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WEAK CONVERGENCE OF THE VARIATIONS, ITERATED INTEGRALS AND DOLÉANS–DADE EXPONENTIALS OF SEQUENCES OF SEMIMARTINGALES[1]

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If $X^{(n)}$ is a sequence of semimartingales, converging to a semimartingale X, and such that $[X^{(n)}, X^{(n)}]$ converges to $[X, X]$, then all higher-order variations and all the iterated integrals of $X^{(n)}$ converge jointly to the respective functionals of X.

1. Introduction.

A. Let $X_t^{(n)}$ be a sequence of semimartingales, with $t \in [0,1]$, such that

$$X^{(n)} \to_w X, \tag{1.1}$$

where X is a semimartingale, and $\to_w$ denotes weak convergence on $D[0,1]$ with respect to the J_1-Skorohod topology.

We investigate the convergence of the variations, iterated integrals and Doléans–Dade exponentials of $X^{(n)}$, which are defined as follows: For Y a semimartingale,

$$V_k(Y)_t = \begin{cases} Y_t, & \text{for } k = 1, \\ [Y, Y]_t = \langle Y^c, Y^c \rangle_t + \sum_{s \le t} (\Delta Y_s)^2, & \text{for } k = 2, \\ \sum_{s \le t} (\Delta Y_s)^k, & \text{for } k \ge 3, \end{cases} \tag{1.2}$$

$$I_k(Y)_t = \begin{cases} Y_t, & \text{for } k = 1, \\ \int_0^t I_{k-1}(Y)_{s-}\, dY_s, & \text{for } k \ge 2, \end{cases} \tag{1.3}$$

$$E(\lambda Y)_t = \exp\left[\lambda Y_t - \frac{\lambda^2}{2}[Y, Y]_t\right] \prod_{s \le t} l(\lambda\, \Delta Y_s), \tag{1.4}$$

where $l(x) = (1 + x)e^{-x + x^2/2}$.

$V_k(Y)$, $I_k(Y)$ and $E(\lambda Y)$ are called, respectively, the variations, the iterated integrals and the Doléans–Dade exponentials of the semimartingale Y. It is known that V_k, I_k and E are well defined for any semimartingale Y [see Meyer (1976)]. These quantities are important in the theory of multiple integration

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with respect to Y_t.

B. When $X_t^{(n)} = \sum_{i=1}^{[nt]} X_{i,n}$, with $X_{i,n}$ a triangular array, then

$$V_k(X^{(n)})_t = \sum_{i=1}^{[nt]} X_{i,n}^k,$$

$$I_k(X^{(n)})_t = \sum_{1 \le i_1 < \cdots < i_k \le [nt]} X_{i_1,n} \cdots X_{i_k,n},$$

and

$$E(\lambda X^{(n)})_t = \prod_{i=1}^{[nt]} (1 + \lambda X_{i,n}) = \sum_{k=0}^{[nt]} \lambda^k I_k(X^{(n)})_t.$$

The problem of the convergence of these "moments," "symmetric statistics" and generating function of the symmetric statistics has been studied in [1], [3–5], [7] and [9].

C. From formula 41.1 of Meyer (1976), it follows that in the semimartingale context, just as in the discrete deterministic case, I_k, $k = 1, \ldots, m$, and V_k, $k = 1, \ldots, m$, can be represented as polynomials of n variables in one another (the Newton polynomials which relate sums of powers to the sums of products). Thus, the issue of the joint convergence of I_k, $k = 1, \ldots, m$, and that of the convergence of V_k, $k = 1, \ldots, m$, are equivalent.

D. $X^{(n)} \to_w X$ does not imply in general $[X^{(n)}, X^{(n)}] \to [X, X]$, as the following deterministic example from Jacod (1983) shows:

$$X_t^{(n)} = \sum_{k=1}^{[n^2 t]} \frac{(-1)^k}{n} \text{ converges uniformly to 0, but } [X^{(n)}, X^{(n)}]_t = \sum_{k=1}^{[n^2 t]} \frac{1}{n^2} \to t.$$

E. However, the following result holds.

Theorem 1. *The following three statements are equivalent*:

$$(1.5)\qquad (X^{(n)}, [X^{(n)}, X^{(n)}]) \to_w (X, [X, X]), \quad \text{as } n \to \infty;$$

$$(1.6)\qquad (V_1(X^{(n)}), \ldots, V_m(X^{n})) \to_w (V_1(X), \ldots, V_m(X)), \quad \text{as } n \to \infty, \quad \forall\, m \ge 2;$$

$$(1.7)\qquad (I_1(X^{(n)}), \ldots, I_m(X^{(n)})) \to_w (I_1(X), \ldots, I_m(X)), \quad \text{as } n \to \infty, \quad \forall\, m \ge 2.$$

They also imply

$$(1.8)\qquad E(\lambda X^{(n)}) \to_w E(\lambda X), \quad \forall\, \lambda.$$

Corollary. *If*

$$(1.9)\qquad X^{(n)} \to_w X$$

and the condition of Jacod (1983) *holds*:

$$\lim_{b\to\infty} \sup_{n\to\infty} P\{\mathrm{Var}(B^{h,n})_1 > b\} = 0 \tag{1.10}$$

[*where h is a truncation function and* $(B^{h,n})_t$ *is the finite variation term in the canonical decomposition of the truncated semimartingale* $X^{(n)}$], *then* (1.5)—(1.8) *hold*.

