The Annals of Probability
1988, Vol. 16, No. 1, 1–57

SPECIAL INVITED PAPER

SELF-INTERSECTION GAUGE FOR RANDOM WALKS AND FOR BROWNIAN MOTION[1]

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A class of random fields associated with multiple points of a random walk in the plane is studied. It is proved that these fields converge in distribution to analogous fields measuring self-intersections of the planar Brownian motion. The concluding section contains a survey of literature on intersection local times and their renormalizations. A brief look through the first pages of this section could provide the reader with additional motivation for the present work.

1. Introduction.

1.1. A *random walk* on a d-dimensional lattice $\mathbb{Z}^d$ is a sequence

$$S_0 = 0, \qquad S_n = \xi_1 + \cdots + \xi_n \quad \text{for } n = 1, 2, \ldots, \tag{1.1}$$

where $\xi_1, \ldots, \xi_n, \ldots$ are i.i.d. random variables with values in $\mathbb{Z}^d$. We assume that:

1.1.A. ξ_i have mean 0 and finite second moments.
1.1.B. The covariance matrix for ξ_i is the identity matrix.
1.1.C. The set of m such that $P\{S_m = 0\} > 0$ generates the group $\mathbb{Z}$ of integers (aperiodicity).
1.1.D. The set of x such that $P\{\xi_1 = x\} > 0$ generates $\mathbb{Z}^d$.

Only condition 1.1.A and an assumption that the covariance matrix is nondegenerate are substantial. The rest of the conditions are imposed only to simplify notations.

The formula

$$X_t^\alpha = \sqrt{\alpha}\, S_{t/\alpha}, \qquad t \in \mathbb{Q}_\alpha = \{0, \alpha, 2\alpha, \ldots, n\alpha, \ldots\} \tag{1.2}$$

defines a random walk on the lattice $\mathbb{Z}_\alpha = \sqrt{\alpha}\,\mathbb{Z}^d$. As $\alpha \downarrow 0$, the finite-dimensional distributions of X_t^α converge to those of the Brownian motion W_t in $\mathbb{R}^d$.

Let $\tilde{X}_t^\alpha$ be a function on $[0, \infty)$ which is linear on each interval $[n\alpha, (n+1)\alpha]$ and coincides with X_t^α on $\mathbb{Q}_\alpha$. According to Donsker's invariance principle, if F is a continuous functional on $C([0, c], \mathbb{R}^d)$, then

$$F(\tilde{X}^\alpha) \to_d F(W) \tag{1.3}$$

Received October 1986.

[1]Partially supported by National Science Foundation Grant DMS-85-05020. This paper is based on a Special Invited Lecture delivered at the IMS Annual Meeting in Chicago, August 18, 1986.

AMS 1980 *subject classifications*. Primary 60G60; secondary 60J55, 60J65.

Key words and phrases. Self-intersection local times, self-intersection gauges, the invariance principle, multiple points of random walks and of the Brownian motion.

(writing $\to_d$ means convergence in distribution). If ρ is a continuous function with compact support, then the formula

$$(1.4)\qquad F(W) = \int_0^c dt_1 \cdots \int_0^c dt_k\, \rho(t_1, W_{t_1}; \cdots; t_k, W_{t_k})$$

defines a continuous functional on $C([0,c], \mathbb{R}^d)$ and $F(\tilde{X}^\alpha) \to_d F(W)$ by the invariance principle. Let

$$(1.5)\qquad F_\alpha(X^\alpha) = \alpha^k \sum \rho(t_1, X^\alpha_{t_1}; \cdots; t_k, X^\alpha_{t_k}),$$

where the sum is taken over all $t_1, \dots, t_k \in \mathbb{Q}_\alpha \cap [0, c]$. We note that $E|F_\alpha(X^\alpha) - F(\tilde{X}^\alpha)| \to 0$ and, therefore,

$$(1.6)\qquad F_\alpha(X^\alpha) \to_d F(W) \quad \text{as } \alpha \downarrow 0.$$

1.2. Formula (1.6) can be extended to a certain class of discontinuous functionals by using the following lemma, which follows immediately from Lemma A.1 in the Appendix.

LEMMA 1.1. *We have*

$$(1.7)\qquad \lim_{\alpha \to 0} F_\alpha(X^\alpha) = \lim_{\varepsilon \to 0} F^\varepsilon(W) \quad \textit{in distribution},$$

if there exist functionals $F^\varepsilon_\alpha = F^\varepsilon_\alpha(X^\alpha)$ *such that*

$$(1.8)\qquad F^\varepsilon_\alpha \to_d F^\varepsilon \quad \textit{as } \alpha \to 0,$$

$$(1.9)\qquad F^\varepsilon_\alpha \to_d F_\alpha \quad \textit{as } \varepsilon \to 0$$

and

$$(1.10)\qquad \rho(F^\varepsilon_\alpha, F_\alpha) \to 0 \quad \textit{as } \alpha, \varepsilon \to 0,$$

where ρ *is an arbitrary metric with the property* $U_n \to_d U$ *if* $\rho(U_n, U) \to 0$ (*which is true, for instance, for an* L^p *metric* $\rho(U, V) = [E|U - V|^p]^{1/p}$).

We call F^ε a *Brownian shadow* for F_α and F^ε_α a *link* between F_α and F^ε.

1.3. One can use Lemma 1.1 for proving limit theorems by a modified method of moments.

Suppose that for all positive integers p,

$$(1.11)\qquad \lim_{\alpha \to 0} E(F_\alpha)^p = E(F)^p < \infty.$$

If

$$(1.12)\qquad \sum E[|F|^p]^{-1/p} = \infty,$$

then $F_\alpha \to_d F$ [see, e.g., Feller (1966), pages 224 and 262]. Condition (1.12) is not satisfied for the functionals we are going to investigate. We overcome the difficulty by proving, instead of (1.11), a stronger relation: For all integers $k, l \geq 0$,

$$(1.13)\qquad \lim_{\alpha, \varepsilon \to 0} E\left[(F_\alpha)^k (F^\varepsilon_\alpha)^l\right] = EF^{k+l} < \infty,$$

which implies (1.10) for the L^p metric. In our case,

$$F = \lim_{\varepsilon \to 0} F^\varepsilon \quad \text{in } L^p \text{ for all } p \geq 1 \tag{1.14}$$

and F_α, F^ε and F_α^ε satisfy conditions (1.8) and (1.9). We conclude from Lemma 1.1 that $F_\alpha \to_d F$.

1.4. We are interested in limit distributions for random fields $\rho_\alpha(t_1, X_{t_1}^\alpha; \cdots; t_k, X_{t_k}^\alpha)$. This means that we consider functionals

$$\begin{aligned} F_\alpha(f) &= \alpha^k \sum \rho_\alpha\big(t_1, X_{t_1}^\alpha; \cdots; t_k, X_{t_k}^\alpha\big) f(t) \\ &= \int \rho_\alpha\big(t_1, X_{t_1}^\alpha; \cdots; t_k, X_{t_k}^\alpha\big) f(t) \nu_\alpha(dt), \end{aligned} \tag{1.15}$$

where $t = (t_1, \ldots, t_k)$, ν_α is the uniform measure on $\mathbb{Q}_\alpha^k$ which charges each point with α^k and f is a test function. We want to find a generalized random field $F(f)$ associated with the Brownian motion such that $F_\alpha(f) \to_d F(f)$ for all test functions f. To this end, we need to define a shadow field $\rho^\varepsilon(t_1, W_{t_1}; \cdots; t_k, W_{t_k})$ and a link $\rho_\alpha^\varepsilon(t_1, X_{t_1}^\alpha; \cdots; t_k, X_{t_k}^\alpha)$ in such a way that

$$F^\varepsilon(f) = \int \rho^\varepsilon\big(t_1, W_{t_1}; \cdots; t_k, W_{t_k}\big) f(t)\, dt \tag{1.16}$$

and

$$F_\alpha^\varepsilon(f) = \int \rho_\alpha^\varepsilon\big(t_1, X_{t_1}^\alpha; \cdots; t_k, X_{t_k}^\alpha\big) f(t) \nu_\alpha^k(dt) \tag{1.17}$$

satisfy conditions (1.8)–(1.10).

1.5. Additive functionals of order k studied in Dynkin (1986a) are an example of generalized random fields which appear in this context. Basically, an additive functional of the Brownian motion is a random measure $A(\omega; dt_1, \ldots, dt_k)$ on $\mathbb{R}_+^k$ such that for any open intervals $I_1, \ldots, I_k$, the restriction of A to $I_1 \times \cdots \times I_k$ is a kernel from $(\Omega, \mathscr{F}(I_1 \cup \cdots \cup I_k)^*)$ to $(I_1 \times \cdots \times I_k, \mathscr{B}(I_1 \times \cdots \times I_k))$. Here $\mathscr{B}$ stands for the Borel σ-algebra, $\mathscr{F}(U)$ means the σ-algebra in Ω generated by W_t, $t \in U$, and an asterisk means the universal completion of a σ-algebra. (The exact definition includes some finiteness and continuity assumptions.) The corresponding random field is defined by the formula

$$A(f) = \int f(t_1, \ldots, t_k) A(dt_1, \ldots, dt_k) \tag{1.18}$$

for all positive Borel functions. Moreover, f can depend on ω. We put

$$A(\varphi) = \int \varphi\big(t_1, W_{t_1}; \cdots; t_k, W_{t_k}\big) A(dt_1, \ldots, dt_k) \tag{1.19}$$

for every positive Borel function $\varphi(t_1, x_1; \cdots; t_k, x_k)$. It turns out that

$$EA(\varphi) = \int \varphi(t_1, x_1; \cdots; t_k, x_k) p(t_1, x_1; \cdots; t_k, x_k) \mu(dt, dx), \tag{1.20}$$

where p is the joint probability density for $W_{t_1}, \ldots, W_{t_k}$ and μ is a measure on $(\mathbb{R}_+ \times \mathbb{R}^d)^k$. The measure μ determines A uniquely up to equivalence. We call it the *spectral measure* of A. If a spectral measure has the form

$$\lambda(dx_1, \ldots, dx_k)\, dt_1 \cdots dt_k, \tag{1.21}$$

then λ is called the *characteristic measure* of A. We can rewrite (1.19) in the form

$$\begin{aligned} A(\varphi) &= \int \varphi(t_1, W_{t_1}; \cdots; t_k, W_{t_k}) \rho(W_{t_1}, \ldots, W_{t_k})\, dt_1 \cdots dt_k \\ &= \int \varphi(t_1, W_{t_1}; \cdots; t_k, W_{t_k}) \lambda(dz_1, \ldots, dz_k) \\ &\quad \times \delta_{z_1}(W_{t_1}) \cdots \delta_{z_k}(W_{t_k})\, dt_1 \cdots dt_k, \end{aligned} \tag{1.22}$$

where ρ is the (generalized) Radon–Nikodym derivative of λ with respect to the Lebesgue measure and δ_z is Dirac's function (which can be interpreted as the derivative of the unit mass concentrated at z with respect to the Lebesgue measure).

1.6. Which σ-finite measures λ are characteristic measures? An answer is well known for additive functionals of order 1: λ corresponds to an additive functional if and only if it does not charge any polar set. In the general case, Dynkin (1986a) proved that λ is a characteristic measure if, for every compact set B,

$$\begin{aligned} &\int_B p(t_1, y_1 - x_1; \cdots; t_k, y_k - x_k) \\ &\quad \times \lambda(dx_1, \ldots, dx_k) \lambda(dy_1, \ldots, dy_k)\, dt_1 \cdots dt_k < \infty. \end{aligned} \tag{1.23}$$

This condition is satisfied if $d = 1$ and if $\lambda(B) < \infty$ for all compact B. An additive functional

$$T(z, \Gamma) = \int_\Gamma \delta_z(W_t)\, dt, \tag{1.24}$$

corresponding to a unit measure at point z, is concentrated on the set $\{t\colon W_t = z\}$ and it is called the local time at z. This functional is not defined for $d > 1$.

The *self-intersection local time* is an additive functional

$$T(k, \Gamma) = \int_{\mathbb{R}^d} dz \int_\Gamma \delta_{z_1}(W_{t_1}) \cdots \delta_{z_k}(W_{t_k})\, dt_1 \cdots dt_k, \tag{1.25}$$

corresponding to the measure

$$\lambda(B) = \int_{\mathbb{R}^d} 1_B(z, \ldots, z)\, dz. \tag{1.26}$$

(Here $dz = dz_1 \cdots dz_k$.) The measure $T(k, \cdot)$ is concentrated on the set $\{t\colon W_{t_1} = \cdots = W_{t_k}\}$. Note that $T(1, dt) = dt$. The measure (1.26) satisfies the condition (1.23) for all k if $d = 2$ and for $k = 2$ if $d = 3$. This condition does not hold for all other pairs (d, k) with $d > 1$ and $k > 1$. This, of course, is closely related to the fact that, with probability 1, the Brownian motion in $\mathbb{R}^d$ has multiple points of every multiplicity k for $d \le 2$, has only double but not triple points for $d = 3$ and has no self-intersections for $d \ge 4$.

We concentrate on the case $d = 2$. *Starting from this point on, a Brownian motion is always a Brownian motion in* $\mathbb{R}^2$ *and a random walk is a random walk in* $\mathbb{Z}^2$.

1.7. Suppose that $k > 1$. By symmetry, it is sufficient to study the self-intersection local time $T(k, \cdot\,)$ in the region $D_k = \{0 \le t_1 \le \cdots \le t_k\}$. We denote by $\partial_1 D_k$ the part of the boundary which is located on hyperplanes $\{t\colon t_i = t_{i+1}\}$, $i = 1, \dots, k-1$. It turns out that $T(k, U) = \infty$ for every neighborhood U of every point $t \in \partial_1 D_k$. We construct a one-parameter family of random fields $\mathscr{T}(\kappa; k, \varphi)$ which coincides with $T(k, \cdot\,)$ outside of $\partial_1 D_k$. We call it the *self-intersection gauge for the Brownian motion.* [One of these fields appears in Dynkin (1986b, 1988). It corresponds to $\kappa = C/2\pi$, where $C = 0.5772\dots$ is Euler's constant.] The fields $\mathscr{T}(\kappa; k, \cdot\,)$ are defined for test functions of the form $\varphi(W_{t_1}; t_1, \dots, t_k)$, where φ is a function on $\mathbb{R}^2 \times D_k$ with the following properties:

1.7.A. φ has a compact support.
1.7.B. For every $x \in \mathbb{R}^2$, $\varphi(x, \cdot\,)$ is infinitely differentiable in t on D_k and all partials are continuous in x, t.

Conditions 1.7.A, B define a functional space $\mathscr{L}_k$ in which we introduce a natural topology. For every κ and every $p \ge 1$, $\mathscr{T}(\kappa; k, \cdot\,)$ is a continuous linear operator from $\mathscr{L}_k$ to L^p. The fields $\mathscr{T}(\kappa; k, \varphi)$ can be continued to functions which satisfy condition 1.7.B only near $\partial_1 D_k$. Moreover, they can be continued to functions $\varphi(t_1, W_{t_1}; \cdots; t_k, W_{t_k})$ by the formula $\mathscr{T}(k, \varphi) = \mathscr{T}(k, \hat{\varphi})$, where

$$\hat{\varphi}(x; t_1, \dots, t_k) = \varphi(t_1, x; \cdots; t_k, x).$$

Our main goal is to show that a certain class of random fields related to multiple points of random walks converges in distribution to $\mathscr{T}(\kappa; k, \cdot\,)$.

1.8. Put

$$1(a_1, \dots, a_k) = 1, \quad \text{if } a_1 = \cdots = a_k,$$
$$= 0, \quad \text{otherwise}.$$

Let $h = \{h_n\}$ be a sequence of real numbers. We call the family of functionals

$$\mathscr{T}(h; k, n, \varphi) = n^{-1} \sum_{0 \le m_1 \le \cdots \le m_k} \Big\{ \varphi\big(S_{m_1}/\sqrt{n}\,;\, m_1/n, \dots, m_k/n\big) \times \prod_{i=2}^{k} \big[1(S_{m_{i-1}}, S_{m_i}) + h_n 1(m_{i-1}, m_i)\big] \Big\} \tag{1.27}$$

the *self-intersection gauge for the random walk* S_m. We assume that φ has a compact support and, therefore, the sum on the right side contains only a finite number of nonzero terms. [If $k = 1$, then $\mathscr{T}(h; 1, n, \varphi) = n^{-1}\sum_{m\ge 0}\varphi(S_m/\sqrt{n})$ does not depend on h.]

By Lemma A.4 (see also the remark following the lemma),

$$\mathscr{T}(h+\tilde h; k, n, \varphi) = \sum_{l=1}^{k} \tilde h_n^{k-l}\mathscr{T}(h; l, n, B_k^l\varphi), \tag{1.28}$$

where

$$(B_k^l\varphi)(t_1,\dots,t_l) = \sum_\sigma \varphi(t_{\sigma_1},\dots,t_{\sigma_k}), \tag{1.29}$$

with σ running over all mappings from $\{1,2,\dots,k\}$ onto $\{1,2,\dots,l)$ which satisfy the condition $\sigma_i \le \sigma_j$ for $i \le j$.

Put

$$T(k,n,\varphi) = n^{-1}\sum_{0\le m_1<\cdots<m_k}\varphi\left(S_{m_1}/\sqrt{n}\,;\, m_1/n,\dots,m_k/n\right) \times 1(S_{m_1},\dots,S_{m_k}). \tag{1.30}$$

Note that $T(k,n,\varphi) = \mathscr{T}(-1; k, n, \varphi)$ and by (1.28),

$$\mathscr{T}(h; k, n, \varphi) = \sum_{l=1}^{k}(h_n+1)^{k-l}T(l, n, B_k^l\varphi) \tag{1.31}$$

and

$$T(k,n,\varphi) = \sum_{l=1}^{k}(-h_n-1)^{k-l}\mathscr{T}(h; l, n, B_k^l\varphi). \tag{1.32}$$

If $\varphi = 0$ on $\partial_1 D_k$, then $\mathscr{T}(h; k, n, \varphi) = T(k, n, \varphi)$ for all h.

1.9. We put $\varphi \in \mathscr{L}_k^c$ if φ belongs to $\mathscr{L}_k$ and if the support of φ is contained in the region

$$\{|x| \le c, 0 \le t_1 \le \cdots \le t_k \le c\}. \tag{1.33}$$

We define a sequence of norms $\|\varphi\|_l$, $l = 0,1,2,\dots$, in $\mathscr{L}_k$ by setting $\|\varphi\|_0 = \|\varphi\| = \sup|f(x,t)|$ over all $x\in\mathbb{R}^2$, $t \in D_k$, and by defining $\|\varphi\|_m$ as the maximum of the uniform norms $\|\cdot\|$ of all partial derivatives of order less than or equal to m relative to t. We introduce a topology in $\mathscr{L}_k$ by setting $\varphi_n \to \varphi$ if all φ_n belong to the same $\mathscr{L}_k^c$ and if $\|\varphi_n - \varphi\|_m \to 0$ for all m.

Put $\varphi \in \mathscr{E}_k^{\beta c}$ if φ is a bounded Borel function on $\mathbb{R}^2\times D_k$ and if the support of φ is contained in the intersection of the set (1.33) with the region $\{t_i - t_{i-1} \ge \beta$ for $i = 1,\dots,k\}$.

Denote by $\mathscr{E}_k$ the union of $\mathscr{E}_k^{\beta c}$ over all positive β and c and set $\varphi_n \to \varphi$ in $\mathscr{E}_k$ if all φ_n and φ belong to the same $\mathscr{E}_k^{\beta c}$ and if $\|\varphi_n - \varphi\| \to 0$.

Put $\mathscr{S}_k = \mathscr{L}_k + \mathscr{E}_k$. It follows from Lemma A.5 that $\varphi \in \mathscr{S}_k$ if and only if it is a bounded Borel function with a compact support which satisfies, for some

$\beta > 0$, condition 1.7.B with D_k replaced by the region

$$U_\beta = \bigcup_{i=2}^{k} \{0 \le t_i - t_{i-1} \le \beta\}. \tag{1.34}$$

Note that $B_k^l(\mathscr{L}_k) \subset \mathscr{L}_l$ and $B_k^l(\mathscr{E}_k) = 0$ for $l < k$.

THEOREM 1.1. *Let q be a probability density in $\mathbb{R}^2$ which is bounded and has a compact support. Put*

$$q^\varepsilon(y) = \varepsilon^{-2} q(y/\varepsilon), \tag{1.35}$$

$$T^\varepsilon(k, \varphi) = \int_{D_k} \varphi(W_{t_1}; t_1, \dots, t_k) \prod_{i=2}^{k} q^\varepsilon(W_{t_i} - W_{t_{i-1}})\, dt, \tag{1.36}$$

$$\mathscr{T}^\varepsilon(h; k, \varphi) = \sum_{l=1}^{k} (h^\varepsilon)^{k-l} T^\varepsilon(l, B_k^l \varphi), \tag{1.37}$$

where h^ε are constants and B_k^l are operators given by (1.29). *For every $\varphi \in \mathscr{S}_k$ and every real κ, there exists a random variable $\mathscr{T}(\kappa; k, \varphi)$, independent of q and such that*

$$E|\mathscr{T}^\varepsilon(h; k, \varphi) - \mathscr{T}(\kappa; k, \varphi)|^p \to 0 \tag{1.38}$$

for every $p \ge 1$ if

$$h^\varepsilon + E\int_0^1 q^\varepsilon(W_t)\, dt \to \kappa. \tag{1.39}$$

For every c, k, p and β, there exists a constant C such that

$$\begin{aligned} \|\mathscr{T}(k, \varphi)\|_{L^p} &\le C\|\varphi\|_{(k-1)p} \quad \text{for all } \varphi \in \mathscr{L}_k^c, \\ &\le C\|\varphi\| \quad \text{for all } \varphi \in \mathscr{E}_k^{\beta c}. \end{aligned} \tag{1.40}$$

For all κ and a,

$$\mathscr{T}(\kappa + a; k, \varphi) = \sum_{l=1}^{k} a^{k-l} \mathscr{T}(\kappa; l, B_k^l \varphi). \tag{1.41}$$

REMARK. Condition (1.39) is satisfied (with κ depending on q) if and only if $h^\varepsilon + 1/\pi \ln(1/\varepsilon)$ tends to a finite limit as $\varepsilon \to 0$.

1.10. We say that a function in $\mathbb{R}^n$ is *almost continuous* if it is continuous outside of a closed set of the Lebesgue measure 0.

THEOREM 1.2. *Let q be an almost continuous density function subject to the conditions of Theorem* 1.1. *If*

$$h_n + \sum_{m=0}^{n} P\{S_m = 0\} \to \kappa, \tag{1.42}$$

then for every $k = 1, 2, \ldots$ *and every almost continuous* $\varphi \in \mathscr{S}_k$,

$$\mathscr{T}(h; n, k, \varphi) \to_d \mathscr{T}(\kappa; k, \varphi). \tag{1.43}$$

Moreover, for all almost continuous $\varphi_i \in \mathscr{S}_{k_i}$, *the joint probability distribution of* $\mathscr{T}(h; k_1, n, \varphi_1), \ldots, \mathscr{T}(h; k_r, n, \varphi_r)$ *converges weakly to the distribution of* $\mathscr{T}(\kappa; k_1, \varphi_1), \ldots, \mathscr{T}(\kappa; k_r, \varphi_r)$ *and*

$$E \prod_{i=1}^{r} \mathscr{T}(h; k_i, n, \varphi_i) \to E \prod_{i=1}^{r} \mathscr{T}(\kappa; k_i, \varphi_i). \tag{1.44}$$

REMARK. Since $P\{S_m = 0\} \sim (2\pi m)^{-1}$ as $m \to \infty$ [see, e.g., Spitzer (1964), P7.9], we have

$$\sum_{m=0}^{n} P\{S_m = 0\} \sim \ln n / 2\pi \quad \text{as } n \to \infty \tag{1.45}$$

and $h_n \sim -\ln n/2\pi$ for every h subject to the condition (1.42).

