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SPREADING AND PREDICTABLE SAMPLING IN EXCHANGEABLE SEQUENCES AND PROCESSES[1]

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Ryll-Nardzewski has proved that an infinite sequence of random variables is exchangeable if every subsequence has the same distribution. We discuss some restatements and extensions of this result in terms of martingales and stopping times. In the other direction, we show that the distribution of a finite or infinite exchangeable sequence is invariant under sampling by means of a.s. distinct (but not necessarily ordered) predictable stopping times. Both types of result generalize to exchangeable processes in continuous time.

1. Introduction. A finite sequence of random variables $\xi = (\xi_1, \dots, \xi_n)$ is said to be *exchangeable*, if every permutation has the same distribution, i.e., if

$$(\xi_{k_1}, \dots, \xi_{k_n}) =_d (\xi_1, \dots, \xi_n) \tag{1.1}$$

for every permutation $(k_1, \dots, k_n)$ of $(1, \dots, n)$. For infinite sequences, we require the same property for every finite subsequence. It is easy to see that exchangeability of an infinite sequence $\xi = (\xi_1, \xi_2, \dots)$ implies that

$$(\xi_{k_1}, \xi_{k_2}, \dots) =_d (\xi_1, \xi_2, \dots), \qquad k_1 < k_2 < \cdots . \tag{1.2}$$

A sequence satisfying (1.2) is said to be *spreadable*. (Kingman [13] calls (1.2) the *selection property*, while Aldous [1] refers to (1.2) as the property of *spreading-invariance*.)

de Finetti's [3] celebrated theorem states that an infinite exchangeable sequence is mixed i.i.d., in the sense that its distribution is a mixture of distributions of i.i.d. sequences. Ryll-Nardzewski [14] noticed that the same conclusion follows from the weaker assumption of spreadability. Both results are in fact simple (though remarkable) corollaries of the mean ergodic theorem. (Yet the latter is somewhat deeper than the martingale convergence theorem which the standard proofs use.) In Proposition 2.1, we shall show that the same argument yields an even stronger result.

We proceed in Proposition 2.2 to restate the preceding results in terms of stopping times and martingales. In particular, a sequence ξ is spreadable iff $\Theta_\tau \circ \xi =_d \xi$ for every Z_+-valued stopping time τ (extensive use of this result was made in [10]), or equivalently, iff the *prediction sequence*

$$\pi_n = P[\Theta_n \circ \xi \in \cdot \,|\mathscr{F}_n], \qquad n \in Z_+, \tag{1.3}$$

is a measure-valued martingale. Here $\mathscr{F} = (\mathscr{F}_0, \mathscr{F}_1, \dots)$ is the filtration induced

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by ξ (so $\mathscr{F}_0$ is trivial), stopping times are defined with respect to $\mathscr{F}$ and $\Theta_0, \Theta_1, \dots$ denote the shift operators on R^∞.

The preceding stopping time condition characterizes exchangeability in terms of certain *randomly* selected subsequences. More generally, one may look for conditions on the random indices $\tau_1, \tau_2, \dots,$ such that

$$(1.4) \qquad (\xi_{\tau_1}, \xi_{\tau_2}, \dots) =_d (\xi_1, \xi_2, \dots)$$

implies that ξ is exchangeable. Another instance when (1.4) implies exchangeability (for stationary ξ) is that of *thinning*, where the elements of ξ are selected independently with a fixed probability $p \in (0,1)$. This result (Proposition 2.3) is closely related to a result in point process theory (cf. [12]), where mixed Poisson processes are characterized in terms of thinning.

Section 3 deals with the converse problem of finding general conditions on $\tau_1, \tau_2, \dots,$ such that (1.4) holds for a given exchangeable sequence ξ. If ξ is infinite and i.i.d., we may, e.g., take $\tau_1, \tau_2, \dots$ to be any strictly increasing sequence of predictable stopping times. (Recall that a stopping time τ is *predictable* if $\tau - 1$ is a stopping time in the usual sense.) This result is well known to gamblers (or at least it ought to be). The first formal proof appears in Doob [4]. Our main result in Section 3 states that (1.4) is true for arbitrary *a.s. distinct* predictable stopping times $\tau_1, \tau_2, \dots,$ whenever ξ is a *finite or infinite* exchangeable sequence. Note in particular that the τ_j may form a random (but predictable) permutation of the indices of ξ, since no requirement is made on the order.

The preceding result, which generalizes Theorem 5.1 in [10], has the most surprising consequences for finite games (e.g., card games, lotteries, sampling from finite populations), as shown by examples in [11]. For the sake of applications (but also for the proof), it is useful to introduce the associated *allocation sequence* $\alpha_1, \alpha_2, \dots,$ given by

$$(1.5) \qquad \alpha_k = \inf\{j: \tau_j = k\}, \qquad k = 1, 2, \dots.$$

(Here inf $\varnothing$ means ∞, as usual.) Note that the finite values of $\alpha_1, \alpha_2, \dots$ are a.s. distinct and that α_k is $\mathscr{F}_{k-1}$-measurable by assumption for each k. Informally, the element ξ_k is moved to a new position α_k, which is only allowed to depend on the past history $(\xi_1, \dots, \xi_{k-1})$. Note that ξ_k is discarded for the new sequence if $\alpha_k = \infty$.

Sections 4 and 5 deal with the corresponding problems in continuous time. A process X defined on $I = [0,1]$ or R_+ is said to be *exchangeable* if $X_0 = 0$, if X is continuous in probability at every $t \in I$ and if the increments of X over an arbitrary set of disjoint intervals of equal length form an exchangeable sequence. In that case, we may (and will) choose a version of X which is right-continuous with left-hand limits. If $I = R_+$, the analogue of de Finetti's theorem states that X is a mixture (again in the distributional sense) of Lévy processes. For $I = [0,1]$, we have instead the more general representation (cf. [8])

$$(1.6) \qquad X_t = \alpha t + \sigma B_t + \sum_{j=1}^{\infty} \beta_j \big(1\{\tau_j \le t\} - t\big), \qquad t \in [0,1]$$

($1\{\cdot\}$ denoting the indicator function of the event within brackets), where B is a Brownian bridge, while $\tau_1, \tau_2, \dots$ are i.i.d. random variables uniformly distributed on $[0,1]$ and $\alpha, \sigma, \beta_1, \beta_2, \dots$ are arbitrary random variables satisfying $\sigma \geq 0$ and $\Sigma\beta_j^2 < \infty$, the three objects B, $(\tau_1, \tau_2, \dots)$ and $(\alpha, \sigma, \beta_1, \beta_2, \dots)$ being independent. We shall write β for the point process $\Sigma\delta_{\beta_j}$ and say that X is *directed* by the triple $(\alpha, \sigma^2, \beta)$. Note that X is a mixture of *ergodic* exchangeable processes (1.6), where α, σ^2 and β are nonrandom.

Exchangeable processes will be seen to be semimartingales. In Section 4, we shall essentially characterize the exchangeability of a semimartingale X in terms of its local characteristics (as defined in [6, 7]). If X is exchangeable and integrable, the latter will be absolutely continuous, with densities which form martingales with respect to the filtration induced by X. Conversely, a semimartingale X on R_+ with the preceding property can be shown to be exchangeable, provided that X has stationary increments, and a similar result (related also to Theorem 3.3 in [10]) will be proved for processes on $[0,1]$. A related characterization of mixed Poisson processes has been obtained independently by Heller and Pfeifer [5].

The continuous time counterpart of the predictable sampling theorem of Section 3 is stated in Section 5 in terms of stochastic integrals. More precisely, the allocation sequence in (1.5) is now replaced by an *allocation process* V, which is predictable and a.s. measure preserving, at least on some suitable subinterval J of the index set I. (Thus $\lambda V^{-1} = \lambda$ on J a.s., where λ denotes Lebesgue measure.) Given X and V, we may define a new process XV^{-1} on J by

$$(XV^{-1})_t = \int_I 1\{V_s \leq t\}\, dX_s, \qquad t \in J. \tag{1.7}$$

The main result of Section 5 (which generalizes Theorem 5.2 in [10]) states that X and (a suitable version of) XV^{-1} have the same distribution on J, whenever X is exchangeable. As in the discrete time case, there are some rather surprising applications of this result, which are discussed in [11]. The result has also proved useful in establishing representations of stable integrals, but this will be discussed elsewhere.

We now turn to discuss some technical extensions. Our first point concerns the choice of filtration. For many purposes, one needs to introduce some more general filtration $\mathscr{F}$ than that generated by the sequence or process under consideration. Following [10], we shall then say that a sequence ξ is $\mathscr{F}$-*exchangeable* if ξ is adapted to $\mathscr{F}$ and if $\Theta_n \cdot \xi$ is conditionally exchangeable, given $\mathscr{F}_n$, for every $n \in Z_+$. The latter condition means of course that the shifted sequence should a.s. be exchangeable under the conditional probability law. It is easy to check that an $\mathscr{F}$-exchangeable sequence is exchangeable and that the two notions are equivalent in the case when $\mathscr{F}$ is induced by ξ. Most of the results previously described extend without effort to the more general setting just described. In particular, this is true for the predictable sampling theorem, where one may hence allow for independent randomizations in each step in the construction of (α_k).

The continuous time case is similar. For technical reasons, we shall only consider *standard filtrations* $\mathscr{F}$, satisfying the usual conditions of right-continuity and completeness (so that $\mathscr{F}_t = \mathscr{F}_{t+}$ for all t, while $\mathscr{F}_0$ contains all null sets in a completion of $\mathscr{F}_\infty = \bigvee \mathscr{F}_t$). In particular, the filtration induced by X is defined as the smallest standard filtration making X adapted. Defining $\mathscr{F}$-exchangeability as before, we have the same relationship to the usual notion of exchangeability (cf. [10]).

A second point concerns the predictable sampling theorem already discussed. In many applications, the sample size is random, and there may be no obvious way of extending the given sequence of stopping times to a sequence of fixed length. In that case, we can still prove that the sampled sequence η can be *embedded in distribution* into the original sequence ξ (which we denote by $\eta \subset_d \xi$; cf. [10]). This means that η can be continued, by randomization or otherwise, to a sequence η' of the same length as ξ and such that $\eta' =_d \xi$. A corresponding extension exists in the continuous time case, with a similar definition of embedding. Note that the preceding construction of η' may require an extension of the original probability space.