PROOF. Cf. Jacod (1983), Theorem 5.1.1. (1.9) and (1.10) imply (1.5). □

2. Proofs. Introduce the following notation: For any real number x,

$$x^{>a} := x \cdot 1_{\{|x| > a\}},$$

$$x^{\le a} := x \cdot 1_{\{|x| \le a\}}.$$

We establish now the following.

LEMMA 1. (a) *Suppose* $X^{(n)}$ *are semimartingales such that*

$$\lim_{b\to\infty} \limsup_{n\to\infty} P\{[X^{(n)}, X^{(n)}]_1 > b\} = 0, \tag{2.1}$$

and let $f(x)$ *be any real function such that* $f(x) = o(x^2)$, *as* $x \to 0$. *Then, for all* ε,

$$\lim_{a\to 0} \limsup_{n\to\infty} P\left\{ \sum_{s\le 1} \left|f\left(\left(\Delta X_s^{(n)}\right)^{\le a}\right)\right| \ge \varepsilon \right\} = 0. \tag{2.2}$$

(b) *If the assumptions of* (a) *hold*, $X^{(n)} \to_w X$ *and* f *is a continuous, vector-valued function, then*

$$\sum_{s\le t} f\left(\Delta X_s^{(n)}\right) \to_w \sum_{s\le t} f(\Delta X_s). \tag{2.3}$$

PROOF. (a) Note first that $\Sigma_{s\le t}\left| f\left(\Delta X_s^{(n)}\right)\right| < \infty$, since $\Sigma_{s\le t}(\Delta X_s^{(n)})^2 < \infty$. Let now $g(a) = \sup_{|x|\le a}|f(x)|/x^{-2}$. Then

$$P\left\{ \sum_{s\le 1} \left|f\left(\left(\Delta X_s^{(n)}\right)^{\le a}\right)\right| > \varepsilon \right\} \le P\left\{ \sum_{s\le 1} \left(\left(\Delta X_s^{(n)}\right)^{\le a}\right)^2 g(a) > \varepsilon \right\}$$
$$\le P\{[X^{(n)}, X^{(n)}]_1 > \varepsilon/g(a)\}.$$

Since $g(a) \to 0$, (2.2) follows from (2.1).

(b) Let $U(X) = \{u > 0: P\{|\Delta X_t| \ne u, \text{ for all } t\} = 1\}$. $U(X)$ is dense in R_+. For any $a \in U(X)$, and f continuous, the functional

$$S_f^a(Z)_t = \sum_{s\le t} f(\Delta Z_s^{>a})$$

is J_1-continuous a.s. [dist(X)]. Thus, $X^{(n)} \to_w X$ implies for $a \in U(X)$,

$$S_f^a(X^{(n)}) \to_w S_f^a(X).$$

Also,

$$S_f^a(X)_t \to_w S_f(X)_t := \sum_{s \le t} f(\Delta X_s), \quad \text{as } a \to 0 \text{ a.s. } (J_1).$$

The result follows now by (2.2) and Theorem 4.2 of Billingsley (1968). □

PROOF OF THEOREM 1. By Lemma 1(b), we have (1.5) ⇒ (1.6), and in fact the same type of argument yields (1.5) ⇒ (1.8), as follows: Assume for convenience $\lambda = 1$ and $1 \in U(X)$, let

$$f(x) = \left[\ln(1+x) - x + \frac{x^2}{2} \right],$$

and let $T\colon D_{[0,1]} \to D_{[0,1]}$ be defined by

$$T(Z)_t := \prod_{s \le t} l(\Delta Z_s^{>1}) = \prod_{s \le t} (1 + \Delta Z_s^{>1}) \exp\left\{ -\Delta Z_s^{>1} + \tfrac{1}{2} (\Delta Z_s^{>1})^2 \right\}.$$

Since the Doléans–Dade exponential

$$E(X)_t = \exp\left\{ X_t - \tfrac{1}{2}[X, X]_t + \sum_{s \le t} f\left[\Delta X_s^{\le 1}\right] \right\} T(X)_t,$$

it remains only to note that the functional

$$X^a\colon D^{(2)}[0,1] \to D^{(4)}[0,1],$$
$$X(Z_1, Z_2) = \left(Z_1, Z_2, S_f^a(Z_1), T(Z_1)\right)$$

is continuous a.s., if both spaces are endowed with the respective J_1-topologies. Letting then $a \to 0$, as in the proof of Lemma 1, one gets

$$\left(X_t^{(n)}, [X^{(n)}, X^{(n)}]_t, \sum_{s \le t} f\left((\Delta X_s^{(n)})^{\le 1} \right), \prod_{s \le t} l\left((\Delta X_s^{(n)})^{>1} \right) \right)$$
$$\to_w \left(X_t, [X, X]_t, \sum_{s \le t} f(\Delta X_s^{\le 1}), \prod_{s \le t} l(\Delta X_s^{>1}) \right),$$

since $\ln(1+x) - x + x^2/2 = o(x^2)$, and since (1.5) implies (2.1). Finally, applying the continuous functional

$$\rho\colon D_{[0,1]}^{(4)} \to D_{[0,1]},$$
$$\rho(Z_1, Z_2, Z_3, Z_4) = \exp\left[Z_1 - \tfrac{1}{2} Z_2 + Z_3\right] Z_4,$$

we get that

$$E(\lambda X^{(n)}) \to_w E(\lambda X).$$

Since (1.6) is equivalent to (1.7) (by the use of the polynomial mapping), and (1.6) trivially implies (1.5), Theorem 1 is proved. □

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