1.11. Because of (1.28) and an analogous relation for $\mathscr{T}^\varepsilon(h; k, \varphi)$ which follows from (1.37), it is sufficient to prove Theorems 1.1 and 1.2 for the case $\kappa = 0$ and

$$h^\varepsilon = -E \int_0^1 q^\varepsilon(W_t)\, dt, \tag{1.46}$$

$$h_n = -\sum_{m=0}^{n} P\{S_m = 0\}. \tag{1.47}$$

To prove Theorem 1.2, we express $\mathscr{T}(h; k, n, \varphi)$ in terms of the process X_t^α. Let

$$\hat{h}_\alpha = -\sum_{t \in \mathbb{Q}_\alpha} 1_{t \le 1} P\{X_t^\alpha = 0\} = -E \int 1_{t \le 1} \delta_\alpha(X_t^\alpha) \nu_\alpha(dt), \tag{1.48}$$

$$\begin{aligned} \mathscr{T}_\alpha(\hat{h}, k, \varphi) &= \int_{D_k} \varphi(X_{t_1}^\alpha; t_1, \ldots, t_k) \\ &\quad \times \prod_{i=2}^{k} \left(\delta_\alpha(X_{t_i}^\alpha - X_{t_{i-1}}^\alpha) + \hat{h}_\alpha \delta_\alpha(t_i - t_{i-1})\right) \nu_\alpha(dt). \end{aligned} \tag{1.49}$$

Here

$$\delta_\alpha(0) = 1/\alpha, \qquad \delta_\alpha(y) = 0 \quad \text{for } y \ne 0,$$

is a discrete analogue of Dirac's function. Obviously,

$$h_n = \hat{h}_{1/n}, \qquad \mathscr{T}(h; k, n, \varphi) = \mathscr{T}_{1/n}(\hat{h}; k, \varphi). \tag{1.50}$$

We show that $\mathscr{T}^\varepsilon(h; k, \varphi)$ is a Brownian shadow of $\mathscr{T}_\alpha(\hat{h}; k, \varphi)$ with the link

$$\begin{aligned} \mathscr{T}_\alpha^\varepsilon(h; k, \varphi) &= \int_{D_k} \varphi(X_{t_1}^\alpha; t_1, \ldots, t_k) \\ &\quad \times \prod_{i=2}^{k} \left(q_\alpha^\varepsilon(X_{t_i}^\alpha - X_{t_{i-1}}^\alpha) + h_\alpha^\varepsilon \delta_\alpha(t_i - t_{i-1})\right) \nu_\alpha(dt). \end{aligned} \tag{1.51}$$

Here

$$h_\alpha^\varepsilon = -E\int 1_{t\le 1} q_\alpha^\varepsilon(X_t^\alpha)\nu_\alpha(dt) \tag{1.52}$$

and

$$q_\alpha^\varepsilon(y) = \alpha^{-1}\int_{B(\alpha,\, y)} q^\varepsilon(x)\, dx, \tag{1.53}$$

with

$$B(\alpha, y) = \left[y_1 - \sqrt{\alpha}/2, y_1 + \sqrt{\alpha}/2\right] \times \left[y_2 - \sqrt{\alpha}/2, y_2 + \sqrt{\alpha}/2\right). \tag{1.54}$$

If Y is an $\mathbb{R}^2$-valued random variable with the density q and if ψ_α is a mapping which sends each point of $\mathbb{R}^2$ to a point of the lattice $\mathbb{Z}_\alpha$ at the minimal distance from x, then $q_\alpha^\varepsilon(y) = \alpha^{-1}P\{\psi_\alpha(\varepsilon Y) = x\}$, $x \in \mathbb{Z}_\alpha$. Note that $\psi_\alpha(x) = 0$ for $|x| < \sqrt{\alpha}/2$. If a is an upper bound for $|Y|$, then $q_\alpha^\varepsilon = \delta_\alpha$, $h_\alpha^\varepsilon = \hat{h}_\alpha$ and $\mathscr{T}_\alpha^\varepsilon = \mathscr{T}_\alpha$ for $\varepsilon < \sqrt{\alpha}/(2a)$. Hence, $\mathscr{T}_\alpha^\varepsilon(k, \varphi)$ with an arbitrary φ satisfies condition (1.9). Assumptions which are sufficient for (1.8) are given by

LEMMA 1.2. *If q and φ are bounded almost continuous Borel functions with compact supports, then $\mathscr{T}_\alpha^\varepsilon(h; k, \varphi) \to_d \mathscr{T}^\varepsilon(0; k, \varphi)$ as $\alpha \to 0$.*

The hard part is to establish the relationship (1.10).

Obviously, (1.10) holds for $\varphi_1 + \varphi_2$ if it holds for φ_1 and for φ_2. Therefore, it is sufficient to deal with φ which belongs either to $\mathscr{L}_k$ or to $\mathscr{E}_k$. The proof of Theorem 1.3 occupies the main part of the paper.

THEOREM 1.3. *Suppose that q is almost continuous and satisfies the conditions of Theorem 1.1. Suppose that either $\varphi_i \in \mathscr{L}_{k_i}$ for all $i = 1, \dots, p$ or φ_i belong to $\mathscr{E}_{k_i}$ and are almost continuous for all $i = 1, \dots, p$. Let $\{1, 2, \dots, p\} = \mathscr{A} \cup \mathscr{B}$ be a partition of $\{1, \dots, p\}$ into two disjoint subsets. Put*

$$\wp_\alpha^\varepsilon = \prod_{i\in\mathscr{A}}\left[\mathscr{T}_\alpha(\hat{h}; k_i, \varphi_i)\prod_{i\in\mathscr{B}}\mathscr{T}_\alpha^\varepsilon(h; k_i, \varphi_i)\right], \tag{1.55}$$

where $\hat{h}_\alpha$, $\mathscr{T}_\alpha$, h_α^ε and $\mathscr{T}_\alpha^\varepsilon$ are defined by (1.48), (1.49), (1.52) *and* (1.51).

There exists a finite limit

$$\lim_{\alpha,\, \varepsilon \downarrow 0} E\wp_\alpha^\varepsilon. \tag{1.56}$$

For every c, k, p, β, there exists a constant C such that, for all sufficiently small α, ε,

$$\begin{aligned} |E\wp_\alpha^\varepsilon| &\le C\prod_1^p \|\varphi_i\|_m, \quad \text{if } \varphi_i \in \mathscr{L}_{k_i}^c, \\ &\le C\prod_1^p \|\varphi_i\|, \quad \text{if } \varphi_i \in \mathscr{E}_k^{\beta c}. \end{aligned} \tag{1.57}$$

Here, $m = \sum_1^p(k_i - 1)$.

1.12. An explicit description of the limits (1.56) involves some combinatorics and certain operators which map functions of several variables to functions of a part of these variables. We give such a description in Theorems 2.2 and 2.3 after necessary tools are introduced. In Section 2, we also outline the main steps for proving results stated in Section 1. The first step—an investigation of the limit behaviour of Green's function and some other integrals related to the process X_t^α— is done in Section 3. The proof of the main results is divided between Section 4, which contains the combinatorial part, and Section 5, where we do a passage to the limit. In Section 6, we review the literature and mention some open problems. A few auxiliary propositions used in the paper are proved in the Appendix.

At each stage, we explain the objective first and present technicalities later. For this reason, propositions are not proved in the same order they are stated. We hope that the following diagram of the logical relations between theorems (T.) and lemmas (L.) will be helpful for the reader:

$$\begin{array}{ccccccccc}
 & & \text{L.1.1} & & & & & & \\
 & & \Downarrow & & & & & & \\
\text{L.1.2} & \Rightarrow & \text{T.1.2} & \Rightarrow & \text{T.1.1} & & & & \\
 & & & & \Uparrow & & & & \\
\Uparrow\ \text{L.3.3} & \Rightarrow & \text{T.3.3} & & \text{T.1.3} & \Leftarrow & \text{T.2.3} & & \\
 & & & \searrow & \Uparrow & & \Uparrow & & \\
\text{L.3.2} & \Rightarrow & & & \text{T.2.2} & \Leftarrow & \text{T.2.1} & \Leftarrow & \text{T.5.1} \\
 & & & & \Uparrow & & \Uparrow & & \\
 & & & & \text{L.2.2} & \Leftarrow & \text{T.5.2} & & \\
 & & & & \Uparrow & & \Uparrow & & \\
 & & & & \text{T.3.2} & \Rightarrow & \text{L.5.1} & \Leftarrow & \text{L.3.1} \\
 & & & & & & & & \Uparrow \\
 & & & & & & & & \text{T.3.1}
\end{array}$$

2. Outline of proofs.

2.1. To evaluate $E\wp_\alpha^\varepsilon$, we consider a finite set S partitioned into disjoint ordered subsets S^i, $i \in \mathscr{A} \cup \mathscr{B}$, with $|S^i| = k_i$. Let A and B stand for the unions of S^i over $\mathscr{A}$ and over $\mathscr{B}$. We say that elements of S^i have color i and we put $s \sim s'$ if s and s' are of the same color. Denote by l_i the smallest element of S^i and put $L = \{l_i,\ i \in \mathscr{A} \cup \mathscr{B}\}$, $M = S \setminus L$. Define a mapping $s \to [s]$ from S onto L by the formula $[s] = l_i$ for $s \in S^i$. For every set $\Lambda \subset S$, put $\Lambda_A = \Lambda \cap A$ and $\Lambda_B = \Lambda \cap B$. If $\Lambda \subset S$, then $\prod_\Lambda$ means the product over $s \in \Lambda$. Writing t_Λ means the same as $\{t_s,\ s \in \Lambda\}$ and

$$\nu_\alpha(dt_\Lambda) = \prod_\Lambda \nu_\alpha(dt_s). \tag{2.1}$$

Put

$$\varphi(z_L, t_S) = \prod_{i \in \mathscr{A} \cup \mathscr{B}} \varphi_i(z_{l_i}, t_{S_i}). \tag{2.2}$$

By (1.49) and (1.51),

$$(2.3)\quad \begin{aligned}\wp_\alpha^\varepsilon = \int_{\mathbb{D}} \nu_\alpha(dt_S)\varphi(X_{t_L}^\alpha, t_S)\prod_{M_A}\left[\delta_\alpha(X_{t_s}^\alpha - X_{t_{s^*}}^\alpha) + h_\alpha\delta_\alpha(t_s - t_{s^*})\right]\\ \times\prod_{M_B}\left[q_\alpha^\varepsilon(X_{t_s}^\alpha - X_{t_{s^*}}^\alpha) + h_\alpha^\varepsilon\delta_\alpha(t_s - t_{s^*})\right],\end{aligned}$$

where $\mathbb{D}$ is the product of the regions D_{k_i} and s^* means the biggest element of the same color as s which is smaller than s.

Let λ_α be the uniform measure on $\mathbb{Z}_\alpha$ which charges each point with α and let p_t^α be the probability density for X_t^α relative to λ_α. If $\Lambda = \{s_1, \dots, s_r\}$ and if $t_{s_1} \le \cdots \le t_{s_r}$, then the joint probability density for $X_{t_s}^\alpha$, $s \in \Lambda$, is given by the formula

$$(2.4)\quad p^\alpha(t_\Lambda, x_\Lambda) = \prod_{i=1}^r p^\alpha(t_{s_{i-1}}, x_{s_{i-1}}; t_{s_i}, x_{s_i}),$$

where $t_{s_0} = x_{s_0} = 0$ and $p^\alpha(t, x; t', x') = p_{t'-t}^\alpha(x' - x)$. Therefore, investigating $E\wp_\alpha^\varepsilon$ involves various orderings of the set S and its subsets.

Denote by Γ the set of all orderings γ of S which are compatible with the order within each set S^i. Let $\gamma(s)$ mean the subordinate of s relative to γ (that is the largest element which is smaller than s). Consider an extra element ∂ and put $\gamma(s) = \partial$ for the smallest element of S. Denote by $\mathbb{D}(S,\gamma)$ the set of all t_S such that $t_{\gamma(s)} \le t_s$ for all $s \in S$ (with $t_\partial = 0$) and put $\varphi \in \mathscr{L}(S, \gamma, L)$ if φ is a function on $(\mathbb{R}^2)^L \times \mathbb{D}(S,\gamma)$ satisfying conditions 1.7.A, B [with $\mathbb{R}^2$ replaced by $(\mathbb{R}^2)^L$]. Put $\varphi \in \mathscr{L}^c(S, \gamma, L)$ if the support of φ is contained in

$$(2.5)\quad Q_c = \{|x_s| \le c \text{ for all } s \in L;\ 0 \le t_s \le c \text{ for all } s \in S\}.$$

Note that the restriction of φ given by (2.2) to $(\mathbb{R}^2)^L \times \mathbb{D}(S,\gamma)$ belongs to $\mathscr{L}(S,\gamma,L)$ and that, for every integer $m \ge 0$,

$$(2.6)\quad \|\varphi\|_m \le \prod_{i=1}^p \|\varphi_i\|_m.$$

Moreover, if $\varphi_i \in \mathscr{L}_{k_i}^c$ for all i, then $\varphi \in \mathscr{L}^c(S,\gamma,L)$.

2.2. Formula

$$(2.7)\quad u_s = t_s - t_{\gamma(s)}, \qquad s \in S,$$

defines a 1–1 linear mapping from $\mathbb{D}(S,\gamma)$ onto $\mathbb{R}_+^S$. We denote by C_γ the inverse mapping and we put, for every $\varphi \in \mathscr{L}(S,\gamma,L)$,

$$(2.8)\quad \tilde{\varphi}_\gamma(x_L, u_S) = (C_\gamma\varphi)(x_L, u_S) = \varphi(x_L, C_\gamma u_S).$$

We work with the space $\mathscr{D}(S)$ of all infinitely differentiable functions $f(u_S)$, $u_S \in \mathbb{R}_+^S$, with compact supports. Put $f \in \mathscr{D}^c(S)$ if $f \in \mathscr{D}(S)$ and if $f = 0$ outside of the cube $\{u_S\colon 0 \le u_s \le c \text{ for all } s \in S\}$.

We denote by $D_s f$ the partial derivative of f with respect to u_s and we put

$$D_k = \prod_S D_s^{k_s} \quad \text{for } k \in \mathbb{Z}_+^S.$$

To every $l \in \mathbb{Z}_+^S$, there corresponds a norm in $\mathbb{D}(S)$ given by the formula

$$\|f\|_l = \max_{k \le l} \|D_k f\|. \tag{2.9}$$

A sequence f_n converges to f in $\mathscr{D}(S)$ if all f_n belong to the same $\mathscr{D}^c(S)$ and if $\|f_n - f\|_l \to 0$ for all l. For every $\varphi \in \mathscr{L}(S, \gamma, L)$, formula (2.8) determines an element $\tilde{\varphi}_\gamma$ of $\mathscr{D}(S)$ depending on parameter x_L. If $\varphi \in \mathscr{L}^c(S, \gamma, L)$, then $\tilde{\varphi}_\gamma \in \mathscr{D}^c(S)$ for all x_L. Note that

$$\|D_k \tilde{\varphi}_\gamma\| \le |S|^{|k|} \|\varphi\|_{|k|}, \tag{2.10}$$

where

$$|k| = \sum_S k_s. \tag{2.11}$$

By (2.9) and (2.10),

$$\|\tilde{\varphi}_\gamma\|_l \le |S|^{|l|} \|\varphi\|_{|l|} \tag{2.12}$$

(the left side depends on x_L, but the right side does not).

2.3. To every $\Lambda \subset S$ there corresponds a linear topological space $\mathscr{D}(\Lambda)$. [For the empty set $\varnothing$, we put $\mathscr{D}(\varnothing) = \mathbb{R}$.]

Let $M \subset \Lambda \subset S$. We denote by $\mathscr{R}(\Lambda, M)$ the set of all continuous linear operators η: $\mathscr{D}(\Lambda) \to \mathscr{D}(M)$. In particular, $\mathscr{R}(\Lambda, \varnothing) = \mathscr{D}'(\Lambda)$ is the space of all generalized functions of u_Λ.

If $\Lambda \cap K = \varnothing$, then $\varphi \in \mathscr{D}(\Lambda \cup K)$ can be considered as an element of $\mathscr{D}(\Lambda)$ depending on a parameter $u_K \in \mathbb{R}_+^K$. Using this remark, an operator $\eta \in \mathscr{R}(\Lambda, M)$ can be applied to a $\varphi \in \mathscr{D}(\Lambda \cup K)$ yielding an element of $\mathscr{D}(M)$ depending on parameter u_K, which can be interpreted as a function of $u_{M \cup K}$.

LEMMA 2.1. *The operator induced on $\mathscr{D}(\Lambda \cup K)$ by $\eta \in \mathscr{R}(\Lambda, M)$ belongs to $\mathscr{R}(\Lambda \cup K, M \cup K)$. For all $\varphi \in \mathscr{D}(\Lambda \cup K)$ and all $s \in K$,*

$$D_s \eta(\varphi) = \eta(D_s \varphi). \tag{2.13}$$

Lemma 2.1 was proved in Schwartz [(1950), pages 103–105] in the case $M = \varnothing$. We give a proof for the general case in the Appendix.

By Lemma 2.1, we can treat $\mathscr{R}(\Lambda, M)$ as a subset of $\mathscr{R}(\Lambda \cup K, M \cup K)$.

Let $\Lambda = \Lambda_1 \cup \Lambda_2$ and $M = M_1 \cup M_2$ be partitions of Λ and M into disjoint sets and let $M_1 \subset \Lambda_1$ and $M_2 \subset \Lambda_2$. Suppose that $\eta_a \in \mathscr{R}(\Lambda_a, M_a)$, $a = 1, 2$. According to Lemma 2.1, η_2 can be interpreted as an element of $\mathscr{R}(\Lambda_1 \cup \Lambda_2, \Lambda_1 \cup M_2)$ and η_1 can be considered as an element of $\mathscr{R}(\Lambda_1 \cup M_2, M_1 \cup M_2)$. The operator

$$\eta(\varphi) = \eta_1(\eta_2(\varphi)), \qquad \varphi \in \mathscr{D}(\Lambda_1 \cup \Lambda_2), \tag{2.14}$$

belongs to $\mathscr{D}(\Lambda_1 \cup \Lambda_2, M_1 \cup M_2)$. We write $\eta = \eta_1 \times \eta_2$ and we call η the *direct product* of η_1 and η_2. The operation is commutative and associative and we use $\prod_{a \in \mathscr{A}} \eta_a$ for the direct product of a family of operators η_a.

Let $\eta \in \mathscr{R}(\Lambda, M)$. For every $c > 0$ and every $l \in \mathbb{Z}_+^\Lambda$, we define a norm

(2.15) $\|\eta\|_{l,c} = \sup\|\eta(\varphi)\|$ over all $\varphi \in \mathscr{D}^c(\Lambda)$ such that $\|\varphi\|_l = 1$.

Obviously,

(2.16) $$\|\eta(\varphi)\| \le \|\eta\|_{l,c}\|\varphi\|_l \quad \text{for all } \varphi \in \mathscr{D}^c(\Lambda).$$

LEMMA 2.2. *Let*

$$\Lambda = \bigcup_{a \in \mathscr{A}} \Lambda_a, \qquad M = \bigcup_{a \in \mathscr{A}} M_a$$

be partitions of Λ *and* M *into disjoint sets and let* $M_a \subset \Lambda_a$ *for all* $a \in \mathscr{A}$. *If* $\eta_a \in \mathscr{R}(\Lambda_a, M_a)$, *then* $\eta = \prod_{a \in \mathscr{A}} \eta_a \in \mathscr{R}(\Lambda, M)$ *and, for every* $c > 0$, *there exist constants* c_a, $a \in \mathscr{A}$, *such that, for an arbitrary* $l \in \mathbb{Z}_+^\Lambda$,

(2.17) $$\|\eta\|_{l,c} \le \prod_{a \in \mathscr{A}} \|\eta_a\|_{l_a, c_a},$$

where l_a *is the restriction of* l *to* Λ_a.

Lemma 2.2 will be proved in the Appendix. Usually, we apply it in the situation when all Λ_a are one-point sets and, therefore, $M_a = \Lambda_a$ or $M_a = \varnothing$. Typical examples of operators $\eta_s \in \mathscr{D}(s, s)$ are D_s and their discrete analogue

(2.18) $$\Delta_s^\alpha \varphi(u_s) = [\varphi(u_s + \alpha) - \varphi(u_s)]/\alpha.$$

Note that

(2.19) $$\|D_s\|_{1,c} \le 1, \qquad \|\Delta_s^\alpha\|_{1,c} \le 1 \qquad \text{for all } c.$$

We also deal with the following elements of $\mathscr{D}(s, \varnothing) = \mathscr{D}'(s)$ (generalized functions):

(2.20) $$\delta_s(\varphi) = \varphi(0) \qquad \text{(Dirac's } \delta\text{-function)},$$

(2.21) $$\xi_s(\varphi) = \int_{\mathbb{R}_+} du_s\, \hat{\varphi}(u_s)/2\pi u_s,$$

(2.22) $$\xi_s^\alpha(\varphi) = \int_{\mathbb{R}_+} p_{u_s}^\alpha(0)\hat{\varphi}(u_s)\nu_\alpha(du_s),$$

(2.23) $$\xi_s^{\alpha\varepsilon}(\varphi) = \int q_\alpha^\varepsilon(x) p_{u_s}^\alpha(x)\hat{\varphi}(u_s)\nu_\alpha(du_s)\lambda_\alpha(dx),$$

where

(2.24) $$\hat{\varphi}(u_s) = \varphi(u_s) - 1_{u \le 1}\varphi(0).$$

Obviously,

(2.25) $$\|\delta_s\|_{0,c} \le 1 \quad \text{for all } c.$$

We prove in Section 3, that there exists a function $\mathbb{F}(c)$ (depending only on the random walk) such that

$$\|\xi\|_{1,c} + \|\xi^{\alpha}\|_{1,c} + \|\xi^{\alpha\varepsilon}\|_{1,c} \le \mathbb{F}(c) \tag{2.26}$$

for all sufficiently small α and ε.