A simple way of proving the embedding $\eta \subset_d \xi$ is to construct, on some suitable probability space, a sequence $\xi' =_d \xi$ and a $\overline{Z}_+$-valued random variable ν', such that

$$(\xi'_1, \dots, \xi'_{\nu'}) =_d \eta, \tag{1.8}$$

where the left-hand side should be interpreted as ξ' when $\nu' = \infty$. In continuous time, it is convenient first to extend the definition of the sampled process Y, originally given on some random interval $[0, \zeta)$, by putting $Y_t = \partial$ for $t \geq \zeta$, where ∂ denotes an auxiliary *coffin state*. We may further define the *killing operators* k_s by

$$(k_s f)_t = \begin{cases} f_t, & s < t \in I, \\ \partial, & s \geq t \in I, \end{cases}$$

defined for functions f on $I = [0,1]$ or R_+, and for numbers $s \in I \cup \{\infty\}$. In order to prove that $Y \subset_d X$, it is then enough to construct, on some suitable probability space, a process $X' =_d X$ and a random variable $\zeta' =_d \zeta$, such that $k_{\zeta'} \circ X' =_d Y$.

The preceding statements are simple consequences of the following randomization lemma.

LEMMA 1.1. *Let ξ and η be random elements on some probability space (Ω, P) and taking values in the spaces S and T, where S is separable metric while T is Polish. Assume that $\xi =_d f(\eta)$ for some measurable function f: $T \to S$. Then there exists some random element $\eta' =_d \eta$ on $(\Omega \times [0,1], P \times \lambda)$, such that $\xi = f(\eta')$ a.s. $P \times \lambda$.*

PROOF. It is enough to prove the result for $T = R$, since it will then extend immediately to the case of linear Borel sets, and next, by Borel isomorphism (cf.

[1], page 50), to arbitrary Polish spaces. For $T = R$, we may choose a regular version of the conditional probabilities

$$\mu_s = P[\eta \in \cdot | f(\eta) \in ds], \qquad s \in S,$$

and define

$$\eta'(\omega, x) = \sup\{y: \mu_{\xi(\omega)}(-\infty, y] \le x\}, \qquad \omega \in \Omega,\ x \in [0,1].$$

It is easy to check that η' is measurable and satisfies $(\xi, \eta') =_d (f(\eta), \eta)$. Since S is separable, the diagonal in S^2 is measurable, so we get

$$1\{\xi = f(\eta')\} =_d 1\{f(\eta) = f(\eta)\} = 1,$$

which shows that $\xi = f(\eta')$ a.s. □

Let us conclude with some remarks on literature. Though the present paper is formally self-contained as far as exchangeability theory is concerned, we recommend Kingman's paper [13] and Aldous' lecture notes [1] for introductory reading. Some further background on the continuous time theory may be found in [8–10]. Standard results from stochastic calculus and weak convergence theory will often be used without explicit references and for these, the reader may, e.g., consult Jacod [6, 7] and Billingsley [2].

2. Spreading characterizations. Let us first show how de Finetti's and Ryll-Nardzewski's results follow easily from the mean ergodic theorem. Assume that $\xi = (\xi_1, \xi_2, \dots)$ is spreadable and let the functions $f_1, f_2, \dots: R \to R$ be bounded and measurable. Write $\mathscr{I}$ for the shift invariant σ-field in R^∞ and let μ be a regular version of $P[\xi_1 \in \cdot | \xi^{-1}\mathscr{I}]$. Then

$$E\prod_{j=1}^{k} f_j(\xi_j) = E\prod_{j=1}^{k}\left\{\frac{1}{n}\sum_{i=1}^{n} f_j(\xi_{jn+i})\right\} \to E\prod_{j=1}^{k}\mu f_j$$

as $n \to \infty$, by the L_2 ergodic theorem (where the convergence is clearly uniform under shifts) plus dominated convergence. Here and to follow, $\mu f = \int f\,d\mu$. The proof is completed by a monotone class argument. (Essentially the same proof yields the usual conditional forms of de Finetti's theorem; cf. [1].)

We shall use the same method to prove the following stronger result.

PROPOSITION 2.1. *Let $\xi = (\xi_1, \xi_2, \dots)$ be a stationary sequence of random variables satisfying*

$$(\xi_1, \dots, \xi_n, \xi_{n+2}) =_d (\xi_1, \dots, \xi_n, \xi_{n+1}), \qquad n \in Z_+. \tag{2.1}$$

Then ξ is exchangeable.

FIRST PROOF. Extend ξ to a doubly infinite stationary sequence $\dots, \xi_{-1}, \xi_0, \xi_1, \dots$ and conclude from (2.1) that

$$(\dots, \xi_n, \xi_{n+2}) =_d (\dots, \xi_n, \xi_{n+1}), \qquad n \in Z.$$

Iterating this result yields

$$(\dots, \xi_n, \xi_{n+k}) =_d (\dots, \xi_n, \xi_{n+1}), \qquad n \in Z,\ k \in N.$$

Letting g and $f_1, f_2, \dots$ be bounded measurable functions on R^∞ and R, and writing $\xi^- = (\dots, \xi_{-1}, \xi_0)$, we get by the L_2 ergodic theorem,

$$Eg(\xi^-)\prod_{j=1}^{k} f_j(\xi_j) = Eg(\xi^-)\prod_{j=1}^{k-1} f_j(\xi_j)\left(\frac{1}{n}\sum_{i=k}^{k+n-1} f_k(\xi_i)\right)$$

$$\to Eg(\xi^-)\prod_{j=1}^{k-1} f_j(\xi_j)\mu f_k.$$

Since μ is ξ^--measurable by the law of large numbers, we may continue recursively until we get, after k steps,

$$Eg(\xi^-)\prod_{j=1}^{k} f_j(\xi_j) = Eg(\xi^-)\prod_{j=1}^{k} \mu f_j.$$

Thus the conclusion follows as before. □

We may also give a simple martingale proof, in the spirit of Aldous ([1], page 22).

SECOND PROOF. It is convenient to reflect the index set in the origin, so we may assume instead that ξ is stationary and satisfies $\xi =_d (\xi_1, \Theta_2 \circ \xi)$. By iteration and stationarity, we get

$$\Theta_{k-1} \circ \xi =_d (\xi_k, \Theta_n \circ \xi) =_d (\xi_1, \Theta_n \circ \xi), \qquad k \le n,$$

so

$$E[f(\xi_k)|\Theta_k \circ \xi] =_d E[f(\xi_k)|\Theta_n \circ \xi] = E[f(\xi_1)|\Theta_n \circ \xi], \qquad k \le n,$$

for any bounded and measurable function f. By Lemma 3.4 in [1], the left equality must also hold in the a.s. sense and we get as $n \to \infty$,

$$E[f(\xi_k)|\Theta_k \circ \xi] = E[f(\xi_1)|\mathscr{T}] \equiv \mu f \quad \text{a.s.},$$

where $\mathscr{T}$ denotes the tail σ-field of ξ. Letting $f_1, \dots, f_n$ be bounded and measurable, we hence obtain by iterated conditioning

$$E\left[\prod_{k=1}^{n} f(\xi_k)\middle|\mathscr{T}\right] = E\left[\prod_{k=1}^{n} E[f_k(\xi_k)|\Theta_k \circ \xi]\middle|\mathscr{T}\right] = E\left[\prod_{k=1}^{n} \mu f_k\middle|\mathscr{T}\right] = \prod_{k=1}^{n} \mu f_k,$$

which proves that ξ is conditionally i.i.d., given $\mathscr{T}$. □

It is useful to restate the previous conditions in terms of stopping times and martingales. For the sake of simplicity, these will here be defined with respect to the induced filtration

$$\mathscr{F}_n = \sigma(\xi_1, \dots, \xi_n), \qquad n \in Z_+.$$

Define the measure-valued processes (π_n) and (λ_n) by

$$(2.2)\qquad \pi_n = P[\Theta_n \circ \xi \in \cdot\,|\mathscr{F}_n], \qquad \lambda_n = P[\xi_{n+1} \in \cdot\,|\mathscr{F}_n], \qquad n \in Z_+,$$

and note that these formulas remain true when n is a finite stopping time. All functions that follow are assumed to be measurable.

PROPOSITION 2.2. *Let $\xi = (\xi_j)$ be an infinite sequence of random variables and define (π_n) and (λ_n) by* (2.2). *Then properties* (i), (ii) *and* (iii) *are equivalent*:

(i) *ξ is spreadable.*
(ii) *$\Theta_\tau \circ \xi =_d \xi$ for every finite stopping time τ.*
(iii) *$(\pi_n f)$ is a martingale for every bounded $f\colon R^\infty \to R$.*

Properties (i′), (ii′) *and* (iii′) *are also equivalent*:

(i′) *ξ satisfies* (2.1).
(ii′) *$\xi_{\tau+1} =_d \xi_1$ for every finite stopping time τ.*
(iii′) *$(\lambda_n f)$ is a martingale for every bounded $f\colon R \to R$.*

The fact that (ii) with a general filtration $\mathscr{F}$ is equivalent to $\mathscr{F}$-exchangeability was noted with a direct proof in [10], Theorem 2.1. Condition (iii′) is interesting mainly because of its analogy with the continuous time conditions of Section 4.

