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PROBABILITY ESTIMATES FOR MULTIPARAMETER BROWNIAN PROCESSES[1]

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Let F be a distribution function on $[0,1]^d$, and let W_F be the Gaussian process that is the weak limit of the empirical process determined by F. If G is a function on $[0,1]^d$, upper and lower bounds are found for $P(\sup_{t\in[0,1]^d}|W_F(t) - G(t)| \le \varepsilon)$.

1. Introduction. Let F be a distribution function on $[0,1]^d$ and let X_i, $i = 1, 2, \dots,$ be independent random vectors taking values in $[0,1]^d$ with distribution function F. If one forms the empirical distribution function

$$F_n(\mathbf{t}) = \frac{1}{n}\sum_{i=1}^{n} 1_{[\mathbf{0},\mathbf{t}]}(X_i),$$

it is well known that $\sqrt{n}(F_n(\cdot) - F(\cdot))$ converges in law (with respect to the supremum norm $\|\cdot\|$) to the mean 0 Gaussian process W_F, where

$$\mathrm{Cov}(W_F(\mathbf{s}), W_F(\mathbf{t})) = F([\mathbf{0},\mathbf{s}] \cap [\mathbf{0},\mathbf{t}]) - F([\mathbf{0},\mathbf{s}])F([\mathbf{0},\mathbf{t}]).$$

We refer to W_F as the tied-down Brownian process determined by F. Let G be a function on $[0,1]^d$. The main purpose of this paper is to obtain upper and lower bounds for $P(\|W_F - G\| \le \varepsilon)$. We also obtain analogous bounds for $P(\|B - G\| \le \varepsilon)$, where B is the standard Brownian sheet.

The principal motivation is as follows. Given samples X_i as above, one wants to test whether they could have law F. One forms F_n as above, one defines the Kolmogorov–Smirnov statistic $D_n = \sqrt{n}(F_n - F)$, and one would like to reject the hypothesis that the X_i's have law F if $\|D_n\|$ is too large. The distribution of D_n cannot be calculated exactly, so instead one looks at $\|W_F\|$. Even here, in the case that the dimension $d > 1$, the distribution of the tail of W_F is not known, although successively better estimates have been obtained by Goodman (1976), Cabaña and Wschebor (1982) and Adler and Brown (1986). Now suppose one wants to determine the power of this test. [Blum, Kiefer and Rosenblatt (1961) have argued that tests of this type should have very good power.] This means one wants to estimate $P(\|D_n\| \le \alpha)$ when the X_i's are in fact from another distribution H. Write

$$\begin{aligned} P\big(\|\sqrt{n}(F_n - F)\| \le \alpha\big) &= P\big(\|\sqrt{n}(F_n - H) + \sqrt{n}(H - F)\| \le \alpha\big) \\ &\approx P(\|W_H - K\| \le \alpha), \end{aligned}$$

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where $K = \sqrt{n}(F - H)$. Consequently, the estimates of this paper (Theorems 3.4 and 4.5) will give information on the power of the Kolmogorov–Smirnov test. Our bounds are clearly not sharp, and hopefully future research will refine them.

Our bounds may also be viewed as being of large deviation type. Using standard techniques, one can recover known results on the probability of large deviations for the Brownian sheet and the tied-down Brownian process [cf. Varadhan (1984), Theorem 5.1, for the one-parameter case]. Moreover, in a very rough sense, these estimates give some information as to what the second-order term should look like in Sanov-type inequalities [see Sanov (1957), Borovkov (1967) and Groeneboom, Oosterhoff and Ruymgaart (1979)]. Of course, there one really wants estimates for $\|D_n - G\|$ and not $\|W_F - G\|$.

A third application of our estimates is related to the extension of Chung's law of the iterated logarithm to the Brownian sheet. Our estimates can be easily used to show that if there exists a function $\varphi(t)$ such that

$$\liminf_{t\to\infty} \sup_{s\in[\mathbf{0},(t,\dots,t)]} |B(\mathbf{s})|/\varphi(t)$$

equals a constant, a.s., then $\varphi(t)$ must lie between $t^{d/2}(\log\log\log t)^{(d-1)/2}/(\log\log t)^{1/2}$ and $t^{d/2}(\log\log\log t)^{3(d-1)/2}/(\log\log t)^{1/2}$. This improves a result of Révész (1981) who showed in the case $d = 2$ that $\varphi(t)$ must lie between $t(\log\log\log t)^{1/2}/(\log\log t)^{1/2}$ and $t(\log\log\log t)^{5/2}/(\log\log t)^{1/2}$. The presence of the $\log\log\log t$ is rather surprising, and it would be an interesting but quite difficult problem to determine if an exact normalizing function $\varphi(t)$ exists, if it can be expressed in terms of an exact power of $\log\log\log t$, and if so, what power.

We obtain the upper bound by reducing the problem for the tied-down Brownian process determined by F to the corresponding problem for the untied process determined by F, then using stochastic integrals to reduce to the case of Brownian sheet, and finally using the Haar function decomposition of the Brownian sheet. This use of stochastic integrals is, as far as we know, new. The lower bound proceeds by the same sequence of reductions and also uses the Haar function representation, but, in addition, we use the Cameron–Martin–Girsanov transformation. Section 2 contains the necessary preliminaries concerning Haar functions and stochastic integrals, Section 3 contains the derivation of the upper bound, and Section 4 contains the derivation of the lower bound.

To simplify the notation, throughout we consider only the case $d = 2$, pointing out the changes necessary for $d > 2$ when they are not straightforward. We say F: $[0,1]^d \to \mathbb{R}$ has a density f if $F(\mathbf{t}) = \int_{[\mathbf{0},\mathbf{t}]} f(\mathbf{s})\,d\mathbf{s}$ for all $\mathbf{t}$. The letter c, with or without subscripts, denotes constants whose value may change from line to line.

2. Preliminaries. Let F be a continuous distribution function on $[0,1]^2$. We may, of course, view F as either a function or a measure. Define the Brownian sheet determined by F, B_F, to be the mean 0 Gaussian process with continuous paths and covariance given by $\mathrm{Cov}(B_F(\mathbf{s}), B_F(\mathbf{t})) = F([\mathbf{0},\mathbf{s}] \cap [\mathbf{0},\mathbf{t}])$. Define the tied-down Brownian process determined by F, W_F, to be the mean 0 Gaussian process with continuous paths and covariance given by

$\mathrm{Cov}(W_F(\mathbf{s}), W_F(\mathbf{t})) = F([\mathbf{0},\mathbf{s}] \cap [\mathbf{0},\mathbf{t}]) - F([\mathbf{0},\mathbf{s}])F([\mathbf{0},\mathbf{t}])$. When F is the uniform distribution, i.e., when $F([\mathbf{0},\mathbf{s}]) = |[\mathbf{0},\mathbf{s}]|$, where $|\cdot|$ denotes Lebesgue measure, we write simply B for the standard Brownian sheet.