It follows easily from Taylor's formula that

$$\|\Delta_s^{\alpha} - D_s\|_{2,c} \le \alpha/2 \quad \text{for all } c \text{ and } \alpha. \tag{2.27}$$

We prove in Section 3 that

$$\|\xi^{\alpha} - \xi\|_{2,c} \le \mathbb{F}_1(c,\alpha), \qquad \|\xi^{\alpha\varepsilon} - \xi\|_{2,c} \le \mathbb{F}_2(c,\alpha,\varepsilon), \tag{2.28}$$

where

$$\lim_{\alpha\to 0} \mathbb{F}_1(c,\alpha) = \lim_{\alpha,\varepsilon\to 0} \mathbb{F}_2(c,\alpha,\varepsilon) = 0 \quad \text{for all } c > 0. \tag{2.29}$$

2.4. The following result is the main step in proving Theorem 1.3:

THEOREM 2.1. *Let $\wp_\alpha^\varepsilon$ be given by formula* (1.55) *with $\varphi_i \in \mathscr{L}_{k_i}^c$, and let φ and $\tilde{\varphi}$ be defined by* (2.2) *and* (2.8). *Then*

$$E\wp_\alpha^\varepsilon = \sum_{\gamma\in\Gamma} \int \left[\lambda_\alpha(dx_L)\nu_\alpha(du_I)\phi_\gamma^{\alpha\varepsilon}(x_L,u_I)p_{u_I}^{\alpha}(x_L) + R_1^{\alpha\varepsilon} + R_2^{\alpha\varepsilon}, \right. \tag{2.30}$$

where

$$\phi_\gamma^{\alpha\varepsilon} = \left[\prod_{J_A} \xi_s^{\alpha} \prod_{J_B} \xi_s^{\alpha\varepsilon}\right] \tilde{\varphi}_\gamma, \tag{2.31}$$

$$p_{u_I}^{\alpha}(x_L) = \prod_I p_{u_s}^{\alpha}(x_{[s]} - x_{[\gamma(s)]}), \tag{2.32}$$

$$I = \{s: \gamma(s) \nsim s\}, \qquad J = S\setminus I = \{s: \gamma(s) \sim s\} \tag{2.33}$$

(*with $s \nsim \partial$ for all $s \in S$*) *and*

$$|R_1^{\alpha\varepsilon}| \le \mathbb{F}(c)\alpha\left(\ln\frac{1}{\alpha}\right)^l \|\varphi\|, \tag{2.34}$$

$$|R_2^{\alpha\varepsilon}| \le \mathbb{F}(c)\varepsilon\left(\ln\frac{1}{\varepsilon}\right)^l \|\varphi\|_{|S\setminus L|} \tag{2.35}$$

for all sufficiently small α and ε. A positive constant l and a function $\mathbb{F}(c)$ do not depend on α, ε and φ.

Proof of Theorem 2.1 is presented in Sections 4 and 5. Section 4 contains a combinatorial part which involves removing brackets in formula (2.3), evaluating the mathematical expectations by formula (2.4) and combining certain groups of terms to compensate for infinities. In this way, we establish (2.30) and preliminary bounds for $R_1^{\alpha\varepsilon}$ and $R_2^{\alpha\varepsilon}$. An estimate for $R_1^{\alpha\varepsilon}$ involves certain integrals relative to λ_α over hyperplanes $\{t_s = t_{s'}\}$ with $s \nsim s'$. The term $R_2^{\alpha\varepsilon}$ is bounded by "a decoupling error." The bounds (2.34) and (2.35) are proved in Section 5.

We use many times a discrete analogue of integration by parts:

$$(2.36)\quad \int_{[0,u]} \varphi(t)h(t)\nu_\alpha(dt) = \varphi(u(\alpha))H(u) - \int_{[0,u]} \Delta^\alpha\varphi(t)H(t)\nu_\alpha(dt),$$

where

$$(2.37)\qquad H(u) = \int_{[0,u]} h(t)\nu_\alpha(dt)$$

and $u(\alpha)$ is the smallest element of $\mathbb{Q}_\alpha$ which is bigger than u. In particular, if $\varphi(u)$ has a compact support, then

$$(2.38)\qquad \int\varphi(t)h(t)\nu_\alpha(dt) = \int(-\Delta^\alpha)\varphi(t)H(t)\nu_\alpha(dt).$$

Let $\varphi(t_\Lambda)$ have a compact support. By applying (2.38) several times, we get

$$(2.39)\quad \int\varphi(u_\Lambda)\prod_\Lambda h_s(u_s)\nu_\alpha(du_\Lambda) = \int\prod_\Lambda(-\Delta_s^\alpha)\varphi(u_\Lambda)\prod_\Lambda H_s(u_s)\nu_\alpha(du_\Lambda),$$

with

$$(2.40)\qquad H_s(u_s) = \int_{[0,u_s]} h_s(t_s)\nu_\alpha(dt_s).$$

2.5. Formula (2.39) is used also to pass to the limit in (2.30). We call

$$(2.41)\qquad G_u^\alpha(x) = \int_{[0,u]} p_t^\alpha(x)\nu_\alpha(dt)$$

the (*truncated*) *Green's function* for X_t^α. By (2.30) and (2.39),

$$(2.42)\quad E\wp_\alpha^\varepsilon = \sum_{\gamma\in\Gamma}\int\left[\lambda_\alpha(dx_L)\nu_\alpha(du_I)f_\gamma^{\alpha\varepsilon}(x_L,u_I)G_{u_I}^\alpha(x_L) + R_1^{\alpha\varepsilon} + R_2^{\alpha\varepsilon},\right.$$

where

$$(2.43)\qquad f_\gamma^{\alpha\varepsilon} = \left[\prod_I(-\Delta_s^\alpha)\right]\phi_\gamma^{\alpha\varepsilon},$$

$$(2.44)\qquad G_{u_I}^\alpha(x_L) = \prod_I G_{u_s}^\alpha(x_{[s]} - x_{[\gamma(s)]}).$$

As $\alpha\to 0$, the measures $\lambda_\alpha(dx_L)$ and $\nu_\alpha(du_I)$ converge weakly to the Lebesgue measures on $(\mathbb{R}^2)^L$ and on $\mathbb{R}_+^I$, respectively. By the local central limit theorem, p_t^α converges to the Brownian density

$$(2.45)\qquad p_t(x) = (2\pi t)^{-1}e^{-|x|^2/2t}.$$

It will be shown in Section 3 that $G_u^\alpha(x)$ converges to the truncated Brownian Green's function

$$(2.46)\qquad G_u(x) = \int_0^u p_t(x)\,dt.$$

We also get bounds for $G_u^\alpha(x)$ which justify a passage to the limit under the integral sign in (2.42). Using Lemma 2.2, bounds (2.26)–(2.29) and formula (2.39), we get

THEOREM 2.2. *Under the assumptions and in the notation of Theorem* 2.1,

$$\lim_{\alpha,\varepsilon\to 0} E\wp_\alpha^\varepsilon = \sum_{\gamma\in\Gamma}\int f_\gamma(x_L, u_I)G_{u_I}(x_L)\,dx_L\,du_I \tag{2.47}$$

$$= \sum_{\gamma\in\Gamma}\int \phi_\gamma(x_L, u_I)p_{u_I}(x_L)\,dx_L\,du_I,$$

where

$$\phi_\gamma = \left[\prod_J \xi_s\right]\tilde{\varphi}_\gamma, \tag{2.48}$$

$$f_\gamma = \left[\prod_I D_s\right]\phi_\gamma, \tag{2.49}$$

$$p_{u_I}(x_L) = \prod_I p_{u_s}(x_{[s]} - x_{[\gamma(s)]}). \tag{2.50}$$

Moreover, for all sufficiently small α, ε *and for all* φ,

$$|E\wp_\alpha^\varepsilon| \le C\prod_{i=1}^{p}\|\varphi_i\|_{|S\setminus L|}, \tag{2.51}$$

with a constant C depending on c.

In a similar way, but much simpler, we prove

THEOREM 2.3. *If* $\varphi_i \in \mathscr{E}_{k_i}$, $i = 1,\dots,p$, *then*

$$\lim_{\alpha,\varepsilon\to 0} E\wp_\alpha^\varepsilon = \sum_{\gamma\in\Gamma}\int \phi_\gamma(x_L, u_I)p_{u_I}(x_L)\,dx_L\,du_I, \tag{2.52}$$

where p_{u_I} *is given by* (2.50) *and*

$$\phi_\gamma(x_L, u_I) = \int\prod_{s\in J} p_{u_s}(0)\tilde{\varphi}_\gamma(u_S)\,du_J. \tag{2.53}$$

Moreover, if $\varphi_i \in \mathscr{E}_{k_i}^c$, *then*

$$|E\wp_\alpha^\varepsilon| \le C\prod_{i=1}^{p}\|\varphi_i\|, \tag{2.54}$$

where C is a constant depending on c.

Note that under the conditions of Theorem 2.3,

$$\xi_s(\varphi) = \int p_{u_s}(0)\varphi(u_s)\,du_s \tag{2.55}$$

and, therefore, (2.48) defines the same function as (2.53). In general, for $\varphi_i \in \mathscr{L}_{k_i}$ the integral (2.55) diverges and (2.21) is its regularization.

Theorems 2.2 and 2.3 imply Theorem 1.3. In Section 5, we prove Theorem 1.1 using Theorem 1.3 and Lemmas A.2 and A.3. The same lemmas are used to prove Lemma 2.2. As we already know, Theorem 1.2 follows from Theorem 1.3 and Lemmas 1.1 and 1.2.

3. Asymptotic behaviour of Green's function and functions ξ^α and $\xi^{\alpha\varepsilon}$.

3.1. The probability density for X_t^α relative to λ_α is given by the formula

$$p_t^\alpha(x) = \alpha^{-1}P\{X_t^\alpha = x\} = \alpha^{-1}p_{t/\alpha}^1(x/\sqrt{\alpha}), \qquad t \in \mathbb{Q}_\alpha,\ x \in \mathbb{Z}_\alpha, \tag{3.1}$$

where $p_m^1(x) = P\{S_m = x\}$.

We investigate the asymptotic behaviour, as $\alpha \to 0$, of Green's function

$$g^\alpha(x) = \int e^{-t}p_t^\alpha(x)\nu_\alpha(dt) = \alpha \sum_{m=0}^{\infty} e^{-\alpha m}p_{m\alpha}^\alpha(x), \tag{3.2}$$

the truncated Green's function G_u^α defined by (2.41) and the generalized functions ξ^α and $\xi^{\alpha\varepsilon}$ [see (2.22) and (2.23)].

We note that

$$h_\alpha^\varepsilon = -EG_1^\alpha(Y_\alpha^\varepsilon), \qquad h^\varepsilon = -EG_1(\varepsilon Y), \qquad \hat{h}_\alpha = -G_1^\alpha(0), \tag{3.3}$$

where $Y_\alpha^\varepsilon = \psi_\alpha(\varepsilon Y)$ (Y and ψ_α are introduced in Section 1.11) and G_1 is defined by (2.46).

3.2. The characteristic function of S_m is equal to Φ^m, where $\Phi(\theta) = Ee^{i\theta\xi}$, and by the inversion formula

$$4\pi^2 p_m^1(x) = \int_\Pi e^{-i\theta x}\Phi(\theta)^m\, d\theta, \tag{3.4}$$

where $\Pi = [-\pi, +\pi]^2$. By (3.1) and (3.4),

$$4\pi^2 p_t^\alpha(x) = \int_{\Pi_\alpha} e^{-i\theta x}\Phi(\theta\sqrt{\alpha})^{t/\alpha}\, d\theta, \tag{3.5}$$

where $\Pi_\alpha = [-\pi/\sqrt{\alpha}, +\pi/\sqrt{\alpha}]^2$. Under the assumptions 1.1.A–D, Φ has the following properties:

3.2.A. Φ is twice continuously differentiable, $\Phi(\theta + 2\pi z) = \Phi(\theta)$ for all $z \in \mathbb{Z}^2$, $\nabla\Phi(0) = 0$ and $\Phi(\theta) = 1 - \frac{1}{2}|\theta|^2 + o(|\theta|)^2)$ as $\theta \to 0$.

3.2.B. There exists a constant $\beta > 0$ such that $|\Phi(\theta)| \le e^{-\beta|\theta|^2}$ for all $\theta \in \Pi$.

3.2.C. There exists a constant $\gamma > 0$ such that $\mathrm{re}[1 - \Phi(\theta)] \ge \gamma|\theta|^2$ for all $\theta \in \Pi$.

Property 3.2.A is obvious; 3.2.B follows from 3.2.A and the fact that $|\Phi(\theta)| < 1$ for all $\theta \in \Pi$ except 0. A proof of this fact and a proof of 3.2.C can be found in Spitzer [(1964), P7.5 and P7.8].

We use the following elementary inequalities:

$$|1 - e^{ia}| \le |a| \quad \text{for all real } a, \tag{3.6}$$

$$|1 - e^{-a}| \ge a/2 \quad \text{for } 0 \le a \le 1. \tag{3.7}$$

3.3.

THEOREM 3.1. *Let*

$$g(x) = \int_0^\infty e^{-t} p_t(x)\, dt \tag{3.8}$$

be Green's function of the Brownian motion [$p_t(x)$ *is defined by* (2.45)]. *We have*

$$g^\alpha(x_\alpha) \to g(x) \quad \text{as } \alpha \downarrow 0,\ x_\alpha \to x \tag{3.9}$$

and

$$g^\alpha(x) \le C\Upsilon(|x|) \quad \text{for all } 0 < \alpha \le 1,\ x \in \mathbb{Z}_\alpha, \tag{3.10}$$

where C is a constant and

$$\begin{aligned} \Upsilon(r) &= 4\log r^{-1}, \quad \text{for } 0 < r \le \tfrac{1}{2}, \\ &= (\ln 2) r^{-2}, \quad \text{for } \tfrac{1}{2} < r, \end{aligned} \tag{3.11}$$

is a monotone decreasing function.

PROOF. Let

$$a_\alpha = (1 - e^{-\alpha})/\alpha, \qquad \Phi_\alpha(\theta) = [1 - \Phi(\theta\sqrt{\alpha})]/\alpha \tag{3.12}$$

and note that

$$\begin{aligned} &\alpha[1 - e^{-\alpha}\Phi(\theta\sqrt{\alpha})]^{-1} \\ &\quad = (a_\alpha + e^{-\alpha}\Phi_\alpha)^{-1} = \int_0^\infty \exp[-s(a_\alpha + e^{-\alpha}\Phi_\alpha)]\, ds. \end{aligned} \tag{3.13}$$

By (3.2), (3.5) and (3.13)

$$4\pi^2 g^\alpha(x) = \int_0^\infty ds \exp[-a_\alpha s] A^\alpha(e^{-\alpha}s, x), \tag{3.14}$$

where

$$A^\alpha(s, x) = \int_{\Pi_\alpha} \exp[-s\Phi_\alpha(\theta) - ix\theta]\, d\theta. \tag{3.15}$$

By 3.2.C,

$$|\exp[-s\Phi_\alpha(\theta)]| \le \exp[-\gamma s|\theta|^2]. \tag{3.16}$$

Hence,

$$|A^\alpha(s, x)| \le C_1 s^{-1}. \tag{3.17}$$

Let Δ be the Laplacian in the θ-plane. Since Φ_α is periodical,

$$(3.18)\qquad \int_{\Pi_\alpha} \Delta \exp[-s\Phi_\alpha(\theta)]\, e^{-i\theta x}\, d\theta = \int_{\Pi_\alpha} \exp[-s\Phi_\alpha(\theta)]\, \Delta e^{-i\theta x}\, d\theta = -|x|^2 A^\alpha(s,x).$$

We have

$$(3.19)\qquad \begin{aligned} \Delta \exp(-s\Phi_\alpha) &= \exp(-s\Phi_\alpha)\big(s^2|\nabla\Phi_\alpha|^2 - s\Delta\Phi_\alpha\big),\\ \nabla\Phi_\alpha(\theta) &= -\alpha^{-1/2}\nabla\Phi(\theta\sqrt{\alpha}), \qquad \Delta\Phi_\alpha(\theta) = -\Delta\Phi(\theta\sqrt{\alpha}). \end{aligned}$$

By 3.2.A, $|\nabla\Phi_\alpha(\theta)| \le C_2|\theta|$, $|\Delta\Phi_\alpha(\theta)| \le C_3$ and, by (3.19),

$$(3.20)\qquad |\Delta \exp(-s\Phi_\alpha)| \le C_4 |\exp(-s\Phi_\alpha)|\big(s^2|\theta|^2 + s\big).$$

It follows from (3.18), (3.20) and (3.16) that

$$(3.21)\qquad |A^\alpha(s,x)| \le C_5|x|^{-2},$$

Formulas (3.7), (3.14), (3.17) and (3.21) imply that, for $0 \le \alpha \le 1$, $4\pi^2 g^\alpha(x) \le C_6 F(|x|^2)$, where

$$F(u) = \int_0^\infty e^{-s/2}(s^{-1} \wedge u^{-1})\, ds = 2(1 - e^{-u/2})/u + \int_u^\infty e^{-s/2}s^{-1}\, ds$$

and (3.10) follows.

Now we note that $a_\alpha \to 1$, $\Phi_\alpha(x_\alpha) \to |\theta|^2/2$ as $\alpha \downarrow 0$ and $x \to \alpha$. The estimate (3.16) justifies passage to the limit in (3.15) and, therefore,

$$(3.22)\qquad A^\alpha(s, x_\alpha) \to \int_\Pi \exp(-s|\theta|^2/2 - ix\theta)\, d\theta.$$

Because of (3.7), (3.17) and (3.21), it is legitimate to pass to the limit in (3.14) and we get (3.9). □

3.4. Consider a mapping $f_\alpha\colon \mathbb{R} \to \mathbb{R}$ such that $f_\alpha(-u) = -f_\alpha(u)$ and

$$f_\alpha(u) = k\sqrt{\alpha} \quad \text{for } 0 \le (k-1)\sqrt{\alpha} < u \le k\sqrt{\alpha}.$$

The formula $\iota_\alpha(z^1, z^2) = (f_\alpha(z^1), f_\alpha(z^2))$ defines a mapping from $\mathbb{R}^2$ onto $\mathbb{Z}_\alpha$ such that $\iota_\alpha(z) \to z$ as $\alpha \downarrow 0$. By Theorem 3.1,

$$(3.23)\qquad g^\alpha(\iota_\alpha(x)) \to g(x),$$

$$(3.24)\qquad g^\alpha(\iota_\alpha(x)) \le C\Upsilon(x) \quad \text{for all } 0 < \alpha \le 1.$$

The inverse image $\iota_\alpha^{-1}(z^1, z^2)$ is a square of area α if $z^1 \ne 0$ and $z^2 \ne 0$. It is an interval of length $\sqrt{\alpha}$ if $z^1 \ne 0$, $z^2 = 0$ or if $z^1 = 0$, $z^2 \ne 0$. Finally, $\iota_\alpha^{-1}(0,0) = 0$. Therefore,

$$(3.25)\qquad \int F(z)\lambda_\alpha(dz) = \alpha F(0) + \sqrt{\alpha}\int F(\iota_\alpha(x))\gamma(dx) + \int F(\iota_\alpha(x))\, dx,$$

where γ is the linear Lebesgue measure on the coordinate axes.

3.5.

LEMMA 3.1. *For every $k \geq 1$, $0 < \alpha \leq 1$ and for an arbitrary positive monotone decreasing function $h(r)$,*

$$\int g^\alpha(z)^k h(|z|)\lambda_\alpha(dz) \leq 2\pi\int_0^\infty \Upsilon(r)^k h(r) r\,dr + Ch(0), \tag{3.26}$$

where C is a constant independent of h.

PROOF. We apply (3.25) to $F = h(g^\alpha)^k$. By (3.1) and (3.2),

$$g^\alpha(0) = \sum_{m=0}^{\infty} e^{-m\alpha} P\{S_m = 0\}. \tag{3.27}$$

It follows from (1.45) that

$$g^\alpha(0) \sim \frac{1}{2\pi}\ln\frac{1}{\alpha}. \tag{3.28}$$

Hence, the first term in (3.25) does not exceed $C_1 h(0)$. The second term is dominated by $\sqrt{\alpha}\, h(0)\int\Upsilon(|z|)^k\gamma(dz)$. The third term is dominated by the integral in the right side of (3.26). □

3.6.

LEMMA 3.2. *If $u_\alpha \to u \neq 0$ and $x_\alpha \to x \neq 0$, then*

$$G^\alpha_{u_\alpha}(x_\alpha) \to G_u(x). \tag{3.29}$$

PROOF. By applying (2.36) to $\varphi(t) = e^t$, $h(t) = e^{-t}p^\alpha_t(x)$, we get

$$G^\alpha_u(x) = e^{u(\alpha)}g^\alpha_u(x) + (1 - e^\alpha)\alpha^{-1}\int_{[0,\,u]} e^t g^\alpha_t(x)\nu_\alpha(dt), \tag{3.30}$$

where

$$g^\alpha_u(x) = \int e^{-t}1_{t\leq u}p^\alpha_t(x)\nu_\alpha(dt). \tag{3.31}$$

On the other hand,

$$G_u(x) = e^u g_u(x) - \int_0^u e^t g_t(x)\,dt, \tag{3.32}$$

where

$$g_u(x) = \int_0^u e^{-t}p_t(x)\,dt. \tag{3.33}$$

Formula (3.29) will be proved if we show that

$$g^\alpha_{u_\alpha}(x_\alpha) \to g_u(x) \tag{3.34}$$

and

$$\int_{[0,u]} e^t g_t^\alpha(x_\alpha)\nu_\alpha(dt) \to \int_0^u e^t g_t(x)\,dt. \tag{3.35}$$

Note that the left side in (3.35) is equal to

$$\int 1_{0\le t\le u}(\exp t_\alpha) g_{t_\alpha}^\alpha(x_\alpha)\,dt + o(1),$$

where

$$t_\alpha = m\alpha \quad \text{for } (m-1)\alpha < t \le m\alpha,\ m = 1,2,\dots. \tag{3.36}$$

Since $g_{t_\alpha}^\alpha(x_\alpha) \le g^\alpha(x_\alpha) \le C\Upsilon(x_\alpha) \to C\Upsilon(x) < \infty$, formula (3.35) follows from (3.34) and the dominant convergence theorem. Thus, we need only to establish (3.34). Note that

$$g_u(x) - g(x) + e^{-u}\int g(y)p_u(x-y)\,dy = 0 \tag{3.37}$$

and

$$g_{u_\alpha}^\alpha(x_\alpha) - g^\alpha(x_\alpha) + \exp(-u_\alpha)\int g^\alpha(y)p_{u_\alpha}^\alpha(x_\alpha - y)\lambda_\alpha(dy) \to 0 \quad \text{as } \alpha \to 0. \tag{3.38}$$

By (3.9), $g^\alpha(x_\alpha) \to g(x)$. By the local central limit theorem [Spitzer (1964), page 77],

$$\sup_{z\in\mathbb{Z}_\alpha} |p_{u_\alpha}^\alpha(z) - p_u(z)| \to 0 \quad \text{as } \alpha \to 0. \tag{3.39}$$

Put

$$a_\alpha(y) = g^\alpha(\iota_\alpha(y)), \qquad b_\alpha(y) = p_{u_\alpha}^\alpha(x_\alpha - \iota_\alpha(y)).$$

By (3.25), the integral in (3.38) is equal to

$$\alpha a_\alpha(0)b_\alpha(0) + \sqrt{\alpha}\int a_\alpha b_\alpha\,d\gamma + \int a_\alpha b_\alpha\,dy.$$

The first term tends to 0 by (3.28) and (3.39). The same is true for the second term since $\int\Upsilon(|y|)\gamma(dy) < \infty$ and $p_u(y) \le (2\pi u)^{-1}$. Note that

$$\begin{aligned}\left|\int(a_\alpha - g)b_\alpha\,dy\right| &\le \int|a_\alpha - g|\,dy \sup b_\alpha \\ &\le \text{const.}\int|a_\alpha - g|\,dy.\end{aligned} \tag{3.40}$$

Since

$$g(y) \le \text{const.}\,\Upsilon(|y|) \quad \text{for all } y \in \mathbb{R}^2 \tag{3.41}$$

[see, e.g., Dynkin (1984b), Appendix], the integrand in the right side of (3.40) is dominated by an integrable function. By (3.23), the integrand tends to 0 as $\alpha \downarrow 0$.