PROOF. Condition (iii) means that

$$E[f(\Theta_{n+1} \circ \xi); A] = E[f(\Theta_n \circ \xi); A], \qquad A \in \mathscr{F}_n,\ n \in Z_+,$$

for bounded $f\colon R^\infty \to R$. By a monotone class argument, this is equivalent to

$$(\xi_1, \dots, \xi_n, \xi_{n+2}, \xi_{n+3}, \dots) =_d \xi, \qquad n \in Z_+,$$

from which (i) follows by iteration. Thus (i) $\Leftrightarrow$ (iii). Condition (iii) is further equivalent to

$$E\pi_\tau f = E\pi_0 f$$

for bounded $f\colon R^\infty \to R$ and for finite stopping times τ. This may be rewritten as

$$Ef(\Theta_\tau \circ \xi) = Ef(\xi),$$

which is equivalent to (ii). Thus (ii) $\Leftrightarrow$ (iii), so (i)–(iii) are equivalent. A similar argument proves the equivalence of (i′)–(iii′). □

It should be noted that Proposition 2.1 is false without the hypothesis of stationarity. For a simple counterexample, let $\xi_1, \xi_2, \dots$ take the values 0 or 1 and choose $P\{\xi_1 = 1\} = \frac{1}{4}$. Let us further assume that $\xi_2, \xi_3, \dots$ are conditionally i.i.d., given ξ_1, with

$$P[\xi_n = 1|\xi_1] = \tfrac{1}{6} + \tfrac{1}{3}\xi_1, \qquad n > 1.$$

Then

$$E[\xi_{n+1}|\mathscr{F}_n] = \begin{cases} \frac{1}{4}, & n = 0, \\ \frac{1}{6} + \frac{1}{3}\xi_1, & n > 0, \end{cases}$$

is a martingale and hence so is $(\lambda_n f)$ for every $f: \{0,1\} \to R$. Thus (2.1) holds by Proposition 2.2. But ξ is not exchangeable since $P\{\xi_1 = \xi_2 = 1\} = \frac{1}{8}$ while $P\{\xi_2 = \xi_3 = 1\} = \frac{1}{12}$.

We turn to the thinning characterization of exchangeability mentioned in the introduction. For a formal definition of thinning, let ξ be an infinite random sequence and let the random variables $\varkappa_1, \varkappa_2, \dots$ be i.i.d. and independent of ξ with

$$P\{\varkappa_i = 1\} = 1 - P\{\varkappa_i = 0\} = p, \qquad i \in N,$$

for some $p \in (0,1]$. Then the random variables

$$\tau_j = \inf\left\{k \in N: \sum_{i=1}^{k} \varkappa_i = j\right\}, \qquad j \in N,$$

are a.s. finite, so the sequence

$$\eta = (\xi_{\tau_1}, \xi_{\tau_2}, \dots)$$

is a.s. well defined and will be called a *p-thinning* of ξ.

PROPOSITION 2.3. *Fix $p \in (0,1)$ and let ξ be a stationary sequence of random variables with p-thinning η. Then ξ is exchangeable iff $\xi =_d \eta$.*

FIRST PROOF. By iteration, we get the same property with p replaced by p^n, $n \in N$, so we may take p arbitrarily small. Fix $m, n \in N$ with $m \le n$ and note that

$$P\left[\bigcap_{j=1}^{m} \{\tau_j = k_j\} \middle| \tau_{m+1} = n + 1\right] = \binom{n}{m}^{-1},$$

for all $k_1 < \cdots < k_m \le n$. Letting $f_1, \dots, f_m: R \to R$ be bounded, we get as in Kingman ([13], page 188),

$$\lim_{n\to\infty} E\left[\prod_{j=1}^{m} f_j(\eta_j) \middle| \tau_{m+1} = n + 1\right] = E\prod_{j=1}^{m} \mu f_j,$$

where $\mu f = E[f(\xi_1)|\mathscr{I}]$. Since $\tau_{m+1} \to_P \infty$ as $p \to 0$, it follows that

$$E\prod_{j=1}^{m} f_j(\xi_j) = E\prod_{j=1}^{m} f_j(\eta_j) = EE\left[\prod_{j=1}^{m} f_j(\eta_j) \middle| \tau_{m+1}\right] \to E\prod_{j=1}^{m} \mu f_j,$$

which implies that ξ is exchangeable. □

For readers aquainted with random measure theory, we shall outline an alternative proof, exhibiting the relationship with thinning of point processes.

Here and for the remainder of this section, we shall use the terminology and notation of [12].

SECOND PROOF. Introduce the marked point process

$$\tilde{\xi} = \sum_{j=1}^{\infty} \delta_{(1,\xi_j)}$$

and construct another point process $\tilde{\eta}_p$ from $\tilde{\xi}$ by a p-thinning followed by a scale contraction by a factor p^{-1}. Note that the successive marks of $\tilde{\eta}_p$ are given by $\eta = \eta_p$. Let us further construct $\tilde{\zeta}_p$ by a p^{-1}-contraction of the random measure $p\tilde{\xi}$. As before, we may let $p \to 0$ along a sequence. By the ergodic theorem, we get $\tilde{\zeta}_p \to_v \mu \times \lambda$ a.s. for some random probability measure μ on R, so Theorem 8.4 in [12] yields $\tilde{\eta}_p \to_d \tilde{\eta}'$, where $\tilde{\eta}'$ is a Cox process directed by $\mu \times \lambda$. It follows by continuous mapping that $\xi =_d \eta_p \to_d \eta'$, where η' is the sequence of successive marks of $\tilde{\eta}'$. It remains to notice that η' is conditionally i.i.d. μ. □

We conclude this section by stating an analogous point process result, which follows easily by a similar argument. Recall that a marked point process on R_+ is exchangeable (in the sense of Chapter 9 in [12]) iff it is a mixture of stationary Poisson processes.

PROPOSITION 2.4. *Fix $p \in (0,1)$, let ξ be a stationary marked point process on R_+ and let η be obtained from ξ by a p-thinning followed by a scale contraction by a factor p^{-1}. Then ξ is exchangeable iff $\xi =_d \eta$.*

3. Predictable sampling. Here we shall prove the fact, already mentioned in the introduction, that the distribution of an exchangeable sequence is invariant under predictable sampling. To facilitate access, we begin with the special case when the sampled sequence has fixed length. Fix a filtration $\mathscr{F} = (\mathscr{F}_0, \mathscr{F}_1, \dots)$.

THEOREM 3.1. *Let ξ be a finite or infinite $\mathscr{F}$-exchangeable sequence with index set I and let $\tau_1, \dots, \tau_k$ be a.s. distinct I-valued predictable stopping times. Then*

$$(\xi_{\tau_1}, \dots, \xi_{\tau_k}) =_d (\xi_1, \dots, \xi_k). \tag{3.1}$$

PROOF. Let us first consider the case when $\xi = (\xi_1, \dots, \xi_n)$ is finite and $k = n$. Let $\alpha_1, \dots, \alpha_n$ be the allocation sequence associated with $\tau_1, \dots, \tau_n$, and note that the two sequences are inverse random permutations of the integers $1, \dots, n$. Define for each $m \in \{0, \dots, n\}$ another random permutation (α_{mj}) by putting $\alpha_{mj} = \alpha_j$ for $j \le m$ and then recursively,

$$\alpha_{mj} = \min\bigl(N \setminus \{\alpha_1, \dots, \alpha_{j-1}\}\bigr), \qquad j = m+1, \dots, n.$$

Note that α_{mj} is $\mathscr{F}_{m-1}$-measurable for each j and that $\alpha_{mj} = \alpha_{m-1,j} = \alpha_j$

whenever $j < m$. Using the $\mathcal{F}$-exchangeability of ξ, we get for any bounded measurable functions $f_1, \ldots, f_n$,

$$\begin{aligned}
E\prod_{j=1}^{n} f_{\alpha_{mj}}(\xi_j) &= EE\left[\prod_{j=1}^{n} f_{\alpha_{mj}}(\xi_j)\middle|\mathcal{F}_{m-1}\right] \\
&= E\prod_{j<m} f_{\alpha_{mj}}(\xi_j)E\left[\prod_{j\ge m} f_{\alpha_{mj}}(\xi_j)\middle|\mathcal{F}_{m-1}\right] \\
&= E\prod_{j<m} f_{\alpha_{m-1,j}}(\xi_j)E\left[\prod_{j\ge m} f_{\alpha_{m-1,j}}(\xi_j)\middle|\mathcal{F}_{m-1}\right] \\
&= E\prod_{j=1}^{n} f_{\alpha_{m-1,j}}(\xi_j).
\end{aligned}$$

Summing over $m = 1, \ldots, n$ and noting that $\alpha_{nj} \equiv \alpha_j$ while $\alpha_{0j} \equiv j$, we hence obtain

$$E\prod_{j=1}^{n} f_j(\xi_{\tau_j}) = E\prod_{j=1}^{n} f_{\alpha_j}(\xi_j) = E\prod_{j=1}^{n} f_j(\xi_j).$$

The assertion now follows by a monotone class argument.

If instead $k < n$, we may extend the sequence (τ_j) by putting, recursively,

$$\tau_j = \min(N\setminus\{\tau_1, \ldots, \tau_{j-1}\}), \qquad j = k+1, \ldots, n.$$

The assumptions are then fulfilled by the extended sequence, so

$$(\xi_{\tau_1}, \ldots, \xi_{\tau_n}) =_d (\xi_1, \ldots, \xi_n),$$

which implies the same result for the first k components.

Let us finally assume that ξ is infinite. We then define the predictable stopping times τ_{nj}, for $n \in N$ and $j = 1, \ldots, k$, by

$$\tau_{nj} = \begin{cases} \tau_j, & \tau_j \le n, \\ n + j, & \tau_j > n. \end{cases} \tag{3.2}$$

Since $\tau_{n1}, \ldots, \tau_{nk}$ are a.s. distinct and bounded by $n + k$, we may apply the result in the finite case to the subsequence $(\xi_1, \ldots, \xi_{n+k})$, to obtain

$$(\xi_{\tau_{n1}}, \ldots, \xi_{\tau_{nk}}) =_d (\xi_1, \ldots, \xi_k).$$

But then the same relation must be true for $\tau_1, \ldots, \tau_k$, since $\tau_{nj} \to \tau_j$ for each j, as we let $n \to \infty$. □

We turn to the general result, where the length of the sampled sequence is allowed to be random. Recall that the *graph* of a random time τ is the random set $\{t < \infty: t = \tau\}$. Recall also the definition of $\subset_d$ from Section 1.