Next, we recall the definition of the Haar functions. For $m = 1, 2, \dots$, let $\Gamma(m) = \{1, 2, \dots, 2^{m-1}\}$. Let $\Gamma(0) = \{0\}$. For $m \geq 1$, $j \in \Gamma(m)$, define $\phi_{jm}: [0,1] \to \mathbb{R}$ by

$$\phi_{jm}(t) = \begin{cases} 2^{(m-1)/2}, & t \in \left[\dfrac{2j-2}{2^m}, \dfrac{2j-1}{2^m}\right), \\ -2^{(m-1)/2}, & t \in \left[\dfrac{2j-1}{2^m}, \dfrac{2j}{2^m}\right), \\ 0, & \text{otherwise}. \end{cases}$$

For $j = 0$, $m = 0$, let $\phi_{jm}(t) = 1$ for $t \in [0,1]$. It is well known that the Haar functions ϕ_{jm} form a complete orthonormal system for $L^2([0,1], dt)$.

Now if $\mathbf{m} = (m_1, m_2)$, let $\Gamma(\mathbf{m}) = \{(j_1, j_2): j_1 \in \Gamma(m_1),\ j_2 \in \Gamma(m_2)\}$. If $\mathbf{j} = (j_1, j_2) \in \Gamma(\mathbf{m})$ and $\mathbf{t} = (t_1, t_2) \in [0,1]^2$, define

$$\Phi_{\mathbf{jm}}(\mathbf{t}) = \phi_{j_1, m_1}(t_1)\phi_{j_2, m_2}(t_2).$$

The $\Phi_{\mathbf{jm}}$ form a complete orthonormal system for $L^2([0,1]^2, d\mathbf{t})$.

Let $\psi(\mathbf{m})$ be defined so that the cardinality of $\Gamma(\mathbf{m})$ is $2^{\psi(\mathbf{m})}$. Thus

$$\psi(\mathbf{m}) = (m_1 - 1)1_{(m_1 > 0)} + (m_2 - 1)1_{(m_2 > 0)}.$$

Also, observe that the number of $\mathbf{m}$'s for which $\psi(\mathbf{m}) = k$ is $k + 3$ for $k > 0$, 4 if $k = 0$.

Let $\langle f, g \rangle$ denote $\int_{[0,1]^2} f(\mathbf{t})g(\mathbf{t})\, d\mathbf{t}$, and let $\| f \|$ denote $\sup_{\mathbf{t} \in [0,1]^2} |f(\mathbf{t})|$. Let

$$\alpha_{\mathbf{jm}}(\mathbf{t}) = \langle 1_{[\mathbf{0},\mathbf{t}]}, \Phi_{\mathbf{jm}} \rangle.$$

A simple calculation shows that $\|\alpha_{\mathbf{jm}}\| \leq 2(2^{-\psi(\mathbf{m})/2})$, while $\|\Phi_{\mathbf{jm}}\| = 2^{\psi(\mathbf{m})/2}$ is immediate from the definitions.

The support of $t \to \int_0^t \phi_{jm}(s)\, ds$ is the same as the support of ϕ_{jm}. Since by Fubini,

$$\alpha_{\mathbf{jm}}(\mathbf{t}) = \left(\int_0^{t_1} \phi_{j_1, m_1}(s_1)\, ds_1\right)\left(\int_0^{t_2} \phi_{j_2, m_2}(s_2)\, ds_2\right),$$

it follows that the support of $\alpha_{\mathbf{jm}}$ is the same as the support of $\Phi_{\mathbf{jm}}$. For each $\mathbf{m}$, the functions $\Phi_{\mathbf{jm}}$, $\mathbf{j} \in \Gamma(\mathbf{m})$, have disjoint support, and hence for each $\mathbf{m}$, each point $\mathbf{t}$ is in the support of at most one $\alpha_{\mathbf{jm}}$. So for fixed $\mathbf{m}$ and constants $d_{\mathbf{jm}}$, we have

$$\left\| \sum_{\mathbf{j} \in \Gamma(\mathbf{m})} d_{\mathbf{jm}} \alpha_{\mathbf{jm}} \right\| \leq \sup_{\mathbf{j} \in \Gamma(\mathbf{m})} |d_{\mathbf{jm}}|\, \|\alpha_{\mathbf{jm}}\|. \tag{2.1}$$

We also will need Parseval's identities,

$$\sum_{\mathbf{j},\mathbf{m}} \langle f, \Phi_{\mathbf{jm}} \rangle^2 = \langle f, f \rangle \tag{2.2}$$

and

$$\sum_{\mathbf{j},\mathbf{m}} \langle f, \Phi_{\mathbf{jm}} \rangle \langle g, \Phi_{\mathbf{jm}} \rangle = \langle f, g \rangle, \tag{2.3}$$

where the sum is over all $\mathbf{m}$ and all $\mathbf{j} \in \Gamma(\mathbf{m})$.

We now recall the construction of Brownian sheet by means of Haar functions [see Park (1970)]. Let $Z_{\mathbf{jm}}$ be independent standard normal variables. For each $\mathbf{t}$, $N \geq 1$, define

$$B_N(\mathbf{t}) = \sum_{\{\mathbf{m}:\, \psi(\mathbf{m}) \leq N\}} \sum_{\mathbf{j} \in \Gamma(\mathbf{m})} Z_{\mathbf{jm}} \alpha_{\mathbf{jm}}(\mathbf{t}). \tag{2.4}$$

Clearly, $B_N(\mathbf{t})$ is a mean 0 Gaussian process with continuous paths. It is known that $\|B_N - B\| \to 0$ in $L^2(\Omega, P)$, where B is the process defined by

$$B(\mathbf{t}) = \sum_{\mathbf{m}} \sum_{\mathbf{j} \in \Gamma(\mathbf{m})} Z_{\mathbf{jm}} \alpha_{\mathbf{jm}}(\mathbf{t}). \tag{2.5}$$

The fact that B is a mean 0 Gaussian process with continuous paths follows from the corresponding fact about B_N. Using (2.3), we have

$$\operatorname{Cov}(B(\mathbf{s}), B(\mathbf{t})) = \sum_{\mathbf{m}} \sum_{\mathbf{j}} \alpha_{\mathbf{jm}}(\mathbf{t}) \alpha_{\mathbf{jm}}(\mathbf{s}) = \sum_{\mathbf{m},\mathbf{j}} \langle 1_{[0,\mathbf{t}]}, \Phi_{\mathbf{jm}} \rangle \langle 1_{[0,\mathbf{s}]}, \Phi_{\mathbf{jm}} \rangle$$

$$= \langle 1_{[0,\mathbf{t}]}, 1_{[0,\mathbf{s}]} \rangle = |[\mathbf{0},\mathbf{t}] \cap [\mathbf{0},\mathbf{s}]|.$$