We conclude from (3.40) that $\int a_\alpha b_\alpha\,dy - \int g b_\alpha\,dy \to 0$ and, by (3.39), $\int a_\alpha b_\alpha\,dy$ tends to the integral in (3.37). This proves (3.34). □

COROLLARY. $h^\varepsilon_\alpha \to h^\varepsilon$ *as* $\alpha \to 0$.

Indeed, by (3.3), $h^\varepsilon_\alpha = -EG^\alpha_1(Y^\varepsilon_\alpha)$. Obviously, $Y^\varepsilon_\alpha \to \varepsilon Y$ as $\alpha \to 0$ and, by (3.29), $G^\alpha_1(Y^\varepsilon_\alpha) \to G_1(\varepsilon Y)$ if $Y \neq 0$. Since q^ε_α is bounded above as $\alpha \to 0$,

$$EG^\alpha_1(Y^\varepsilon_\alpha)^2 \le e^2 Eg^\alpha(Y^\varepsilon_\alpha)^2 \le C\int g^\alpha(z)^2\lambda_\alpha(dz).$$

By (3.26), the right side is bounded as $\alpha \to 0$. Hence, $G^\alpha_1(Y^\varepsilon_\alpha)$ are uniformly integrable and $EG^\alpha_1(Y^\varepsilon_\alpha) \to EG_1(\varepsilon Y)$.

3.7.

THEOREM 3.2. *For every* $U > 0$, *there exists a constant* C *such that*

$$\int [G^\alpha_u(z) - G^\alpha_u(z+y)]^2\lambda_\alpha(dz) \le C|y| \tag{3.42}$$

for all $\alpha \ge 0$, $y \in \mathbb{Z}_\alpha$ *and* $u \in \mathbb{Q}^\alpha \cap [0, U]$.

PROOF. By (2.41), (3.5) and (3.12),

$$4\pi^2 G^\alpha_u(x) = \int_{\Pi_\alpha} H_\alpha(\theta)e^{-i\theta x}\,d\theta, \tag{3.43}$$

where

$$H_\alpha(\theta) = \int_{[0,u]} \Phi(\theta\sqrt{\alpha})^{t/\alpha}\nu_\alpha(dt) = \left[1 - \Phi(\theta\sqrt{\alpha})^{m+1}\right] + \Phi_\alpha(\theta)^{-1} \tag{3.44}$$

and $m = u/\alpha$. By 3.2.C, $|\Phi_\alpha(\theta)| \ge \gamma|\theta|^2$ and, by 3.2.A, $|1 - \Phi(\theta\sqrt{\alpha})^{m+1}| \le C_0(|\theta|^2 \wedge 1)$ for all $\alpha m = u \le U$ (C_0 depends on U.) Therefore,

$$|H_\alpha(\theta)| \le C_1(1 \wedge |\theta|^{-2}). \tag{3.45}$$

By (3.43),

$$4\pi^2[G^\alpha_u(z) - G^\alpha_u(z+y)] = \int_{\Pi_\alpha} H^y_\alpha(\theta)e^{-i\theta z}\,d\theta, \tag{3.46}$$

with $H^y_\alpha(\theta) = H_\alpha(\theta)(1 - e^{-i\theta y})$. By (3.6) and (3.45),

$$\int_{\Pi_\alpha} |H^y_\alpha(\theta)|^2\,d\theta < C_2|y|\int_{\mathbb{R}^2}(|\theta| \wedge |\theta|^{-3})\,d\theta. \tag{3.47}$$

The functions $h_z(\theta) = e^{iz\theta}$, $z \in \mathbb{Z}_\alpha$, form an orthonormal basis in the space $L^2(\Pi_\alpha, \mu_\alpha)$, where $\mu_\alpha(d\theta) = d\theta\,\alpha(4\pi)^{-2}$. Therefore,

$$\int_{\Pi_\alpha} |H^y_\alpha(\theta)|^2\mu_\alpha(d\theta) = \sum_{z\in\mathbb{Z}_\alpha}\left|\int_{\Pi_\alpha} e^{iz\theta}H^y_\alpha(\theta)\mu_\alpha(d\theta)\right|^2$$

and, taking into account (3.46),

$$\int_{\Pi_\alpha} |H_\alpha^y(\theta)|^2\,d\theta = \int \lambda_\alpha(dz)\big[G_u^\alpha(z) - G_u^\alpha(z+y)\big]^2. \tag{3.48}$$

Formula (3.42) follows from (3.47) and (3.48). □

3.8.

Lemma 3.3. *For all $0 < \alpha \le 1$, $t \in \mathbb{Q}_\alpha$ and $x \in \mathbb{Z}_\alpha$,*

$$p_t^\alpha(x) \le bt^{-1}, \tag{3.49}$$

$$|p_t^\alpha(0) - p_t^\alpha(x)| \le b|x|t^{-3/2}, \tag{3.50}$$

where

$$b = (4\pi^2)^{-1}\int_{\mathbb{R}^2}(1 + |\theta|)e^{-\beta|\theta|^2}\,d\theta < \infty. \tag{3.51}$$

(β is the constant in 3.2.B.)

Proof. Using (3.5), 3.2.B and (3.6), we get

$$4\pi^2 p_t^\alpha(x) \le \int_{\Pi_\alpha} e^{-\beta t|\theta|^2}\,d\theta,$$

$$4\pi^2|p_t^\alpha(0) - p_t^\alpha(x)| \le |x|\int_{\Pi_\alpha}|\theta|e^{-\beta t|\theta|^2}\,d\theta$$

and we arrive at (3.49) and (3.50) by introducing a new variable $\theta\sqrt{t}$. □

3.9.

Theorem 3.3. *Let ξ, ξ^α and $\xi^{\alpha\varepsilon}$ be given by* (2.21)–(2.23). *Then,* (2.26) *holds with $\mathbb{F}(c) = 3bc_1$, where $c_1 = c \vee 1$ and b is the constant in Lemma* 3.2. *The bounds* (2.28) *are valid with*

$$\mathbb{F}_1(c,\alpha) = \int_0^c |tp_t(0) - t_\alpha p_{t_\alpha}(0)|\,dt + 2(c_1+1)b\alpha \tag{3.52}$$

and

$$\mathbb{F}_2(c,\alpha,\varepsilon) = \mathbb{F}_1(c,\alpha) + b(a\varepsilon + \sqrt{\alpha})(6+\alpha) \tag{3.53}$$

if $q(x)$ vanishes for $|x| > a$.

Proof. Note that

$$\begin{aligned} \xi(\varphi) &= \int tp_t(0)h(t)\,dt,\\ \xi^\alpha(\varphi) &= \int t_\alpha p_{t_\alpha}^\alpha(0)h(t_\alpha)\,dt, \qquad (3.54)\\ \xi^{\alpha\varepsilon}(\varphi) &= E\int t_\alpha p_{t_\alpha}^\alpha(Y_\alpha^\varepsilon)h(t_\alpha)\,dt, \end{aligned}$$

where Y_α^ε is defined in Section 3.1, t_α is determined by (3.36) and

$$h(t) = [\varphi(t) - 1_{t\le 1}\varphi(0)]/t \quad \text{for } t > 0. \tag{3.55}$$

Taking into account that $h(t) = 0$ for $t > c_1$, we conclude from (3.54) and (3.49) that

$$|\xi(\varphi)| + |\xi^\alpha(\varphi)| + |\xi^{\alpha\varepsilon}(\varphi)| \le 3bc_1\|h\|. \tag{3.56}$$

By (3.49),

$$\begin{aligned} |\xi(\varphi) - \xi^\alpha(\varphi)| &= \left| \int dt\left[tp_t(0) - t_\alpha p_{t_\alpha}(0)\right] h(t) \right. \\ &\qquad \left. + \int dt\, t_\alpha p_{t_\alpha}^\alpha(0)\left[h(t) - h(t_\alpha)\right] \right| \\ &\le \|h\|\tilde{r}_\alpha(c) + b\int dt|h(t) - h(t_\alpha)|, \end{aligned} \tag{3.57}$$

with

$$\tilde{r}_\alpha(c) = \int_0^c |tp_t(0) - t_\alpha p_{t_\alpha}(0)|\, dt. \tag{3.58}$$

By (3.39) and (3.49), $\tilde{r}_\alpha(c) \to 0$ as $\alpha \to 0$. Denote by $\|h'\|$ the supremum of $|h'(t)|$ on the open set $B = (0,1) \cup (1, +\infty)$. If $[t, t_\alpha] \subset B$, then $|h(t) - h(t_\alpha)| \le \|h'\|(t_\alpha - t)$ and, therefore,

$$\begin{aligned} \int dt|h(t) - h(t_\alpha)| &\le \int 1_{t\le 1 < t_\alpha}|h(t) - h(t_\alpha)|\, dt + \|h'\|\int_0^{c_1}(t_\alpha - t)\, dt \\ &\le \alpha(2\|h\| + c_1\|h'\|). \end{aligned} \tag{3.59}$$

Using Taylor's formula for φ, we establish bounds

$$\begin{aligned} |h(t)| &\le \|\varphi'\| \quad \text{for } 0 < t < 1, \\ |h(t)| &\le \|\varphi\|/t \quad \text{for } t > 1, \\ |h'(t)| &\le \|\varphi''\| \quad \text{for } 0 < t < 1, \\ |h'(t)| &\le \|\varphi\| + \|\varphi'\| \quad \text{for } t > 1, \end{aligned} \tag{3.60}$$

which imply that $\|h\| \le \|\varphi\|_1$ and $\|h'\| \le 2\|\varphi\|_2$. The bound (2.26) with $\mathbb{F}(c) = 3bc_1$ follows from (3.56) and (2.15). The bound (3.52) for $\|\xi^\alpha - \xi\|_{c,2}$ is an implication of (3.57)–(3.59).

Note that

$$\xi^{\alpha\varepsilon}(\varphi) = \xi^\alpha(\varphi) + ES_\alpha(Y_\alpha^\varepsilon), \tag{3.61}$$

where

$$S_\alpha(x) = \int t_\alpha\left[p_{t_\alpha}^\alpha(x) - p_{t_\alpha}^\alpha(0)\right] h(t_\alpha)\, dt. \tag{3.62}$$

By using the estimate (3.49) on the interval $(0, |x|)$ and (3.50) on $(|x|, \infty)$, we get

$$|S_\alpha(x)| \le b|x|\left[2\|h\| + \int_{|x|}^{\infty} t_\alpha^{-1/2}|h(t_\alpha)|\,dt\right]. \tag{3.63}$$

By (3.60),

$$t_\alpha^{-1/2}|h(t_\alpha)| \le 1_{0<t<1}\|\varphi'\|t^{-1/2} + 1_{t>1/2}\|\varphi\|t^{-3/2} + 1_{t<1<t_\alpha}. \tag{3.64}$$

Therefore,

$$\int_0^{\infty} t_\alpha^{-1/2}|h(t_\alpha)|\,dt \le (4+\alpha)\|\varphi\|_1$$

and, since $\|\mathbf{h}\| \le \|\varphi\| \le \|\varphi\|_1$,

$$|S_\alpha(x)| \le b|x|(6+\alpha)\|\varphi\|_1. \tag{3.65}$$

Since $|Y_\alpha^\varepsilon| \le |\varepsilon Y| + \sqrt{\alpha/2} \le a\varepsilon + \sqrt{\alpha}$, we get the bound (3.53) for $\|\xi^{\alpha\varepsilon} - \xi\|_{2,c}$ from (3.61) and (3.65). □

REMARK. It follows from (2.22), (2.23), (3.3) and the relation

$$\int \delta_\alpha(u)\varphi(u)\nu_\alpha(du) = \varphi(0)$$

that

$$\xi^\alpha(\varphi) = \int \left[p_u^\alpha(0) + \hat{h}_\alpha\delta_\alpha(u)\right]\varphi(u)\nu_\alpha(du), \tag{3.66}$$

$$\xi^{\alpha\varepsilon}(\varphi) = E\int \left[p_u^\alpha(Y_\alpha^\varepsilon) + h_\alpha^\varepsilon\delta_\alpha(u)\right]\varphi(u)\nu_\alpha(du). \tag{3.67}$$

4. Proof of Theorem 2.1: Combinatorial part.

4.1. In this section, we manipulate the product (1.55) keeping α and ε frozen and, to simplify formulas, *we drop all subscripts and superscripts α and ε*. For the sake of brevity, we put

$$\begin{aligned} \Delta_s(t) &= \hat{h}_\alpha\delta_\alpha(t), \qquad q_s(x) = \delta_\alpha(x) \quad \text{for } s \in A; \\ \Delta_s(t) &= h_\alpha^\varepsilon\delta_\alpha(t), \qquad q_s(x) = q_\alpha^\varepsilon(x) \quad \text{for } s \in B. \end{aligned} \tag{4.1}$$

For every $K \subset M$, we set

$$\Delta_K(t_S) = \prod_K \Delta_s(t_s - t_{s^*}), \qquad q_K(x_S) = \prod_K q_s(x_s - x_{s^*}). \tag{4.2}$$

(See Section 2.1 for the definition of M and s^*.) In this notation,

$$\begin{aligned} \wp &= \int_{\mathbb{D}} \nu(dt_S)\varphi(X_{t_L}, t_S)\prod_M \left[q_s(X_{t_s} - X_{t_{s^*}}) + \Delta_s(t_s - t_{s^*})\right] \\ &= \int_{\mathbb{D}} \nu(dt_S)\varphi(X_{t_L}, t_S) \sum_{\Lambda \in \mathscr{S}} q_{\Lambda\setminus L}(X_{t_S})\Delta_{S\setminus\Lambda}(t_S), \end{aligned} \tag{4.3}$$

where $\mathscr{S} = \{\Lambda\colon L \subset \Lambda \subset S\}$.

Note that

$$(4.4)\qquad \Delta_{S\setminus\Lambda}(t_S)q_{\Lambda\setminus L}(X_{t_S}) = \Delta_{S\setminus\Lambda}(t_S)\tilde{q}_{\Lambda\setminus L}(X_{t_\Lambda}),$$

where

$$(4.5)\qquad \tilde{q}_{\Lambda\setminus L}(x_\Lambda) = \prod_{\Lambda\setminus L} q_s(x_s - x_{s^*\Lambda})$$

and $s^*\Lambda = s^*$ if $s^* \in \Lambda$; otherwise, $s^*\Lambda$ is the maximal element of Λ which has the same color as s^* and is smaller than s^*. By (4.4),

$$(4.6)\qquad \begin{aligned} &\Delta_{S\setminus\Lambda}(t_S)E\varphi(X_{t_L}, t_S)q_{\Lambda\setminus L}(X_{t_S}) \\ &= \Delta_{S\setminus\Lambda}(t_S)\int\lambda(dx_\Lambda)p(t_\Lambda, x_\Lambda)\varphi(x_L, t_S)\tilde{q}_{\Lambda\setminus L}(x_\Lambda) \\ &= \Delta_{S\setminus\Lambda}(t_S)E\varphi(V_L, t_S)p(t_\Lambda, V_\Lambda), \end{aligned}$$

where $p(t_\Lambda, x_\Lambda)$ is the probability density for X_{t_Λ} relative to $\lambda(dx_\Lambda)$ and V_Λ is a family of random variables with the law $\tilde{q}_{\Lambda\setminus L}(x_\Lambda)\lambda(dx_\Lambda)$. The family V_Λ has the following properties:

4.1.A. V_s, $s \in L$, and $V_s - V_{s^*\Lambda}$, $s \in \Lambda\setminus L$, are independent.
4.1.B. V_s, $s \in L$, are distributed with the law λ.
4.1.C. $V_s - V_{s^*\Lambda}$, $s \in \Lambda\setminus L$, are distributed with the law $q_s(x)\lambda(dx)$.

It follows from (4.1), (4.2) and 4.1.C that:

4.1.D. $V_s = V_{s^*\Lambda}$ and, therefore, $V_s = V_{[s]}$ for $s \in \Lambda_A$ (the mapping $[s]$ is defined at the beginning of Section 2.1).
4.1.E. $V_s - V_{s^*\Lambda}$, $s \in \Lambda_B\setminus L_B$, are identically distributed (with the probability density $q = q_\alpha^\varepsilon$).

For every $\Lambda \in \mathscr{S}$ and every function $f(x_\Lambda, t_S)$, we put

$$(4.7)\qquad \mathscr{M}_\Lambda(f) = \int_{\mathbb{D}}\nu(dt_S)\Delta_{S\setminus\Lambda}(t_S)Ef(V_L, t_S)p(t_\Lambda, V_\Lambda).$$

By (4.3) and (4.6),

$$(4.8)\qquad E\wp = \sum_{\Lambda\in\mathscr{S}}\mathscr{M}_\Lambda(\varphi).$$

To each ordering γ of Λ, there corresponds a set

$$(4.9)\qquad \mathbb{D}(\Lambda, \gamma) = \{t_S\colon t_{\gamma(s)} \le t_s \text{ for all } s \in \Lambda\}.$$

By (2.4),

$$(4.10)\qquad p(t_\Lambda, x_\Lambda) = \prod_{s\in\Lambda} p(t_{\gamma(s)}, x_{\gamma(s)}; t_s, x_s) \quad \text{for all } t_S \in \mathbb{D}(\Lambda, \gamma)$$

(with $t_\partial = x_\partial = 0$). Put $\gamma \in \Gamma_\Lambda$ if γ agrees with the ordering of each S^i within Λ.

Set

$$\mathscr{M}(\Lambda,\gamma)=\mathscr{M}_\Lambda(\varphi 1_{\mathbb{D}(\Lambda,\gamma)}), \tag{4.11}$$

$$\mathscr{N}=\sum_{\Lambda\in\mathscr{S}}\sum_{\gamma\in\Gamma_\Lambda}\mathscr{M}(\Lambda,\gamma). \tag{4.12}$$

Obviously,

$$\mathbb{D}=\bigcup_{\gamma\in\Gamma_\Lambda}\mathbb{D}(\Lambda,\gamma).$$

Denote by $\mathbb{D}^*(\Lambda,\gamma)$ the subset of $\mathbb{D}(\Lambda,\gamma)$ defined by the condition $t_{\gamma(s)}<t_s$ if $\gamma(s)\nrightarrow s$. The sets $\mathbb{D}^*(\Lambda,\gamma)$, $\gamma\in\Gamma_\Lambda$, are disjoint and $\mathbb{D}(\Lambda,\gamma)\setminus\mathbb{D}^*(\Lambda,\gamma)$ is covered by the sets $\mathbb{D}(\Lambda,\gamma,r)=\mathbb{D}(\Lambda,\gamma)\cap\{t_{\gamma(r)}=t_r\}$, with $r\in\Lambda$ such that $\gamma(r)\nrightarrow r$ and $\gamma(r)\neq\partial$. Therefore,

$$\mathscr{N}\geq E\wp\geq\sum_{\Lambda\in\mathscr{S}}\sum_{\gamma\in\Gamma_\Lambda}\mathscr{M}_\Lambda(\varphi 1_{\mathbb{D}^*(\Lambda,\gamma)})$$

and

$$0\leq\mathscr{N}-E\wp\leq R_1, \tag{4.13}$$

where

$$R_1=\sum_{\Lambda\in\mathscr{S}}\sum_{\gamma\in\Gamma_\Lambda}\sum_{r}\left|\mathscr{M}_\Lambda(\varphi 1_{\mathbb{D}(\Lambda,\gamma,r)})\right|, \tag{4.14}$$

with r running over all $r\in\Lambda$ such that $\gamma(r)\nrightarrow r$ and $\gamma(r)\neq\partial$.

We set $p(s,x;t,y)=0$ for $s>t$. By (4.11), (4.7) and (4.10),

$$\mathscr{M}(\Lambda,\gamma)=\int_{\mathbb{D}}\nu(dt_S)\Delta_{S\setminus\Lambda}(t_S)E\varphi(V_L,t_S)\prod_\Lambda p(t_{\gamma(s)},V_{\gamma(s)};t_s,V_s). \tag{4.15}$$

4.2. The *characteristic set* I for a pair (Λ,γ) is a subset of Λ which consists of all $s\in\Lambda$ such that $\gamma(s)\nrightarrow s$. We denote by γ_I the ordering of I induced by γ. Note that $I\supset L$ and $\gamma_I(s)\nrightarrow s$ for all $s\in I$. Let $\mathscr{I}$ stand for the family of all pairs (I,γ_I) with these two properties. For every $(I,\gamma_I)\in\mathscr{I}$, we denote by $\mathscr{S}(I,\gamma_I)$ the set of all pairs $\Lambda\in\mathscr{S}$, $\gamma\in\Gamma_\Lambda$, with the characteristic set (I,γ_I). Note that if (I,γ_I) is given, then for every $\Lambda\supset I$ there exists exactly one ordering $\gamma\in\Gamma_\Lambda$ such that $(\Lambda,\gamma)\in\mathscr{S}(I,\gamma_I)$. It follows from (4.12) that

$$\mathscr{N}=\sum_{(I,\gamma_I)\in\mathscr{I}}\mathscr{N}(I,\gamma_I), \tag{4.16}$$

where

$$\mathscr{N}(I,\gamma_I)=\sum_{(\Lambda,\gamma)\in\mathscr{S}(I,\gamma_I)}\mathscr{M}(\Lambda,\gamma). \tag{4.17}$$

There exists a 1–1 correspondence between the orderings $\gamma\in\Gamma$ of the set S and the elements (I,γ_I) of $\mathscr{I}$: The set I corresponding to γ is determined by (2.33) and γ_I is induced by γ. Therefore, we can rewrite (4.16) and (4.17) as

$$\mathcal{N} = \sum_{\gamma\in\Gamma} \mathcal{N}_\gamma, \tag{4.18}$$

$$\mathcal{N}_\gamma = \sum_{\Lambda\supset I} \mathcal{M}(\Lambda, \gamma_\Lambda), \tag{4.19}$$

where γ_Λ is the ordering of Λ induced by γ. We claim that:

4.2.A. For $s \in \Lambda$,

$$\Delta_{S\setminus\Lambda}(t_S)p\big(t_{\gamma_\Lambda(s)}, x_{\gamma_\Lambda(s)}; t_s, x_s\big) = \Delta_{S\setminus\Lambda}(t_S)p\big(t_{\gamma(s)}, x_{\gamma_\Lambda(s)}; t_s, x_s\big).$$

4.2.B. For $s \in I$, $\gamma_\Lambda(s) \sim \gamma_I(s)$ and $V_{\gamma_\Lambda(s)} = V_{\gamma_N(s)}$, where $N = I_A \cup \Lambda_B$.
4.2.C. For $s \in \Lambda \setminus I$,

$$\begin{aligned} V_s - V_{\gamma_\Lambda(s)} &= 0, && \text{if } s \in A,\\ &= V_s - V_{s^*\Lambda_B}, && \text{if } s \in B. \end{aligned}$$

4.2.D. For every $s \in N$, $\gamma_N(s) \sim \gamma(s)$.

To prove 4.2.A, note that, for $s \in \Lambda$, the interval $(\gamma_\Lambda(s), \gamma(s)]$ is contained in $S \setminus \Lambda$ and is monochromatic. Analogously, 4.2.D holds because the interval $[\gamma_N(s), \gamma(s)] \subset S \setminus N$ is monochromatic.

Let $s \in I$. If $\gamma_\Lambda(s) \nsim \gamma_I(s)$, then there exists $s' \in \Lambda$ such that $\gamma_I(s) < s' \le \gamma_\Lambda(s)$ and $\gamma(s') \nsim s'$. Since I is the characteristic set for Λ, this is impossible. Hence, $\gamma_\Lambda(s) \sim \gamma_I(s)$. If $\gamma_\Lambda(s) \in N$, then the second part of 4.2.B is obvious. If $\gamma_\Lambda(s) \in \Lambda \setminus N = \Lambda_A \setminus I_A$, then $\gamma_N(s) \sim \gamma_\Lambda(s)$ (same arguments as at the beginning of this proof) and 4.2.B follows from 4.1.D.