THEOREM 3.2. *Let ξ be a finite or infinite $\mathcal{F}$-exchangeable sequence with index set I, and let $\tau_1, \tau_2, \ldots$ be $(I \cup \{\infty\})$-valued predictable stopping times* *with a.s. disjoint graphs. Put* $\nu = \inf\{j \geq 0: \tau_{j+1} = \infty\}$. *Then*

$$(\xi_{\tau_1}, \dots, \xi_{\tau_\nu}) \subset_d \xi. \tag{3.3}$$

Note that the left-hand side of (3.3) should be interpreted as the infinite sequence $(\xi_{\tau_1}, \xi_{\tau_2}, \dots)$ when $\nu = \infty$. The preceding result was obtained for *increasing* (τ_j) in Theorem 5.1 of [10] by a cumbersome direct argument.

PROOF. We may clearly assume that $\tau_j = \infty$ for $j > \nu$. Consider first the case when $\xi = (\xi_1, \dots, \xi_n)$ is finite. Define a new allocation sequence (α'_k) recursively by

$$\alpha'_k = \begin{cases} \alpha_k, & \alpha_k < \infty, \\ \max(\{1, \dots, n\} \setminus \{\alpha'_1, \dots, \alpha'_{k-1}\}), & \alpha_k = \infty. \end{cases}$$

The inverse permutation $(\tau'_1, \dots, \tau'_n)$, given by

$$\{\tau'_j = k\} = \{\alpha'_k = j\}, \qquad j, k = 1, \dots, n,$$

will then satisfy the requirements of Theorem 3.1 and, moreover, $\tau'_j = \tau_j$ for $j \leq \nu$, so we get

$$(\xi_{\tau_1}, \dots, \xi_{\tau_\nu}, \xi_{\tau'_{\nu+1}}, \dots, \xi_{\tau'_n}) =_d (\xi_1, \dots, \xi_n),$$

proving (3.3).

It remains to consider the case when ξ is infinite. Defining τ_{nj} as in (3.2), we get by Theorem 3.1,

$$\eta_n \equiv (\xi_{\tau_{n1}}, \xi_{\tau_{n2}}, \dots) =_d \xi. \tag{3.4}$$

Let us further write

$$\nu_n = \inf\{j \geq 0: \tau_{j+1} > n\}, \qquad n \in N,$$

and note that $\nu_n \to \nu$. Note also that

$$(\eta_{n1}, \dots, \eta_{n\nu_n}, \partial, \partial, \dots) = (\xi_{\tau_1}, \dots, \xi_{\tau_{\nu_n}}, \partial, \partial, \dots), \tag{3.5}$$

since $\tau_{nj} = \tau_j$ for $j \leq \nu_n$. The sequence of pairs (η_n, ν_n) is trivially tight in $R^\infty \times \overline{N}$, so $(\eta_n, \nu_n) \to_d$ some (η', ν') along a suitable subsequence, where $\eta' =_d \xi$ by (3.4). Letting $n \to \infty$ in (3.5) and noting that

$$(x_1, x_2, \dots; k) \to (x_1, \dots, x_k, \partial, \partial, \dots)$$

defines a continuous mapping from $R^\infty \times \overline{N}$ to $(R \cup \{\partial\})^\infty$, we get in the limit

$$(\eta'_1, \dots, \eta'_{\nu'}, \partial, \partial, \dots) =_d (\xi_{\tau_1}, \dots, \xi_{\tau_\nu}, \partial, \partial, \dots),$$

with the usual interpretation in case of infinite ν or ν'. Thus (3.3) holds by Lemma 1.1. □

4. Martingale characterizations in continuous time. The main purpose of the present section is to discuss a continuous time counterpart of condition (iii′) in Proposition 2.2 and its bearing on the exchangeability of a process on

$[0,1]$ or R_+. Recall from Propositions 2.1 and 2.2 that (iii′) is equivalent to exchangeability for a stationary sequence of random variables.

Before indulging in the main theme, we remark that most methods and results related to the notion of spreadability carry over rather easily to the context of processes on R_+. In particular, the continuous time ergodic theorem yields an easy direct approach to the continuous time analogue of de Finetti's theorem (though under the assumption of measurability). Much deeper is the spreading characterization of ergodic exchangeable processes on $[0,1]$ in Theorem 3.3 of [10], whose proof employed some martingale techniques akin to those that follow.

Recall (e.g., from [6]) that a process X on some interval I is a *semimartingale* (with respect to a standard filtration $\mathscr{F}$) if X is right-continuous and adapted and if $X = M + V$ for some local martingale M and some process V with locally finite variation and $V_0 = 0$. Moreover, X is a *special semimartingale* if V can be chosen to be predictable. In that case, the preceding decomposition is unique and will be called the *canonical decomposition* of X. Associated with a semimartingale is a marked point process ξ_t and a continuous increasing process σ_t^2, given (for Borel sets $A \subset R$ with $0 \notin \bar{A}$) by

$$\xi_t A = \sum_{s \le t} 1_A(\Delta X_s), \qquad \sigma_t^2 = \langle X^c, X^c \rangle_t, \quad t \in I, \tag{4.1}$$

where $X^c = M^c$ is the unique continuous component of the martingale part M. The *compensator* (dual predictable projection) of ξ will be denoted by $\hat{\xi}$. For special semimartingales, the processes V, σ^2 and $\hat{\xi}$ will be called the *local characteristics* of X. (Note the slight deviation from common practice, in our definition of the first characteristic V.)

The continuous time counterpart of condition (iii′) is to assume that X is a special semimartingale, whose local characteristics are absolutely continuous in time with respect to Lebesgue measure λ and such that suitable versions of the associated densities form martingales. (For brevity, we shall then say that V, σ^2 and $\hat{\xi}$ are *absolutely continuous with martingale densities*.) In case of $\hat{\xi}$, this means that

$$\hat{\xi}_t A = \int_0^t \mu_s A \, ds, \qquad t \in I, \tag{4.2}$$

for some measure-valued process μ, such that $\mu_t A$ is a martingale in t for every fixed A. All martingales in this section are with respect to a fixed standard filtration $\mathscr{F}$ and we shall always choose their right-continuous versions.

Our plan for this section is first to show in Theorem 4.1 that the preceding condition is fulfilled for an exchangeable process, under suitable moment conditions. (We shall actually prove slightly more, in preparation for the next section.) We then show in Theorems 4.3 and 4.4 that the stated condition is also sufficient, under appropriate additional assumptions, for a process on R_+ or $[0,1]$, respectively, to be exchangeable. As in the case of Proposition 2.1, the sufficiency assertion fails without such extra conditions.

In what follows, we shall avoid using the explicit representation of exchangeable processes stated in Section 1, since the results of this section will then provide a martingale approach to the basic representation formula, at least under moment restrictions.

THEOREM 4.1. *Any $\mathscr{F}$-exchangeable process X on $[0,1]$ is a semimartingale, such that σ^2 and $\hat{\xi}$ are absolutely continuous. If, moreover, $E|X_t| < \infty$, then X is a special semimartingale on $[0,1)$, such that $X - V$ is a martingale on $[0,1)$, while V is absolutely continuous with a martingale density on $[0,1)$. If $EX_t^2 < \infty$, then even σ^2 and $\hat{\xi}$ have martingale densities on $[0,1)$ and $X - V$ is an L_2-martingale on $[0,1]$, while $E(\int|dV|)^2 < \infty$.*

We shall need the following simple lemma.

LEMMA 4.2. *If X is $\mathscr{F}$-exchangeable on $[0,1]$, we have for any Borel set $A \subset R$ with $0 \notin \bar{A}$,*

$$E[\xi_1 A | \mathscr{F}_t] < \infty \quad a.s., \quad t \in (0,1].$$

PROOF. This is trivial for $t = 1$, so we may fix a $t \in (0,1)$. Letting $k \in Z_+$ be arbitrary, we get

$$\begin{aligned} E[\xi_1 A; \xi_t A = k] &= \sum_{n \geq k} nP\{\xi_1 A = n\}\binom{n}{k} t^k (1-t)^{n-k} \\ &\leq \sum_{n \geq k} n\binom{n}{k}(1-t)^{n-k} = \sum_{n \geq k} a_n < \infty, \end{aligned}$$

since

$$\frac{a_n}{a_{n-1}} = \frac{n^2(1-t)}{(n-1)(n-k)} \to 1 - t < 1.$$

Hence

$$E\big[E[\xi_1 A|\mathscr{F}_t]|\xi_t A\big] = E[\xi_1 A|\xi_t A] < \infty \quad \text{a.s.},$$

so

$$P\big[E[\xi_1 A|\mathscr{F}_t] < \infty|\xi_t A\big] = 1 \quad \text{a.s.}$$

and the assertion follows by taking expectations on both sides. □

PROOF OF THEOREM 4.1. Let us first assume that $E|X_t| < \infty$. Write M for a right-continuous version of the process

$$M_t = E[X_1 - X_t|\mathscr{F}_t]/(1-t), \qquad t \in [0,1).$$

Letting $s \leq t$ with $1 - s$ and $1 - t$ rationally dependent and using the exchangeability of X, we get

$$E[M_t|\mathscr{F}_s] = E[X_1 - X_t|\mathscr{F}_s]/(1-t) = E[X_1 - X_s|\mathscr{F}_s]/(1-s) = M_s,$$

which extends by right-continuity to arbitrarily related s and t. Thus M is a martingale on $[0,1)$. In particular,

$$E[X_t - X_s|\mathscr{F}_s] = (1-s)M_s - (1-t)E[M_t|\mathscr{F}_s] = (t-s)M_s, \qquad 0 \le s \le t < 1.$$

Writing

$$V_t = \int_0^t M_s\,ds, \qquad t \in [0,1), \tag{4.3}$$

and noting that

$$E\int_0^t |M_s|\,ds = \int_0^t E|M_s|\,ds \le tE|M_t| < \infty, \qquad t \in [0,1),$$

since $|M|$ is a submartingale, it is further seen that

$$E[V_t - V_s|\mathscr{F}_s] = E\left[\int_s^t M_u\,du\middle|\mathscr{F}_s\right] = (t-s)M_s, \qquad 0 \le s \le t < 1.$$

Thus $X - V$ is a martingale on $[0,1)$. Since V is predictable, this shows that X is a special semimartingale on $[0,1]$ with canonical decomposition $X = (X - V) + V$.