We next turn to stochastic integrals. If $h(\mathbf{t}) = \sum_{i=1}^{I} d_i 1_{[0,\mathbf{r}_i]}(\mathbf{t})$ is simple, the stochastic integral $\int h\,dB$ is defined to be $\sum_{i=1}^{I} d_i B(\mathbf{r}_i)$. Then $\int h\,dB$ is mean 0, Gaussian, $\mathbf{t} \to \int 1_{[0,\mathbf{t}]} h\,dB$ has continuous paths, and using the independent increments property of the Brownian sheet, an easy calculation gives

$$E\left(\int h\,dB\right)^2 = \int h^2(\mathbf{t})\,d\mathbf{t}. \tag{2.6}$$

Applying (2.6) to h, k and $h + k$, we get by polarization that

$$E\left(\int h\,dB\right)\left(\int k\,dB\right) = \int h(\mathbf{t}) k(\mathbf{t})\,d\mathbf{t}. \tag{2.7}$$

An argument using Doob's inequality applied twice [cf. Cairoli (1970)] shows that if $Y(\mathbf{t}) = \int 1_{[0,\mathbf{t}]} h\,dB$, then

$$E\|Y\|^2 \leq 16 \int h(\mathbf{t})^2\,d\mathbf{t}. \tag{2.8}$$

For $h \in L^2([0,1]^2, d\mathbf{t})$, choose h_n simple converging in L^2 to h, and define $\int h\,dB$ as the $L^2(\Omega, dP)$ limit of $\int h_n\,dB$. The equality (2.6) is used to show that the limit exists and is independent of the choice of h_n. Again, $\int h\,dB$ is mean 0, Gaussian, and (2.6) and (2.7) hold. Just as in the one-parameter case, (2.8) is used to show that $t \to \int 1_{[0,\mathbf{t}]} h\,dB$ has continuous paths a.s.

If F is a distribution function with a density f, then the process $Y(\mathbf{t}) = \int 1_{[0,\mathbf{t}]}\sqrt{f}\, dB$ is a mean 0 continuous Gaussian process. Applying (2.7) with $h = 1_{[0,\mathbf{s}]}\sqrt{f}$, $k = 1_{[0,\mathbf{t}]}\sqrt{f}$, we get

$$\operatorname{Cov}(Y(\mathbf{s}), Y(\mathbf{t})) = \int 1_{[0,\mathbf{s}]}1_{[0,\mathbf{t}]} f = F([0,\mathbf{s}] \cap [0,\mathbf{t}]).$$

Hence $Y(\mathbf{t})$ has the same law as $B_F(\mathbf{t})$.

We need the following simple properties of stochastic integrals.

LEMMA 2.1. *If $h \in L^2([0,1]^2, d\mathbf{t})$, then $E(\int h\, dB_N - \int h\, dB)^2 \to 0$ as $N \to \infty$.*

PROOF. Note $B_N(\mathbf{t}) = \int_{[0,\mathbf{t}]} b_N(\mathbf{s})\, d\mathbf{s}$, where

$$b_N(\mathbf{s}) = \sum_{\{\mathbf{m}:\, \psi(\mathbf{m}) \le N\}} \sum_{\mathbf{j} \in \Gamma(\mathbf{m})} Z_{\mathbf{jm}} \Phi_{\mathbf{jm}}(\mathbf{s}). \tag{2.9}$$

Hence the first integral in the statement of the lemma makes sense. For any $H \in L^2$, $\int H\, dB_N = \int H b_N\, d\mathbf{s}$ is mean 0, and recalling (2.2),

$$\begin{aligned} E\left(\int H b_N\, d\mathbf{s}\right)^2 &= \sum_{\{\mathbf{m}:\, \psi(\mathbf{m}) \le N\}} \sum_{\mathbf{j} \in \Gamma(\mathbf{m})} \left(\int H \Phi_{\mathbf{jm}}\, d\mathbf{s}\right)^2 \\ &\le \sum_{\mathbf{m},\mathbf{j}} \langle H, \Phi_{\mathbf{jm}} \rangle^2 = \langle H, H \rangle. \end{aligned} \tag{2.10}$$

Next, if $K = \sum d_i 1_{[0,\mathbf{r}_i]}$ is simple,

$$\int K\, dB_N = \sum d_i B_N(\mathbf{r}_i) \to \sum d_i B(\mathbf{r}_i) = \int K\, dB,$$

as $N \to \infty$, the convergence being in $L^2(\Omega, dP)$.

Given ε, choose K simple so that $\langle K - h, K - h \rangle \le \varepsilon$. Let $H = K - h$. Then by the triangle inequality, (2.6), and (2.10),

$$\begin{aligned} &\limsup_{N \to \infty} \left(E\left(\int h\, dB_N - \int h\, dB \right)^2 \right)^{1/2} \\ &\quad \le 2\varepsilon + \limsup_{N \to \infty} \left(E\left(\int K\, dB_N - \int K\, dB \right)^2 \right)^{1/2} = 2\varepsilon. \qquad \square \end{aligned}$$

LEMMA 2.2. *Suppose M has density m in $L^2([0,1]^2, d\mathbf{t})$ and H has density h in $L^1([0,1]^2, d\mathbf{t})$. Suppose $K \in L^2([0,1]^2, d\mathbf{t})$. Then*

$$\left| \int_{[0,\mathbf{t}]} HK\, dB - \int_{[0,\mathbf{t}]} HK\, dM \right| \le 4\left(\int |h|(\mathbf{r})\, d\mathbf{r} \right) \left\| \int_{[0,\cdot]} K\, dB - \int_{[0,\cdot]} K\, dM \right\|.$$

PROOF. It clearly suffices to take h bounded, the general case following by taking a limit. Let

$$Y_N(\mathbf{t}) = \int_{[\mathbf{0},\mathbf{t}]} K\,d(B_N - M), \qquad Y(\mathbf{t}) = \int_{[\mathbf{0},\mathbf{t}]} K\,dB - \int_{[\mathbf{0},\mathbf{t}]} K\,dM.$$