If $s \in \Lambda \setminus I$, then $\gamma_\Lambda(s) \sim s$; hence, $\gamma_\Lambda(s) = s^*\Lambda$ and 4.2.C follows from 4.1.C because $s^*\Lambda = s^*\Lambda_B$.

Taking into account 4.2.A, B, C, we conclude from (4.15) that

$$\mathcal{M}(\Lambda, \gamma_\Lambda) = \int_{\mathbb{D}} \nu(dt_S)\Lambda_{S\setminus\Lambda}(t_S)E\varphi(V_L, t_S)\mathcal{P}_S, \tag{4.20}$$

where

$$\mathcal{P}_s = \begin{cases} p\big(t_{\gamma(s)}, V_s; t_s, V_s\big) = p_{t_s - t_{\gamma(s)}}(0) & \text{for } s \in \Lambda_A \setminus I_A,\\ p_{t_s - t_{\gamma(s)}}\big(V_s - V_{s^*\Lambda_B}\big) & \text{for } s \in \Lambda_B \setminus I_B,\\ p\big(t_{\gamma(s)}, V_{\gamma_N(s)}; t_s, V_s\big) & \text{for } s \in I \end{cases} \tag{4.21}$$

and

$$\mathcal{P}_K = \prod_K \mathcal{P}_s \tag{4.22}$$

for every $K \subset S$.

4.3. Fix a set $I \supset L$. Let $\mathscr{S}_A = \{\Lambda_A\colon I_A \subset \Lambda_A \subset A\}$ and $\mathscr{S}_B = \{\Lambda_B\colon I_B \subset \Lambda_B \subset B\}$. Formula $\Lambda \to (\Lambda_A, \Lambda_B)$ establishes a 1–1 correspondence between $\mathscr{S}$ and $\mathscr{S}_A \times \mathscr{S}_B$. By (4.19),

$$\mathscr{N}_\gamma = \sum_{\Lambda_B \in \mathscr{S}_B} \mathscr{K}_\gamma(\Lambda_B), \tag{4.23}$$

with

$$\begin{aligned} \mathscr{K}_\gamma(\Lambda_B) &= \sum_{\Lambda_A \in \mathscr{S}_A} \mathscr{M}(\Lambda, \gamma_\Lambda) \\ &= \int_{\mathbb{D}} \nu(dt_S)\Delta_{B\setminus\Lambda_B}(t_B)E\varphi(V_L, t_S)\mathscr{P}_N \mathscr{Q}_{J_A}, \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} \mathscr{Q}_{J_A} &= \sum_{\Lambda_A \in \mathscr{S}_A} \Delta_{A\setminus\Lambda_A}(t_S)\mathscr{P}_{\Lambda_A\setminus I_A} \\ &= \prod_{J_A}\left[p_{t_s - t_{\gamma(s)}}(0) + \Delta_s(t_s - t_{\gamma(s)})\right] \end{aligned} \tag{4.25}$$

[cf. (2.33)]. Let $\tilde{\mathscr{K}}_\gamma(\Lambda_B)$ be obtained from $\mathscr{K}_\gamma(\Lambda_B)$ by replacing the factors $\mathscr{P}_s$ with

$$\tilde{\mathscr{P}}_s = p\left(t_{\gamma(s)}, V_{[\gamma_N(s)]}; t_s, V_{[s]}\right) = p\left(t_{\gamma(s)}, V_{[\gamma(s)]}; t_s, V_{[s]}\right) \tag{4.26}$$

for all $s \in I$ ($[\gamma_N(s)] = [\gamma(s)]$ by 4.2.D). Put

$$\tilde{\mathscr{N}}_\gamma = \sum_{\Lambda_B \in \mathscr{S}_B} \tilde{\mathscr{K}}_\gamma(\Lambda_B), \qquad \tilde{\mathscr{N}} = \sum_{\gamma \in \Gamma} \tilde{\mathscr{N}}_\gamma. \tag{4.27}$$

We have

$$|\mathscr{N} - \tilde{\mathscr{N}}| \le \sum_{\gamma \in \Gamma} \sum_{\Lambda_B \in \mathscr{S}_B} |\mathscr{K}_\gamma(\Lambda_B) - \tilde{\mathscr{K}}_\gamma(\Lambda_B)| = R_2. \tag{4.28}$$

Note that

$$\begin{aligned} \tilde{\mathscr{N}}_\gamma &= \sum_{\Lambda_B \in \mathscr{S}_B} \tilde{\mathscr{K}}_\gamma(\Lambda_B) \\ &= \int_{\mathbb{D}} \nu(dt_S)\mathscr{Q}_{J_A} E\varphi(V_L, t_S)\tilde{\mathscr{P}}_I \mathscr{Q}_{J_B}, \end{aligned} \tag{4.29}$$

with

$$\mathscr{Q}_{J_B} = \sum_{\Lambda_B \in \mathscr{S}_B} \Delta_{B\setminus\Lambda_B}(t_S)\mathscr{P}_{\Lambda_B\setminus I_B}. \tag{4.30}$$

By 4.1.A and C, $\mathscr{P}_s$, $s \in J_B$, are mutually independent and independent of $\varphi(V_L, t_S)\tilde{\mathscr{P}}_I$ and, taking into account 4.1.E, we get

$$\tilde{\mathscr{N}}_\gamma = \int \nu(dt_S)\mathscr{Q}_{J_A}\tilde{\mathscr{Q}}_{J_B} E\varphi(V_L, t_S)\tilde{\mathscr{P}}_I, \tag{4.31}$$

where

$$\tilde{\mathcal{Q}}_{J_B} = E\mathcal{Q}_{J_B} = \sum_{\Lambda_B \in \mathcal{S}_B} \Delta_{B\setminus\Lambda_B}(t_S) \prod_{\Lambda_B \setminus I_B} E\mathcal{P}_s$$

(4.32)

$$= \prod_{J_B} \left[\int \lambda(dx) q(x) p_{t_s - t_{\gamma(s)}}(x) + \Delta_s(t_s - t_{\gamma(s)}) \right].$$

By comparing (4.31), (4.32), (4.25), (4.26) and (4.1) with (3.68), (3.69), (2.31) and (2.32), we get

$$\tilde{\mathcal{N}}_\gamma = \int \nu(du_I) E\phi_\gamma(V_L, u_I) p^\alpha_{u_I}(V_L), \tag{4.33}$$

with the variables u_s defined by (2.7). Formula (2.30) follows from (4.13), (4.28) and (4.33).

5. Passage to the limit.

5.1. A *random germ* is a family of random variables $W = \{W^{\alpha\varepsilon}\}$ defined for sufficiently small positive α and ε. Two germs are considered as indistinguishable if they coincide for all sufficiently small α, ε and all statements about germs should be followed by the words "for all sufficiently small α, ε." We drop these words when it causes no confusion.

We call a germ W *standard* if random variables $W^{\alpha\varepsilon}$ are $\mathbb{Z}_\alpha$-valued and if there exist constants a, b independent of α, ε such that

$$|W^{\alpha\varepsilon}| \le a\varepsilon, \tag{5.1}$$

$$P\{W^{\alpha\varepsilon} = x\} \le b\alpha\varepsilon^{-2} \quad \text{for all } x \ne 0. \tag{5.2}$$

We say that W is *perfect* if (5.2) is satisfied also for $x = 0$. If W and $\tilde{W}$ are standard and if $W^{\alpha\varepsilon}$ and $\tilde{W}^{\alpha\varepsilon}$ are independent, then $W + \tilde{W}$ is also standard. If, in addition, W is perfect, then $W + \tilde{W}$ is perfect as well.

The germ Y^ε_α, described in Section 3.1, is perfect. Indeed, if $|Y| \le a/3$, then $Y^\varepsilon_\alpha = 0$ for $2a\varepsilon \le 3\sqrt{\alpha}$. For $2a\varepsilon > 3\sqrt{\alpha}$, we have $|Y^\varepsilon_\alpha| \le a\varepsilon/3 + \sqrt{\alpha} \le a\varepsilon$. Formula (5.2) follows from (1.35) and (1.53).

Consider the family of random germs $V_s = \{V^{\alpha\varepsilon}_s\}$, $s \in \Lambda$, described in Section 4.1 and put

$$U^{\alpha\varepsilon}_{s\sigma} = V^{\alpha\varepsilon}_s - V^{\alpha\varepsilon}_\sigma, \qquad Z^\alpha_s = V_{[s]}. \tag{5.3}$$

By 4.1.A–E we have:

5.1.A. If $s \sim \sigma$, then $U_{s\sigma}$ is standard.

5.1.B. If $s \nsim \sigma$, then $\tilde{U}_{s\sigma} = U_{s\sigma} - Z_s + Z_\sigma$, Z_s, Z_σ are independent, $\tilde{U}_{s\sigma}$ is a standard germ and Z_s and Z_σ have the law λ_α.

5.1.C. If $s \sim \sigma$, $s \ne \sigma$ and if $s \in B$, then $\tilde{U}_{s\sigma}$ is perfect.

It follows from 5.1.B that if $s \nsim \sigma$, then

$$EF(U^{\alpha\varepsilon}_{s\sigma}, Z_\sigma, Z_s - Z_\sigma) = E\int F(\tilde{U}^{\alpha\varepsilon}_{s\sigma} + x, z, x)\lambda_\alpha(dz)\lambda_\alpha(dx). \tag{5.4}$$

In particular,

$$E[F_1(U_{s\sigma}^{\alpha\varepsilon})F_2(Z_s^\alpha)] = \lambda_\alpha(F_1)\lambda_\alpha(F_2). \tag{5.5}$$

We use letters C and l for constants independent of α and ε.

5.2.

LEMMA 5.1. *If W is a standard germ, then*

$$E\int[G_u^\alpha(z) - G_u^\alpha(z + W^{\alpha\varepsilon})]^2\lambda_\alpha(dz) \le C_1\varepsilon, \tag{5.6}$$

$$EG_c^\alpha(W^{\alpha\varepsilon})^k \le C_2\left(\log\frac{1}{\alpha}\right)^k. \tag{5.7}$$

If W is perfect, then

$$EG_c^\alpha(W^{\alpha\varepsilon})^k \le C_2\left(\log\frac{1}{\varepsilon}\right)^k. \tag{5.8}$$

The constant C_1 is independent of u (but C_2 depends on k).

PROOF. Formula (5.6) follows from Theorem 3.2 and (5.1).

By (5.1) and (5.2), the density function $q^{\alpha\varepsilon}$ of $W^{\alpha\varepsilon}$ relative to λ_α satisfies the conditions

$$q^{\alpha\varepsilon}(x) = 0 \quad \text{for } |x| \ge a\varepsilon, \qquad q^{\alpha\varepsilon}(x) \le b\varepsilon^{-2} \quad \text{for } 0 < |x| < a\varepsilon. \tag{5.9}$$

By Lemma 3.1 [with $h(r) = 1_{r < a\varepsilon}$] and (3.28),

$$\begin{aligned} Eg^\alpha(W_\alpha^\varepsilon)^k &\le g^\alpha(0)^k + b\varepsilon^{-2}\int 1_{|x|<a\varepsilon}g^\alpha(x)^k\lambda_\alpha(dx) \\ &\le C_2'\left[\left(\ln\frac{1}{\alpha}\right)^k + \varepsilon^{-2}\int_0^{a\varepsilon}\Upsilon(r)^k r\,dr + 1\right]. \end{aligned} \tag{5.10}$$

If $\varepsilon \le (2a)^{-1}$, then by (3.11) the integral in the right side is equal to

$$4^k\int_0^{a\varepsilon}(-\ln r)^k r\,dr = 4^k\varepsilon^2\int_0^a(-\log\varepsilon\rho)^k\rho\,d\rho.$$

Since

$$G_c^\alpha(x) \le e^c g^\alpha(x), \tag{5.11}$$

this implies that

$$EG_c^\alpha(W^{\alpha\varepsilon})^k \le C_2''\left[\left(\ln\frac{1}{\alpha}\right)^k + \left(\ln\frac{1}{\varepsilon}\right)^k\right]. \tag{5.12}$$

If $a\varepsilon \ge \sqrt{\alpha}$, then $\ln(1/\varepsilon) \le \ln(a/\sqrt{\alpha})$ and (5.7) holds. If $a\varepsilon < \sqrt{\alpha}$, then by (5.1), $W^{\alpha\varepsilon} = 0$ and (5.7) follows from (5.11) and (3.28).

If W is perfect, then the first term in the right side of (5.10) can be dropped and we get (5.8). □

REMARK. By (3.3), (5.7) and (5.8),

$$|\hat{h}_\alpha| \le C_3 \ln\frac{1}{\alpha}, \qquad |h^\varepsilon_\alpha| \le C_3\left(\ln\frac{1}{\varepsilon}\right) \wedge \left(\ln\frac{1}{\alpha}\right). \tag{5.13}$$

5.3. We reserve the letter $\mathbb{F}$ for functions independent of α, ε and φ.

THEOREM 5.1. *Let $R_1 = R_1^{\alpha\varepsilon}$ be given by* (4.14). *If $\varphi(x_L, t_S) = 0$ outside of the set* (2.5), *then*

$$|R_1^{\alpha\varepsilon}| \le \mathbb{F}_1(c)\alpha\left(\ln\frac{1}{\alpha}\right)^l \|\varphi\|. \tag{5.14}$$

PROOF. Put

$$U_s = V_s - V_{\gamma(s)}, \qquad Z_s = V_{[s]}, \tag{5.15}$$

where V_s, $s \in \Lambda$, are random germs introduced in Section 4.1. Note that

$$\int \prod_{S\setminus\Lambda} \delta_\alpha(t_s - t_{s^*})\nu_\alpha(dt_{S\setminus\Lambda}) = 1. \tag{5.16}$$

Changing variables in (4.7) and (4.10) by the formula

$$u_s = t_s - t_{\gamma(s)}, \qquad s \in \Lambda, \tag{5.17}$$

and taking into account (4.1), (4.2) and (5.16), we get

$$|\mathscr{M}_\Lambda(\varphi 1_{\mathbb{D}(\Lambda,\gamma,r)})| \le \hat{h}^{|A\setminus\Lambda_A|}|h^\varepsilon_\alpha|^{|B\setminus\Lambda_B|}Q^{\alpha\varepsilon}\|\varphi\|, \tag{5.18}$$

where

$$Q^{\alpha\varepsilon} = E\int \nu_\alpha(du_\Lambda)1(u_r)\prod_\Lambda p^\alpha_{u_s}(U_s^{\alpha\varepsilon})1_{0\le u_s\le c}\prod_L 1_{|Z_s^\alpha|\le c}, \tag{5.19}$$

where $1(u) = 1$ for $u = 0$ and $1(u) = 0$ for $u \ne 0$. By (3.1),

$$\int_0^c 1(u)p^\alpha_u(x)\nu_\alpha(du) = 1(x). \tag{5.20}$$

By (2.41) and (5.20),

$$Q^{\alpha\varepsilon} = E1(U_r^{\alpha\varepsilon})\prod_M G^\alpha_c(U_s^{\alpha\varepsilon})\prod_L 1_{|Z_s^\alpha|\le c}, \tag{5.21}$$

where $M = \Lambda \setminus r$. By (5.21) and Hölder's inequality,

$$Q^{\alpha\varepsilon} \le \Phi_r \prod_M \Phi_s, \tag{5.22}$$

where

$$\Phi_r^2 = P\{|Z_r^\alpha| \le c,\ U_r^{\alpha\varepsilon} = 0\}, \tag{5.23}$$

$$\Phi_s^k = EG^\alpha_c(U_s^{\alpha\varepsilon})^k 1_{|Z_s^\alpha|\le c} \quad \text{for } s \in M, \tag{5.24}$$

with $k = 2|M|$.

By (5.5),

$$\Phi_r^2 \le 4c^2\alpha. \tag{5.25}$$

Let $s \nsim \gamma(s)$. Then by (5.5), Lemma 3.1 and (5.11),

$$\Phi_s^k \le 4c^2 \int \lambda_\alpha(dy) G_c^\alpha(y)^k \le \mathbb{F}_3(c) \quad \text{for } s \in M. \tag{5.26}$$

If $s \sim \gamma(s)$, then by 5.1.A, U_s is standard and by (5.7),

$$\Phi_s^k \le 4c^2 \left(\ln \frac{1}{\alpha}\right)^k. \tag{5.27}$$

The estimate (5.14) follows from (4.14), (5.18), (5.13), (5.22) and (5.25)–(5.27). □

5.4.

THEOREM 5.2. *Let $R_2 = R_2^{\alpha\varepsilon}$ be given by* (4.28). *If $\varphi \in \mathscr{L}^c(s, \gamma, L)$ and if $\tilde{\varphi}$ is defined by* (2.8), *then*

$$|R_2^{\alpha\varepsilon}| \le \tilde{\mathbb{F}}(c)\varepsilon \left(\ln \frac{1}{\varepsilon}\right)^l \|\tilde{\varphi}\|_{1_{S \setminus L}}. \tag{5.28}$$

PROOF. We change variables in (4.24), (4.25), (4.21) and (4.26) by formula (2.7) and then integrate by parts by (2.39). Taking into account (4.1), (3.68) and (2.41), we get

$$\begin{aligned} \mathscr{K}^{\alpha\varepsilon} &= \mathscr{K}_\gamma(\Lambda_B) \\ &= (h_\alpha^\varepsilon)^{|B \setminus \Lambda_B|} E \int \nu_\alpha(du_N) f_\alpha(Z_L^\alpha, u_N) \Pi^\alpha(u_L) \prod_{N \setminus L} G_{u_s}^\alpha(U_s^{\alpha\varepsilon}), \end{aligned} \tag{5.29}$$

where

$$\begin{aligned} \Pi^\alpha(u_L) &= \prod_L p_{u_s}^\alpha(U_s^{\alpha\varepsilon}), \\ f_\alpha &= \left[\prod_{N \setminus L} (-\Delta_s^\alpha) \prod_{J_A} \xi_s^\alpha \prod_{B \setminus \Lambda_B} \delta_s\right] \tilde{\varphi} \end{aligned} \tag{5.30}$$

and

$$U_s = \begin{cases} V_s - V_{s * \Lambda_B} & \text{for } s \in \Lambda_B \setminus I_B, \\ V_s - V_{\gamma_N(s)} & \text{for } s \in I. \end{cases} \tag{5.31}$$

An analogous formula holds for $\tilde{\mathscr{K}}^{\alpha\varepsilon}$ with $U_s^{\alpha\varepsilon}$ replaced by $Z_s^\alpha - Z_{\gamma_N(s)}^\alpha$ for $s \in I$. Therefore,

$$\mathscr{K}^{\alpha\varepsilon} - \tilde{\mathscr{K}}^{\alpha\varepsilon} = (h_\alpha^\varepsilon)^{|B \setminus \Lambda_B|} E \int \nu_\alpha(du_N) f_\alpha(Z_L^\alpha, u_N) \Pi^\alpha(u_L) F, \tag{5.32}$$

with

$$F = \left[\prod_{I\setminus L} G^\alpha_{u_s}(U^{\alpha\varepsilon}_s) - \prod_{I\setminus L} G^\alpha_{u_s}\left(Z^\alpha_s - Z^\alpha_{\gamma_N(s)}\right)\right] \prod_{\Lambda_B\setminus I_B} G^\alpha_{u_s}(U^{\alpha\varepsilon}_s). \tag{5.33}$$

The product (5.33) can be decomposed into terms of the form

$$\Psi = \left[G^\alpha_{u_r}(U^{\alpha\varepsilon}_r) - G^\alpha_{u_r}\left(Z^\alpha_r - Z^\alpha_{\gamma_N(r)}\right)\right] \prod_{K_1} G^\alpha_{u_s}(U^{\alpha\varepsilon}_s) \prod_{K_2} G^\alpha_{u_s}\left(Z^\alpha_s - Z^\alpha_{\gamma_N(s)}\right), \tag{5.34}$$

where $N\setminus L = \{r\} \cup K_1 \cup K_2$ is a partition of N into disjoint subsets, $r \in I$ and $K_1 \supset \Lambda_B \setminus I_B$.

If $\varphi \in \mathscr{L}^c(S, \gamma, L)$, then $\tilde{\varphi} \in \mathscr{D}^c(S)$ for all x_L and by (2.16), (2.19), (2.25), (2.26) and Lemma 2.2,

$$|f_\alpha(x_L, u_N)| \le \tilde{\mathbb{F}}_1(c)\|\tilde{\varphi}\|_{1_{S\setminus L}} \prod_L 1_{|x_s| \le c} \prod_N 1_{u_s \le c}. \tag{5.35}$$

By (5.35) and Hölder's inequality,

$$\begin{aligned} &\left|E\int \nu_\alpha(du_N) f_\alpha(Z^\alpha_L, u_N)\Psi\Pi^\alpha(u_L)\right| \\ &\quad\le \tilde{\mathbb{F}}_1(c)\|\tilde{\varphi}\|_{1_{S\setminus L}} E\int \nu_\alpha(du_{N\setminus L})|\Psi| \prod_L \left[1_{|Z^\alpha_s| \le c} G^\alpha_c(U^{\alpha\varepsilon}_s)\right] \prod_{N\setminus L} 1_{u_s \le c} \\ &\quad\le \tilde{\mathbb{F}}_1(c)\|\tilde{\varphi}\|_{1_{S\setminus L}} \prod_N \Phi_s, \end{aligned} \tag{5.36}$$

where

$$\Phi_r^2 = \int \nu_\alpha(du) 1_{u \le c} E\left[G^\alpha_u(U^{\alpha\varepsilon}_r) - G^\alpha_u\left(Z^\alpha_r - Z^\alpha_{\gamma_N(r)}\right)\right]^2 1_{|Z^\alpha_r| \le c}, \tag{5.37}$$

Φ_s^k is given by (5.24) for $s \in K_1 \cup L$ and

$$\Phi_s^k = E\left[G^\alpha_c\left(Z^\alpha_s - Z^\alpha_{\gamma_N(s)}\right)^k 1_{|Z^\alpha_s| \le c}\right] \quad \text{for } s \in K_2. \tag{5.38}$$

Let $s \in I$. By 4.2.B, $V_{\gamma_N(s)} = V_\sigma$, where $\sigma = \gamma_\Lambda(s) \nrightarrow s$. By (5.31) and 5.1.B, $U_s = V_{s\sigma} = \tilde{U}_{s\sigma} + Z_s - Z_\sigma$. Therefore, the bound (5.26) is valid for Φ_s^k, $s \in K_1 \cap I$.

By (5.37) and (5.4),

$$\Phi_r^2 = 4c^2 \int \nu_\alpha(du) 1_{u \le c} E\left[G^\alpha_u(x + U^{\alpha\varepsilon}_r) - G^\alpha_u(x)\right]^2 \lambda_\alpha(dx)$$

[cf. (5.25)] and by (5.6),

$$\Phi_r^2 \le \tilde{\mathbb{F}}_2(c)\varepsilon. \tag{5.39}$$

If $s \in \Lambda_B \setminus I_B$, then $s^*\Lambda \sim s$ and by (5.31) and 5.1.C, U_s is perfect. By (5.8),

$$\Phi_s^k \le C_2\left(\log\frac{1}{\varepsilon}\right)^k. \tag{5.40}$$

Finally, if $s \in K_2$, then by (5.38),

$$(5.41)\qquad \Phi_s^k = \int \lambda_\alpha(dz)\lambda_\alpha(dz')G_c^\alpha(z-z')^k 1_{|z|\le c} \le 4c^2\int \lambda_\alpha(dx)G_c^\alpha(x)^k \le \mathbb{F}_3(c).$$

The bound (5.28) follows from (5.32), (5.33), (5.36), (5.13), (5.26), (5.39) and (5.40).

