The Annals of Probability
1988, Vol. 16, No. 3, 1306–1332

REGENERATIVE SYSTEMS ON THE REAL LINE

BY H. KASPI AND B. MAISONNEUVE

Technion-Israel Institute of Technology and IMSS

We consider regenerative systems on the real line and study their structure. Local times and exit systems are defined. This leads to time changes and last exit decompositions. We establish a correspondence between stationary regenerative systems and Markov additive processes which are stationary in the space component.

1. Introduction. The theory of regenerative systems on the real line unifies the notions of strong Markov processes indexed by $\mathbb{R}$, renewal process on the real line, regenerative processes, semi-Markov processes, etc. A regenerative system, as defined in this paper, consists of a closed random set $M \subset \mathbb{R}$ and a process $(X_t)_{t\in\mathbb{R}}$ such that, with $D_t = \inf\{s > t: s \in M\}$, $((M - D_t) \cap (0, \infty), (X_{D_t+s})_{s\geq 0})$ and $(M \cap (-\infty, D_t], (X_{s\wedge D_t})_{s\in\mathbb{R}})$ are conditionally independent on $\{D_t < \infty\}$ given X_{D_t}, for each $t \in \mathbb{R}$. If the state space of X is reduced to one point, this definition reduces to the definition of a regenerative set on the real line given in [20].

This paper is devoted to a study of the structure of (M, X). We shall define a local time for M and an exit system which provides the structure of the excursions of X outside of M. In the stationary case this will lead to a last exit decomposition of the invariant measure of the incursion process. We also establish a correspondence between stationary regenerative systems and invariant measures of the process X time changed by using the local time of M (after some modifications of the jumps of the local time to the effect that the time changed process be Markov; similar procedures were already used in [1] and [5] in the context of Markov processes). In particular we extend the formulas for the stationary distributions of the backward and forward recurrence times which appear in the classical renewal theory and in the context of regenerative sets [3, 20 and 24]. Finally, we give a general result for constructing stationary regenerative systems via Markov additive processes (MAP)—these results rely on representations of invariant measures obtained in [11, 12] for Markov processes and are somewhat connected with [8].

2. General definitions and preliminary results. We first define a canonical setting for regenerative systems. Consider the set Ω^0 of all closed subsets of $\mathbb{R}$, the set Ω^1 of all functions from $\mathbb{R}$ into a Lusinian space E and the set Ω of all $(\omega^0, \omega^1) \in \Omega^0 \times \Omega^1$ such that the restrictions of ω^1 to ω^0_r and to $\mathbb{R} \setminus \omega^0_r$ are right continuous, where ω^0_r denotes the set of right accumulation points of ω^0. The space E is equipped with its Borel field $\mathscr{E}$. The projections $(\omega^0, \omega^1) \to \omega^0$ and

Received October 1986.

AMS 1980 *subject classifications*. 60K05, 60J55, 60J25, 60J50, 60G17.

Key words and phrases. Regenerative systems, local times, exit systems, time changes, last exit decompositions, Markov additive processes, stationarity.

$(\omega^0, \omega^1) \to \omega^1$ are denoted by M and X, respectively, so that (M, X) is just the canonical map on Ω. For $t \in \mathbb{R}$ we set (with the convention $\inf \varnothing = +\infty$)

$$D_t = \inf\{s > t: s \in M\}, \qquad R_t = D_t - t, \qquad X_t = X(t),$$

$$\mathscr{H}_t^0 = \sigma\{D_{s \wedge t}, X_{s \wedge D_t}, s \in \mathbb{R}\}, \qquad \mathscr{H}^0 = \sigma\{D_t, X_t, t \in \mathbb{R}\},$$

$$\varphi_t = \big(((M - t) \cap (0, \infty))^-, X_{t+s^+}, s \in \mathbb{R}\big), \qquad \varphi(t) = \varphi_t,$$

where A^- denotes the closure of $A \subset \mathbb{R}$ and $s^+ = s \vee 0$. The Ω valued random variable φ_t represents the future of t for our system (M, X). More explicitly,

$$\varphi_t(\omega^0, \omega^1) = \big(\{s - t: s \in \omega^0, s > t\}^-, s \to \omega^1(t + s^+)\big).$$

Note that the set M is $(\mathscr{H}_t^0)$ predictable since $M = \{t: R_{t-} = 0\}$ and that the processes (D_t), (R_t), $(X_{D_t}, t < \sup M)$ are right continuous and $(\mathscr{H}_t^0)$ adapted. The process (X_t) is $(\mathscr{H}_t^0)$ optional, since for every continuous real function f on E one has

$$f(X_t) = f(X_{D_t}) I_{\{R_t = 0\}} + Z_t \quad \text{with} \quad Z_t = \sum_{s \in \mathbb{R}} f(X_t) I_{\{R_{s-} = 0,\, R_s > 0,\, s \le t < D_s\}}.$$

[Z is a limit of right continuous and $(\mathscr{H}_t^0)$ adapted processes.] The process (φ_t), with values in $(\Omega, \mathscr{H}^0)$, is measurable and satisfies $\varphi_{t+s} = \varphi_s \circ \varphi_t$, $t \in \mathbb{R}$, $s \in \mathbb{R}_+$.

We now introduce a Borel subset E^0 of E, equipped with its Borel field $\mathscr{E}^0 = E^0 \cap \mathscr{E}$ and restrict Ω (without changing the notation) to the set of all ω's such that $X_{D_t}(\omega) \in E^0$ for $t < \sup M(\omega)$. We consider a measurable family $P^{\cdot} = (P^x)_{x \in E^0}$ of probability measures on $(\Omega, \mathscr{H}^0)$ such that

$$\text{(2.1)} \qquad \varphi_0(P^x) = P^x, \qquad P^x(X_0 = x) = 1 \qquad \forall x \in E^0.$$

The previous measurability assumption is in most examples too restrictive (only universal measurability of $P^{\cdot}$ should be required), but it is made for simplicity. We shall state in the sequel the conditions under which it may be relaxed.

(2.2) DEFINITIONS. (1) Given a stochastic basis $(\overline{\Omega}, \overline{\mathscr{H}}, (\overline{\mathscr{H}}_t)_{t \in \mathbb{R}}, \overline{P})$ and a random variable $(\overline{M}, \overline{X})$ from $(\overline{\Omega}, \overline{\mathscr{H}})$ into $(\Omega, \mathscr{H}^0)$, the system $(\overline{\Omega}, \overline{\mathscr{H}}, (\overline{\mathscr{H}}_t)_{t \in \mathbb{R}}, \overline{M}, \overline{X}, \overline{P})$ is called regenerative with regeneration laws $(P^x)_{x \in E^0}$ provided [for every function Z on Ω we set $\overline{Z} = Z \circ (\overline{M}, \overline{X})$]: (i) $(\overline{M}, \overline{X})$ is $\overline{\mathscr{H}}_t | \mathscr{H}_t^0$ measurable for each $t \in \mathbb{R}$ [equivalently $\overline{D}_t$ and $\overline{X(s \wedge D_t)}$ are $\overline{\mathscr{H}}_t$ measurable for all $t \in \mathbb{R}$, $s \in \mathbb{R}$] and (ii) for every $t \in \mathbb{R}$,

$$\text{(2.3)} \qquad \overline{P}_{\overline{\varphi(D_t)} | \overline{\mathscr{H}}_t} = P^{\overline{X(D_t)}} \quad \text{on } \{\overline{D}_t < \infty\}.$$

(2) Let $\mathscr{R}(P^{\cdot})$ be the family of all probability measures P on $(\Omega, \mathscr{H}^0)$ such that $(\Omega, \mathscr{H}^0, \mathscr{H}_t^0, M, X, P)$ is regenerative relative to $(P^x)_{x \in E^0}$. For $P \in \mathscr{R}(P^{\cdot})$ we shall denote by $(\mathscr{H}_t^P)$ the P completion of the filtration $(\mathscr{H}_t^0)$ ($\mathscr{H}_t^P$ includes all P null sets).

(3) The family $(P^x)_{x \in E^0}$ is called regenerative provided $P^x \in \mathscr{R}(P^{\cdot})$ for each $x \in E^0$. The system $(\Omega, \mathscr{H}, \mathscr{H}_t, M, X, \varphi_t, \mathscr{R}(P^{\cdot}))$, with $\mathscr{H}_t = \bigcap\{\mathscr{H}_t^P$: $P \in$

$\mathscr{R}(P^{\cdot})\}$, $\mathscr{H} = \bigvee_t \mathscr{H}_t$, is then called the canonical regenerative system associated with $P^{\cdot}$. In this context a.s. means P a.s. for each $P \in \mathbb{R}(P^{\cdot})$.