By Lemma 2.1, $E(Y_N(\mathbf{t}) - Y(\mathbf{t}))^2 \to 0$ for each $\mathbf{t}$, and moreover, by (2.10), $E(Y_N(\mathbf{t}) - Y(\mathbf{t}))^2 \le 4\langle K, K\rangle$. So by dominated convergence,

$$(2.11)\quad E\left|\int [Y_N(\mathbf{t}) - Y(\mathbf{t})]\,h(\mathbf{t})\,d\mathbf{t}\right| \le \|h\| \int \left(E\left(Y_N(\mathbf{t}) - Y(\mathbf{t})\right)^2\right)^{1/2} d\mathbf{t} \to 0.$$

Recalling the definition of $b_N(\mathbf{s})$ in (2.9), we have

$$\begin{aligned}
&\left|\int_{[\mathbf{0},\mathbf{t}]} H(\mathbf{s})K(\mathbf{s})[b_N(\mathbf{s}) - m(\mathbf{s})]\,d\mathbf{s}\right| \\
&\quad= \left|\int_{[\mathbf{0},\mathbf{t}]} K(\mathbf{s})[b_N(\mathbf{s}) - m(\mathbf{s})]\int_{[\mathbf{0},\mathbf{s}]} h(\mathbf{r})\,d\mathbf{r}\,d\mathbf{s}\right| \\
(2.12)\qquad &\quad= \left|\int_{[\mathbf{0},\mathbf{t}]} h(\mathbf{r})\int_{[\mathbf{r},\mathbf{t}]} K(\mathbf{s})[b_N(\mathbf{s}) - m(\mathbf{s})]\,d\mathbf{s}\,d\mathbf{r}\right| \\
&\quad= \left|\int [Y_N(\mathbf{t}) + Y_N(\mathbf{r}) - Y_N(r_1, t_2) - Y_N(t_1, r_2)]\,h(\mathbf{r})\,d\mathbf{r}\right|.
\end{aligned}$$

Let $N \to \infty$. Then $\int Y_N(\mathbf{r})h(\mathbf{r})\,d\mathbf{r} \to \int Y(\mathbf{r})h(\mathbf{r})\,d\mathbf{r}$ in $L^1(\Omega, dP)$ by (2.11). The other terms may be treated similarly, and so the right-hand side of (2.12) converges in $L^1(\Omega, dP)$ to

$$\left|\int [Y(\mathbf{t}) + Y(\mathbf{r}) - Y(r_1, t_2) - Y(t_1, r_2)]\,h(\mathbf{r})\,d\mathbf{r}\right| \le 4\|Y\| \int |h(\mathbf{r})|\,d\mathbf{r}.$$

By Lemma 2.1, the left-hand side of (2.12) converges to $|\int_{[\mathbf{0},\mathbf{t}]} HK\,dB - \int_{[\mathbf{0},\mathbf{t}]} HK\,dM|$, which completes the proof. □

Finally, we need to recall the Cameron–Martin–Girsanov theorem. There are several proofs in the literature, but using the Haar decomposition of the Brownian sheet, we can give the following simple proof.

THEOREM 2.3. *Suppose $h \in L^2([0,1]^2, d\mathbf{t})$. Define Q by*

$$dQ/dP = \exp\left(\int h\,dB - \tfrac{1}{2}\int h^2\,d\mathbf{t}\right).$$

Then under Q, B is a Gaussian process with $\mathrm{Cov}_Q(B(\mathbf{s}), B(\mathbf{t})) = |[\mathbf{0},\mathbf{s}] \cap [\mathbf{0},\mathbf{t}]|$ *and* $E_Q B(\mathbf{t}) = \int_{[\mathbf{0},\mathbf{t}]} h$.

PROOF. Write h as $\Sigma_{\mathbf{j},\mathbf{m}}\langle h, \Phi_{\mathbf{jm}}\rangle\Phi_{\mathbf{jm}}$. Then $\int h\,dB = \Sigma_{\mathbf{j},\mathbf{m}}\langle h, \Phi_{\mathbf{jm}}\rangle Z_{\mathbf{jm}}$. Recalling the independence of the $Z_{\mathbf{jm}}$'s, (2.2) and (2.3),

$$\begin{aligned} E_Q\exp(iuB(\mathbf{t})) &= E_P\exp\Big(\sum_{\mathbf{j},\mathbf{m}}(iu\langle 1_{[\mathbf{0},\mathbf{t}]}, \Phi_{\mathbf{jm}}\rangle + \langle h, \Phi_{\mathbf{jm}}\rangle)Z_{\mathbf{jm}}\Big)\exp\Big(-\tfrac{1}{2}\int h^2\Big) \\ &= \prod_{\mathbf{j},\mathbf{m}}\exp\Big(\tfrac{1}{2}(iu\langle 1_{[\mathbf{0},\mathbf{t}]}, \Phi_{\mathbf{jm}}\rangle + \langle h, \Phi_{\mathbf{jm}}\rangle)^2\Big)\exp\Big(-\tfrac{1}{2}\int h^2\Big) \\ &= \exp\Big(-\tfrac{1}{2}u^2\sum_{\mathbf{j},\mathbf{m}}\langle 1_{[\mathbf{0},\mathbf{t}]}, \Phi_{\mathbf{jm}}\rangle^2 + iu\sum_{\mathbf{j},\mathbf{m}}\langle 1_{[\mathbf{0},\mathbf{t}]}, \Phi_{\mathbf{jm}}\rangle\langle h, \Phi_{\mathbf{jm}}\rangle \\ &\qquad + \tfrac{1}{2}\sum_{\mathbf{j},\mathbf{m}}\langle h, \Phi_{\mathbf{jm}}\rangle^2\Big)\exp\Big(-\tfrac{1}{2}\int h^2\Big) \\ &= \exp\big(-\tfrac{1}{2}u^2\langle 1_{[\mathbf{0},\mathbf{t}]}, 1_{[\mathbf{0},\mathbf{t}]}\rangle + iu\langle 1_{[\mathbf{0},\mathbf{t}]}, h\rangle\big). \end{aligned}$$

So under Q, $B(\mathbf{t})$ is Gaussian with mean $\int h 1_{[\mathbf{0},\mathbf{t}]}$ and variance $\|[\mathbf{0},\mathbf{t}]\|$. A similar calculation for $E_Q\exp(i\Sigma_k u_k B(\mathbf{t}_k))$ shows that the finite-dimensional distributions of B under Q are Gaussian and that the covariances under Q are the same as those under P. □

3. Upper bound.

We proceed by a series of reductions.

PROPOSITION 3.1. *Suppose G is continuous and $G(\mathbf{1}) = 0$. Then*

$$P(\|W_F - G\| \le \varepsilon) \le c(\varepsilon)P(\|B_F - G\| \le 2\varepsilon),$$

where $c(\varepsilon) = O(1/\varepsilon)$ as $\varepsilon \to 0$.