Theorems 5.1 and 5.2 and formula (2.12) imply the bounds (2.34) and (2.35). □

5.5. The proof of Theorem 2.2 has been sketched in Section 2.4. Now, we fill the gaps.

The notation $\mathbb{F}(c, \alpha, \varepsilon)$ with subscripts $1, 2, \dots$ is used for functions with the property, for every $c > 0$, $\mathbb{F}(c, \alpha, \varepsilon) \to 0$ as $\alpha, \varepsilon \to 0$.

The restriction of φ to $(\mathbb{R}^2)^L \times \mathbb{D}(S, \gamma)$ belongs to $\mathscr{L}(S, \gamma, L)$. Thus, for every x_L, $\tilde{\varphi}_\gamma \in \mathscr{D}^c(S)$, and f_γ, $f_\gamma^{\alpha\varepsilon}$ given by (2.48), (2.49), (2.43) and (2.31) belong to $\mathscr{D}^c(I)$.

First, we find a bound for $f_\gamma^{\alpha\varepsilon} - f_\gamma$. Note that

$$(5.42)\qquad f_\gamma^{\alpha\varepsilon} = \eta^{\alpha\varepsilon}\tilde{\varphi}_\gamma, \qquad f_\gamma = \eta\tilde{\varphi}_\gamma,$$

where

$$(5.43)\qquad \eta^{\alpha\varepsilon} = \prod_S \eta_s^{\alpha\varepsilon}, \qquad \eta = \prod_S \eta_s$$

and $\eta_s^{\alpha\varepsilon}$ equals $-\Delta_s^\alpha$ on I, ξ_s^α on J_A and $\xi_s^{\alpha\varepsilon}$ on J_B; η_s equals $-D_s$ on I and ξ_s on J. Put $\rho_s^{\alpha\varepsilon} = \eta_s^{\alpha\varepsilon} - \eta_s$. By (2.19) and (2.26)–(2.28),

$$(5.44)\qquad \|\eta_s\|_{1,c} < \infty, \qquad \|\rho_s^{\alpha\varepsilon}\|_{2,c} \le \mathbb{F}_1(c, \alpha, \varepsilon) \quad \text{for all } s \in S \text{ and all } c > 0.$$

Note that

$$(5.45)\qquad \eta^{\alpha\varepsilon} - \eta = \sum\left[\prod_\Lambda \rho_s^{\alpha\varepsilon}\prod_{S\setminus\Lambda}\eta_s\right],$$

with the sum taken over all nonempty subsets Λ of S. We conclude from Lemma 2.2 that $\|\eta^{\alpha\varepsilon} - \eta\|_{2,c} \le \mathbb{F}_2(c, \alpha, \varepsilon)$ and by (5.42) and (2.16),

$$(5.46)\qquad |f_\gamma^{\alpha\varepsilon}(x_L, u_I) - f_\gamma(x_L, u_I)| \le \mathbb{F}_2(c, \alpha, \varepsilon)\|\tilde{\varphi}_\gamma\|_2.$$

Denote by $\mathscr{N}_\gamma^{\alpha\varepsilon}$ and $\mathscr{N}_\gamma$ the terms in (2.30) and (2.47) corresponding to an ordering γ and put

$$(5.47)\qquad \mathscr{N}_\gamma^\alpha = \int \lambda_\alpha(dx_L)\nu_\alpha(du_I)f_\gamma(x_L, u_I)G_{u_I}^\alpha(x_L).$$

It is easy to see from (2.41) and (3.1) that $\int G_u^\alpha(x)\lambda_\alpha(dx) = \nu_\alpha[0, u] \to u$ as $\alpha \to 0$ and, by (5.46),

$$(5.48)\qquad |\mathscr{N}_\gamma^{\alpha\varepsilon} - \mathscr{N}_\gamma^\alpha| \le C'\mathbb{F}_4(c, \alpha, \varepsilon)\|\tilde{\varphi}_\gamma\|_2.$$

Note that

$$(5.49)\qquad \mathscr{N}_\gamma^\alpha = \int dx_L\, du_I\, f_\gamma(x_L^\alpha, u_I^\alpha)G_{u_I^\alpha}^\alpha(x_L^\alpha),$$

where $x_L^\alpha = \{x_s^\alpha, s \in L\}$ and $u_I^\alpha = \{u_s^\alpha, s \in I\}$, $x_s^\alpha = \iota_\alpha(x_s)$, with ι_α introduced in Section 3.4 and

$$(5.50) \qquad u_s^\alpha = m\alpha \quad \text{for } (m-1)\alpha < u_s \le m\alpha$$

[cf. (3.36)]. As $\alpha \to 0$, the integrand in (5.49) converges to the integrand in (2.47) by Lemma 3.2. By (3.24) and (5.11),

$$(5.51) \qquad G_{u_I^\alpha}^\alpha(x_L^\alpha) \le C_1 \prod_L \Upsilon(x_s)$$

in the region Q_c determined by (2.5) and by the dominated convergence theorem, $\mathscr{N}_\gamma^\alpha \to \mathscr{N}_\gamma$ as $\alpha, \varepsilon \to 0$. Therefore, the first equation (2.47) follows from (5.48). Integration by parts yields the second equation.

By (2.26) and Lemma 2.2, $\|\prod_{J_A} \xi_s^\alpha \prod_{J_B} \xi_s^{\alpha\varepsilon}\|_{1_J, c} = \mathbb{F}_3(c) < \infty$ and by (2.31) and (2.16), $|\phi_\gamma^{\alpha\varepsilon}| \le \mathbb{F}_3(c)\|\tilde{\varphi}_\gamma\|_{1_J}$. Therefore,

$$|\mathscr{N}_\gamma^{\alpha\varepsilon}| \le \mathbb{F}_3(c)\|\tilde{\varphi}_\gamma\|_{1_J} \int \lambda_\alpha(dx_L)\nu_\alpha(du_I) G_{u_I}^\alpha(x_L) 1_{Q_c}(x_L, u_I).$$

By (5.51), $|\mathscr{N}_\gamma^{\alpha\varepsilon}| \le \mathbb{F}_2(c)\|\tilde{\varphi}_\gamma\|_1$ and (2.30), (2.34), (2.12) and (2.6) imply the bound (2.51). □

5.6.

Proof of Theorem 2.3. If $\varphi_i \in \mathscr{E}_k^{\beta c}$, $i = 1, \dots, r$, then $\mathscr{T}_\alpha^\varepsilon(h; k, \varphi)$ and $\mathscr{T}_\alpha(\hat{h}; k, \varphi)$ do not depend on h and $\hat{h}$ and we can put $h_\alpha^\varepsilon = \hat{h}_\alpha = 0$. Therefore, $\Delta_{S \setminus \Lambda}(t_S) = 0$ for $\Lambda \ne S$ and only terms with $\Lambda = S$ remain in (4.8), (4.12) and (4.14). By (4.13) and Theorem 5.1,

$$(5.52) \qquad |E\wp_\alpha^\varepsilon - \mathscr{N}| = |E\wp_\alpha^\varepsilon - \sum_{\gamma \in \Gamma} \mathscr{M}^{\alpha\varepsilon}(S, \gamma)| \le R_1^{\alpha\varepsilon} \to 0 \quad \text{as } \alpha, \varepsilon \to 0.$$

By (4.15) and (5.15),

$$(5.53) \qquad \begin{aligned} \mathscr{M}^{\alpha\varepsilon}(S, \gamma) &= E\int_{\mathbb{D}} \nu_\alpha(dt_S)\varphi(Z_L^\alpha, t_S)\prod_S p_{t_s - t_{\gamma(s)}}^\alpha(U_s^{\alpha\varepsilon}) \\ &= E\int_{\mathbb{R}_+^S} du_S\, \tilde{\varphi}_\gamma(Z_L^\alpha, u_S^\alpha)\prod_S p_{u_s^\alpha}^\alpha(U_s^{\alpha\varepsilon}), \end{aligned}$$

where u_s^α is defined by (5.50).

Note that $\tilde{\varphi}_\gamma(x_L, u_S) = 0$ on the complement of $K_\beta = \{u_S: u_s \ge 0 \text{ for } s \in I; u_s \ge \beta \text{ for } s \in J\}$. For every $0 < \sigma < \beta$, put $K_{\beta\sigma} = K_\beta \cap \{u_S: u_s \ge \sigma \text{ for } s \in I\}$.

It follows from (3.1) and (3.39) that for every $\sigma > 0$,

$$(5.54) \qquad \sup_{x \in \mathbb{R}^2, u \ge \sigma} |p_{u^\alpha}^\alpha(x^\alpha) - p_u(x)| \to 0 \quad \text{as } \alpha \to 0$$

and

$$(5.55) \qquad p_u^\alpha(x) \le C_1$$

for all sufficiently small α and all $x \in \mathbb{Z}_\alpha$, $u \in \mathbb{Q}_\alpha$, $u \geq \sigma$. Put

$$\tilde{U}_s = U_s - Z_s + Z_{\gamma(s)}. \tag{5.56}$$

By (5.54) and 4.1.A, B,

$$\begin{aligned}
&\lim_{\alpha,\varepsilon\to 0} E\int_{K_{\beta\sigma}} du_S\, \tilde{\varphi}_\gamma(Z_L^\alpha, u_S^\alpha)\prod_S p_{u_s^\alpha}^\alpha(U_s^{\alpha\varepsilon})\\
&= \lim_{\alpha,\varepsilon\to 0} E\int_{K_{\beta\sigma}} du_S \int dx_L\, \tilde{\varphi}_\gamma(x_L^\alpha, u_S^\alpha)\prod_S p_{u_s}\big(\tilde{U}_s^{\alpha\varepsilon} + x_{[s]}^\alpha - x_{[\gamma(s)]}^\alpha\big) \qquad (5.57)\\
&= E\int_{K_{\beta\sigma}} du_S \int dx_L\, \tilde{\varphi}(x_L, u_S) p_{u_I}(x_L)\prod_J p_{u_s}(0),
\end{aligned}$$

with $p_{u_I}(x_L)$ determined by (2.50). The second equality holds because (i) $\tilde{U}_s^{\alpha\varepsilon} \to 0$ as $\alpha, \varepsilon \to 0$ by 5.1.A, B and (5.1); (ii) $x_s^\alpha \to x_s$, $u_s^\alpha \to u_s$ for all s as $\alpha \to 0$; (iii) (5.55) justifies passage to the limit under the integral sign.

Note that $K_\beta \setminus K_{\beta\sigma} = \bigcup_{r\in I} H_{\beta\sigma r}$, where

$$H_{\beta\sigma r} = \{u_S\colon u_s \geq \beta \text{ for } s \in J;\ u_s \geq 0 \text{ for } s \in I\setminus r;\ 0 \leq u_r \leq \sigma\}.$$

If $\varphi = 0$ outside of Q_c [see (2.5)], then by (5.55) the part of the integral (5.53) over $H_{\beta\sigma r}$ does not exceed in absolute value

$$C_2 E\left[G_\sigma^\alpha(U_r^{\alpha\varepsilon})\prod_{I\setminus r} G_c^\alpha(U_s^{\alpha\varepsilon})\prod_L 1_{|Z_s^\alpha|\leq c}\right].$$

By Hölder's inequality, this is not larger than $C_2\prod_I \Phi_s$, where

$$\Phi_s^p = E\left[1_{|Z_s^\alpha|\leq c} G_c^\alpha(U_s^{\alpha\varepsilon})^p\right], \qquad \Phi_r^p = E\left[1_{|Z_r^\alpha|\leq c} G_\sigma^\alpha(U_r^{\alpha\varepsilon})^p\right],$$

with $p = |I|$. By (5.5), Lemma 3.1 and (5.11),

$$\Phi_s^p \leq 4C_3 c^2 \quad \text{for } s \in I\setminus r, \qquad \Phi_r^p \leq 4C_3\sigma^2. \tag{5.58}$$

Hence, the part of the integral (5.55) over $K_\beta \setminus K_{\beta\sigma}$ can be made arbitrary small uniformly in α, ε if we choose a sufficiently small σ. This proves that (5.53) tends to the integral in the right side of (2.52), and (2.52) follows from (5.52).

By (5.53), (5.55) and (5.58),

$$\begin{aligned}
|\mathscr{M}^{\alpha\varepsilon}| &\leq \|\tilde{\varphi}_\gamma\| E\int 1_{K_\beta}(u_S)\prod_S p_{u_s}^\alpha(U_s^{\alpha\varepsilon})\nu_\alpha(du_s) 1_{Q_c}(x_L, u_S)\\
&\leq C_2\|\tilde{\varphi}_\gamma\| E\prod_I G_c^\alpha(U_s^{\alpha\varepsilon}) \leq C_3\|\tilde{\varphi}_\gamma\|
\end{aligned} \tag{5.59}$$

and the estimate (2.54) follows from (5.59), (5.52) and (5.14). □

5.7.

Proof of Lemma 1.2. We note that by Lemma A.4

$$\mathscr{T}_\alpha^\varepsilon(h; k, \varphi) = \sum_{l=1}^{k} (h_\alpha^\varepsilon + 1)^{k-l} T_\alpha^\varepsilon(l, B_k^l\varphi) \tag{5.60}$$

[cf. (1.31)], where

$$T_\alpha^\varepsilon(k,\varphi) = \mathcal{T}_\alpha^\varepsilon(-1;k,\varphi) = \int_{D_k} \rho_\alpha(X_t^\alpha, t)\nu_\alpha(dt), \tag{5.61}$$

with

$$\rho_\alpha(x,t) = \varphi(x_1;t)\prod_{i=2}^{k} q_\alpha^\varepsilon(x_i - x_{i-1}). \tag{5.62}$$

We rewrite (1.36) in the form

$$T^\varepsilon(k,\varphi) = \int_{D_k} \rho(W_t, t)\,dt, \tag{5.63}$$

where $\rho(x,t)$ is defined by (5.62) with q_α^ε replaced by q^ε. By the Corollary to Lemma 3.2, $h_\alpha^\varepsilon \to h^\varepsilon$ as $\alpha \to 0$ and by (1.37), Lemma 1.2 will be proved if we show that $T_\alpha^\varepsilon(k,\varphi) \to_d T^\varepsilon(k,\varphi)$.

Put

$$F_\alpha = \int_{D_k} \rho(X_t^\alpha, t)\nu_\alpha(dt) = \int_{D_k} \rho\left(X_{\chi_\alpha(t)}^\alpha, \chi_\alpha(t)\right) dt, \tag{5.64}$$

where

$$\chi_\alpha(t_1,\dots,t_k) = (m_1\alpha,\dots,m_k\alpha) \quad \text{for } m_i\alpha \le t_i < (m_i+1)\alpha,\ i = 1,\dots,k.$$

First, we prove that

$$T_\alpha^\varepsilon(k,\varphi) - F_\alpha \to 0 \quad \text{in probability as } \alpha \to 0 \tag{5.65}$$

and then, using Lemmas A.2 and A.3, we show that

$$F_\alpha \to_d T^\varepsilon(k,\varphi) \quad \text{as } \alpha \to 0. \tag{5.66}$$

The function q^ε is continuous outside a closed set B of the Lebesgue measure 0. We denote by U_δ the δ-neighborhood of B and by V_δ the complement of U_δ. If ∂U_δ has measure 0, then $P\{X_{t_i}^\alpha - X_{t_{i-1}}^\alpha \in U_\delta\} \to P\{W_{t_i} - W_{t_{i-1}} \in U_\delta\}$ as $\alpha \to 0$. Therefore, for every $\beta > 0$ there exists a $\delta > 0$ such that for all sufficiently small α, with probability greater than or equal to $1-\beta$, $x_{t_i}^\alpha - X_{t_{i-1}}^\alpha \in V_\delta$. Since q^ε is continuous on V_δ and has a compact support, it is uniformly continuous on V_δ and

$$\int |\rho_\alpha(X_t^\alpha, t) - \rho(X_t^\alpha, t)| \prod_{i=2}^{k} 1_{V_\delta}(X_{t_i}^\alpha - X_{t_{i-1}}^\alpha)\nu_\alpha(dt) \to 0 \quad \text{as } \alpha \to 0,$$

which implies (5.65).

Put $D_k^c = \{t\colon 0 \le t_1 \le \cdots \le t_k \le c\}$. If c is big enough, then $\varphi(x,t) = 0$ for all $x \in \mathbb{R}^2$ and all $t \notin D_k^c$. We apply Lemmas A.2 and A.3 to $T = D_k^c$ and $X = (\mathbb{R}^2)^k$ with the Lebesgue measures. If Y_t, $t \ge 0$, is a measurable stochastic process in $\mathbb{R}^2$, then $(Y_{t_1},\dots,Y_{t_k})$, $0 \le t_1 \le \cdots \le t_k \le c$, is a stochastic process in X indexed by T and its law is a measure on the space Ω of all measurable mappings from T to X. We denote by P the measure on Ω corresponding to the Brownian motion W_t and by P_α the measure corresponding to the linearly interpolated random walk $\tilde{X}^\varepsilon_\alpha$ introduced in Section 1.1. By the invariance principle, P_α converges weakly to P. By (1.1), $|\tilde{X}^\alpha_{t_i} - \tilde{X}^\alpha_{m_i\alpha}| \le \sqrt{\alpha}\,\xi_{m_i}$ for $m_i\alpha \le t_i < (m_i + 1)\alpha$ and condition (b) of Lemma A.3 holds since $P_\alpha Z^\alpha_t \le k(\alpha + \sqrt{\alpha}\,E|\xi_1|)$ and $E|\xi_1| < \infty$ by 1.1.A. The rest of the conditions are also satisfied. Hence, (5.66) holds. □

REMARK. The same arguments show that

$$\mathscr{T}^\varepsilon_\alpha(k,\varphi) - \mathscr{T}^{\varepsilon'}_\alpha(k,\varphi) \to_d \mathscr{T}^\varepsilon(k,\varphi) - T^{\varepsilon'}(k,\varphi) \quad \text{as } \alpha \to 0. \tag{5.67}$$

5.8.

PROOF OF THEOREM 1.1. Fix a φ which belongs to $\mathscr{S}_k$ and put $\mathscr{T}^\varepsilon_\alpha = \mathscr{T}^\varepsilon_\alpha(k; h, \varphi)$. The abbreviations $\mathscr{T}^\varepsilon$ and $\mathscr{T}_\alpha$ have an analogous meaning. Since $\|\mathscr{T}^\varepsilon_\alpha - \mathscr{T}^{\varepsilon'}_\alpha\|_{L^p} \le \|\mathscr{T}^\varepsilon_\alpha - \mathscr{T}_\alpha\|_{L^p} + \|\mathscr{T}_\alpha - \mathscr{T}^{\varepsilon'}_\alpha\|_{L^p}$, we have, by Theorem 1.3, that

$$E|\mathscr{T}^\varepsilon_\alpha - \mathscr{T}^{\varepsilon'}_\alpha|^p \to 0 \quad \text{as } \varepsilon, \varepsilon', \alpha \to 0. \tag{5.68}$$

Choose a sequence $\alpha_n \to 0$. By (5.67) and Skorohod's lemma, we can construct random variables Y_n, Y on a probability space $\tilde{\Omega}$ such that $Y_n(\tilde{\omega}) \to Y(\tilde{\omega})$ for every $\tilde{\omega} \in \tilde{\Omega}$ and the probability distributions of Y_n and Y coincide with those of $\mathscr{T}^\varepsilon_{\alpha_n} - \mathscr{T}^{\varepsilon'}_{\alpha_n}$ and $\mathscr{T}^\varepsilon - \mathscr{T}^{\varepsilon'}$, respectively. By Theorem 1.3, for every $p \ge 1$, $\|Y_n\|_{L^p} \le \|\mathscr{T}^\varepsilon_{\alpha_n}\|_{L^p} + \|\mathscr{T}^{\varepsilon'}_{\alpha_n}\|_{L^p}$ are bounded as $n \to \infty$. Therefore [see, e.g., Meyer (1966), Chapter 2, T22], $|Y_n|^p$ are uniformly integrable and $E|Y_n|^p \to E|Y|^p = E|\mathscr{T}^\varepsilon - \mathscr{T}^{\varepsilon'}|^p$. By (5.68), $E|\mathscr{T}^\varepsilon - \mathscr{T}^{\varepsilon'}|^p \to 0$ as $\varepsilon, \varepsilon' \to 0$. This proves (1.38). The bounds (1.40) follow from (1.57) and (1.41) follows from (1.37). □

6. Survey of literature.

6.1. Lévy (1940) was the first to prove that almost all Brownian paths in $\mathbb{R}^2$ have double points. The next step was due to Dvoretzky, Erdös and Kakutani (1950) who discovered that the same property holds for the Brownian motion in $\mathbb{R}^3$, but that almost all Brownian paths in $\mathbb{R}^4$ have no double points. Dvoretzky, Erdös and Kakutani (1954) proved that a Brownian path in $\mathbb{R}^2$ has a.s. k-multiple points for every finite k. Finally, the remaining case was settled in Dvoretzky, Erdös, Kakutani and Taylor (1957) by proving that, with probability 1, a Brownian path in $\mathbb{R}^3$ has no triple points. An interesting addition to these pictures was given in Dvoretzky, Erdös and Kakutani (1958): For the planar Brownian motion, there exist a.s. points x such that the set $\{t\colon W_t = x\}$ has cardinality c (continuum).

6.2. Intersection local times for k independent Brownian motions $W^1, \dots, W^k$ can be introduced by a symbolic formula

$$\int_\Gamma \delta_0\left(W^2_{t_2} - W^1_{t_1}\right) \cdots \delta_0\left(W^k_{t_k} - W^{k-1}_{t_{k-1}}\right) dt_1 \cdots dt_k \tag{6.1}$$

[cf. (1.25)]. It can be defined rigorously by replacing δ_0 with q_ε given by (1.35) and by passing to the limit in L^2. This is a much simpler subject than the self-intersection local times for one Brownian motion which we discussed in Sections 1.6 and 1.7. In particular, there is no explosion on hyperplanes $\{t_i = t_{i+1}\}$. The intersection local times (6.1) first appeared in Wolpert (1978). Dynkin (1981) developed a general theory of additive functionals of a family of independent symmetric Markov processes. In Dynkin (1986a), this theory was applied to construct additive functionals of order k for one symmetric Markov process. The self-intersection local times are a particular case.

A different approach to the intersection local times for a family of independent Brownian motions was given in Geman, Horowitz and Rosen (1984). It is based on their theory of occupation densities. Let $X(t)$ be a Borel function from $\mathbb{R}^k_+$ to $\mathbb{R}^N$. The formula

$$\mu_A(B) = \int 1_A(t) 1_B(X(t))\, dt \tag{6.2}$$

determines, for every Borel set A in $\mathbb{R}^k_+$, a measure on the Borel sets of $\mathbb{R}^N$ which the authors call the occupation measure of X. Its Radon–Nikodym derivative $\alpha(x, A)$ with respect to the Lebesgue measure on $\mathbb{R}^N$—if it exists—is called the occupation density or local time on A. Geman, Horowitz and Rosen (1984) proved that, in the case of

$$X(t) = \left(W^2_{t_2} - W^1_{t_1}, \dots, W^k_{t_k} - W^{k-1}_{t_{k-1}}\right), \tag{6.3}$$

a random kernel $\alpha(x, A)$ can be chosen in such a way that $\alpha(x, Q_t)$ is continuous in x, t (here $Q_t = [0, t_1] \times \cdots \times [0, t_k]$). Symbolically,

$$\alpha(x, A) = \int_A \delta_{x_1}\left(W^2_{t_2} - W^1_{t_1}\right) \cdots \delta_{x_k}\left(W^k_{t_k} - W^{k-1}_{t_{k-1}}\right) dt_1 \cdots dt_k. \tag{6.4}$$

By putting $x = 0$, one gets the intersection local time (6.1).