It is convenient to set $X_\infty = \delta$, $\varphi_\infty = \Delta$ with $\delta \notin E$, $\Delta \notin \Omega$. We set $E_\delta = E \cup \{\delta\}$, $\mathscr{E}_\delta = \mathscr{E} \vee \{\delta\}$ and define similarly $(E_\delta^0, \mathscr{E}_\delta^0)$, $(\Omega_\Delta, \mathscr{H}_\Delta^0)$. For $x \in E^0$ the measure P^x extends to $\mathscr{H}_\Delta^0$ so that $P^x(\{\Delta\}) = 0$ and P^δ is defined as the Dirac measure ε_Δ on $\mathscr{H}_\Delta^0$. With these notations the process (X_{D_t}) is E_δ^0 valued and in (2.3) the restriction "on $\{\overline{D}_t < \infty\}$" can be dropped.

It is important to realize that $\mathscr{H}_t$ represents the (possibly enlarged) past of the system $(\overline{M}, \overline{X})$ at time $\overline{D}_t$. The consideration of the past at time t arises in the following definitions, where $p.$ stands for progressively.

(2.4) DEFINITION. Given a stochastic basis $(\overline{\Omega}, \overline{\mathscr{H}}, (\overline{\mathscr{F}}_t)_{t\in\mathbb{R}}, \overline{P})$ with the usual conditions [right continuity and completeness of $(\overline{\mathscr{F}}_t)$] and a random variable $(\overline{M}, \overline{X})$ from $(\overline{\Omega}, \overline{\mathscr{H}})$ into $(\Omega, \mathscr{H}^0)$, the system $(\overline{\Omega}, \overline{\mathscr{H}}, \overline{\mathscr{F}}_t, \overline{M}, \overline{X}, \overline{P})$ is called a $p.$ regenerative system, with regeneration laws $(P^x)_{x\in E^0}$ provided (i) the random set $\overline{M}$ and the process $\overline{X}$ are $(\overline{\mathscr{F}}_t)$ progressive and (ii) $(\overline{\Omega}, \overline{\mathscr{H}}, \overline{\mathscr{F}}_{\overline{D}_t}, \overline{M}, \overline{X}, \overline{P})$ is regenerative with regeneration laws $(P^x)_{x\in E^0}$.

(2.5) DEFINITION. Let $(\Omega, \mathscr{H}, \mathscr{H}_t, M, X, \varphi_t, \mathscr{R}(P^{\cdot}))$ be a canonical regenerative system. We denote by $(\mathscr{F}_t^0)$ the natural filtration of the process (G_t, X_t), where

$$(2.6) \qquad G_t = \sup M \cap (-\infty, t], \qquad t \in \mathbb{R} \quad (\sup \varnothing = -\infty),$$

and for $P \in \mathscr{R}(P^{\cdot})$ we denote by $(\mathscr{F}_t^P)$ the P completion of the filtration $(\mathscr{F}_{t+}^0)$. We shall say that $(\Omega, \mathscr{H}, \mathscr{F}_t, M, X, \varphi_t, \mathscr{R}(P^{\cdot}))$, with $\mathscr{F}_t = \bigcap\{\mathscr{F}_t^P: P \in \mathscr{R}(P^{\cdot})\}$ is a canonical $p.$ regenerative system provided $(\Omega, \mathscr{H}^P, \mathscr{F}_t^P, M, X, P)$ is a $p.$ regenerative system relative to (P^x) for each $P \in \mathscr{R}(P^{\cdot})$.

Note that (G_t) is a right continuous $(\mathscr{F}_t^0)$ adapted process, so that $M = \{t: G_t = t\}$ is $(\mathscr{F}_t^0)$ optional. Hence the requirement of Definition 2.5 is equivalent to $(\mathscr{F}_t^P)$ progressiveness of (X_t) together with

$$(2.7) \qquad P_{\varphi(D_t)|\mathscr{F}_{D_t}} = P^{X(D_t)}, \qquad t \in \mathbb{R},$$

for all $P \in \mathscr{R}(P^{\cdot})$.

(2.8) PROPOSITION. *Let $(\Omega, \mathscr{H}, \mathscr{F}_t, M, X, \varphi_t, \mathscr{R}(P^{\cdot}))$ be a canonical $p.$ regenerative system. Then*

$$(2.9) \qquad \mathscr{F}_{D_t}^P = \mathscr{H}_t^P, \qquad \mathscr{F}_{D_t} = \mathscr{H}_t \quad \textit{for all } P \in \mathscr{R}(P^{\cdot}),\ t \in \mathbb{R}.$$

PROOF. Let $P \in \mathscr{R}(P^{\cdot})$. In order to prove (2.9) it suffices to check that for $H \in b\mathscr{H}^P$,

$$P\big(H|\mathscr{F}_{D_t}^P\big) = P\big(H|\mathscr{H}_t^P\big).$$

But for $H = H_t \cdot f \circ \varphi_{D_t}$, with $H_t \in b\mathscr{H}_t^0$, $f \in b\mathscr{H}^0$, both sides are equal to

$H_t P^{X(D_t)}(f)$ since $\mathscr{H}_t^0 \subset \mathscr{F}_{D_t}^P$. The general case follows by monotone class and completion arguments. □

(2.10) REMARK. If $(\bar{\Omega}, \bar{\mathscr{H}}, \bar{\mathscr{F}}_t, \bar{M}, \bar{X}, \bar{P})$ is a p. regenerative system as defined in Definition 2.4 and if $\bar{\mathscr{H}}$ is generated by $(\bar{M}, \bar{X})$ up to completion, the same argument as before shows that $\bar{\mathscr{F}}_{\bar{D}_t} = \bar{\mathscr{H}}_t$, where $\bar{\mathscr{H}}_t$ denotes the $\bar{P}$ completion of $(\bar{M}, \bar{X})^{-1}(\mathscr{H}_t^0)$.

Here are three classical examples of regenerative systems.

(2.11) EXAMPLE. If $(\bar{\Omega}, \bar{\mathscr{H}}, \bar{\mathscr{H}}_t, \bar{M}, \bar{X}, \bar{P})$ is regenerative with respect to $(P^x)_{x \in E^0}$ and if E^0 is reduced to one point x_0, then $\bar{M}$ is a regenerative set relative to $(\bar{\mathscr{H}}_t)$ in the sense of [20] [provided the definition of τ_t in this paper is replaced by $\tau_t = ((M^0 - t) \cap (0, \infty))^-]$ and on $\{\bar{D}_t < \infty\}$, $\varphi_{\bar{D}_t}$ is independent of $\bar{\mathscr{H}}_t$ with a fixed distribution P^{x_0}. If further $\bar{M}$ is $\bar{P}$ a.s. discrete, then $\bar{X}$ is a regenerative process in the sense of [4].

(2.12) EXAMPLE. Let $(\bar{\Omega}, \bar{\mathscr{H}}, \bar{\mathscr{F}}_s, \bar{X}_s, \bar{\theta}_s, \bar{P}^x;\ s \in \mathbb{R}_+, x \in E)$ be a right process with a Borel semigroup and $\bar{M}$ a homogeneous optional closed random subset of $\mathbb{R}_+$. Then $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathscr{F}}_t, \bar{M}, \bar{X}, \bar{P}^x)$ is a p. regenerative system for each $x \in E$ (we set $\bar{\mathscr{F}}_t = \bar{\mathscr{F}}_0$ and $\bar{X}_t = \bar{X}_0$ for $t < 0$ and $\bar{X} = \bar{X}_.$) with regeneration laws $P^x = ((\bar{M} \setminus \{0\})^-, \bar{X})(\bar{P}^x)$, $x \in E$, $E^0 = E$. Here the strict measurability assumptions are not met, namely $(P^x)_{x \in E}$ is only universally measurable. However the assumptions of Remarks 3.11 of the next section are satisfied and therefore all our results will apply to this example. Note that the family (P^x) can be restricted to $x \in E^0$, for some $E^0 \in \mathscr{E}$ provided $\bar{X}_{\bar{D}_t} \in E^0$ for $t < \sup \bar{M}$ a.s.

(2.13) EXAMPLE. Let $(\bar{\Omega}, \bar{\mathscr{M}}, \bar{\mathscr{M}}_t, \bar{S}_t, \bar{E}_t, \bar{\sigma}_t, \bar{P}^x;\ t \in \mathbb{R}_+, x \in E)$ be a Markov additive process in the sense of Çinlar [2], with $\bar{S}$ nondecreasing additive and continuous or strictly increasing and additive. Let $\bar{C}_t = \inf\{s \in \mathbb{R}_+: \bar{S}_s > t\}$, $t \in \mathbb{R}$ and $\bar{X}_t = \bar{E}_{\bar{C}_t}$, $\bar{X} = \bar{X}_.$. Then $(\bar{\Omega}, \bar{\mathscr{M}}, \bar{\mathscr{M}}_{\bar{C}_t}, \bar{M}, \bar{X}, \bar{P}^x)$ is regenerative for each $x \in E$ with regeneration laws $(P^x)_{x \in E}$ defined as in Example 2.12. This result will be generalized in Section 6.