PROOF. A standard calculation involving conditional distributions shows that the distribution of $B_F(\cdot)$ given $B_F(\mathbf{1}) = \mathbf{0}$ is that of a mean 0 Gaussian process with the same covariance structure as $W_F(\cdot)$. The distribution of $B_F(\cdot)$ given $B_F(\mathbf{1}) = x$ has the same covariance structure, and the mean is readily calculated:

$$\begin{aligned} E(B_F(\mathbf{t})|B_F(\mathbf{1}) = x) &= E(B_F(\mathbf{t}) - F(\mathbf{t})B_F(\mathbf{1})|B_F(\mathbf{1}) = x) + F(\mathbf{t})x \\ &= E(B_F(\mathbf{t}) - F(\mathbf{t})B_F(\mathbf{1})) + F(\mathbf{t})x = F(\mathbf{t})x, \end{aligned}$$

where in the next to the last equality we used the fact that $B_F(\mathbf{t}) - F(\mathbf{t})B_F(\mathbf{1})$ and $B_F(\mathbf{1})$ are Gaussian random variables whose covariance is 0, hence they are independent.

We then have

$$\begin{aligned} P(\|B_F - G\| \le 2\varepsilon) &\ge P(\|B_F - G\| \le 2\varepsilon, |B_F(\mathbf{1})| \le \varepsilon) \\ &= \int_{-\varepsilon}^{\varepsilon} P(\|B_F - G\| \le 2\varepsilon | B_F(\mathbf{1}) = x)P(B_F(\mathbf{1}) \in dx) \\ &\ge \int_{-\varepsilon}^{\varepsilon} P(\|B_F - F(\cdot)x - G\| \le \varepsilon | B_F(\mathbf{1}) = x)P(B_F(\mathbf{1}) \in dx) \\ &\ge (c(\varepsilon))^{-1}P(\|W_F - G\| \le \varepsilon). \end{aligned}$$

□

PROPOSITION 3.2. *Suppose $G(\mathbf{1}) = 0$, G has a density g in $L^2([0,1]^2, d\mathbf{t})$, and F has a density f with*

$$\frac{d(f^{1/2})}{d\mathbf{t}}, \frac{d(f^{-1/2})}{d\mathbf{t}} \in L^1([0,1]^2, d\mathbf{t}).$$

Then

$$P(\|B_F - G\| \le \varepsilon) \le P(\|B - L\| \le c(F)\varepsilon),$$

where L is the function with density $g/\sqrt{f}$.

PROOF. By Section 2, $\int_{[0,\mathbf{t}]} f^{1/2}\, dB$ has the same law as B_F. By Lemma 2.2 (with $H = f^{-1/2}$, $K = f^{1/2}$, $M = L$),

$$|B(\mathbf{t}) - L(\mathbf{t})| \le (c(F))^{-1}\left\| \int_{[0,\cdot]} f^{1/2}\, dB - \int_{[0,\cdot]} f^{1/2}\, dL \right\| = (c(F))^{-1}\|B_F - G\|.$$

The proposition is now immediate. □

PROPOSITION 3.3. *If H has a density $h \in L^2([0,1]^2, d\mathbf{t})$, then*

$$P(\|B - H\| \le \varepsilon) \le c_1 \exp\left(-\tfrac{1}{2}\int_{[0,1]^2} h^2\, d\mathbf{t}\right)\exp\left(-c_2\varepsilon^{-2}\ln(1/\varepsilon)\right).$$

PROOF. By the definition of stochastic integrals,

$$\begin{aligned} &\left|\int 1_{[\mathbf{r},\mathbf{s}]}\, dB - \int 1_{[\mathbf{r},\mathbf{s}]}\, dH\right| \\ &\quad = |(B - H)(\mathbf{s}) + (B - H)(\mathbf{r}) - (B - H)(r_1, s_2) - (B - H)(s_1, r_2)| \\ &\quad \le 4\|B - H\|. \end{aligned}$$

Since any Haar function $\Phi_{\mathbf{jm}}$ can be written as $\Sigma_i d^{(i)}_{\mathbf{jm}} 1_{[\mathbf{r}_i, \mathbf{s}_i]}$, where the sum is from $i = 1$ to $i = 1$, 2 or 4 (depending on $\mathbf{j}, \mathbf{m}$) and $d^{(i)}_{\mathbf{jm}} = \pm\|\Phi_{\mathbf{jm}}\|$, then $|\int \Phi_{\mathbf{jm}}\, d(B - H)| \le 16\|\Phi_{\mathbf{jm}}\|\,\|B - H\|$. On the other hand, $\int \Phi_{\mathbf{jm}}\, d(B - H) = Z_{\mathbf{jm}} - \langle \Phi_{\mathbf{jm}}, h\rangle$. So if $\|B - H\| \le \varepsilon$, then for each $\mathbf{j}$ and $\mathbf{m}$, $|Z_{\mathbf{jm}} - \langle \Phi_{\mathbf{jm}}, h\rangle| \le 16(2^{\psi(\mathbf{m})/2})\varepsilon$. Since the $Z_{\mathbf{jm}}$ are independent, then for each M,

$$P(\|B - H\| \le \varepsilon) \le \prod_{\{\mathbf{m}:\, \psi(\mathbf{M}) \le M\}} \prod_{\mathbf{j} \in \Gamma(\mathbf{m})} p_{\mathbf{jm}}, \tag{3.1}$$

where

$$p_{\mathbf{jm}} = P\left(|Z_{\mathbf{jm}} - \langle \Phi_{\mathbf{jm}}, h\rangle| \le 16(2^{\psi(\mathbf{m})/2})\varepsilon\right).$$

By the trivial inequality

$$P(Z \in A) \le (2\pi)^{-1/2}\left(\sup_{y \in A} e^{-y^2/2}\right)|A|,$$

for Z a standard normal random variable, we have that for each $\mathbf{j}$ and $\mathbf{m}$,

$$p_{\mathbf{jm}} \le (2\pi)^{-1/2} 16(2^{\psi(\mathbf{m})/2})\varepsilon. \tag{3.2}$$

By this same inequality, we have

$$
(3.3)\quad \begin{aligned} p_{\mathbf{jm}} &\le (2\pi)^{-1/2} 16(2^{\psi(\mathbf{m})/2})\varepsilon \exp\Big(-\tfrac{1}{2}\big(|\langle \Phi_{\mathbf{jm}}, h\rangle| - 16(2^{\psi(\mathbf{m})/2})\varepsilon\big)^2\Big) \\ &\le 16(2\pi)^{-1/2} 2^{\psi(\mathbf{m})/2}\varepsilon \exp\big(-\tfrac{1}{2}\langle \Phi_{\mathbf{jm}}, h\rangle^2\big)\exp\big(c(\mathbf{j},\mathbf{m},h)(\varepsilon + o(\varepsilon))\big) \end{aligned}
$$

provided $|\langle \Phi_{\mathbf{jm}}, h\rangle| > 16(2^{\psi(\mathbf{m})/2})\varepsilon$.