6.3. A new interest in self-intersections of the Brownian paths was inspired by Symanzik's idea that a "gas" of such paths can be used to construct Euclidean quantum fields [Symanzik (1969)]. Problems of constructive field theory stimulated, in particular, the work of Wolpert and of Geman, Horowitz and Rosen. They also motivated introduction of regularized self-intersection local times in Varadhan (1969).

Varadhan investigated the limit behaviour, as $\varepsilon \to 0$, of

$$S^\varepsilon(u) = \int_0^u ds \int_0^u dt\, p_\varepsilon(W_t - W_s) - h_\varepsilon u, \tag{6.5}$$

where p_ε is the Brownian density (2.45) and h_ε are suitable constants. He proved that there exists an $L^2(P_0^y)$ limit of $S^\varepsilon(u)$ as $\varepsilon \to 0$; here P_0^y is the measure corresponding to the Brownian bridge, i.e., to the Brownian motion conditioned by the equations $W_0 = 0$, $W_u = y$. In the notation of Theorem 1.1,

$$\lim_{\varepsilon \to 0} S^\varepsilon(u) = \mathscr{T}(\kappa; 2, \varphi),$$

with $\varphi(s, t) = 1_{[0,u]}(s) 1_{[0,u]}(t)$ and with κ depending on the choice of h_ε.

6.4. For $k > 2$, the functionals $\mathscr{T}(k)$ appeared first in Dynkin (1984a, b) as a tool for a probabilistic representation of $P(\varphi)_2$ fields.

Our objective was to define polynomials of the occupation field

$$T_z(\zeta) = \int_0^\zeta \delta_z(X_t)\,dt. \tag{6.6}$$

Here, X_t is a Markov process with a symmetric transition density $p_t(x, y)$ and ζ is the death time of X_t.

We replaced the formal expression (6.6) with a random variable

$$T_{\varepsilon z} = \int_0^\zeta p_\varepsilon(z, X_t)\,dt \tag{6.7}$$

and we put

$$:T_{\varepsilon z}^k:/k! = \sum_{l=0}^{k} B_{kl}(\varepsilon, z) T_{\varepsilon z}^l. \tag{6.8}$$

Our goal was to choose the coefficients B_{kl} in such a way that the limit

$$:T^k:_\lambda = \lim_{\varepsilon \to 0} \int \lambda(dz) :T_{\varepsilon z}^k: \tag{6.9}$$

exists. We were able to do this under the assumption that Green's function $g(x, y)$ of the process X_t has the singularity of the same type as Green's function of the planar Brownian motion [given by (3.8)]. Formula (6.9) holds in $L^p(P)$ for all $p \geq 1$ and for all measures λ subject to the condition

$$\int \lambda(dx) g(x, y)^m \lambda(dy) < \infty \quad \text{for } m = 1, 2. \tag{6.10}$$

The measure P is defined by "the finite-dimensional densities"

$$a_{t_1}(x_1) p_{t_2 - t_1}(x_1, x_2) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n) b(x_n) \tag{6.11}$$

for $0 < t_1 < \cdots < t_n$, with

$$a_t(x) = \int \mu(dz) p_t(z, x), \qquad b(x) = \int g(x, z) \nu(dz). \tag{6.12}$$

This means that the process X_t is conditioned to be born at time 0 with the initial law μ and to die at a random time ζ with the "final law" ν.

The functionals $:T^k:_\lambda$ are closely related to Wick's powers $:\varphi^{2n}:_\lambda$ of the free Gaussian field associated with X. In fact, we have arrived at the renormalization (6.8) by using this relation.

6.5. In Dynkin (1986c), the functionals (6.9) are studied directly, without using Gaussian fields. The process (X_t, P) is specified as the Brownian motion in $\mathbb{R}^2$ with an initial law μ.

We write $Y_u = \mathrm{LIM}_{\varepsilon \to 0} Y_{\varepsilon u}$ if, for every $r > 0$ and every $p \geq 2$,

$$\int_0^\infty du\, e^{-ru} P|Y_{\varepsilon u} - Y|^p \to 0 \quad \text{as } \varepsilon \to 0.$$

Our starting point is the field

$$(6.13)\qquad T(\varepsilon, z, u) = \int_0^u q^\varepsilon(W_t - z)\,dt,$$

where q^ε is defined by (1.35). For every k and every real-valued function $h = h_\varepsilon$, we put

$$(6.14)\qquad \mathscr{T}^k(h; \varepsilon, \lambda, u) = \int \lambda(dz)\mathscr{L}_k(h_\varepsilon, T(\varepsilon, z, u)),$$

where $\mathscr{L}_k(h, v)$ is a polynomial of degree k in h, v such that $k!\mathscr{L}_k(0, v) = v^k$. We show that, with a proper choice of $\mathscr{L}_k$, there exist functionals $\mathscr{T}_k(\lambda, u)$ such that, for a wide class of measures μ, λ and density functions q,

$$(6.15)\qquad \mathscr{T}_k(\lambda, u) = \operatorname*{LIM}_{\varepsilon\to 0}\mathscr{T}^k(h^0, \lambda, u),$$

where

$$(6.16)\qquad h_\varepsilon^0 = \frac{1}{\pi}\ln\varepsilon.$$

The polynomials $\mathscr{L}_k$ are defined by the formula

$$(6.17)\qquad \exp\{v\mathscr{S}[\psi_h(w)]\} = 1 + \sum_{k=1}^\infty \mathscr{L}_k(h, v)w^k,$$

where

$$(6.18)\qquad \mathscr{S}(v) = v + a_2v^2 + \cdots + a_nv^n + \cdots$$

is a power series with coefficients depending on q and

$$(6.19)\qquad \psi_h(w) = w/(1 - hw) = w\sum_0^\infty (hw)^k.$$

[In the notation of Dynkin (1986c), $\mathscr{L}_k(h, v) = \sum_{l=1}^k L_{kl}(-h)v^l/l!$ and $\psi_h = \phi_{-h}$.]

Along with $\mathscr{T}^k(\varepsilon, \lambda, u)$, we have investigated another class of functionals which converge to $\mathscr{T}_k(\lambda, u)$. Let $D_k(u) = \{t\colon 0 \le t_1 \le \cdots \le t_k \le u\}$ and

$$(6.20)\qquad \begin{aligned} T_k(\varepsilon, \lambda, u) = \int_{D_k(u)} &\rho(W_{t_1})q^\varepsilon(W_{t_2} - W_{t_1})\cdots \\ &\times q^\varepsilon(W_{t_k} - W_{t_{k-1}})\,dt_1\cdots dt_k. \end{aligned}$$

Put

$$(6.21)\qquad \mathscr{T}_k(\varepsilon, \lambda, u) = \sum_{l=1}^k \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} \tilde{h}_\varepsilon^{k-l}T_l(\varepsilon, \lambda, u),$$

where ρ is the density of the measure λ relative to the Lebesgue measure and

$$(6.22)\qquad \tilde{h}_\varepsilon = \frac{1}{\pi}\left[\ln\varepsilon + C + \int dy\,q(y)\ln(|y|/\sqrt{2})\right] = -Eg(\varepsilon Y) + o(1),$$

with g defined by (3.8). We prove that

$$\mathcal{T}_k(\lambda, u) = \operatorname*{LIM}_{\varepsilon \to 0} \mathcal{T}_k(\varepsilon, \lambda, u). \tag{6.23}$$

If $q(x) = (2\pi)^{-1} e^{-x^2/2}$, then the integral in (6.22) is equal to $-C/2$ and $\tilde{h}_\varepsilon = 1/\pi(\ln \varepsilon + C/2)$.

6.6. We note that the operators B_k^l defined by (1.29) act on the indicator function I_{ku} of the region $D_k(u)$ by the formula

$$B_k^l I_{ku} = \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} I_{lu}. \tag{6.24}$$

By comparing formulas (6.20) and (6.21) with (1.36) and (1.37), we conclude that $\mathcal{T}_k(\varepsilon, \lambda, u) = \mathcal{T}^\varepsilon(\tilde{h}; k, \phi)$ with $\phi(x, t) = \rho(x) I_{ku}(t)$. Therefore, it is natural to consider the limit (6.23) as the value of a Brownian self-intersection gauge:

$$\mathcal{T}_k(\lambda, u) = \mathcal{T}(\kappa_0; k, \phi) = \int_{D_k(u)} \rho(W_{t_1}) {:}\delta(W_{t_1}, \dots, W_{t_k}){:}_{\kappa_0}\, dt. \tag{6.25}$$

By (1.39),

$$\kappa_0 = \lim_{\varepsilon \to 0} \left[\tilde{h}_\varepsilon + E \int_0^1 q^\varepsilon(W_t)\, dt \right].$$

Taking into account (1.46) and (3.3), we get

$$\begin{aligned} \kappa_0 &= \lim_{\varepsilon \to 0} \left[\tilde{h}_\varepsilon + E G_1(\varepsilon Y) \right] \\ &= \lim_{\varepsilon \to 0} E \left[\int_0^1 p_t(\varepsilon Y)(1 - e^{-t})\, dt - \int_1^\infty e^{-t} p_t(\varepsilon Y)\, dt \right] \\ &= (2\pi)^{-1} \left[\int_0^1 (1 - e^{-t}) t^{-1}\, dt - \int_1^\infty e^{-t} t^{-1}\, dt \right] = C/2\pi. \end{aligned} \tag{6.26}$$

For an arbitrary κ, we put

$$\mathcal{T}_k(\kappa; \lambda, u) = \sum_{l=1}^{k} \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} (\kappa - \kappa_0)^{k-l} \mathcal{T}_l(\lambda, u). \tag{6.27}$$

We claim that:

6.6.A. If

$$h_\varepsilon - \frac{1}{\pi} \ln \varepsilon + \frac{C}{2\pi} \to \kappa \quad \text{as } \varepsilon \to 0, \tag{6.28}$$

then

$$\mathcal{T}_k(\kappa; \lambda, u) = \operatorname*{LIM}_{\varepsilon \to 0} \mathcal{T}^k(h^\varepsilon; \varepsilon; \lambda, u). \tag{6.29}$$

6.6.B. If

$$h_\varepsilon + E G_1(\varepsilon Y) \to \kappa \quad \text{as } \varepsilon \to 0, \tag{6.30}$$

then

$$\mathscr{T}_k(\kappa; \lambda, u) = \operatorname*{LIM}_{\varepsilon\to 0} \mathscr{T}_k(h_\varepsilon; \varepsilon, \lambda, u). \tag{6.31}$$

6.6.C. For every real κ and a,

$$\mathscr{T}_k(\kappa + a; \lambda, u) = \sum_{l=1}^{k} \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} a^{k-l} \mathscr{T}_l(\kappa; \lambda, u). \tag{6.32}$$

Formula (6.30) is just another form of (1.39) and (6.32) is a particular case of (1.41). To prove 6.6.A, we use the expression (6.17) for the generating function. Since $\psi_{a+b}(w) = \psi_b[\psi_a(w)]$, we have

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} \mathscr{L}_k(a+b, v) w^k &= \exp\{v\mathscr{S}[\psi_b(\psi_a(w))]\} \\ &= 1 + \sum_{l=1}^{\infty} \mathscr{L}_l(b, v)\psi_a(w)^l. \end{aligned} \tag{6.33}$$

Taking into account that

$$w^l(1-w)^{-l} = \sum_{k=l}^{\infty} \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} w^k,$$

we get from (6.33),

$$\mathscr{L}_k(a+b, v) = \sum_{l=1}^{k} \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} a^{k-l} \mathscr{L}_l(b, v). \tag{6.34}$$

Therefore, for every function $h = h_\varepsilon$ and for every constant a,

$$\mathscr{T}^k(h + a; \varepsilon, \lambda, u) = \sum_{l=1}^{k} \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} a^{k-l} \mathscr{T}^l(h; \varepsilon, \lambda, u). \tag{6.35}$$

By (6.15) and (6.35), (6.29) holds for h^0 given by (6.16). It follows from (6.27) and (6.35) that it holds for every h subject to condition (6.28).

Formula (6.29) and the identity

$$T(\varepsilon, z, u)^k/k! = \int_{D_k(u)} q^\varepsilon(W_{t_1} - z) \cdots q^\varepsilon(W_{t_k} - z)\, dt_1 \cdots dt_k$$

suggest another symbolism,

$$\mathscr{T}_k(\kappa; \lambda, u) = \int \lambda(dz) \int_{D_k(u)} :\delta_z(W_{t_1}) \cdots \delta_z(W_{t_k}):_\kappa\, dt_1 \cdots dt_k, \tag{6.36}$$

and the fact that the limits (6.29) and (6.31) coincide finds a heuristic explanation in the symbolic identity

$$:\delta_z(W_{t_1}) \cdots \delta_z(W_{t_k}):_\kappa = \delta_z(W_{t_1}) :\delta(W_{t_1}, \ldots, W_{t_k}):_\kappa \tag{6.37}$$

and the formula

(6.38) $$\int \rho(W_t)\varphi(t)\,dt = \int \varphi(t)A_\lambda(dt) = \int \lambda(dz)\int \delta_z(W_t)\varphi(t)\,dt,$$

where A_λ is an additive functional with the characteristic measure λ [cf. (1.21)].

6.7. In Dynkin (1986b, 1988), we introduced a particular case of the Brownian self-intersection gauge. In the present notation

(6.39) $$\mathscr{T}_k(\lambda,\varphi) = \int_{D_k} \rho(W_{t_1}):\delta(W_{t_1},\dots,W_{t_k}):_{\kappa_0}\varphi(t)\,dt.$$

Fields studied in Dynkin (1984a, b) (and described in Section 6.4) can be expressed by the formula

(6.40) $$:T^k:_\lambda/k! = \int \lambda(dz)\int_{D_k(\zeta_r)} dt_1 \cdots dt_k:\delta_z(W_{t_1})\cdots\delta_z(W_{t_k}):_{(C+\ln r)/2\pi}$$

if $X_t = W_t$ is the Brownian motion in $\mathbb{R}^2$ and $\zeta = \zeta_r$ is a random variable independent of W_t, with an exponential probability distribution $P\{\zeta_r > t\} = e^{-rt}$.

6.8. Rosen has studied a different type of regularization of self-intersection local times. He uses the notation $\{Y\} = Y - EY$ for any random variable Y. Rosen (1986b) proved that for every bounded Borel set $B \subset D_k$,

(6.41) $$R^\varepsilon(k,B) = \int_B \prod_{i=2}^k \{p_\varepsilon(W_{t_1} - W_{t_{i-1}})\}\,dt_1\cdots dt_k$$

has an $L^2(P_\mu)$ limit $\gamma^k(B)$ as $\varepsilon \to 0$. It seems natural to write

(6.42) $$\gamma^k(B) = \int_B \prod_{i=2}^k \{\delta(W_{t_i} - W_{t_{i-1}})\}\,dt_1\cdots dt_k.$$

It is established in Rosen (1986c) that, under conditions 1.1.A, B, D,

(6.43) $$R(k,n) = n^{-1}\sum_{0<m_1<\cdots<m_k\le n}\prod_{i=2}^k \{1(S_{m_{i-1}}, S_{m_i})\}$$

converges in distribution to $\gamma^k(D_k(1))$. To get this result, Rosen introduces functionals

(6.44) $$R_\alpha^\varepsilon(k) = \alpha^k \sum_{0<m_1<\cdots<m_k\le n}\prod_{j=2}^k (2\pi)^{-2} \times \int_{\Pi_\alpha} d\theta_j\Big\{\exp\Big[i\alpha\theta_j\big(S_{m_j} - S_{m_{j-1}}\big) - \varepsilon|\theta_j|^2/2\Big]\Big\},$$

where $\alpha = n^{-1}$ and Π_α is defined in Section 3.2. For $\varepsilon = 0$, this expression coincides with $R(k,n)$. Rosen shows that

(6.45) $$E|R_\alpha^\varepsilon(k) - R_\alpha^0(k)|^2 \le C\varepsilon^\beta$$

for some $C < \infty$, $\beta > 0$ independent of α. In our terminology, $R_\alpha^\varepsilon(k)$ is a link between $R_\alpha^0(k) = R(k,n)$ and its Brownian shadow $R^\varepsilon(k, D_k(1))$.

Dynkin (1985) proved the existence of L^p-limits for

$$\int dz\, \rho(z) \int_{D_2} f(s,t) \Psi_z^\varepsilon(s,t)\, ds\, dt, \tag{6.46}$$

where

$$\Psi_z^\varepsilon(s,t) = p_\varepsilon(z, W_s)\left[p_\varepsilon(z, W_t) - E\{p_\varepsilon(z, W_t) | W_s\}\right].$$

Since $\int dz\, \Psi_z^\varepsilon(s,t) = \{p_{2\varepsilon}(W_t - W_s)\}$, the limit for $\rho = 1$ is equal to

$$\int_{D_2} f(s,t)\{\delta(W_t - W_s)\}\, ds\, dt.$$

It is easy to see that

$$\begin{aligned} &\mathscr{T}^\varepsilon(\tilde{h}; 2, \varphi) - \int \{q^\varepsilon(W_t - W_s)\}\varphi(W_s; s, t)\, ds\, dt \\ &\quad = \int ds\, E \int du\, p_u(\varepsilon Y)\hat{\varphi}(s,u), \end{aligned} \tag{6.47}$$

where

$$\hat{\varphi}(s,u) = \varphi(W_s; s, s+u) - 1_{u \le 1}\varphi(W_s; s, s) = \tilde{\varphi}(u) - 1_{u \le 1}\tilde{\varphi}(0),$$

with $\tilde{\varphi}(u) = \varphi(W_s; s, s+u)$. Passing to the limit, we get

$$\begin{aligned} &\mathscr{T}(0; 2, \varphi) - \int \{\delta(W_t - W_s)\}\varphi(W_s; s, t)\, ds\, dt \\ &\quad = \int ds\, du (2\pi u)^{-1}\hat{\varphi}(s,u) \\ &\quad = \int ds\, \xi(\tilde{\varphi})(s), \end{aligned} \tag{6.48}$$

where $\xi(\tilde{\varphi})$ is defined by (2.21). Note that the right side is equal to $E\mathscr{T}(0; 2, \varphi)$ evaluated by (2.47). One expects that, in general, the field (6.42) can be expressed through the fields of the type $:\delta(W_{t_{i_1}}, \dots, W_{t_{i_l}}):$ and ξ_{u_j}, but this remains an open problem.

6.9. For $k = 2$, the Brownian self-intersection local times and gauge can be evaluated in terms of stochastic integrals. The first formulas of this type are due to Rosen [see Rosen (1985a) and (1986a)]. More results in the same direction were obtained by Yor (1985a), (1986a, b) and Dynkin (1987). In particular, it is shown in Dynkin (1987) that

$$\begin{aligned} &\pi \int_D \{\delta(W_t - W_s)\} f(s,t)\, dA_s\, dt \\ &\quad = -\int_D (fU\, dA_s\, dW_t + f_t V\, dA_s\, dt) \\ &\qquad - \int_{\Gamma_1} fV\, dA_s + \frac{C - \ln 2}{2}\int_{\Gamma_2} f\, dA_s. \end{aligned} \tag{6.49}$$

Here,

$$U = (W_t - W_s)/|W_t - W_s|^2,$$

(6.50)
$$V = \ln\{|W_t - W_s|/\sqrt{t-s}\},$$

$$f_t = \partial f(s,t)/\partial t,$$

D is a domain which is contained in $D_2 = \{0 \le s \le t\}$, $\Gamma_1 = \partial D \cap \{s < t\}$, $\Gamma_2 = \partial D \cap \{s = t\}$ and A is an additive functional with the characteristic measure λ. Formula (6.49) is proved for a wide class of domains D, measures λ and functions $f(s,t)$. In particular,

(6.51)
$$\pi \int_{T_u} \{\delta(W_t - W_s)\}\, dA_s\, dt = -\int_{T_u} U\, dA_s\, dW_t + \int_0^u \left[V + \frac{C - \ln 2}{2}\right] dA_s,$$

where $T_u = D_2(u)$.

By applying (6.48) to $\varphi(x; s, t) = \rho(x) 1_{0 \le s \le t \le u}$, we get

(6.52)
$$\int_{T_u} :\delta(W_s, W_t):\, dA_s\, dt = \int_{T_u} \{\delta(W_t - W_s)\}\, dA_s\, dt + \frac{1}{2\pi}\int_0^u \ln|u - s|\, dA_s,$$

with $dA_s = \rho(W_s)\, ds$.

By (1.41),

(6.53)
$$\mathscr{T}(\kappa; 2, \varphi) = \mathscr{T}(0; 2, \varphi) + \kappa \int_0^\infty \varphi(W_s; s, s)\, ds.$$

It follows from (6.50)–(6.53) that

(6.54)
$$\begin{aligned} &\pi \int_{D_2(u)} :\delta(W_s, W_t):_\kappa\, dA_s\, dt \\ &\quad = -\int_{T_u} U\, dA_s\, dW_t + \int_0^u [\ln|W_u - W_s| + (C - \ln 2)/2 + \pi\kappa]\, dA_s \end{aligned}$$

and, by (6.40),

(6.55)
$$\frac{\pi}{2} :T^2:_\lambda = -\int_{D_2(\zeta_r)} U\, dA_s\, dW_t + \int_0^{\zeta_r} \left[-\ln|W_{\zeta_r} - W_s| + C + \frac{1}{2}\ln\frac{r}{2}\right] ds.$$

Yor (1985c) has proved that

(6.56)
$$\begin{aligned} &\int_{D_3(u)} \{\delta(W_{t_2} - W_{t_1})\}\{\delta(W_{t_3} - W_{t_2})\}\, dt_1\, dt_2\, dt_3 \\ &\quad = \int_0^u dW_{s_1} \int_0^{s_1} d_- W_{s_2} \left[\int_{s_2}^{s_1} dt\, Q_{s_2}(W_t - W_{s_2}) Q_{u - s_1}(W_{s_1} - W_t),\right. \end{aligned}$$

with

$$Q_s(x) = -x\pi^{-1}|x|^{-2}\exp(-|x|^2/2u) \quad \text{for } x \neq 0, \tag{6.57}$$
$$= 0 \quad \text{for } x = 0.$$

Here $d_- W_{s_2}$ refers to the backward Itô integral.

6.10. Le Gall discovered another type of functional which converges to the self-intersection gauge of the Brownian motion. With every compact set $K \subset \mathbb{R}^2$, a Wiener sausage S_ε^K is associated: this is the set of all points $y \in \mathbb{R}^2$ such that $W_t - y \in K$ for some $t \le 1$. Let m be the Lebesgue measure on $\mathbb{R}^2$ and let h_ε^0 be given by (6.16). Le Gall (1986c) has proved that

$$h_\varepsilon^0[1 + h_\varepsilon^0 m(S_\varepsilon^K)] \to -\int_{T_1}\{\delta(W_t - W_s)\}\, ds\, dt + \left(C + 1 + \ln\frac{\operatorname{cap} K}{2}\right)\Big/ 2\pi \tag{6.58}$$

in L^2, where $\operatorname{cap} K$ is the logarithmic capacity of K. For the case $K = \{x\colon |x| \le 1\}$, this was obtained earlier in Le Gall (1985, 1986b).