Let us next assume that $EX_t^2 < \infty$. By Jensen's inequality,

$$EM_t^2 \le E(X_1 - X_t)^2/(1-t)^2 = E\alpha^2 + \frac{t}{1-t}E\left(\sigma^2 + \sum \beta_j^2\right),$$

so by the Schwarz inequality,

$$E\left(\int_0^1 |dV|\right)^2 = E\int_0^1\int_0^1 |M_s M_t|\,ds\,dt \le \left(\int_0^1 (EM_t^2)^{1/2}\,dt\right)^2 < \infty.$$

Thus

$$\sup_t E(X_t - V_t)^2 \le 2EX_{1/2}^2 + 2E\left(\int |dV|\right)^2 < \infty,$$

so $X - V$ is uniformly integrable and extends to an L_2-martingale on $[0,1]$. In particular,

$$E\left[\sigma_1^2 + \int x^2 \xi_1(dx)\right] = E[X - V, X - V]_1 = E(X_1 - V_1)^2 < \infty,$$

which implies that $E\xi_1 A < \infty$ for Borel sets A with $0 \notin \bar{A}$.

The $\mathscr{F}$-exchangeability of X is clearly inherited by the processes σ_t^2 and ξ_t. We may thus conclude as before that there exists a martingale M' on $[0,1)$ making the process

$$M_t'' = \sigma_t^2 - \int_0^t M_s'\,ds, \qquad t \in [0,1),$$

a martingale. Since M'' is continuous with locally finite variation, it follows that $M'' = 0$, which proves the desired representation for σ_t^2. Similarly, ξ_t is compensated by the process $\hat{\xi}_t$ in (4.2), with μ_t chosen as the measure-valued martingale

$$\mu_t A = E[\xi_1 A - \xi_t A | \mathscr{F}_t]/(1-t), \qquad t \in [0,1). \tag{4.4}$$

Let us finally turn to the general case when there are no moment restrictions. Let us then assume that X is directed by some triple $(\alpha, \sigma^2, \beta)$ and define a new filtration $\mathscr{G}$ by $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{I}$, where $\mathscr{I} = \sigma(\alpha, \sigma^2, \beta)$. (The threefold meaning of σ should not cause any confusion.) Then X remains $\mathscr{G}$-exchangeable and, moreover, $E[X_t^2 | \mathscr{I}] < \infty$ for all t, so it may be seen as before that X is a special $\mathscr{G}$-semimartingale on $[0,1]$ with canonical decomposition $X = (X - V) + V$, where V is now given by (4.3) with

$$M_t = E[X_1 - X_t | \mathscr{G}_t]/(1-t), \qquad t \in [0,1).$$

Since $\mathscr{F}_t \subset \mathscr{G}_t$ for all t, we may conclude from Theorem 9.19 of Jacod [6] that X remains a semimartingale with respect to $\mathscr{F}$. Note also that the process σ_t^2 is absolutely continuous, since this is conditionally true, given $\mathscr{I}$.

To see that $\hat{\xi}$ remains absolutely continuous in the general case, fix a Borel set $A \subset R$ with $0 \notin \bar{A}$, and let $\mu_t A$ be given by (4.4) for $t > 0$. Then $\mu_t A < \infty$ a.s. by Lemma 4.2 and we get, as before,

$$E[\xi_t A - \xi_s A | \mathscr{F}_s] = E\left[\int_s^t \mu_u A \, du \middle| \mathscr{F}_s\right] = (t-s)\mu_s A, \qquad 0 < s \le t \le 1,$$

which shows that

$$\hat{\xi}_t A - \hat{\xi}_s A = \int_s^t \mu_u A \, du, \qquad 0 < s \le t \le 1.$$

Letting $s \to 0$ and noting that $\hat{\xi}_0 = 0$, we obtain the representation (4.2). □

We turn to the results in the opposite direction and begin with the case of processes on R_+.

Theorem 4.3. *Let X be a special semimartingale on R_+ with stationary increments and with $X_0 = 0$ and $E[X, X]_1 < \infty$. Assume that V, σ^2 and $\hat{\xi}$ are absolutely continuous with martingale densities. Then X is exchangeable.*

Proof. Assume that X is compensated by the process V in (4.3), for some martingale M. Then V is integrable and $X - V$ is a martingale since $E[X, X]_1 < \infty$, so we get for $s \le t$,

$$\begin{aligned} E[X_t - X_s | \mathscr{F}_s] &= E[V_t - V_s | \mathscr{F}_s] \\ &= E\left[\int_s^t M_u \, du \middle| \mathscr{F}_s\right] = \int_s^t E[M_u | \mathscr{F}_s] \, du = (t-s)M_s. \end{aligned}$$

In particular,

$$E[X_{s+h} - X_s | \mathscr{G}_s] = hE[X_{t+1} - X_t | \mathscr{G}_s], \qquad 0 \le s \le t, \quad h > 0, \tag{4.5}$$

where $\mathscr{G} = (\mathscr{G}_s)$ denotes the standard filtration generated by X.

Let us now extend X to a right-continuous process X' on R with stationary increments and define

$$\mathscr{H}_s = \sigma\{X'_s - X'_{s-h}, h > 0\}, \qquad s \geq 0.$$

Let us further write $\mathscr{S}$ for the σ-field generated by all shift-invariant functions of the increments of X' and note that $\mathscr{S} \subset \mathscr{H}_s$ a.s. for all s. From the stationarity of the increments, it is easily seen that (4.5) remains true with $\mathscr{G}_s$ replaced by $\mathscr{G}_s \vee \mathscr{H}_s$ and hence also by $\mathscr{G}'_s = \mathscr{G}_s \vee \mathscr{S}$.

Applying the ergodic theorem to the right-hand side of (4.5) yields

$$E[X_{s+h} - X_s | \mathscr{G}'_s] = hE[X_1 | \mathscr{S}], \qquad s, h \geq 0.$$

A similar argument shows that

$$E[\xi_{s+h}A - \xi_s A | \mathscr{G}'_s] = hE[\xi_1 A | \mathscr{S}], \qquad s, h \geq 0,$$

$$E\left[\sigma^2_{s+h} - \sigma^2_s | \mathscr{G}'_s\right] = hE\left[\sigma^2_1 | \mathscr{S}\right], \qquad s, h \geq 0.$$

Thus the processes

$$X_t - tE[X_1 | \mathscr{S}], \qquad \xi_t A - tE[\xi_1 A | \mathscr{S}], \qquad t \geq 0,$$

are $\mathscr{G}'$-martingales, while

$$\sigma^2_t = tE\left[\sigma^2_1 | \mathscr{S}\right], \qquad t \geq 0.$$

This means that X is a special $\mathscr{G}'$-martingale with linear and $\mathscr{S}$-measurable local characteristics. It then follows as in Theorem 3.57 of Jacod [6] (cf. [7]) that X is conditionally a Lévy process, given $\mathscr{S}$. Hence X is a mixture of Lévy processes and therefore exchangeable. □

We turn to the case of processes on $[0, 1]$. Here the stationarity assumption in Theorem 4.3 will be replaced by a suitable constraint at the terminal point. The following result, in conjunction with Theorem 4.1, yields a complete martingale characterization of *ergodic* exchangeable processes on $[0, 1]$. The corresponding characterization of finite exchangeable sequences is the martingale version of Proposition 2.3 in [10].

THEOREM 4.4. *Let X be a uniformly integrable special semimartingale on $[0, 1]$ with $X_0 = 0$ and nonrandom X_1, σ^2_1 and ξ_1, such that V, σ^2 and $\hat{\xi}$ are absolutely continuous with martingale densities on $[0, 1)$. Then X is ergodic $\mathscr{F}$-exchangeable.*

Two lemmas will be needed for the proof.

LEMMA 4.5. *Let B be both a Brownian motion and an $\mathscr{F}$-martingale and let ξ be an $\mathscr{F}$-adapted marked point process whose $\mathscr{F}$-compensator depends predictably on ξ. Then B and ξ are independent.*

PROOF. It is clearly enough to show that $Ef(B)g(\xi) = 0$ for any bounded measurable functions f and g with $Ef(B) = Eg(\xi) = 0$. By Theorems 11.16 and

12.23 in Jacod [6], there exist predictable processes V and W with

$$\int V_s^2\,ds < \infty, \qquad \iint W_{s,x}^2\,d\hat{\xi}_{s,x} < \infty,$$

and such that $f(B) = M_\infty$ while $g(\xi) = N_\infty$, where M and N denote the martingales

$$M_t = \int_0^t V_s\,dB_s, \qquad N_t = \int_0^t \int W_{s,x}\,d(\xi - \hat{\xi})_{s,x}, \qquad t \geq 0.$$

In fact, this is all true with respect to the filtrations generated by B and ξ, respectively. But by assumptions, $\hat{\xi}$ remains the $\mathscr{F}$-compensator of ξ while B remains an $\mathscr{F}$-martingale. Moreover, M is continuous while N is purely discontinuous, so $M \perp N$, and we get

$$Ef(B)g(\xi) = EM_\infty N_\infty = 0. \qquad \square$$

Lemma 4.6. *Let X be an exchangeable process on $[0,1]$ directed by $(\alpha, \sigma^2, \beta)$. Then $[X, X]_1 = \sigma^2 + \Sigma\beta_j^2$.*

This result was obtained in [9] by cumbersome arguments. Here is a simple martingale proof.