In these three examples the regenerative systems are *strong* in the sense that the filtration $(\bar{\mathscr{H}}_t)$ is right continuous and that

$$\bar{P}_{\varphi_S | \bar{\mathscr{H}}_T} = P^{\bar{X}_S} \quad \text{with } S = \bar{D}_T \tag{2.14}$$

for every $(\bar{\mathscr{H}}_t)$ stopping time T (all our stopping times are $(-\infty, +\infty]$ valued). We end this section with a useful left regeneration property satisfied by strong regenerative systems. Let D be the set of all right end points in $\mathbb{R}$ of M contiguous intervals. Note that D is $(\mathscr{H}_t^0)$ predictable since

$$D = \bigcup_{n \geq 1} \left\{ t \in \mathbb{R}: R_{t-} = 0,\ R_{(t-1/n)^-} = \frac{1}{n} \right\}.$$

(2.15) THEOREM. *Let* $(\overline{\Omega}, \mathscr{H}, \overline{\mathscr{H}}_t, \overline{M}, \overline{X}, \overline{P})$ *be a strong regenerative system relative to* $(P^x)_{x\in E^0}$ *and let* T *be an* $(\overline{\mathscr{H}}_t)$ *predictable time in* $\overline{D}$ $(\cup\{+\infty\})$. *Then* $\overline{X}_T$ *is* $\overline{\mathscr{H}}_{T-}$ *measurable and*

$$\overline{P}_{\overline{\varphi}_T|\overline{\mathscr{H}}_{T-}} = P^{\overline{X}_T}. \tag{2.16}$$

PROOF. For simplicity we drop the bars from the notations $\overline{\mathscr{H}}_t$, $\overline{X}_t$, $\overline{\varphi}_t$, $\overline{P}$. If the time T is announced by a sequence (T_n), one has $D_{T_n} = T$ for large n on $\{T < \infty\}$. Therefore X_T is $\bigvee_n \mathscr{H}_{T_n} = \mathscr{H}_{T-}$ measurable and letting $n \to \infty$ in the equality

$$P\big(f\big(\varphi(D_{T_n})\big)|\mathscr{H}_{T_n}\big) = P^{X(D_{T_n})}(f), \qquad f \in b\mathscr{H}^0,$$

yields (2.16). □

(2.17) REMARK. If $(\overline{\Omega}, \mathscr{H}, \overline{\mathscr{F}}_t, \overline{M}, \overline{X}, \overline{P})$ is a p. regenerative system relative to (P^x) and if we set $\overline{\mathscr{H}}_t = \overline{\mathscr{F}}_{\overline{D}_t}$, it is equivalent to require (2.14) for every $(\overline{\mathscr{H}}_t)$ stopping time T or to require that

$$\overline{P}_{\overline{\varphi}_S|\overline{\mathscr{F}}_S} = P^{\overline{X}_S} \tag{2.18}$$

for every $(\overline{\mathscr{F}}_t)$ stopping time S in $\underline{M}$, where $\underline{M}$ denotes the minimal right closed set with closure $\overline{M}$ (see [17] and note that $\overline{\mathscr{H}}_T = \overline{\mathscr{F}}_{\overline{D}_T}$ for every $(\overline{\mathscr{H}}_t)$ stopping time T by (2.10) and [7] or [15]). Note also that every $(\overline{\mathscr{H}}_t)$ predictable time T in $\overline{D}$ is also an $(\overline{\mathscr{F}}_t)$ stopping time and that $\overline{\mathscr{H}}_{T-} = \overline{\mathscr{F}}_T$, so that (2.16) writes $\overline{P}_{\overline{\varphi}_T|\overline{\mathscr{F}}_T} = P^{\overline{X}_T}$ in this case. A p. regenerative system $(\overline{M}, \overline{X})$ which satisfies (2.18) as indicated previously will be called a *strong p. regenerative system*.

(2.19) COROLLARY. *In the conditions of Theorem* 2.15 *and if the filtration* $(\overline{\mathscr{H}}_t)$ *is complete, the set* $\bar{I}$ *of isolated points of* $\overline{M}$ *is* $(\overline{\mathscr{H}}_t)$ *predictable and included a.s. in* $\{t: \overline{X}_t \in E^0 \setminus F\}$, *where* $F = \{P^{\cdot}(R = 0) = 1\}$ *with* $R = D_0$.

PROOF (without bars). The set D is predictable and contained in $\{D_r, r \text{ rational}\}$. By [6, Chapter 4, Theorem 17] there exists a sequence (T_n) of predictable times with disjoint graphs such that $D = \bigcup_n [\![T_n]\!]$. By (2.16) applied at T_n one has

$$P\big(R(\varphi(T_n)) > 0|\mathscr{H}_{T_n-}\big) = P^{X(T_n)}(R > 0) = 0 \quad \text{on } \{X(T_n) \in F\}.$$

Since $X(T_n)$ is $\mathscr{H}_{T_n-}$ measurable, this shows that $R(\varphi(T_n)) = 0$ a.s. on $\{X(T_n) \in F\}$. Similarly using the fact that $P^{\cdot}(R = 0) = 0$ on $E^0 \setminus F$ (by Proposition 1.4 of [16]), one shows that $R(\varphi(T_n)) > 0$ a.s. on $\{X(T_n) \in E^0 \setminus F\}$. Hence $I = \bigcup_n [\![T_n']\!]$ a.s. P, where $T_n' = T_n$ on $\{X(T_n) \notin F\}$, $+\infty$ elsewhere, and I is (indistinguishable from) a predictable set. □

3. The incursion process.

(3.1) DEFINITION. Let $R = D_0$. For $t \in \mathbb{R}$ we set

$$R_t = D_t - t = R \circ \varphi_t, \qquad i_t = a_R \circ \varphi_t,$$

where $a_r(\omega^0, \omega^1) = \{\omega^1(s \wedge r),\ s \in \mathbb{R}_+\}$ for $r \in \mathbb{R}_+$.

The process $((R_t, i_t),\ t \in \mathbb{R})$ is called the incursion process and is denoted by (Y_t).

We note that the processes (R_t), (i_t) and hence (Y_t) are homogeneous. For (Y_t) this means that $Y_{t+s} = Y_s \circ \varphi_t$ for $t \in \mathbb{R}$, $s \in \mathbb{R}_+$. Note also that (Y_t) is $\mathbb{R}_+ \times \tilde{\Omega}$ valued, where $\tilde{\Omega}$ denotes the set of all right continuous functions from $\mathbb{R}_+$ into E, and that (Y_t) is right continuous provided $\tilde{\Omega}$ is equipped with the metric

$$d(i, i') = \int_{\mathbb{R}_+} e^{-t}\delta(i(t), i'(t))\, dt, \qquad i, i' \in \tilde{\Omega}, \tag{3.2}$$

where δ is a bounded metric on E, compatible with its topology (see [21] for details concerning this metric).

We now consider a regenerative family $(P^x)_{x \in E^0}$ as defined in Definition 2.2(2). The state space of (Y_t) can be restricted to the set

$$U = \{(r, i) \in \mathbb{R}_+ \times \tilde{\Omega} \colon i(r) \in E^0,\ i(\cdot \wedge r) = i\} \cup \{(+\infty, i) \colon i \in \tilde{\Omega}\}.$$

For $(r, i) \in U$ let $P^{r,i}$ be the probability measure on $(\Omega, \mathscr{H}^0)$ defined by

$$P^{r,i} = \begin{cases} \varepsilon_{(\phi, i)}, & \text{if } r = \infty, \\ \psi_{r,i}(P^{i(r)}), & \text{if } r < +\infty, \end{cases} \tag{3.3}$$

where for $u \in U$, ε_u is the Dirac measure concentrated at u and

$$\psi_{r,i} \colon (\omega^0, \omega^1) \to (\{r\} \cup (\omega^0 + r), i/r/\omega^1). \tag{3.4}$$

As usual $i/r/\omega^1$ is the element of Ω^1 which agrees with i [extended by setting $i(t) = i(0)$ for $t < 0$] on $(-\infty, r)$ and with $\omega^1(\cdot - r)$ on $[r, \infty)$. Note that $P^x = P^{0,i}$ whenever $i(0) = x$. The same argument that led to Theorems 2.3 and 2.5 of [20] proves

(3.5) **Theorem.** (1) *$(\Omega, \mathscr{H}^0, \mathscr{H}_t^0, Y_t, \varphi_t, P^y;\ t \in \mathbb{R},\ y \in U)$ is an a.s. U valued right continuous Markov process with shifts (φ_t). That is, for each $y \in U$ and $t \in \mathbb{R}$ one has*

$$P^y_{\varphi_t | \mathscr{H}_t^0} = P^{Y_t}.$$

(2) *A probability P on $(\Omega, \mathscr{H}^0)$ is in $\mathscr{R}(P^\cdot)$ if and only if $P_{\varphi_t | \mathscr{H}_t^0} = P^{Y_t}$ for all $t \in \mathbb{R}$. In particular $P^\mu \in \mathscr{R}(P^\cdot)$ for every probability measure μ on U.*

Note that, as a consequence of this result, a property which holds a.s. [in the sense of Definition 2.2(3)] holds a.s. P^μ for every probability μ on U. Note also that the semigroup $(P_s)_{s \in \mathbb{R}_+}$ of (Y_t), given by ($\mathscr{U}$ denotes the Borel field on U)

$$P_s(y, B) = P^y(Y_s \in B), \qquad y \in U,\ B \in \mathscr{U}, \tag{3.6}$$

admits branch points, namely the points $(0, i) \in U$ such that $i(0) \notin F$. As usual F is the set of regular points for M,

$$F = \{x \in E^0 \colon P^x(R = 0) = 1\}. \tag{3.7}$$

We shall be able to eliminate these branch points and obtain a right process under the following assumption.