A discrete analogue of the Wiener sausage is the set of all point of $\mathbb{Z}^2$ visited by a random walk $S_1, \dots, S_n$. Let R_n be the cardinality of this set. It is proved in Le Gall (1986a) that, under assumptions 1.1.A, B, D,

$$(\ln n)^2 n^{-1}(R_n - ER_n) \to_d -4\pi^2\int_{T_1}\{\delta(W_t - W_s)\}\, ds\, dt. \tag{6.59}$$

In the same paper, the case of k independent random walks $S^1, \dots, S^k$ is investigated. Let I_n^k be the number of points $z \in \mathbb{Z}^2$ visited by all random walks before n. Then all moments of $n^{-1}(\ln n/2\pi)^k I_n^k$ converge to the corresponding moments of the intersection local time (6.1) with $\Gamma = [0,1]^k$.

Le Gall (1986b) proved that (6.1) is also the L^p-limit of $L_\varepsilon^k = m(S_\varepsilon^1 \cap \cdots \cap S_\varepsilon^k)$, where $S_\varepsilon^1, \dots, S_\varepsilon^k$ are Wiener sausages for $W_t^1, \dots, W_t^k$. These functionals seem to be "Brownian shadows" for I_n^k. But the work of establishing links between I_n^k and L_ε^k remains to be done. At present, the convergence of I_n^k in distribution is proved only for $k = 2$ and 3 in which cases it follows from the convergence of the moments.

It looks plausible that an asymptotic expansion for $m(S^\varepsilon)$ should involve the Brownian self-intersection gauges of higher order. However this is also an open problem. [Recently Le Gall (1987b) succeeded in giving a positive solution to this problem. See the Addendum for the exact statement of his result.]

6.11. Le Gall (1985) proved that for every bounded Borel set B, there exists a version of

$$\alpha(x, B) = \int_B \delta_x(W_t - W_s)\, ds\, dt, \tag{6.60}$$

which is continuous a.s. on $\mathbb{R}^2 \setminus \{0\}$ and a version of

$$\gamma(x, B) = \{\alpha(x, B)\} = \int_B \{\delta_x(W_t - W_s)\}\, ds\, dt, \tag{6.61}$$

which is continuous a.s. on $\mathbb{R}^2$. Since

$$E\alpha(x, T_u) = \int_{T_u} p_{t-s}(x)\, ds\, dt = \frac{u}{\pi}\ln\frac{1}{|x|} + \frac{1}{2}u(1 + C - \ln 2), \tag{6.62}$$

there exists a.s.

$$\tilde{\gamma}(0, T_u) = \lim_{x \to 0} \tilde{\alpha}(x, T_u), \tag{6.63}$$

where

$$\tilde{\alpha}(x, T_u) = \alpha(x, T_u) - \frac{u}{\pi}\ln\frac{1}{|x|}. \tag{6.64}$$

Yor (1986b) demonstrated that

$$\gamma(x, T_u) = -\int_{T_u} Q_{u-t}(W_t - W_s - x)\, ds\, dW_t \tag{6.65}$$

for all $x \in \mathbb{R}^2$ [here Q is defined by (6.57)].

In Yor (1986a), a Hölder condition for $\tilde{\alpha}(y, T_u)$ has been established:

$$\lim_{\delta \to 0} \delta^{-1}\left[\ln\frac{1}{\delta}\right]^{-7/2} \sup_{u \le t, |x-y| \le \delta} |\tilde{\alpha}(x, T_u) - \tilde{\alpha}(y, T_u)| < \infty \quad \text{a.s.} \tag{6.66}$$

Yor (1985b) investigated the limit behaviour of the stochastic process

$$Y_x(u) = \frac{\tilde{\alpha}(x, T_u) - \tilde{\gamma}(0, T_u)}{|x|\ln|x|} \tag{6.67}$$

as $x \to 0$. He proved, in particular, its convergence in distribution to a Brownian motion. An analogous result was established for

$$Z_\varepsilon(u) = \left[\int_{T_u} \{q_\varepsilon(W_t - W_s)\}\, ds\, dt - \gamma(0, T_u)\right] \Big/ \left(\varepsilon\sqrt{\ln 1/\varepsilon}\right) \tag{6.68}$$

as $\varepsilon \to 0$.

6.12. The functionals (6.60) and (6.61) exhibit similar properties for some other stochastic processes:

(a) the Brownian motion in $\mathbb{R}^3$ [see Rosen (1983, 1985a), Yor (1985a, d, 1986b) and Le Gall (1986a, b)];
(b) the diffusion processes in $\mathbb{R}^2$ and $\mathbb{R}^3$ [Rosen (1987)];
(c) the fractional Brownian motion in $\mathbb{R}^2$ which is a Gaussian non-Markovian process [Rosen (1985d)];
(d) certain stable processes in $\mathbb{R}^2$ [Rosen (1985b, c, 1988)];
(e) certain $\mathbb{R}^d$-valued Gaussian random fields over $\mathbb{R}^N$ $(2N \le d < 4N)$ ["Brownian sheets"; Rosen (1984, 1985e)].

Rosen (1985c, 1986c) investigated also functionals (6.1) and (6.42) with $k > 2$ for stable processes in $\mathbf{R}^2$ of index greater than or equal to $2 - 2/(2k - 1)$. In particular, he proved that the invariance principle holds for these functionals if a random walk belongs to the normal domain of attraction of the corresponding stable law.

Le Gall (1986a) investigated the limit distributions of I_n^k (see Section 6.10) for random walks on $\mathbb{Z}^d$ (with the finite second moments) for all pairs (d, k), such that $I_n^k \to \infty$ as $n \to \infty$. He proved that

$$n^{-1/2} I_n^2 \to_d c\alpha(0, T_2), \quad \text{if } d = 3,$$

$$(\ln n)^{-1} I_n^2 \to_d c'\Gamma_{1/2}, \quad \text{if } d = 4,$$

$$(\ln n)^{-1} I_n^3 \to_d c''\Gamma_{1/4}, \quad \text{if } d = 3,$$

where constants c, c', c'' depend on the random walk and Γ_a means a random variable which has the gamma distribution with parameter a.

For the range R_n of a random walk on $\mathbb{Z}^d$, the limit distribution is normal if $d \geq 3$ [see Jain and Orey (1968) and Jain and Pruitt (1971, 1974)].

6.13. A remarkable connection between the self-intersection local times and Hausdorff measures was established in Le Gall (1987a). Let D_k be the set of all x such that $W_{t_1} = \cdots = W_{t_k} = x$ for some $0 < t_1 < \cdots < t_k$. Let $h_k - m(B)$ be the Hausdorff measure corresponding to the function

$$h_k(r) = r^2 \left(\ln \frac{1}{r} \ln \ln \ln \frac{1}{r} \right)^k. \tag{6.69}$$

Then there exist two positive constants C_k, C_k' such that

$$C_k T(k, B) \leq h_k - m(B \cap D_k) \leq C_k' T(k, B) \tag{6.70}$$

for all Borel sets B. Here $T(k, B)$ are local times defined by (1.25).

In Le Gall (1987c) the measure $T(2, \cdot)$ has been used to give a precise meaning to the following heuristic statement: Between the two times when it hits a double point a Brownian motion behaves like a Brownian bridge.

APPENDIX

1. Here we prove various auxiliary lemmas used in the article.

LEMMA A.1. *Let L be a complete metric space with a metric ρ. Suppose that an element F_α^ε of L is given for every sufficiently small $\varepsilon > 0$ and $\alpha > 0$ and suppose that*

(A.1) *for every sufficiently small ε, $F_\alpha^\varepsilon \to F^\varepsilon$ as $\alpha \downarrow 0$;*

(A.2) *for every sufficiently small α, $F_\alpha^\varepsilon \to F_\alpha$ as $\varepsilon \downarrow 0$;*

$$\rho(F_\alpha^\varepsilon, F_\alpha) \to 0 \quad \text{as } \alpha, \varepsilon \to 0. \tag{A.3}$$

Then there exists a limit

$$F = \lim_{\varepsilon, \alpha \downarrow 0} F^\varepsilon_\alpha \tag{A.4}$$

and

$$F = \lim_{\varepsilon \downarrow 0} F^\varepsilon = \lim_{\alpha \downarrow 0} F_\alpha. \tag{A.5}$$

PROOF. By (A.3), for every $\delta > 0$ there exists $\beta(\delta) > 0$ such that

$$\rho(F^\varepsilon_\alpha, F^\gamma_\alpha) < \delta \quad \text{if } \varepsilon, \gamma, \alpha \le \beta(\delta). \tag{A.6}$$

By (A.1), there exists $\psi(\delta, \gamma) > 0$ such that

$$\rho(F^\gamma_\alpha, F^\gamma_{\alpha'}) < \delta \quad \text{if } \alpha, \alpha' \le \psi(\delta, \gamma). \tag{A.7}$$

Let $\alpha, \alpha', \varepsilon, \varepsilon' \le \eta(\delta) = \min[\beta(\delta), \psi(\delta, \beta(\delta))]$. Then, for $\gamma = \beta(\delta)$, $\rho(F^\varepsilon_\alpha, F^\gamma_{\alpha'})$ and $\rho(F^{\varepsilon'}_{\alpha'}, F^\gamma_{\alpha'})$ are smaller than δ and, therefore, $\rho(F^\varepsilon_{\alpha'}, F^{\varepsilon'}_{\alpha'}) < 3\delta$. This implies the existence of the limit (A.4). Formula (A.5) follows from (A.1), (A.2) and (A.4). □

2.

LEMMA A.2. *Let Ω be a space of measurable mappings from a measure space $(T, \mathscr{A}, \mu)$ to a metric measure space $(X, \mathscr{B}, \nu)$ and let μ be finite and ν be σ-finite. Suppose that $\mathscr{F}$ is the σ-algebra in Ω generated by the sets $\{\omega\colon \omega(t) \in B\}$, $t \in T$, $B \in \mathscr{B}$, and that P is a measure on $\mathscr{F}$ whose "one-dimensional distributions" $\alpha_t(A) = P\{\omega\colon \omega(t) \in A\}$ are absolutely continuous relative to ν. Let ρ be a bounded Borel function on $X \times T$ with the following property: There exists a set $D \in \mathscr{B} \times \mathscr{A}$ such that $(\nu \times \mu)(D) = 0$ and ρ is continuous in x on the complement of D. Then the functional*

$$F(\omega) = \int_T \rho(\omega(t), t)\mu(dt)$$

is continuous P-almost everywhere with respect to the pointwise convergence in Ω.

PROOF. Choose a density function $\rho(x, t)$ of α_t with respect to ν which is $\mathscr{B} \times \mathscr{A}$-measurable. We have

$$\int \mu(dt) \int P(d\omega) 1_D(\omega(t), t) = \int \mu(dt) \int \nu(dx) \rho(x, t) 1_D(x, t) = 0. \tag{A.8}$$

By Fubini's theorem,

$$\int \mu(dt) 1_D(\omega(t), t) = 0 \tag{A.9}$$

for P-almost all ω. Suppose that $\omega_n(t) \to \omega(t)$ for all t. Then $\rho(\omega_n(t), t) \to$

$\rho(\omega(t), t)$ for $(\omega(t), t) \notin D$. By (A.9) and the dominated convergence theorem, $F(\omega_n) \to \varphi(\omega)$ for P-almost all ω. □

LEMMA A.3. *Suppose that conditions of Lemma* A.2 *are satisfied and that, in addition*:

(a) P_α *converges weakly to* P *as* $\alpha \to 0$ (*relative to the uniform norm in* Ω);
(b) χ_α *is a family of measurable transformations in* T *such that* $P_\alpha Z_t^\alpha \to 0$, *where* Z_t^α *is the distance in* $X \times T$ *between* $(\omega(t), t)$ *and* $(\omega(t^\alpha), t^\alpha)$ *with* $t^\alpha = \chi_\alpha(t)$;
(c) ρ *has a compact support*;
(d) D *is a closed set.*

Then the probability distribution of $F_\alpha(\omega) = \int \rho(\omega(t^\alpha), t^\alpha)\mu(dt)$ *relative to* P_α *converges weakly to the probability distribution of* $F(\omega) = \int \rho(\omega(t), t)\mu(dt)$ *relative to* P.

PROOF. It follows from Lemma A.3 and (a) that the probability distribution of F relative to P_α converges weakly to its probability distribution relative to P [Ethier and Kurtz (1986), page 103]. Thus, Lemma A.3 will be proved if we show that

$$P_\alpha \int \Delta_\alpha(t)\mu(dt) \to 0 \quad \text{as } \alpha \to 0, \tag{A.10}$$

where $\Delta_\alpha(t) = |\rho(\omega(t), t) - \rho(\omega(t^\alpha), t^\alpha)|$.

Let C be an upper bound for $|\rho|$. By Fubini's theorem, Chebyshev's inequality and (b),

$$P_\alpha \int \Delta_\alpha(t) 1_{Z_t^\alpha \ge \beta} \mu(dt) \le 2C \int P_\alpha\{Z_t^\alpha \ge \beta\}\mu(dt) \to 0 \quad \text{as } \alpha \to 0 \tag{A.11}$$

for every $\beta > 0$.

Let U_δ be the δ-neighborhood of D. The function

$$f(\delta) = \int P\{(\omega(t), t) \in U_\delta\}\mu(dt)$$

is monotone decreasing and by (A.9), $f(\delta) \to 0$ as $\delta \to 0$. If $f(\delta+) = f(\delta)$, then $f_\alpha(\delta) = \int P_\alpha\{(\omega(t), t) \in U_\delta\}\mu(dt) \to f(\delta)$ as $\alpha \to 0$ by (a) and

$$P_\alpha \int \Delta_\alpha(t) 1_{U_\delta}(\omega(t), t)\mu(dt) \le \varepsilon \tag{A.12}$$

for all sufficiently small α.

Since ρ is continuous on the complement V_δ of U_δ, it is uniformly continuous by (c). We choose a $\beta < \delta$ in such a way that $|\rho(x, t) - \rho(x', t')| < \varepsilon$ if (x, t) and (x', t') belong to V_δ and the distance between them is smaller than β. With such a choice,

$$\Delta_\alpha(t) 1_{Z_{\alpha t} < \beta} 1_{V_{2\delta}}(\omega(t), t) < \varepsilon. \tag{A.13}$$

Formula (A.10) follows from (A.11)–(A.13). □

3.

LEMMA A.4. *Let A be a finite set and r be a mapping from A^2 to $\mathbb{R}$. Put*

$$\mathscr{T}(h; k, \phi) = \sum_{t \in A^k} \phi(t) \prod_{i=2}^{k} [r(t_{i-1}, t_i) + h1(t_{i-1}, t_i)], \tag{A.14}$$

where ϕ is a function on A^k and h is a constant. We have

$$\mathscr{T}(h + \tilde{h}; k, \phi) = \sum_{l=1}^{k} \tilde{h}^{k-l} \mathscr{T}(h; l, B_k^l \phi), \tag{A.15}$$

where B_k^l is an operator given by formula (1.31) *with σ running over all mappings from $\{1, 2, \dots, k\}$, onto $\{1, 2, \dots, l\}$.*

PROOF. Note that

$$\begin{aligned}\prod_{i=2}^{k} &[r(t_{i-1}, t_i) + (h + \tilde{h})1(t_{i-1}, t_i)] \\ &= \sum_{\Lambda} \prod_{i \in M} [r(t_{i-1}, t_i) + h1(t_{i-1}, t_i)] \prod_{i \notin \Lambda} \tilde{h}1(t_{i-1}, t_i),\end{aligned}$$

where Λ runs over all subsets of the set $\{1, \dots, k\}$ which contain 1 and $M = \Lambda \setminus \{1\}$. Therefore,

$$\begin{aligned}\mathscr{T}(h + \tilde{h}; k, \phi) = \sum_{\Lambda} \tilde{h}^{k-l} \sum_{t \in A^k} \Big\{ &\phi_\Lambda(t_\Lambda) \prod_{i \in M} \big[r(t_{i_\Lambda}, t_i) + h1(t_{i_\Lambda}, t_i)\big] \\ &\times \prod_{i \notin \Lambda} 1(t_{i-1}, t_i) \Big\},\end{aligned} \tag{A.16}$$

where $l = |\Lambda|$, i_Λ is the largest element of Λ which is smaller than i and $\phi_\Lambda(t_\Lambda)$ is obtained from ϕ by replacing t_i, $i \notin \Lambda$, by t_{i_Λ}. Formula (A.15) follows from (A.16). □

REMARK. Formula (1.28) follows from (A.15) if we take a sufficiently large N and put $A = \{1, 2, \dots, N\}$ and

$$\phi(m_1, \dots, m_k) = 1_{0 \le m_1 \le \cdots \le m_k} \varphi\big(S_{m_1}/\sqrt{n}\,; m_1/n, \dots, m_k/n\big), \qquad m_1, \dots, m_k \in A.$$

4.

LEMMA A.5. *Suppose that φ is a bounded Borel function which satisfies condition* 1.7.B *with D_k replaced by the region U_β [see* (1.34)*]. If $0 < \sigma < \beta$, then there exist functions $\varphi_1 \in \mathscr{L}_k^c$ and $\varphi_2 \in \mathscr{E}_k^{\sigma c}$ such that $\varphi = \varphi_1 + \varphi_2$ and $\|\varphi_1\| \le C\|\varphi\|_{m,\beta}$, where $\|\varphi\|_{m,\beta}$ is defined in the same way as $\|\varphi\|_m$ with D_k replaced by U_β and C is a constant independent of φ and c (but dependent on m, σ, β).*

PROOF. Let $\sigma < \kappa < \beta$. There exists a function $h(t)$ of class C^∞ which is equal to 1 on $U_\sigma \cap \{0 \le t_1 \le \cdots \le t_k \le c\}$ and is equal to 0 on $W = \bigcap_{i=2}^k \{t_i - t_{i-1} \ge \kappa\}$ [see Gel'fand and Shilov (1966), Chapter 1, Appendix 1]. The functions $\varphi_1(x, t) = \varphi(x, t)h(t)$ and $\varphi_2(x, t) = \varphi(x, t)(1 - h(t))$ have the properties stated in Lemma A.5. □

5. To prove Lemmas 2.1 and 2.2, we use the following property of a continuous linear operator $\eta\colon \mathscr{D}(\Lambda) \to \mathscr{D}(M)$: For every $c > 0$, there exists a $c' > 0$ such that

$$\eta[\mathscr{D}^c(\Lambda)] \subset \mathscr{D}^{c'}(M). \tag{A.17}$$

[See Gel'fand and Shilov (1968), Chapter 1, Section 8.]

PROOF OF LEMMA 2.1. For every $\varphi \in \mathscr{D}^c(\Lambda \cup K)$, we consider a family $\varphi_{u_K}(u_\Lambda) = \varphi(u_\Lambda, u_K)$. Obviously, $\varphi_{u_K} \in \mathscr{D}^c(\Lambda)$ and, therefore, $\psi_{u_K} = \eta(\varphi_{u_K}) \in \mathscr{D}^{c'}(M)$. Let $s \in K$ and let Δ_s^α be defined by (2.18). We note that $\Delta_s^\alpha \psi_{u_K} = \eta(\Delta_s^\alpha \varphi_{u_K})$ and that $\Delta_s^\alpha \varphi_{u_K} \to D_s \varphi_{u_K}$ in $\mathscr{D}(\Lambda)$. Hence, $\Delta_s^\alpha \psi_{u_K} \to \eta(D_s \varphi_{u_K})$ in $\mathscr{D}(M)$ and $D_s \psi_{u_K} = \eta(D_s \varphi_{u_K}) \in \mathscr{D}(M)$. This implies that $D_k \psi_{u_K} = \eta(D^k \varphi_{u_K})$ for every $k \in \mathbb{Z}_+^\Lambda$. We conclude that $\psi(u_M, u_K) = \psi_{u_K}(u_M)$ belongs to $\mathscr{D}^{c \vee c'}(\Lambda \cup K)$ and (2.13) holds. □

PROOF OF LEMMA 2.2. We prove the lemma for two factors and then apply the induction. Let $\eta = \eta_1 \times \eta_2$, $\eta_1 \in \mathscr{R}(\Lambda_1, M_1)$, $\eta_2 \in \mathscr{R}(\Lambda_2, M_2)$, and let $\varphi \in \mathscr{D}^c(\Lambda_1 \cup \Lambda_2)$. By (A.17), $\psi = \eta_2(\varphi) \in \mathscr{D}^{c'}(\Lambda_1 \cup M_2)$ and by Lemma 2.1, $f = \eta(\varphi) = \eta_1(\psi) \in \mathscr{D}(M_1 \cup M_2)$. By (2.16),

$$\|f\| \le \|\eta_1\|_{l_1, c'} \|\psi\|_{l_1'}, \tag{A.18}$$

where $l_1 = l_1' = l$ on Λ_1 and $l_1 = 0$ on M_2. For an arbitrary $m \in \mathbb{Z}_+^{\Lambda_1}$, $D_m \psi = \eta_2(D_m \varphi)$ by (2.13) and $\|D_m \psi\| \le \|\eta_2\|_{l_2, c} \|D_m \varphi\|_{l_2'}$ with $l_2 = l_2' = l$ on Λ_2 and $l_2' = 0$ on Λ_1. It follows from here and from (A.18) that

$$\|\eta(\varphi)\| \le \|\eta_1\|_{l_1, c'} \|\eta_2\|_{l_2, c} \|\varphi\|_l.$$

This implies the inequality $\|\eta_1 \times \eta_2\|_{l, c} \le \|\eta_1\|_{l_1, c'} \|\eta_2\|_{l_2, c}$. □

Acknowledgments. The author is indebted to R. Durrett, J.-F. Le Gall, J. Rosen and M. Yor for stimulating discussions. The idea of using a family F_α^ε to overtake the rapid growth of the moments of F_α is due to J. Rosen. In proving Theorem 3.1, we followed the proof of Theorem 3.6 in Le Gall [(1986a), page 488].

J.-F. Le Gall has read the manuscript carefully and made a number of valuable suggestions. He also has written the Addendum. To him go my foremost thanks.

Addendum. Recall (cf. Section 6.10) that the Wiener sausage S_u^ε of the radius ε associated with the Brownian motion W on the interval $[0, u]$ is the set $\{y\colon |W_t - y| \le \varepsilon \text{ for some } t \in [0, u]\}$. Let $\lambda(dx) = \rho(x)\,dx$ be a measure on $\mathbb{R}^2$

which is absolutely continuous with respect to the Lebesgue measure and such that $\rho(x)$ is locally bounded. Then, for any integer $n \geq 1$,

$$(*) \qquad \lambda(S_u^\varepsilon) = -\sum_{k=1}^{n} (h_\varepsilon)^{-k} \mathscr{T}_k(\lambda, u) + R_n(\varepsilon, u),$$

where $h_\varepsilon = -g(\varepsilon) = 1/\pi(\log \varepsilon + C - \log 2/2)$ (C is Euler's constant), $\mathscr{T}_k(\lambda, u)$ can be defined by (6.15) or (6.23), and the remainder $R_n(\varepsilon, u)$ satisfies

$$\lim_{\varepsilon \to 0} (h_\varepsilon)^n R_n(\varepsilon, u) = 0 \quad \text{a.s.}$$

Formula (*) gives an asymptotic expansion for $\lambda(S_u^\varepsilon)$, which involves the negative powers of $\log \varepsilon$. The kth term of this expansion is a corrective term which takes into account the fact that W hits a k multiple point at k distinct times. Similar expansions can be obtained for the sausage associated with a general compact set K, thus extending (6.58).

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