Proof. We may clearly take $(\alpha, \sigma^2, \beta)$ to be nonrandom with $\alpha = 0$. In that case,

$$M(t) = \frac{X_t}{1-t} = \sigma\frac{B_t}{1-t} + \sum_{j=1}^{\infty} \beta_j \frac{1\{\tau_j \leq t\} - t}{1-t} = \sum_{j=0}^{\infty} M_j(t)$$

is an orthogonal decomposition of the L_2-martingale on the left, and we get

$$[M, M]_t = \sum_{j=0}^{\infty} [M_j, M_j]_t,$$

which yields a corresponding decomposition of $[X, X]_t$. It remains to notice that $[B, B]_t = t$, since $B_t = W_t - tW_1$ for some Brownian motion W. $\square$

Proof of Theorem 4.4. Let N be a right-continuous version of the martingale density of σ_t^2. Fixing $s \in [0,1]$, we get a.s.

$$\begin{aligned}\int_s^1 N_t\,dt &= \sigma_1^2 - \sigma_s^2 = E\left[\sigma_1^2 - \sigma_s^2 \middle| \mathscr{F}_s\right] \\ &= E\left[\int_s^1 N_t\,dt \middle| \mathscr{F}_s\right] = \int_s^1 E[N_t|\mathscr{F}_s]\,dt = (1-s)N_s.\end{aligned}$$

Hence N is a.s. continuously differentiable and satisfies the differential equation

$$-N_s = (1-s)N_s' - N_s, \qquad 0 \leq s \leq 1,$$

so $N_s' \equiv 0$ a.s., and we get

$$\sigma_t^2 = t\sigma_1^2, \qquad t \in [0,1], \quad \text{a.s.} \tag{4.6}$$

This shows that X^c is a Brownian motion with diffusion rate σ_1^2.

Let us next assume that ξ is compensated by the process $\hat{\xi}$ in (4.2) for some measure-valued martingale μ. Letting $A \subset R$ be a Borel set with $0 \notin \bar{A}$, we get a.s. for any $s \in [0,1]$,

$$\xi_1 A - \xi_s A = E[\xi_1 A - \xi_s A | \mathscr{F}_s] = E\left[\int_s^1 \mu_t A \, dt \middle| \mathscr{F}_s\right]$$
$$= \int_s^1 E[\mu_t A | \mathscr{F}_s] \, dt = (1-s)\mu_s A,$$

so by right-continuity,

$$\mu_s = (\xi_1 - \xi_s)/(1-s), \qquad s \in [0,1], \text{ a.s.} \tag{4.7}$$

By Lemma 4.5, it follows in particular that ξ and X^c are independent.

Let us finally assume that X is compensated by the process V in (4.3), for some martingale M on $[0,1)$. Then V has integrable variation on compact subintervals of $[0,1)$, and $X - V$ is a martingale on $[0,1)$ since

$$[X - V, X - V]_1 = \sigma_1^2 + \int x^2 \xi_1(dx) < \infty,$$

so we get a.s., for any $0 \le s \le t < 1$,

$$E[X_t - X_s | \mathscr{F}_s] = E[V_t - V_s | \mathscr{F}_s] = E\left[\int_s^t M_u \, du \middle| \mathscr{F}_s\right]$$
$$= \int_s^t E[M_u | \mathscr{F}_s] \, du = (t-s)M_s.$$

By the continuity of X at 1, the uniform integrability of X and the right-continuity of X and M, it follows that

$$X_1 - X_s = (1-s)M_s, \qquad s \in [0,1), \text{ a.s.},$$

so

$$dX_s = -(1-s)\, dM_s + M_s \, ds = d(X_s - V_s) + dV_s$$

and, therefore,

$$V_t = \int_0^t M_s \, ds = -\int_0^t ds \int_0^s \frac{d(X_u - V_u)}{1-u}, \qquad t \in [0,1), \text{ a.s.} \tag{4.8}$$

Let us now consider instead an ergodic exchangeable process X' on $[0,1]$ directed by (X_1, σ_1^2, ξ_1). Theorem 4.1 shows that X is a special semimartingale with respect to the induced standard filtration and that the local characteristics of X' are absolutely continuous with martingale densities on $[0,1)$. Since X' is further L_2-bounded and hence uniformly integrable, everything already said for X applies equally to X'. In particular, (4.6)–(4.8) remain true for the processes σ'^2, μ', ξ' and V' associated with X'.

As for previous X, it is seen that X'^c is a Brownian motion independent of ξ', and since $\sigma_1'^2 = \sigma_1^2$ by Lemma 4.6, the diffusion rate is the same as for X^c. Since the functional dependence in (4.7) is the same for ξ and ξ', it may further be seen from Theorem 3.42 of Jacod [6] that $\xi' =_d \xi$, so $\xi' - \hat{\xi}' =_d \xi - \hat{\xi}$, and hence

$X' - V' =_d X - V$. We may next infer from the two versions of (4.8) that $(X' - V', V') =_d (X - V, V)$, which implies that $X' =_d X$. Thus X is exchangeable.

To reach the stronger conclusion of $\mathscr{F}$-exchangeability, it suffices to fix an arbitrary $s \in [0,1)$ and to check that the preceding arguments apply to the conditional distribution of X on the interval $[s,1]$, given the σ-field $\mathscr{F}_s$. We omit the details of this verification. □

We conclude this section with some remarks. First we show by an example that the last two theorems are false without the additional assumptions of stationarity of the increments or of nonrandomness of the local characteristics at the terminal point. Let us then take ξ to be a simple point process on R_+, such that the restriction to $[0,1]$ is a mixture of Poisson processes with intensities 1 or 0, where each possibility is chosen with probability $\frac{1}{2}$. On the remaining interval, we choose ξ to be Poisson with intensity 1 or $(1+e)^{-1}$, depending on whether $\xi_1 > 0$ or not. It is then easy to verify that the density of $\hat{\xi}$ is a martingale. But ξ fails to be exchangeable, since $P\{\xi_1 = 0\} = \frac{1}{2}(1 + e^{-1})$, while

$$P\{\xi_2 - \xi_1 = 0\} = \tfrac{1}{2}(1 - e^{-1})e^{-1} + \tfrac{1}{2}(1 + e^{-1})e^{-(1+e)^{-1}}.$$

As a second remark, we shall sketch how the preceding results may be combined to yield a simple martingale approach to the representation theorem for exchangeable processes on $[0,1]$. Let us then assume that the process X on $[0,1]$ is exchangeable, integrable and continuous in probability at every fixed point. Then $Y_t = (X_1 - X_t)/(1-t)$ is seen to be a martingale on $[0,1)$, so X must have a version in $D[0,1]$. Note also that $[X,X]_1 < \infty$ a.s., since the exchangeability of X carries over to $[X,X]$. Since X remains conditionally exchangeable, given the triple $(X_1, [X,X]_1, \xi_1)$, we may assume that X_1, $[X,X]_1$ and ξ_1 are all fixed. Then $EY_t^2 < \infty$, so even $EX_t^2 < \infty$. It may hence be seen as in Theorem 4.1 that X is a special semimartingale on $[0,1]$, whose local characteristics are absolutely continuous with martingale densities on $[0,1)$. Note also that X is uniformly integrable on $[0,1]$, since EX_t^2 is bounded. The hypotheses of Theorem 4.4 are then fulfilled, so the desired representation formula follows as in the proof of that theorem.

5. Predictable transformations in continuous time. Our aim in the present section is to prove continuous time versions of Theorems 3.1 and 3.2. Let us then fix a standard filtration $\mathscr{F}$ and recall from Section 1 the definition of an $\mathscr{F}$-exchangeable process. Recall also our definition of the transformed processes XV^{-1}. The stochastic integrals occurring in the definition exist by Lemma 5.2.

THEOREM 5.1. *Fix $I = [0,1]$ or R_+ and a subinterval J containing* 0. *Let X and V be processes on I, such that X is $\mathscr{F}$-exchangeable while V is $\mathscr{F}$-predictable with values in $I \cup \{\infty\}$, and assume that $\lambda V^{-1} = \lambda$ a.s. on J. Then*

$$XV^{-1} =_d X \quad \text{on } J. \tag{5.1}$$

Since $(XV^{-1})_t$ is only defined a.s. for each t, (5.1) should be interpreted as a relation between the finite-dimensional distributions. However, (5.1) implies that XV^{-1} has a right-continuous version with left hand limits and for the latter there is clearly equality between the distributions on the Skorohod space $D(J)$.

Two lemmas will be needed for the proof.

LEMMA 5.2. *Let X be an $\mathscr{F}$-exchangeable process on $I = [0,1]$ or R_+, and let $A \subset I$ be predictable with $\lambda A < \infty$ a.s. Then the stochastic integral $\int 1_A \, dX$ exists. Moreover $\int 1_{A_n} dX \to_P 0$ whenever $A_1, A_2, \ldots \subset I$ are predictable with $\lambda A_n \to_P 0$.*

PROOF. Let us first consider the case of processes on $[0,1]$. Changing the filtration, as in the proof of Theorem 4.1 and applying Theorem 9.26 of Jacod [6], we may reduce the discussion to the case when X is conditionally ergodic exchangeable, given $\mathscr{F}_0$. But then Theorem 4.1 shows that X is a special semimartingale on $[0,1]$, with a canonical decomposition $X = M + V$ such that both $\langle M, M\rangle$ and V are absolutely continuous. The existence of the stochastic integrals $\int 1_A \, dX$ follows immediately from this. To prove the convergence assertion, consider an arbitrary subsequence such that $\lambda A_n \to 0$ a.s. Then

$$\lim_{n\to\infty} \int 1_{A_n} |dV| = \lim_{n\to\infty} \int 1_{A_n} d\langle M, M\rangle = 0 \quad \text{a.s.},$$

which yields the desired conclusion. □

For processes on R_+, we may reduce as before to the case when X is conditionally a Lévy process, given $\mathscr{F}_0$. In this case we get a decomposition $X = M + V + J$, where V is linear, while M is a local martingale such that $\langle M, M\rangle$ is linear, and J is conditionally a compound Poisson process. For integrals with respect to $M + V$, the existence and convergence assertions follow as before, so it remains only to consider integrals with respect to J. Letting N denote the associated mixed Poisson process, it is seen from the results for $M + V$ that $\int 1_A \, dN$ exists and that $\int 1_{A_n} dN \to_P 0$. Since the integrals $\int 1_A \, dN$ and $\int 1_A |dJ|$ are simultaneously finite and simultaneously zero, the corresponding statements then follow for J.