(3.8) ASSUMPTION. For each $s > 0$ and each bounded continuous function f on $\overline{\mathbb{R}}_+ \times \tilde{\Omega}$ the mapping $t \mapsto P^{X(D_t)}(f \circ Y_s)$ is a.s. right continuous.

Under this assumption, using the arguments of (1.9) and (2.3) of [20], we can prove that for every $P \in \mathscr{R}(P^{\cdot})$ and for every stopping time T of $(\mathscr{H}_t)$ [as defined in Definition 2.2(3)], we have

$$P_{\varphi_S|\mathscr{H}_{T+}} = P^{X_S} \quad \text{with } S = D_T, \tag{3.9}$$

$$P_{\varphi_T|\mathscr{H}_{T+}} = P^{Y_T} \quad \text{on } \{T < \infty\}. \tag{3.10}$$

By a classical argument using (3.10) with $T = t$ we have $\mathscr{H}_{t+} = \mathscr{H}_t$ and $\mathscr{H}_{T+}$ can be replaced by $\mathscr{H}_T$ in (3.9) and (3.10). It follows from (3.9) that $X_{D_T} \in F$ a.s. on $\{D_T < \infty,\ R_T = 0\}$ and by the section theorem the set of branch points $\{(0, i) \in U: i(0) \notin F\}$ is polar for the process (Y_t) [under each $P \in \mathscr{R}(P^{\cdot})$].

As a consequence of this discussion, if we get rid of the branch points (without altering the notation U), $(\Omega, \mathscr{H}, \mathscr{H}_s, Y_s, \varphi_s;\ s \in \mathbb{R}_+,\ y \in U)$ becomes a strong Markov process which is a.s. U valued and has no branch points. Since the semigroup (P_s) is Borel, the process (Y_s) satisfies the *right* assumptions of Meyer.

(3.11) REMARKS. (1) If E^0 and $(P^x)_{x \in E^0}$ are only universally measurable, (P_s) is not Borel any more. But if we assume that F is nearly Borel for (X_{D_t}) [or only that $\{t: X_{D_t} \in F\}$ is $(\mathscr{H}_t^P)$ optional for each P], we can still get rid of the branch points. This condition holds in Example 2.12.

(2) Note that the condition [$s \to P_t f(Y_s)$ is a.s. right continuous for all bounded and continuous f], which one usually assumes to make Y a right process, is slightly different from Assumption 3.8. However, the expression of the resolvent of (P_s),

$$U^\alpha f(r, i) = \int_0^r e^{-\alpha s} f(r - s, \theta_s i)\, ds + e^{-\alpha r} \int_0^\infty e^{-\alpha s} P^{i(r)}(f(Y_s))\, ds,$$

where (θ_s) is the usual shift, and arguments used to prove Theorem 7.6 of [23] show that the two conditions are actually equivalent.

4. Local times and exit systems. Let $(\Omega, \mathscr{H}, \mathscr{H}_t, M, X, \varphi_t, \mathscr{R}(P^{\cdot}))$ be a fixed canonical regenerative system as defined in Definition 2.2(3). We assume that (P^x) is measurable and satisfies Assumption 3.8 or more generally that (P^x) is universally measurable and that (Y_t) is right. We shall say that the system is *right*.

Consider the random set

$$G = \{t \in \mathbb{R}: R_{t-} = 0,\ R_t > 0\}, \tag{4.1}$$

which consists of the left end points in $\mathbb{R}$ of the contiguous intervals of M. The set G is $(\mathscr{H}_t^0)$ optional and homogeneous,

$$(4.2) \qquad (G - t) \cap (0, +\infty) = G(\varphi_t) \cap (0, +\infty), \qquad t \in \mathbb{R}.$$

For every homogeneous random subset Γ of G we define the homogeneous random measures

$$(4.3) \qquad \Lambda^\Gamma = \sum_{t \in \Gamma} (1 - e^{-R_t}) \varepsilon_t,$$

$$(4.4) \qquad L^\Gamma = \sum_{t \in \Gamma} \lambda(X_t) \varepsilon_t,$$

where $\lambda(x) = P^x(1 - e^{-R})$ if $x \in E^0$, 0 otherwise.

Note that for every homogeneous random measure N on $\mathbb{R}$ such that $P(N(s, t]) < \infty$ for all $P \in \mathscr{R}(P^\cdot)$, $s, t \in \mathbb{R}$, there exists a homogeneous random measure ΠN which is the $(\mathscr{H}_t^P)$ dual *predictable* projection of N under each $P \in \mathscr{R}(P^\cdot)$. In fact $P \in \mathscr{R}(P^\cdot)$ if and only if P is a probability on $(\Omega, \mathscr{H}^0)$ under which $(Y_t)_{t \in \mathbb{R}}$ is Markov with semigroup (P_s) by Theorem 3.5(2), so that the existence of ΠN follows from a general result for Markov processes indexed by $\mathbb{R}_+$, which extends easily to Markov processes indexed by $\mathbb{R}$.

(4.5) Definition. The random measure $L = \Pi \Lambda$, where $\Lambda = 1_M(t)\, dt + \Lambda^G$, is called the local time of M.

Note that $\operatorname{supp} L = M$ a.s., since $\operatorname{supp} \Lambda = M$ and M is $(\mathscr{H}_t^0)$ predictable $(M = \{t: R_{t-} = 0\})$. Owing to Corollary 2.19 and (2.16), one has $\Pi \Lambda^I = L^I$ (I is the set of isolated points of M) and

$$(4.6) \qquad L = 1_M(t)\, dt + L^I + L', \quad \text{with } L' = \Pi \Lambda^{G \setminus I}.$$

We shall denote by J, J' the sets $\{t: L\{t\} > 0\}$, $\{t: L'\{t\} > 0\}$. One has $J = I \cup J'$ a.s. and clearly the diffuse and point parts of L, L' satisfy the relationships

$$(4.7) \qquad L^c = 1_M(t)\, dt + L'^c, \qquad L^d = L^I + L'^d.$$

The *exit system* $(l, {}_*P^\cdot, L)$ is introduced in the following result.

(4.8) Theorem. (1) *The measure L^c is a.s. carried by $\{t: R_t = 0, X_t \in F\}$. There exist a universally measurable positive function l on E^0, carried by F, and a universally measurable family $({}_*P^x)_{x \in F}$ of measures on $(\Omega, \mathscr{H}^0)$, carried by $\{R > 0\}$, such that* (i) $1_M(t)\, dt = l(X_t) L^c(dt)$, (ii) $P \sum_{s \in G \setminus J} Z_s f \circ \varphi_s = P \int Z_s\, {}_*P^{X_s}(f) L^c(ds)$ *for $P \in \mathscr{R}(P^\cdot)$, $f \in \mathscr{H}_+$ and Z positive and $(\mathscr{H}_t^P)$ predictable and* (iii) $l + {}_*P^\cdot(1 - e^{-R}) \equiv 1$ *on F.*

(2) *Suppose that the following left regeneration property holds: For every $x \in E^0$ and every $(\mathscr{H}_t)$ predictable time T in $(M \setminus D) \cup \{+\infty\}$ X_T is P^x a.s. E_δ^0 valued, $\mathscr{H}_{T-}^x \equiv \mathscr{H}_{T-}^{P^x}$ measurable and satisfies*

$$P^x_{\varphi_T | \mathscr{H}_{T-}^x} = P^{X_T}.$$

Then $J \subset \{s \in G \colon X_s \in E^0 \setminus F\}$ *a.s.,* $L^d = L^J$ *a.s. and*

$$1_M(t)\,dt = l(X_t)L(dt) \quad a.s., \tag{4.9}$$

$$P\sum_{s\in G} Z_s f\circ\varphi_s = P\int Z_s\,{}_*P^{X_s}(f)L(ds), \tag{4.10}$$

with ${}_*P^x(f) = P^x(f)/P^x(1-e^{-R})$ *if* $x \in E^0 \setminus F$ *and for all* P, Z, f *as before.*

