.*

Proof. k of type a implies that for every $\varepsilon > 0$ there exists an $A(\varepsilon)$ such that $|k(z)| \le A(\varepsilon)\exp\{(a+\varepsilon)|z|\}$ and hence if $|k(x)| \le M$ on R the theorem in Young [11], page 82, guarantees that $|k(x+iy)| \le M\exp\{(a+\varepsilon)|y|\}$. Now let $\varepsilon \to 0$, then e_ak is bounded and analytic in the upper half plane. □

Proof of Theorem 3. Let $b_s = P\bar{h}e_s \in H^2$. Since $V = jP\bar{j}$ and $Q_0jb_s = 0$, it follows that $Q_0P_{-T}Vhe_s = Q_0e_{-T}Pe_Tjb_s = Q_0e_{-T}(I-Q)e_Tjb_s = -Q_0e_{-T}Qe_Tjb_s$, from which $\|Q_0e_{-T}Vhe_s\| \le \|Qe_Tjb_s\|$. Now let k be any entire function of exponential type $a \le T$ which is bounded on R. Then $e_Tk \in H^\infty$, because $e_ak \in H^\infty$ from Lemma 3 and the fact that $e_{T-a} \in H^\infty$. Hence we have $Qe_Tkb_s = 0$ and $\|Qe_Tjb_s\| = \|Qe_T(j-k)b_s\| \le \|j-k\|_\infty\|b_s\|$, where $\|\ \|_\infty$ denotes the uniform norm in R. Therefore $\|Qe_Tjb_s\| \le \inf_k\|j-k\|_\infty\|b_s\|$, where the infimum is taken over every entire function of exponential type $\le T$ without restricting it to the ones which are bounded on R by virtue of the boundedness of j and this infimum is denoted by $\varepsilon_T(j)$ as usual. Thus Theorem 3 is proved. □

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DEPARTMENT OF MATHEMATICS
MUSASHI INSTITUTE OF TECHNOLOGY
1-28-1 TAMAZUTUMI, SETAGAYA-KU
TOKYO 158
JAPAN