Now let $\delta \in (0,1)$ be fixed, and choose K large so that

$$
\sum_{\{\mathbf{m}:\, \psi(\mathbf{m}) \le K\}} \sum_{\mathbf{j}} \langle h, \Phi_{\mathbf{jm}}\rangle^2 \ge \langle h, h\rangle - \delta.
$$

Then take ε small enough so that for every $\mathbf{m}$ with $\psi(\mathbf{m}) \le K$ and every $\mathbf{j} \in \Gamma(\mathbf{m})$, either $|\langle \Phi_{\mathbf{jm}}, h\rangle| = 0$ or $> 16(2^{\psi(\mathbf{m})/2})\varepsilon$. Pick M so that $16(2\pi)^{-1/2}(2^{M/2}) \in [\varepsilon^{-1}/16, \varepsilon^{-1}]$. If necessary, choose ε even smaller so that $M > K$.

Observe by our choice of ε and (3.2) that (3.3) holds for every $\mathbf{m}$ such that $\psi(\mathbf{m}) \le K$ and $\mathbf{j} \in \Gamma(\mathbf{m})$. Since K does not depend on ε, we have

$$
(3.4)\quad \prod_{\{\mathbf{m}:\, \psi(\mathbf{m}) \le K\}} \prod_{\mathbf{j} \in \Gamma(\mathbf{m})} \exp\big(c(\mathbf{j},\mathbf{m},h)(\varepsilon + o(\varepsilon))\big) = O(1),
$$

as $\varepsilon \to 0$.

Using (3.3) when $\psi(\mathbf{m}) \le K$ and (3.2) when $K < \psi(\mathbf{m}) \le M$, substituting into (3.1), and recalling (3.4), we have

$$
(3.5)\quad \begin{aligned} P(\|B - H\| \le \varepsilon) \le c_1 &\Bigg(\prod_{\{\mathbf{m}:\, \psi(\mathbf{m}) \le K\}} \prod_{\mathbf{j} \in \Gamma(\mathbf{m})} \exp\big(-\tfrac{1}{2}\langle \Phi_{\mathbf{jm}}, h\rangle^2\big)\Bigg) \\ &\times \Bigg(\prod_{\{\mathbf{m}:\, \psi(\mathbf{m}) \le M\}} \prod_{\mathbf{j} \in \Gamma(\mathbf{m})} 16(2\pi)^{-1/2} 2^{\psi(\mathbf{m})/2}\varepsilon\Bigg). \end{aligned}
$$

By the choice of K, the product of the exponentials is

$$
\le \exp\big(-\tfrac{1}{2}(\langle h, h\rangle - \delta)\big) \le c_3 \exp\big(-\tfrac{1}{2}\langle h, h\rangle\big),
$$

and so to complete the proof, it suffices to bound

$$
\begin{aligned} q &= \prod_{\{\mathbf{m}:\, \psi(\mathbf{m}) \le M\}} \prod_{\mathbf{j} \in \Gamma(\mathbf{m})} 16(2\pi)^{-1/2} 2^{\psi(\mathbf{m})/2}\varepsilon \\ &= \prod_{\{\mathbf{m}:\, \psi(\mathbf{m}) \le M\}} \big(c 2^{\psi(\mathbf{m})/2}\varepsilon\big) 2^{\psi(\mathbf{m})}, \end{aligned}
$$

where by the choice of M, $c2^{\psi(\mathbf{m})/2}\varepsilon \le 1$. Taking logarithms, we get

$$
(3.6)\quad \begin{aligned} \ln q &\le \sum_{k=0}^{M} \sum_{\{\mathbf{m}:\, \psi(\mathbf{m}) = k\}} 2^k \ln\big(c 2^{k/2}\varepsilon\big) \\ &\le \sum_{k=0}^{M} k 2^k \ln\big(c 2^{k/2}\varepsilon\big) \\ &\le c_1 \int_1^{2^M} \ln x \ln\big(c_2 x \varepsilon^2\big)\, dx \\ &= c_1 \varepsilon^{-2} \int_{\varepsilon^2}^{2^M\varepsilon^2} \ln y \ln(c_2 y)\, dy + 2c_1 \varepsilon^{-2} \ln(1/\varepsilon) \int_{\varepsilon^2}^{2^M\varepsilon^2} \ln(c_2 y)\, dy \\ &\le -c_3 \varepsilon^{-2} \ln(1/\varepsilon), \end{aligned}
$$

if $\varepsilon \le 1/256$. To get the last inequality, we used the facts that

$$\int_{\varepsilon^2}^{2M_\varepsilon^2} \ge \int_{1/(256)^2}^{2\pi/(256)^2}$$

and that the integrands do not depend on ε. □

REMARK. For $d > 2$, $\sum_{k=0}^{M}\sum_{\{\mathbf{m}:\ \psi(\mathbf{m})=k\}} 1 \ge \sum_{k=0}^{M} k^{d-1}$, provided $\psi(\mathbf{m})$ is defined in the obvious way. With this change in (3.6), we obtain $-c_3\varepsilon^{-2}(\ln(1/\varepsilon))^{d-1}$.

THEOREM 3.4. *Suppose $G(\mathbf{1}) = 0$, G has a density g, F has a density f,*

$$\frac{d(f^{1/2})}{d\mathbf{t}}, \frac{d(f^{-1/2})}{d\mathbf{t}} \in L^1([0,1]^d, d\mathbf{t}) \quad \text{and} \quad g/\sqrt{f} \in L^2([0,1]^d, d\mathbf{t}).$$

Then

$$P(\|W_F - G\| \le \varepsilon) \le c_1 \exp\left(-\tfrac{1}{2}\int_{[0,1]^d} (g^2/f)\, d\mathbf{t} - c_2\varepsilon^{-2}(\ln(1/\varepsilon))^{d-1}\right),$$

where c_1 and c_2 are constants depending only on F.