PROOF. (1) The measure L^c is a.s. carried by $M = \{t\colon R_{t-} = 0\}$. Since $\{t\colon R_{t-} = 0, R_t > 0\}$ is countable, L^c is also carried a.s. by $\{t\colon R_t = 0\}$, which is a.s. equal to $\{t\colon R_t = 0, X_t \in F\}$ by the discussion following Assumption 3.8. By Motoo's theorem there exists a universally measurable function $\tilde{l}$ on U such that $1_M(t)\,dt = \tilde{l}(Y_t)L^c(dt)$. Since L^c is a.s. carried by $\{t\colon R_t = 0, X_t \in F\} = \{t\colon \tilde{Y}_t \in \tilde{F}\}$, where $\tilde{F} = \{(0,[x])\colon x \in F\}$ ($[x]$ is the constant map x), $\tilde{l}$ can be chosen so that $\tilde{l} = 0$ on $U \setminus \tilde{F}$. The desired function l is $l = \tilde{l}(0,[\cdot])$ on E^0. By the arguments of [17] or [18] we also establish the existence of a kernel N from F to Ω such that

$$\Pi \sum_{s\in G\setminus J} (1-e^{-R_s}) f\circ\varphi_s\varepsilon_s = N^{X_s}(f)L^c(ds), \qquad f \in \mathscr{H}_+^0.$$

If we modify N so that $l + N^{\cdot}(1) \equiv 1$ on F and set ${}_*P^{\cdot}(f) = N^{\cdot}(f/1 - e^{-R})$ for $f \in \mathscr{H}_+^0$, we get (ii) and (iii).

(2) Let us first argue under P^x, $x \in E^0$. Let (S_n) be a sequence of $(\mathscr{H}_t)$ predictable times in $M \setminus D$, with disjoint graphs such that $J' = \bigcup_n [\![S_n]\!]$ P^x a.s. [note that $L' = \Pi(\Lambda^{G\setminus D})$ is a.s. carried by $M \setminus D$]. By left regeneration at the S_n's one has $J' \cap G = J' \cap \{t\colon X_t \in E^0 \setminus F\}$ P^x a.s. and $L^{G\cap J'}$ is the $(\mathscr{H}_t)$ dual predictable projection of $\Lambda^{G\cap J'}$ under P^x. But $\Pi\Lambda^{G\cap J'} = L'^d$, and so $J' \subset G \cap J'$ P^x a.s., proving that $J' \subset \{s \in G\colon X_s \in E^0 \setminus F\}$ P^x a.s. and that $L^{J'}$ is the $(\mathscr{H}_t)$ dual predictable projection of $\Lambda^{J'}$ under P^x. Therefore $L^d = L^I + L^{J'} = L^J$ P^x a.s. and we have proved that $J \subset \{s \in G\colon X_s \in E^0 \setminus F\}$ and $L^d = L^J$ a.s. under P^x; the extension to $P \in \mathscr{R}(P^{\cdot})$ follows from $P_{\varphi(D_t)|\mathscr{H}_t} = P^{X(D_t)}$, $t \in \mathbb{R}$, by homogeneity. (4.9) is now immediate.

In order to prove (4.10) it suffices to consider f such that $0 \le f \le 1 - e^{-R}$. For such an f we already know that $\Pi\sum_{s\in G\setminus J} f\circ\varphi_s\varepsilon_s = {}_*P^{X_s}(f)L^c(ds)$, and so we only need to prove that $\Pi A = B$, where

$$A = \sum_{s\in J} f\circ\varphi_s\varepsilon_s, \qquad B = {}_*P^{X_s}(f)L^J(ds) = \sum_{s\in J} P^{X_s}(f)\varepsilon_s \text{ a.s.}$$

[Note that $P(A(s,t]) < \infty$ for $s, t \in \mathbb{R}$.] Let $x \in E^0$. By left regeneration on $J = I \cup J'$ the $(\mathscr{H}_t^x)$ dual predictable projection of A is B under P^x. From this, from the definition of $P^{r,i}$ and from the assumption $P^x = \varphi_0(P^x)$ it follows that for all $(r,i) \in U$ and $u \in \mathbb{R}_+$ [with the convention $(0,t] = \varnothing$ for $t \le 0$]

$$\begin{aligned} P^{r,i}(A(0,u]) &= P^{i(r)}\big(f\circ\varphi_0 I_{\{0<r\le u,\, i(r)\in E^0\setminus F\}} + A(0,u-r]\big) \\ &= P^{i(r)}\big(P^{X_0}(f)I_{\{0<r\le u,\, i(r)\in E^0\setminus F\}} + B(0,u-r]\big) \\ &= P^{r,i}(B(0,u]). \end{aligned}$$

By the Markov property of (Y_t) under P it follows that

$$P(A(t,t+u]|\mathscr{H}_t) = P(B(t,t+u]|\mathscr{H}_t),$$

proving that the $(\mathscr{H}_t^P)$ dual predictable projection of A is B under P. □

The following result does not depend on any left regeneration property. The filtration $(\mathscr{F}_t)$ was defined in Definition 2.5.

(4.11) PROPOSITION. *The local time L is $(\mathscr{F}_t)$ optional.*

PROOF. The measures L^I and $1_M(t)\,dt$ are $(\mathscr{F}_t)$ optional, since the sets I and M are $(\mathscr{F}_t^P)$ optional for each $P \in \mathscr{R}(P^\cdot)$. The $(\mathscr{F}_t)$ optionality of L follows from (4.6) and Lemma 4.12. □

(4.12) LEMMA. *The measure L' is $(\mathscr{F}_t)$ predictable.*

PROOF. Let $P \in \mathscr{R}(P^\cdot)$. The measure L' is $(\mathscr{F}_{D_t}^P)$ predictable, since $\mathscr{H}_t^P \subset \mathscr{F}_{D_t}^P$, and satisfies $L'(t, D_t] = 0$ a.s. for each $t \in \mathbb{R}$, since L' is carried by $M \setminus D$. By the argument of [22, page 229] it follows that L' is $(\mathscr{F}_t^P)$ predictable. □

(4.13) REMARKS. (1) Suppose that for each $P \in \mathscr{R}(P^\cdot)$, the P outer measure of the set $\{X_{t-}^D$ exists in E^0 for all $t \in M \setminus D\}$ is 1, with $X_t^D = X_{D_t}$, and let us restrict Ω to this set without changing the notation, so that on Ω the process (X_{t-}^D) is everywhere defined and E_δ^0 valued (the existence of X_{t-}^D is clear for $t \notin M \setminus D$). We also assume that for every $x \in E^0$ and every $(\mathscr{H}_t)$ predictable time T in $M \setminus D$ one has

$$P^x_{\varphi_T|\mathscr{H}_{T-}} = P^{X_{T-}^D} \quad \text{on } \{T < \infty\}. \tag{4.14}$$

Note that (4.14) implies the left regeneration property of (4.8), since (4.14) combined with the assumption $P^x(X_0 = x) = 1$ implies that $X_T = X_{T-}^D$ a.s. on $\{T < \infty\}$. Hence the results of Theorem 4.8 hold in this case. We also have an interesting characterization of the predictable part J and the totally inaccessible part $G \setminus J$ of G in this case. In fact let us prove that the equalities

$$\begin{aligned} G \setminus J = G^{r-} \quad &\text{with } G^{r-} = \{t \in G\colon X_{t-}^D \in F\},\\ J = G^{i-} \quad &\text{with } G^{i-} = \{t \in G\colon X_{t-}^D \in E^0 \setminus F\} \end{aligned} \tag{4.15}$$

hold a.s. P^x for each x (and hence a.s.). The set G^{r-} is totally inaccessible under P^x, since for all $(\mathscr{H}_t)$ predictable time T one has by Theorem 2.15 and (4.14),

$$\begin{aligned} P^x(T \in G^{r-}) &= P^x\big(R \circ \varphi_T > 0,\, T \in M,\, X_{T-}^D \in F\big)\\ &= P^x\big(P^{X_{T-}^D}(R > 0);\, T \in M,\, X_{T-}^D \in F\big) = 0. \end{aligned}$$

Let us now check that G^{i-} is P^x a.s. equal to the predictable set $\Gamma = \{t \in \mathbb{R}\colon R_{t-} = 0,\, X_{t-}^D \notin F\}$. The set Γ is a.s. contained in the set of jump times of the process (R_t, X_t^D), since $\{0\} \times F^c$ is polar for (R_t, X_t^D). By [6 Chapter 4, Theorem 17], Γ can be written $\bigcup_n [\![T_n]\!]$, where (T_n) is a sequence of $(\mathscr{H}_t^x)$ predictable times with disjoint graphs. By (4.14) we have $\Gamma \subset G$ and $G^{i-} = \Gamma$ P^x a.s. This remark and the results of Theorem 4.8 and Proposition 4.11 provide a full generalization of the results obtained in [14] for Hunt processes.