LEMMA 5.3. *Let $A_1, \ldots, A_n$ be disjoint predictable sets in $[0,1]$ of equal length n^{-1} and fix an $\varepsilon > 0$. Then there exist some integer $m \in N$ and some disjoint predictable sets $A_1', \ldots, A_n'$ of equal length n^{-1}, such that each A_j' is a union of intervals $((j-1)m^{-1}, jm^{-1}]$, and such that, moreover,*

$$\sum_{j=1}^{n} (\lambda \times P)(A_j \Delta A_j') < \varepsilon. \tag{5.2}$$

Proof. Recall that the restriction of the predictable σ-field to the interval $(0,1]$ is generated by the stochastic intervals of the form $(\sigma,\tau]$, where σ and τ are rational-valued stopping times in $[0,1]$. From this, it follows easily by a monotone class argument that any predictable set in $[0,1]$ can be approximated arbitrarily closely in measure $\lambda \times P$ by a predictable union of intervals $I_j = ((j-1)m^{-1}, jm^{-1}]$, with m a fixed multiple of n. This implies in particular that the process $\sum j 1_{A_j}$ can be approximated in $L_1(\lambda \times P)$ by a process of the form $\sum j 1_{U_j}$, where $U_1, \dots, U_n$ are disjoint predictable interval unions as before, with union $(0,1]$. Taking the error to be less than ε/n, we get

$$E \sum_{j=1}^{n} |\lambda U_j - n^{-1}| \le \sum_{j=1}^{n} (\lambda \times P)(A_j \Delta U_j) < \varepsilon/n. \tag{5.3}$$

Let us now define the variables $\alpha_1, \dots, \alpha_m$ by the condition

$$\alpha_j = k \quad \text{if} \quad I_j \subset U_k, \qquad j = 1, \dots, m, \quad k = 1, \dots, n,$$

and put, recursively,

$$\alpha'_j = \begin{cases} \alpha_j, & \text{if } \sum_{i<j} 1\{\alpha'_i = \alpha_j\} < m/n, \\ \min\left\{k: \sum_{i<j} 1\{\alpha'_i = k\} < m/n\right\}, & \text{otherwise.} \end{cases}$$

It is then easily seen that the sets

$$A'_k = \bigcup_{j=1}^{m} \{I_j: \alpha'_j = k\}, \qquad k = 1, \dots, n,$$

are disjoint predictable unions of $I_1, \dots, I_m$ of equal length n^{-1}. Moreover, (5.2) follows from (5.3) and the fact that, by construction

$$\sum_{j=1}^{n} \lambda(U_j \Delta A'_j) \le (n-1) \sum_{j=1}^{n} |\lambda U_j - n^{-1}|. \qquad \square$$

Proof of Theorem 5.1. Let us first assume that $I = J = [0,1]$. By the right-continuity of X and by dominated convergence for stochastic integrals, it is then enough to prove that, for fixed n,

$$(\xi_{n1}, \dots, \xi_{nn}) =_d (\eta_{n1}, \dots, \eta_{nn}), \tag{5.4}$$

where ξ_{nj} and η_{nj} denote the increments of X and XV^{-1}, respectively, over the interval $I_{nj} = ((j-1)n^{-1}, jn^{-1}]$. Note that

$$\eta_{nj} = \int 1_{A_{nj}} \, dX, \qquad j = 1, \dots, n,$$

where A_{nj} denotes the predictable random set

$$A_{nj} = \{t \in I: V_t \in I_{nj}\}, \qquad j = 1, \dots, n.$$

Consider first the case when each set A_{nj} is a union of m/n randomly selected intervals I_{mk}, for some multiple m of n. Then

$$\eta_{nj} = \sum_{k=1}^{m/n} \xi_{m,\tau_{jk}}, \qquad j = 1, \dots, n,$$

for some functions

$$\tau_{j1} < \cdots < \tau_{j,m/n}, \qquad j = 1, \dots, n.$$

Write $\mathscr{G}$ for the discrete filtration $\mathscr{G}_j = \mathscr{F}_{jm^{-1}}$, $j = 0, \dots, m$, and note that $(\xi_{m1}, \dots, \xi_{mm})$ is $\mathscr{G}$-exchangeable, while the τ_{jk} are $\mathscr{G}$-predictable stopping times. Hence Theorem 3.1 yields

$$\left(\xi_{m,\tau_{11}}, \dots, \xi_{m,\tau_{n,m/n}}\right) =_d (\xi_{m1}, \dots, \xi_{mm}),$$

and (5.4) follows by a suitable summation on each side.

In the case of general sets A_{nj}, it is seen from Lemma 5.3 that $A_{n1}, \dots, A_{nn}$ can be approximated in $(\lambda \times P)$-measure by disjoint predictable sets $B_{m1}, \dots, B_{mn}$ of equal length n^{-1}, and such that each B_{mj} is a union of randomly selected intervals I_{mk}. As previously shown, we get for each m,

$$\left(\int 1_{B_{m1}} dX, \dots, \int 1_{B_{mn}} dX\right) =_d (\xi_{n1}, \dots, \xi_{nn}). \tag{5.5}$$

Moreover, it is seen from Lemma 5.2 that

$$\int 1_{B_{mj}} dX \to_P \int 1_{A_{nj}} dX \quad \text{as } m \to \infty, \qquad j = 1, \dots, n.$$

Hence (5.5) remains true with the sets B_{mj} replaced by A_{nj}, and the assertion follows.

Retaining $I = [0, 1]$, we turn to the case when $J = [0, p]$ for some $p < 1$. We may then construct another predictable process U on I by putting

$$U_t = \begin{cases} V_t, & V_t \le p, \\ 1 - \lambda\{s \le t\colon V_t > p\}, & V_t > p. \end{cases}$$

Noting that $\lambda U^{-1} = \lambda$, we may conclude as before that $XU^{-1} =_d X$. Since moreover $XU^{-1} = XV^{-1}$ on J, the assertion follows.

If instead $I = R_+$ while $J = [0, 1]$, say, we may define the processes

$$U_n(t) = \begin{cases} V_t, & t \le n, \\ \inf\{s \in J\colon s - \lambda\{r \le n\colon V_r \le s\} = t - n\}, & t > n. \end{cases}$$

Then each U_n is predictable with $\lambda U_n^{-1} = \lambda$ on J and $U_n^{-1} J \subset [0, n + 1]$, so the result for processes on finite intervals yields $XU_n^{-1} =_d X$ on J. Since moreover

$$\lambda\{s > n\colon U_n(s) \le t\} = \lambda\{s > n\colon V_s \le t\} \to 0 \quad \text{a.s.}$$

by dominated convergence, as $n \to \infty$ for fixed $t \in J$, we get by Lemma 5.2,

$$\left(XU_n^{-1}\right)_t - \left(XV^{-1}\right)_t = \int_n^\infty \left(1\{U_n(s) \le t\} - 1\{V_s \le t\}\right) dX_s \to_P 0,$$

so the finite-dimensional distributions of XU_n^{-1} on J tends weakly to those of XV^{-1} and the assertion follows again. □

Unfortunately, Theorem 5.1 is insufficient for most applications, since the process V is usually measure preserving only on an interval of *random* length. For a simple example, let A be a predictable set in I and define a new process

$$Z_t = \int_0^t 1_A(t)\, dX_t, \qquad t \in I.$$

If λA is fixed, then Theorem 5.1 can be used to show that $Z = X' \circ \tau$ a.s. for some process $X' =_d X$, where the random time change process τ is given by

$$\tau_t = \lambda(A \cap [0, t]), \qquad t \in I.$$

But the result remains true when λA is random (cf. Theorem 5.2 in [10]). Similarly, a more general version is needed to obtain representations of stochastic integrals with respect to nonsymmetric stable Lévy processes.

Recall from Lemma 5.2 that (XV^{-1}) is well defined for fixed t if $\lambda\{s: V_s \le t\} < \infty$ a.s. In general, it may then be extended by localization to the set where $\lambda\{s: V_s \le t\} < \infty$.

THEOREM 5.4. *Let X be an $\mathscr{F}$-exchangeable process on $I = [0,1]$ or R_+, and let the process V on I be $\mathscr{F}$-predictable with values in $I \cup \{\infty\}$. Put*

$$\zeta = \sup\{t \ge 0: \lambda\{s: V_s \le t\} = t\},$$

and let Y denote the restriction of XV^{-1} to $[0, \zeta)$. Then $Y \subset_d X$.

Note that the process Y is well defined on $[0, \zeta)$, since $\{\zeta > t\} \subset A_t$ for each t. The theorem states that Y can be extended to a process on R_+ with the same finite-dimensional distributions as X. As before, this yields the existence on $[0, \zeta)$ of a right-continuous version with left-hand limits.

The core of our proof consists in constructing a measure preserving process V', to replace V in the definition of Y. This will essentially be accomplished by the next two lemmas.

LEMMA 5.5. *Let $f: R_+ \to \overline{R}_+$ be measurable with $\lambda f^{-1} \le \lambda$ on some interval $[0, p)$. Define*

$$g_t = \sup\{x \ge 0: \lambda\{s \le t: f_s \in \cdot\} \le \lambda \text{ on } [0, x)\}, \qquad t \ge 0,$$

and put $h = f + \infty \cdot 1\{f \ge g\}$. Then $\lambda h^{-1} \le \lambda$ on R_+ and we have $h = f$ on the set where $f < p$. If f is a predictable process, then so is h.

LEMMA 5.6. *Let $f: [0,1] \to [0,1] \cup \{\infty\}$ be measurable with $\lambda f^{-1} \le \lambda$ on $[0,1]$ and $\lambda f^{-1} = \lambda$ on some interval $[0, p]$. Define*

$$g_t = \inf\{x \in [0,1]: x = 1 - \lambda\{s \le t: f_s > x\}\}, \qquad t \in [0,1],$$

and put $h = f \wedge g$. Then $\lambda h^{-1} = \lambda$ on $[0,1]$, and $h \wedge p = f \wedge p$ a.e. λ. If f is a predictable process, then so is $h_t^- = f_t \wedge g_{t-}$.