PROOF. Combine Propositions 3.1, 3.2 and 3.3. □

4. Lower bound. Again, we proceed by a series of reductions.

PROPOSITION 4.1. *If $G(\mathbf{1}) = 0$, then*

$$P(\|W_F - G\| \le \varepsilon) \ge P(\|B_F - G\| \le \varepsilon/2).$$

PROOF. Define $Y(\mathbf{t}) = B_F(\mathbf{t}) - F(\mathbf{t})B_F(\mathbf{1})$. Clearly, Y is mean 0 Gaussian with continuous paths, and a simple calculation shows that the covariances of Y are the same as those of W_F, hence the law of Y is the same as the law of W_F. But since $\|F\| \le 1$, we have that if $\|B_F - G\| \le \varepsilon/2$, then in particular $|B_F(\mathbf{1})| \le \varepsilon/2$, and so $\|Y - G\| \le \varepsilon$. The proof is now immediate. □

PROPOSITION 4.2. *Suppose F has a density f with $d(f^{1/2})/d\mathbf{t} \in L^1([0,1]^2, d\mathbf{t})$. Suppose G has a density g with $g/\sqrt{f} \in L^2([0,1]^2, d\mathbf{t})$. Let L be the function with density $g/\sqrt{f}$. Then*

$$P(\|B_F - G\| \le \varepsilon) \ge P(\|B - L\| \le c\varepsilon).$$

PROOF. Apply Lemma 2.2 (with $K = 1$, $M = L$ and $H = f^{1/2}$) to get $|B_F(\mathbf{t}) - G(\mathbf{t})| \le c\|B - L\|$. □

PROPOSITION 4.3. *If H has a density $h \in L^2([0,1]^2, d\mathbf{t})$, then*

$$P(\|B - H\| \le \varepsilon) \ge \exp\left(-\tfrac{1}{2}\int h^2\, d\mathbf{t}\right) P(\|B\| \le \varepsilon).$$

PROOF [cf. Csáki (1980)]. By the symmetry of the Gaussian measure, for all $\mathbf{r}$ and $\mathbf{s}_1, \ldots, \mathbf{s}_k$,

$$E(B(\mathbf{r});\ |B(\mathbf{s}_1)| < \varepsilon, \ldots, |B(\mathbf{s}_k)| < \varepsilon) = 0.$$

By taking limits, we get $E(B(\mathbf{r});\ \|B\| < \varepsilon) = 0$. Using this, we have

$$E\left(\int h\,dB;\ \|B\| < \varepsilon\right) = 0, \tag{4.1}$$

if h is simple, and by taking limits, we have (4.1) for all $h \in L^2$.

Define a probability measure Q by $dQ/dP = \exp(-\int h\,dB - \frac{1}{2}\int h^2\,d\mathbf{t})$. By Theorem 2.3, the law of B under Q is that of a Gaussian process, B has continuous paths under Q (since P and Q are equivalent), $\mathrm{Cov}_Q(B(\mathbf{s}), B(\mathbf{t})) = |[\mathbf{0}, \mathbf{s}] \cap [0, \mathbf{t}]|$ and $E_Q B(\mathbf{t}) = -H(\mathbf{t})$. Thus the law of B under Q is the same as the law of $B - H$ under P. Therefore

$$P(\|B - H\| \le \varepsilon) = Q(\|B\| \le \varepsilon) = \int_{(\|B\| \le \varepsilon)} \exp\left(-\int h\,dB - \tfrac{1}{2}\int h^2\,d\mathbf{t}\right) dP. \tag{4.2}$$

Using Jensen's inequality with respect to $P(\cdot\,|\,\|B\| \le \varepsilon)$, we get

$$\begin{aligned} &\int_{(\|B\| \le \varepsilon)} \exp\left(-\int h\,dB\right) dP \\ &\quad \ge P(\|B\| \le \varepsilon)\exp\left(\int_{(\|B\| \le \varepsilon)} \left(\int -\ h\,dB\right) dP/P(\|B\| \le \varepsilon)\right). \end{aligned} \tag{4.3}$$

Combining (4.1), (4.2) and (4.3) completes the proof. □

PROPOSITION 4.4. $P(\|B\| \le \varepsilon) \ge c_1 \exp(-c_2 \varepsilon^{-2}(\ln(1/\varepsilon))^3)$.

PROOF. Fix ε small, and choose N so that $2^N \varepsilon^2 / N^2 \in [1, 16]$. Let $\beta_{\mathbf{m}} = (3/\pi^2(\psi(\mathbf{m}) + 4))(|N - \psi(\mathbf{m})| + 1)^{-2}$, so that $\sum_{\mathbf{m}} \beta_{\mathbf{m}} = \sum_{k=0}^{\infty} \sum_{\{\mathbf{m}:\ \psi(\mathbf{m}) = k\}} \beta_{\mathbf{m}} \le 1$. Let $r_{\mathbf{jm}} = \ln P(|Z_{\mathbf{jm}}| < 2^{\psi(\mathbf{m})/2} \beta_{\mathbf{m}} \varepsilon / 2)$.

By (2.1) with $d_{\mathbf{jm}} = Z_{\mathbf{jm}}(\omega)$, we have that if $\sup_{\mathbf{j} \in \Gamma(\mathbf{m})} \|\alpha_{\mathbf{jm}}\|\,|Z_{\mathbf{jm}}| < \beta_{\mathbf{m}} \varepsilon$ for each $\mathbf{m}$, then

$$\|B\| \le \sum_{\mathbf{m}} \left\| \sum_{\mathbf{j} \in \Gamma(\mathbf{m})} Z_{\mathbf{jm}} \alpha_{\mathbf{jm}} \right\| < \sum_{\mathbf{m}} \beta_{\mathbf{m}} \varepsilon \le \varepsilon.$$

So using the independence of the $Z_{\mathbf{jm}}$'s, we have

$$\begin{aligned} P(\|B\| \le \varepsilon) &\ge \prod_{\mathbf{m}} P\left(\sup_{\mathbf{j} \in \Gamma(\mathbf{m})} \|\alpha_{\mathbf{jm}}\|\,|Z_{\mathbf{jm}}| < \beta_{\mathbf{m}} \varepsilon\right) \\ &= \prod_{\mathbf{m}} P\left(\sup_{\mathbf{j} \in \Gamma(\mathbf{m})} |Z_{\mathbf{jm}}| < 2^{\psi(\mathbf{m})/2} \beta_{\mathbf{m}} \varepsilon / 2\right) \\ &= \prod_{\mathbf{m}} \prod_{\mathbf{j} \in \Gamma(\mathbf{m})} \exp(r_{\mathbf{jm}}). \end{aligned} \tag{4.4}$$