(2) Without assuming any left regeneration on $M \setminus D$, the totally inaccessible part $G \setminus J$ of G equals $\{t \in G: Y_t^- \in U,\ Y_t^- \neq Y_t\}$ a.s., where Y^- is the left limit process in a Ray–Knight compactification of U. It follows from (13.7) of [9] that for $t \in G \setminus J$, Y_{t-} exists in U and is equal to Y_t^-. Hence X_{t-}^D exists, $X_{t-}^D \in F$ and $Y_{t-} = (0, [X_{t-}^D])$ for $t \in G \setminus J$. Therefore one has a.s.

$$(4.16)\qquad G \setminus J = \left\{t \in G: X_{t-}^D \text{ exists, } X_{t-}^D \in F,\ Y_t^- = \left(0, \left[X_{t-}^D\right]\right)\right\}.$$

We next turn our attention to $p.$ regenerative systems, for which we introduce the following notion of *regularity.*

(4.17) DEFINITION. A canonical right $p.$ regenerative system $(\Omega, \mathscr{H}, \mathscr{F}_t, M, X, \varphi_t, \mathscr{R}(P^\cdot))$ is called regular provided for every $x \in E^0$ and every $(\mathscr{F}_t)$ predictable time S in $M \setminus D$ X_S is P^x a.s. E_δ^0 valued and

$$(4.18)\qquad P^x_{\varphi_S | \mathscr{F}_S^x} = P^{X_S},$$

$$(4.19)\qquad \mathscr{F}_{S-}^x = \mathscr{F}_S^x \quad \left(\text{with } \mathscr{F}_t^x = \mathscr{F}_t^{P^x}\right).$$

(4.20) THEOREM. *If $(\Omega, \mathscr{H}, \mathscr{F}_t, M, X, \varphi_t, \mathscr{R}(P^\cdot))$ is a regular $p.$ regenerative system, then*

(a) *the regeneration property* (4.18) *holds at every $(\mathscr{F}_t)$ stopping time S in M_0, where $M_0 = \underline{M} \cup J$ $[\underline{M} = I \cup (M \setminus G)$ is the minimal right closed set with closure $M]$ for $x \in E^0$ or $x \in U$;*

(b) *the left regeneration property of Theorem* 4.8(2) *holds under each P^x, $x \in E^0$.*

PROOF. (a) For this part of the proof we shall not use (4.19). Let (S_n) be a sequence of $(\mathscr{F}_t)$ predictable times in $M \setminus D$, with disjoint graphs and such that $\bigcup_n [\![S_n]\!] = J' (= \{t: L'\{t\} > 0\})$. The existence of such S_n's follows from Lemma 4.12. Given an $(\mathscr{F}_t)$ stopping time S in M_0, the regeneration property (4.18) holds at each S_n and at $S' = S_{\{S \in \underline{M}\}}$ ($S_A = S$ on A, $+\infty$ on A^c) by (2.18). Since $M_0 = \underline{M} \cup (G \cap J')$, $\underline{M} \cap (G \cap J') = \varnothing$ a.s. and since G and $\underline{M}$ are $(\mathscr{F}_t^P)$ progressive for each P, it follows that for $x \in E^0$,

$$\begin{aligned} P^x_{\varphi_S | \mathscr{F}_S^x} &= P^x_{\varphi_{S'} | \mathscr{F}_{S'}^x} I_{\{S \in \underline{M}\}} + \sum_n P^x_{\varphi_{S_n} | \mathscr{F}_{S_n}^x} I_{\{S = S_n \in G\}} \\ &= P^{X_{S'}} I_{\{S \in \underline{M}\}} + \sum_n P^{X(S_n)} I_{\{S = S_n \in G\}} = P^{X_S}. \end{aligned}$$

In order to prove the same formula under $P^{r,i}$, where $(r, i) \in U$, we note that S is $P^{r,i}$ a.s. greater than or equal to r since $M \subset [r, \infty]$ $P^{r,i}$ a.s. and that $S^r = S \circ \psi_{r,i} - r$ is an $(\mathscr{F}_t^{i(r)})$ stopping time $P^{i(r)}$ a.s. in M_0 (the argument is the same as for Lemma IV.1 of [16] or (4.5) of [17]); in addition $\psi_{r,i}^{-1}(\mathscr{F}_S) \subset \mathscr{F}_{S^r}^{i(r)}$.

(b) Let T be an $(\mathscr{H}_t)$ predictable time in $M \setminus D$, announced by a sequence (T_n) a.s. P^x. Then T is also announced a.s. on $\{0 < T < \infty\}$ by the sequence (D_{T_n}) of $(\mathscr{F}_t^x)$ stopping times and one has $\mathscr{H}^x_{T-} = \bigvee_n \mathscr{H}^x_{T_n} = \bigvee_n \mathscr{F}^x_{D_{T_n}} = \mathscr{F}^x_{T-}$, where the second equality follows from Proposition 2.8. By (4.18) and (4.19) we obtain $P^x_{\varphi_T | \mathscr{H}^x_{T-}} = P^{X_T}$. □

In Sections 5 and 7 we shall only consider regular $p_.$ regenerative systems. For such systems Theorem 4.8(2) holds. We shall indicate in Remark 4.21(2) how to modify the results if we only assume (4.18) in the definition of a regular $p_.$ regenerative system.

(4.21) REMARKS. (1) As a consequence of Theorem 4.8(1) and of Theorem 4.20(a) [or simply (4.18)] a regular $p_.$ regenerative system satisfies (4.10) with $P = P^x$, $x \in E^0$, also for a positive process Z of the form $Z_t = U_t I_{\{t \in G \setminus J\}} + V_t I_{\{t \notin G \setminus J\}}$, where (U_t) is $(\mathscr{H}_t)$ predictable and (V_t) is $(\mathscr{F}_t)$ optional.

(2) If we drop the assumption (4.19) from the definition of a regular $p_.$ regenerative system, the property $L\{t\} = P^{X_t}(1 - e^{-R})$ on $\{L\{t\} > 0\}$ [and hence (4.10)] does not necessarily hold any more. Nevertheless one can produce a *modified* local time $\tilde{L}$, which still has this property and *could replace L in all subsequent results*. For this we set

$$\tilde{L} = L^c + L^{G \cap J}. \tag{4.22}$$

This measure is clearly $(\mathscr{F}_t)$ optional. Let us prove that its support is M a.s. P^x (and hence a.s.). Since $\operatorname{supp} L \setminus \operatorname{supp} \tilde{L} \subset J'$ a.s., it suffices to prove that $[\![S]\!] \subset \operatorname{supp} \tilde{L}$ a.s. for each $(\mathscr{F}_t)$ predictable time S in J'. But on $\{X_S \in E^0 \setminus F\}$ one has $S \in G$ and $\tilde{L}\{S\} > 0$ a.s., so it is enough to check that $S \in \operatorname{supp} \tilde{L}$ a.s. on $\{X_S \in F\}$. Let $T = S_{\{X_S \in F\}}$, $T' = \inf\{u > T: \tilde{L}(T, u] > 0\}$. Since $L^{G \cap J}$ is the $(\mathscr{F}_t^x)$ dual optional projection of $\Lambda^{G \cap J}$ under P^x one has

$$P^x \int I_{\{T < s < T'\}} \Lambda^{G \cap J}(ds) = P^x \int I_{\{T < s < T'\}} L^{G \cap J}(ds).$$

On the other hand

$$P^x \int I_{\{T < s \le T'\}} \big(1_M(s)\, ds + \Lambda^{G \setminus J}(ds)\big) = P^x \int I_{\{T < s \le T'\}} L^c(ds).$$

Therefore

$$P^x \int I_{\{T < s < T'\}} \Lambda(ds) \le P^x \int I_{\{T < s < T'\}} \tilde{L}(ds) = 0.$$

But $R_T = 0$ a.s. on $\{T < \infty\}$ by regeneration at S, and so $T \in \operatorname{supp} \Lambda$ a.s. on $\{T < \infty\}$. Hence $T' = T$ a.s.

Note that (4.9) still holds with $\tilde{L}$ in place of L, since the point part of $\tilde{L}$ is carried by $\{t: X_t \in E^0 \setminus F\}$. (4.10) with $\tilde{L}$ and $P = P^x$ follows from Theorem 4.20(a), since every $(\mathscr{H}_t^x)$ predictable time T in J' satisfies $\mathscr{H}^x_{T-} = \mathscr{F}^x_{T-} \subset \mathscr{F}^x_T$.