In most applications, we have $\lambda V^{-1} \leq \lambda$ already at the outset, so only Lemma 5.6 is needed. For this reason, we omit the simple but tedious proof of Lemma 5.5.

PROOF OF LEMMA 5.6. Since $\lambda\{s \leq t: f_s > x\}$ is continuous in x for fixed t, with values $t \leq 1$ at 0 and $\lambda\{s \leq t: f_s = \infty\} \geq 0$ at 1, the set of solutions to the equation

$$x = 1 - \lambda\{s \leq t: f_s > x\} \tag{5.6}$$

forms a nonempty closed set. In particular, g_t solves (5.6) at t. Note also that g_t decreases from 1 to 0. Substituting $x = g_t$ in (5.6) and letting $t \to t'$ from above and below, it follows easily that both $g_{t'+}$ and $g_{t'-}$ solve (5.6) at t' and the same must then be true for every intermediate value. This shows the existence, for every $x \in [0,1]$, of some $t = t_x \in [0,1]$, such that x solves (5.6) at t and, moreover, $g_{t+} \leq x \leq g_{t-}$.

Let us now assume that x is such that $\lambda\{s: g_s = x\} = 0$. Then

$$\lambda\{s: h_s > x\} = \lambda\{s < t: h_s > x\} = \lambda\{s < t: f_s > x\} = 1 - x.$$

Since the set of x's with the preceding property is dense in $[0,1]$, it follows that $\lambda h^{-1} = \lambda$. In particular we get $\lambda(h \wedge p)^{-1} = \lambda(f \wedge p)^{-1}$, and so $\lambda(f \wedge p - h \wedge p) = 0$. Since the integrand is nonnegative, it follows that $h \wedge p = f \wedge p$ a.e. λ.

If f is predictable, then $\lambda\{s \leq t: f_s > x\}$ is $\mathscr{F}_t$-measurable for every t and x, and hence so is the event

$$\{g_t \geq y\} = \bigcap\{x < 1 - \lambda\{s \leq t: f_s > x\}\},$$

where the intersection extends over all rational numbers x in $[0, y]$. Thus g is adapted, so g_{t-} is predictable, like f. Hence so is h^-. □

Next one needs to verify that the new predictable process V obtained through the last two lemmas gives rise to the same process Y on the random interval $[0, \zeta)$.

LEMMA 5.7. *Let U be another predictable process on I, and assume that*

$$\lambda\{s: U_s \neq V_s, U_s \wedge V_s < \zeta\} = 0 \quad a.s.$$

Then XU^{-1} and XV^{-1} represent the same process on $[0, \zeta)$.

PROOF. Fix $t \in I$ and define the stopping time

$$\tau = \inf\{r \geq 0: \lambda\{s \leq r: U_s \neq V_s, U_s \wedge V_s \leq t\} > 0\}.$$

Then

$$\lambda(\{U_s \leq t, s \leq \tau\} \Delta \{V_s \leq t, s \leq \tau\}) = 0 \quad \text{a.s.},$$

so by Lemma 5.2 we get for all $n \in N$,

$$\int_0^{\tau \wedge n} 1\{V_s \leq t\}\, dX_s = \int 1\{V_s \leq t, s \leq \tau \wedge n\}\, dX_s$$
$$= \int 1\{U_s \leq t, s \leq \tau \wedge n\}\, dX_s = \int_0^{\tau \wedge n} 1\{U_s \leq t\}\, dX_s,$$

which shows that $(XU^{-1})_t = (XV^{-1})_t$ a.s. on the set $\{\tau = \infty, \zeta > t\}$. It remains to notice that a.s.

$$\{\tau = \infty\} = \{\lambda\{s\colon U_s \neq V_s, U_s \wedge V_s \leq t\} = 0\} \supset \{\zeta > t\}. \qquad \square$$

We shall also need an extension of Lemma 5.2 to deal with convergence of our specific stochastic integrals on events of the form $\{\zeta > t\}$. Note that the result is trivial when ζ is $\mathscr{F}_n$-measurable for some n. Write $\mathscr{B}_n$ for the Borel σ-field on the interval (n, ∞).

LEMMA 5.8. *Let $I = R_+$ and assume $A_n \in \mathscr{F}_n \times \mathscr{B}_n$, $n \in N$, and $F \in \mathscr{F}_\infty$ to be such that*

$$1_F \lambda A_n \to_P 0.$$

Then

$$1_F \int 1_{A_n} dX \to_P 0.$$

PROOF. Since $1_F \lambda A_n \to_P 0$, there exist some constants $\varepsilon_n \downarrow 0$, such that $P[\lambda A_n > \varepsilon_n;\ F] \to 0$. The random sets

$$A'_n = A_n \cap (\{\lambda A_n \leq \varepsilon_n\} \times R_+), \qquad n \in N,$$

are then predictable with $\lambda A'_n \leq \varepsilon_n \to 0$, so $\int 1_{A'_n} dX \to_P 0$ by Lemma 5.2. Since, moreover,

$$\int 1_{A'_n} dX = \int 1_{A_n} dX \quad \text{a.s. on } \{\lambda A_n \leq \varepsilon_n\},$$

we get for any $\varepsilon > 0$,

$$P\left[\left|\int 1_{A_n} dX\right| > \varepsilon;\ F\right] \leq P\left\{\left|\int 1_{A'_n} dX\right| > \varepsilon\right\} + P[\lambda A_n > \varepsilon_n;\ F] \to 0,$$

as desired. $\square$

We shall finally need an elementary result on weak convergence in the function space $D(R_+)$ with the Skorohod–Stone topology (cf. [7]). Recall that k_t denotes killing at t. The coffin state ∂ is regarded as isolated in $R \cup \{\partial\}$.

LEMMA 5.9. *Let $X, X_1, X_2, \ldots$ and $\tau, \tau_1, \tau_2, \ldots$ be random elements in $D(R_+)$ and R_+, respectively, and assume that $(X_n, \tau_n) \to_d (X, \tau)$ with respect to the Skorohod–Stone topology on $D(R_+)$. Then*

$$k_{p\tau_n} \circ X_n \to_d k_{p\tau} \circ X \tag{5.7}$$

for $p \in [0, 1]$ a.e. λ.

PROOF. It is easy to check that the mapping $(x, t) \to k_t x$ from $D(R_+) \times \overline{R}_+$ to $D(R_+, R \cup \{\partial\})$ is continuous at $t = \infty$, and also at every (x, t) with $t < \infty$ and such that x is continuous at t. Thus (5.7) is true for every $p \in [0, 1]$ such that X is continuous at $p\tau$ a.s. on $\{\tau < \infty\}$. But conditionally on that event, the process $Y_p = X_{p\tau}$, $p \in [0,1]$, has paths in $D[0,1]$, so Y has an at most countable set of fixed discontinuities (cf. [2]). It remains to notice that X is continuous at $p\tau$ iff Y is continuous at p. □

PROOF OF THEOREM 5.4. Let us first assume that $I = [0,1]$. By Lemmas 5.5 and 5.6, there exists a predictable process V' with $\lambda V'^{-1} = \lambda$ a.s. and such that $V_s' = V_s$ a.e. $\lambda \times P$ on the set $\{V_s \wedge V_s' < \zeta\}$. Putting $Y' = XV'^{-1}$, it is seen from Lemma 5.7 that Y and Y' represent the same process on $[0, \zeta)$. Since, moreover, $Y' =_d X$ by Theorem 5.1, it follows that $Y \subset_d X$.

Let us turn to the case when $I = R_+$. By Lemmas 5.5 and 5.7, we may assume that $\lambda V^{-1} \le \lambda$ a.s. on R_+. For every $n \in N$, we define a predictable process U_n by

$$U_n(t) = \begin{cases} V_t, & t \vee V_t \le n, \\ \infty, & t \le n < V_t, \\ \inf\{s \ge 0\colon s - \lambda\{r \le n\colon V_r \le s \wedge n\} = t - n\}, & t > n. \end{cases}$$

Since clearly $\lambda U_n^{-1} = \lambda$ a.s., we have $Y_n = XU_n^{-1} =_d X$ by Theorem 5.1. It follows in particular that the sequence of pairs (Y_n, ζ) is tight in $D(R_+) \times \overline{R}_+$, so $(Y_n, \zeta) \to_d (X', \zeta')$ along some subsequence $N' \subset N$, for some process $X' =_d X$ and some random variable $\zeta' =_d \zeta$. By Lemma 5.9 it follows that, along N',

$$k_{p\zeta} \circ Y_n \to_d k_{p\zeta'} \circ X', \qquad p \in [0,1] \text{ a.e. } \lambda. \tag{5.8}$$

On the other hand, we have for fixed $t \ge 0$

$$\lambda\{s > n\colon U_n(s) \le t\} = \lambda\{s > n\colon V_s \le t\} \to 0 \quad \text{a.s. on } \{t < \zeta\},$$

so Lemmas 5.2 and 5.8 yield

$$1\{t < \zeta\}(Y(t) - Y_n(t)) = \int_n^\infty (1\{V_s \le t\} - 1\{U_n(s) \le t\})\, dX_s \to_P 0,$$

which shows that

$$(k_{p\zeta} \circ Y_n)_t \to_P (k_{p\zeta} \circ Y)_t, \qquad p \in [0,1], \quad t \in R_+.$$

Comparing this with (5.8) yields

$$k_{p\zeta} \circ Y =_d k_{p\zeta'} \circ X', \qquad p \in [0,1] \text{ a.e. } \lambda,$$

so the same relation must be true for $p = 1$. But then $Y \subset_d X' =_d X$ by Lemma 1.1. □

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Note added in proof. The ergodic theorem is actually never needed for the proof of the de Finetti–Ryll-Nardzewski theorem outlined in Section 2, since

Cauchy convergence in L_2 of the time averages follows easily from the covariance structure of a spreadable sequence. The method also yields bounds for the deviation from exchangeability of a finite spreadable sequence.

Some of the martingales considered in Section 4 have been used before in the special case of empirical processes [cf. Shorack, G. R. and Wellner, J. A. (1986), *Empirical Processes with Applications to Statistics*, Wiley, New York].

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