We now proceed to estimate. Recalling that the $r_{\mathbf{jm}}$ are ≤ 0 we have

$$q_1 = \sum_{\{\mathbf{m}:\, \psi(\mathbf{m}) \le N\}} \sum_{\mathbf{j} \in \Gamma(\mathbf{m})} r_{\mathbf{jm}} \tag{4.5}$$
$$\ge \sum_{k=0}^{N} \sum_{\{\mathbf{m}:\, \psi(\mathbf{m}) = k\}} 2^k \ln P\big(|Z| < 2^{k/2}\beta_{\mathbf{m}}\varepsilon/2\big),$$

where Z is a standard normal variable. By our choice of N, $2^{k/2}\beta_{\mathbf{m}}\varepsilon/2$ is bounded above and below by constants independent of N and ε, hence

$$P\big(|Z| < 2^{k/2}\beta_{\mathbf{m}}\varepsilon/2\big) \ge c2^{k/2}\beta_{\mathbf{m}}\varepsilon/2,$$

and so

$$q_1 \ge \sum_{k=0}^{N} (k+4)2^k \ln\left(\frac{c2^{k/2}\varepsilon}{(k+4)(N-k+1)^2}\right)$$
$$\ge c_1 \sum_{k=1}^{N} k2^k \ln\left(\frac{c_2 2^k \varepsilon^2}{k^2(N+1-k)^4}\right) + o(\varepsilon^{-2})$$
$$\ge c_1 \int_2^{2^N} \ln x \ln\left(\frac{c_2 x\varepsilon^2}{(\ln x)^4 (N+2-\ln x)^4}\right) dx + o(\varepsilon^{-2})$$
$$\ge c_1 \int_2^{2^N} \ln x \ln\left(\frac{c_2 x \varepsilon^2}{(\ln x)^2}\right) dx$$
$$-c_1 \int_1^{2^N} \ln x \ln\left(\frac{2^{N+2}}{x}\right) dx + o(\varepsilon^{-2})$$
$$= c_1 I_1 - c_1 I_2 + o(\varepsilon^{-2}).$$

Since $2^N\varepsilon^2/N^2$ is bounded above and below independent of N and ε, we can write

$$I_1 \ge \int_2^{2^N} (\ln x)^2\, dx + \ln c_2 \int_2^{2^N} \ln x\, dx \tag{4.6}$$
$$-\ln\left(\frac{2^N}{N^2}\right) \int_2^{2^N} \ln x\, dx - 2\int_2^{2^N} \ln x \ln\ln x\, dx.$$

Now $\int (\ln x)^2 = x(\ln x)^2 - 2x\ln x + 2x$; $\int \ln x = x\ln x - x$; and $\int \ln x \ln\ln x = x \ln x \ln\ln x - x - \int \ln\ln x$. Carrying out the integrations in (4.6) and using the fact that $|\int_2^{2^N} \ln\ln x\, dx| \le 2^N \ln\ln 2^N = o(N2^N)$, we get

$$I_1 \ge -cN2^N. \tag{4.7}$$

For I_2, we have

$$I_2 \le c2^N \int_{2^{-N}}^{1} \ln(2^N u)\ln(4/u)\, du \tag{4.8}$$
$$= c2^N \ln 2^N \int_{2^{-N}}^{1} \ln(4/u)\, du + c2^N \int_{2^{-N}}^{1} \ln u \ln(4/u)\, du$$
$$\le cN2^N.$$

Combining, we get

$$(4.9)\qquad q_1 \geq -cN2^N = -cN^3 2^N/N^2 \geq -c(\ln(1/\varepsilon))^3 \varepsilon^{-2}.$$

For $\gamma \geq \gamma_0 > 0$,

$$\ln P(|Z| < \gamma) = \ln(1 - 2P(Z > \gamma)) \geq \ln(1 - c_1 \exp(-c_2\gamma^2)) \geq -c_1 \exp(-c_2\gamma^2),$$

for constants depending on γ_0 but not γ. Hence

$$\begin{aligned} q_2 &= \sum_{\{\mathbf{m}:\ \psi(\mathbf{m}) > N\}} \sum_{\mathbf{j} \in \Gamma(\mathbf{m})} r_{\mathbf{jm}} \\ &= \sum_{k=N+1}^{\infty} \sum_{\{\mathbf{m}:\ \psi(\mathbf{m}) = k\}} 2^k \ln P\left(|Z| < \frac{2^{k/2}\beta_{\mathbf{m}}\varepsilon}{2}\right) \\ (4.10)\qquad &\geq -c_1 \sum_{k=N+1}^{\infty} (k+3)2^k \exp\left(\frac{-c_2 2^k \varepsilon^2}{(k+4)^2(k+1-N)^4}\right) \\ &\geq -c_1 N^2 \varepsilon^{-2} \sum_{j=1}^{\infty} (j+N)2^j \exp\left(\frac{-c_2 2^j N^2}{(j+N)^2(j+1)^4}\right) \\ &= -c_1 N^2 \varepsilon^{-2} \left(\sum_{j=1}^{N} + \sum_{j>N}\right) = J_1 + J_2. \end{aligned}$$

Now

$$J_1 \geq -c_1 N^2 \varepsilon^{-2}(2N) \sum_{j=1}^{\infty} 2^j \exp\left(\frac{-c_2 2^j}{(j+1)^4}\right) \geq -c_1 N^3 \varepsilon^{-2},$$

while

$$J_2 \geq -c_1 N^2 \varepsilon^{-2} \sum_{j=1}^{\infty} (2j)2^j \exp\left(\frac{-c_2 2^j}{j^2(j+1)^4}\right) \geq -c_1 N^2 \varepsilon^{-2}.$$

Substituting in (4.10),

$$(4.11)\qquad q_2 \geq -cN^3\varepsilon^{-2} \geq -c\varepsilon^{-2}(\ln(1/\varepsilon))^3.$$

Adding (4.9) and (4.11) and substituting in (4.4) proves the proposition. □

REMARK. Again, for $d > 2$, we get a $(\ln(1/\varepsilon))^{3(d-1)}$ term from the fact that $\sum_{\{\mathbf{m}:\ \psi(\mathbf{m}) = k\}} 1 \approx k^{d-1}$. To use the two different estimates for $P(|Z| < \gamma)$ and yet have $\sum \beta_{\mathbf{m}} \leq 1$, we must choose N so that $2^N/N^{2(d-1)} \approx \varepsilon^{-2}$.

Finally, we have

THEOREM 4.5. *Suppose F and G have densities f and g, respectively, with $d(f^{1/2})/d\mathbf{t} \in L^1([0,1]^d, d\mathbf{t})$ and $g/\sqrt{f} \in L^2([0,1]^d, d\mathbf{t})$. Suppose $G(\mathbf{1}) = 0$.*

Then

$$P(\|W_F - G\| \le \varepsilon) \ge c_1 \exp\left(-\tfrac{1}{2} \int_{[0,1]^d} (g^2/f)\, dt - c_2 \varepsilon^{-2} (\ln(1/\varepsilon))^{3(d-1)} \right),$$

where c_1 and c_2 are constants depending only on F.

PROOF. Combine Propositions 4.1, 4.2, 4.3 and 4.4. □

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