5. Markov additive processes associated with $p_.$ regenerative systems. This section is devoted to the time change of a regular $p_.$ regenerative system (as defined in Definition 4.17) by means of its local time ($L_t = L(0, t]_{t\geq 0}$). This will lead to a MAP in the sense of [2]. Since the local time is not continuous, we shall first have to modify its jumps by exponential amounts defined on an *enlarged* space in the same manner as in [5] (see also [1] for a similar, but more analytical approach). We have inserted some necessary changes in the construction of [5] and we therefore shall repeat it in full detail.

Let $(\Omega, \mathscr{H}, \mathscr{F}_t, M, X, \varphi_t, \mathscr{R}(P^\cdot))$ be our regular $p_\cdot$ regenerative system. For $m, n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ we denote by S_n^m the nth time that $(L_t)_{t\geq 0}$ jumps by an amount in $(1/m, 1/(m-1)]$. For $t \in \mathbb{R}_+$ and $m \in \mathbb{N}^*$ let

$$N_t^m = \sum_{n\geq 1} I_{\{S_n^m \leq t\}}, \qquad N_t = (N_t^m)_{m\geq 1}. \tag{5.1}$$

5.1. *Enlargement.* Let η be the exponential distribution on $\mathbb{R}_+$ with parameter 1 and let $(\Omega', \mathscr{F}', P') = (\mathbb{R}_+, \mathscr{B}_{\mathbb{R}_+}, \eta)^{\mathbb{N}^*\times\mathbb{N}^*}$. The coordinates on Ω' are denoted by U_n^m ($m, n \in \mathbb{N}^*$). For $\nu = (\nu^m) \in \mathbb{N}^{\mathbb{N}^*}$ we define the σ-algebra $\mathscr{F}_\nu' = \sigma(U_n^m\colon n \leq \nu^m)$ and the shift φ_ν' such that

$$U_n^m \circ \varphi_\nu' = U_{\nu^m+n}^m. \tag{5.2}$$

Set $W = \mathbb{R}_+ \times \Omega \times \Omega'$, $\mathscr{G}^0 = \mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{H}^0 \otimes \mathscr{F}'$ and for $x \in E^0$, $Q^x = \eta^x \otimes P^x \otimes P'$, where $\eta^x = \eta$ if $x \in E^0 \setminus F$, ε_0 if $x \in F$. For $y = (r, i) \in U$ we also set $Q^y = \eta^y \otimes P^y \otimes P'$, where $\eta^y = \eta$ if $i(0) \in E^0 \setminus F$, ε_0 otherwise. Note that for any probability λ on E^0, Q^λ is equal to Q^μ, with $\mu = Y_0(P^\lambda)$.

All notation introduced on $\Omega, \Omega', \Omega \times \Omega'$ is extended to W in the obvious manner. In particular $\varphi_t, \varphi_\nu', N_t, \psi_t = (\varphi_t, \varphi_{N_t}')$ are defined on W. Note that $\psi_{t+s} = \psi_s \circ \psi_t$. On W we consider the filtration $(\mathscr{G}_t)_{t\geq 0}$ obtained by completing with respect to the measures Q^μ [μ probability on $(U, \mathscr{U})$] the filtration $(\mathscr{G}_{t+}^0)$, where

$$\mathscr{G}_t^0 = \Big\{A \in \mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{F}_t^0 \otimes \mathscr{F}'\colon \forall\, \omega \in \Omega\ I_A(\cdot, \omega, \cdot) \text{ is } \mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{F}_{N_t(\omega)}' \text{ measurable}\Big\}. \tag{5.3}$$

We set $\mathscr{G} = \bigvee_{t\geq 0} \mathscr{G}_t$ and recall that $M_0 = \underline{M} \cup J$.

(5.4) THEOREM. *If T is a $(\mathscr{G}_t)$ stopping time in M_0, one has for all $y \in U$ (and hence for all $x \in E^0$),*

$$Q^y(f \circ \psi_T | \mathscr{G}_T) = Q^{X_T}(f) \quad \text{on } \{T < \infty\},\ f \in \mathscr{H} \otimes \mathscr{F}'_+.$$

PROOF. By the monotone class theorem it suffices to show that for $Z \in b\mathscr{H}^0$, $Z' \in b\mathscr{F}'$,

$$Q^y\big(Z \circ \varphi_T Z' \circ \varphi_{N_T}' | \mathscr{G}_T\big) = P^{X_T}(Z) P'(Z') \quad \text{on } \{T < \infty\}.$$

Since $(t, w) \to \varphi_t(w)$ is $B_{\mathbb{R}_+} \otimes \mathscr{H}^0 | \mathscr{H}^0$ measurable and since $w \to (T(w), w)$ is $\mathscr{G}_T \vee \mathscr{H}^0 | \mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{H}^0$ measurable, $Z \circ \varphi_T$ is $\mathscr{G}_T \vee \mathscr{H}$ measurable and the proof will follow from the two steps

(a) $Q^y(Z' \circ \varphi_{N_T}' | \mathscr{G}_T \vee \mathscr{H}) = P'(Z')$ on $\{T < \infty\}$,

(b) $Q^y(Z \circ \varphi_T | \mathscr{G}_T) = P^{X_T}(Z)$.

For (a) it is enough to consider $Z' = f(U_{n_1}^{m_1}, \dots, U_{n_p}^{m_p})$, where f is continuous and bounded on $\mathbb{R}_+^p$. Then $Z' \circ \varphi'_{N_t} = f(U_{N_t^{m_1}+n_1}^{m_1}, \dots, U_{N_t^{m_p}+n_p}^{m_p})$ is right continuous in t, and by a classical argument all we have to prove is that

$$Q^y\big(HK \cdot Z' \circ \varphi'_{N_t}\big) = Q^y(HK)P'(Z')$$

for $t \in \mathbb{R}_+$, $H \in b\mathscr{H}$, $K \in b\mathscr{G}_t^0$. The left-hand side is equal to

$$\int \eta^y(dr) \int P^y(d\omega)H(\omega) \int P'(d\omega')K(r, \omega, \omega')Z'\big(\varphi'_{N_t(\omega)}(\omega')\big).$$

But for each r, ω the variable $K(r, \omega, \cdot)$ is $\mathscr{F}'_{N_t(\omega)}$ measurable and so it is P' independent of $\varphi'_{N_t(\omega)}$, which yields the desired equality.

For (b) it suffices to consider the case where T is a $(\mathscr{G}_{t+}^0)$ stopping time. Let $K \in b\mathscr{G}_{T+}^0$. Then for each $(r, \omega') \in \mathbb{R}_+ \times \Omega'$, $T^{r,\omega'} = T(r, \cdot, \omega')$ is an $(\mathscr{F}_{t+}^0)$ stopping time in M_0 and $K^{r,\omega'} = K(r, \cdot, \omega')$ is $\mathscr{F}_{T^{r,\omega'}+}^0$ measurable, since $KI_{\{T<t\}}$ is $\mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{F}_t^0 \otimes \mathscr{F}'$ measurable by the definition of $\mathscr{G}_t^0$. By Theorem 4.20(a) applied at $S = T^{r,\omega'}$ we get

$$P^y\big(K^{r,\omega'}Z(\psi(T^{r,\omega'}))\big) = P^y\Big(K^{r,\omega'}P^{X(T^{r,\omega'})}(Z)\Big)$$

and by integrating against $\eta^y(dr)P'(d\omega')$ we obtain (b). □

5.2. *Time change.* Let U_0^0 denote the projection $(r, \omega, \omega') \to r$ from W onto $\mathbb{R}_+$ and let $S_0^0 \equiv 0$. We set $\lambda(x) = P^x(1 - e^{-R})$ for $x \in E^0$, 0 otherwise, and for $t \geq 0$,

$$C_t = L_t^c + \sum_{m,n} \lambda\big(X_{S_n^m}\big)U_n^m I_{\{S_n^m \leq t\}},$$

$$S_t = \inf\{s : C_s > t\}, \qquad E_t = X_{S_t},$$

$$V_t = \frac{C_{S_t} - t}{\lambda(E_t)}\left(\frac{\cdot}{0} \equiv 0\right), \qquad \sigma_t = \big(V_t, \psi_{S_t}\big) \text{ on } \{S_t < \infty\},$$

$$\mathscr{M}_t^0 = \mathscr{G}_{S_t-} \vee \sigma(E_t), \qquad \mathscr{M}_t = \mathscr{M}_{t+}^0,$$

where $\mathscr{G}_{0-}$ is by convention $\sigma(X_0)$. Note that $C_0 = \lambda(X_0)U_0^0$ and that (S_t, E_t) is right continuous, $(\mathscr{M}_t)$ adapted and such that $Q^x(S_0 = 0,\ E_0 = x) = 1$ for $x \in E^0$ (Q^x a.s. 0 is a right increase point of C if $x \in F$ and $C_0 > 0$ if $x \in E^0 \setminus F$). Note also that a.s. one has for all $t, s \in \mathbb{R}_+$,

$$\sigma_{t+s} = \sigma_s \circ \sigma_t, \qquad S_{t+s} = S_t + S_s \circ \sigma_t, \qquad E_{t+s} = E_s \circ \sigma_t \text{ on } \{S_t < \infty\}.$$

The following result shows in particular that $(W, \mathscr{G}; \mathscr{M}_t, S_t, E_t, \sigma_t, Q^x; t \in \mathbb{R}_+, x \in E^0)$ is a strong MAP with state space E^0.

(5.5) THEOREM. *Let T be an $(\mathscr{M}_t)$ stopping time and let $y \in U$ or $y \in E^0$. Then*

$$Q^y_{\sigma_T|\mathscr{M}_T} = Q^{E_T} \quad \text{on } \{S_T < \infty\}. \tag{5.6}$$

Proof. Since $\mathscr{M}_t \subset \mathscr{G}_{S_t}$ for all $t \in \mathbb{R}_+$, S_T is a $(\mathscr{G}_t)$ stopping time and $\mathscr{M}_T \subset \mathscr{G}_{S_T}$. But $S_T \in M_0 \cup \{+\infty\}$ and by Theorem 5.4 one has

$$(5.7) \qquad Q^y_{\sigma_T|\mathscr{G}_{S_T}} = \varepsilon_{V_T} \otimes P^{E_T} \otimes P' \quad \text{on } \{S_T < \infty\}.$$

Hence (5.6) is satisfied on $\{E_T \in F\} \subset (V_T = 0\}$ a.s. It remains to prove (5.6) on $\{E_T \in E^0 \setminus F\}$, which in view of (5.7) amounts to proving that

$$(5.8) \qquad Q^y_{V_T|\mathscr{M}_T} = \eta \quad \text{on } \{E_T \in E^0 \setminus F\}.$$

It suffices to prove that for f bounded and continuous on $\mathbb{R}_+$, $A \in \mathscr{M}_T$ and $m, n \in \mathbb{N}$,

$$(5.9) \qquad Q^y\left(f\left(U_n^m - \frac{T - C_{S_n^m-}}{\lambda(X_{S_n^m})}\right); A \cap \{S_T = S_n^m < \infty\}\right) = \eta(f) Q^y(A \cap \{S_T = S_n^m < \infty\}).$$

We first consider the case $T \equiv t$ for $t \in \mathbb{R}_+$ and $A \in \mathscr{M}_t^0$. Since $\{S_t = S_n^m < \infty\} = \{C_{S_n^m-} \le t < C_{S_n^m}, S_n^m < \infty\}$ and since $\mathscr{M}_t^0 \subset \mathscr{G}_{S_t-} \vee \mathscr{H}$, it suffices to show that for Z bounded and $(\mathscr{G}_t)$ predictable [and such that Z_0 is $\sigma(X_0)$ measurable], for $H \in b\mathscr{H}$ and with $V = (t - C_{S-})/\lambda(X_S)$ (we drop m, n from the notation),

$$(5.10) \qquad Q^y(f(U - V) Z_S H; C_{S-} \le t, U > V) = k\eta(f) \qquad (Z_\infty = 0)$$

for some $k \in \mathbb{R}_+$ that does not depend on f. But U is independent of (V, Z_S, H, C_{S-}), by Lemma 5.11, and has exponential distribution, which implies (5.10) and (5.9). Now (5.9) for $T \equiv t$, $A \in \mathscr{M}_t = \mathscr{M}_{t+}^0$, follows by noting that $I_{\{S_{t+1/k} = S_n^m < \infty\}} \to I_{\{S_t = S_n^m < \infty\}}$; hence (5.9) holds when T is discrete, in particular when T is replaced by $T_k = \sum_{l \in \mathbb{N}} (l+1)/2^k \, I_{\{l/2^k < T \le (l+1)/2^k\}} + \infty I_{\{T = \infty\}}$. The general case is obtained by noting that $I_{\{S_{T_k} = S_n^m < \infty\}} \to I_{\{S_T = S_n^m < \infty\}}$ as $k \to \infty$. □

(5.11) **Lemma.** *For each (m, n) U_n^m is independent of $\mathscr{G}_{S_n^m-} \vee \mathscr{H}$ under Q^y.*

Proof. The result is clear for $m = n = 0$, since $\mathscr{G}_{0-} = \sigma(X_0)$ is trivial under Q^y. For $m, n \ge 1$, $S_n^m > 0$ a.s. and $\mathscr{G}_{S_n^m-}$ is generated by the class $\mathscr{S}$ of the sets $A \cap \{t < S_n^m\}$, $t \in \mathbb{R}_+$, $A \in \mathscr{G}_t$. Since $\mathscr{S}$ is $\cap$ stable and contains W, we only need to show that

$$(5.12) \qquad Q^y(G \cdot H \circ \varphi_t f \circ U_n^m) = Q^y(G \cdot H \circ \varphi_t) \eta(f),$$

for $G \in b\mathscr{G}_t$ carried by $\{t < S_n^m\}$, $H \in b\mathscr{H}$, $f \in b\mathscr{B}_{\mathbb{R}_+}$. But this follows from Theorem 5.4 with $T = S_{N_t^m}^m$, since $\varphi_t(w) = \varphi_{t - T(w)}(\varphi_T w)$, $U_n^m = U_{n - N_T^m(w)}^m(\varphi'_{N_T} w)$ for $t < S_n^m(w)$ (note that N_T^m is $\mathscr{G}_T$ measurable). □

(5.13) **Remark.** In the results of the present section $(L_t)_{t \ge 0}$ can be replaced by any additive functional $(\bar{L}_t)_{t \ge 0}$ with support $(M \cap (0, \infty))^-$ as long as $\bar{J} \equiv \{t: \Delta \bar{L}_t > 0\} \subset G$ a.s., $\Delta \bar{L}_t = \lambda(X_t)$ for $t \in \bar{J}$ (where λ is some fixed function) and the regeneration property holds for every $(\mathscr{F}_t)$ stopping time T in $\underline{M} \cup \bar{J}$. For instance without assuming (4.19) we can choose the modified local time defined in Remark 4.21(2).

(5.14) THEOREM. *The process* $(S_t)_{t\geq 0}$ *is quasi left continuous relative to* $(\mathscr{M}_t)$ *under each* Q^y [*without assuming* (4.19)].

PROOF. Let U be a finite $(\mathscr{M}_t)$ predictable time and let (U_n) be a sequence which announces U Q^y a.s. We have to prove that the set $A = \{0 < S_{U-} < S_U\}$ has Q^y measure 0. By Lemma 5.15 one has $S_{U-\varepsilon} = S_{U-}$ and $\lambda(X_{S_{U-}}) > \varepsilon$ for some $\varepsilon > 0$ Q^y a.s. on A. Therefore it suffices to show that $Q^y(U = T) = 0$ for each jump time T of (S_t) satisfying $S_{T-\varepsilon} = S_{T-}$, $\lambda(X_{S_{T-}}) > \varepsilon$ for some $\varepsilon > 0$. But for such a T and for all $\alpha > 0$, one has by using Theorem 5.5,

$$\begin{aligned} Q^y(U = T) &\leq Q^y\Big(\liminf\big\{C_0 \circ \sigma_{U_n} \leq \alpha,\ \lambda\big(E_{U_n}\big) > \varepsilon\big\}\Big)\\ &\leq \liminf Q^y\Big(Q^{E_{U_n}}(C_0 \leq \alpha);\ \lambda\big(E_{U_n}\big) > \varepsilon\Big)\\ &= \liminf Q^y\Big(1 - \exp\big\{-\alpha/\lambda\big(E_{U_n}\big)\big\};\ \lambda\big(E_{U_n}\big) > \varepsilon\Big)\\ &\leq 1 - e^{-\alpha/\varepsilon}, \end{aligned}$$

proving that $Q^y(U = T) = 0$. □

LEMMA 5.15. *One has* $S_{U-} \in G \cap J$, Q^y *a.s. on* A.

PROOF. One has $S_{U-} \in G$ Q^y a.s. on A and $S_{U-} \in J$ Q^y a.s. on $\bigcup_n\{S_{U_n} = S_{U-}\}$. It remains to show that $S_{U-} \in J$ Q^y a.s. on $\bigcap_n\{S_{U_n} < S_{U-} < S_U\}$. For this, consider $(\mathscr{G}^0_{t+})$ stopping times V_n, V_-, V in M which are Q^y a.s. equal to S_{U_n}, S_{U-}, S_U, respectively, and set $B = \bigcap_n\{V_n < V_- < V\}$. For $\eta^y(dr)P'(d\omega')$ a.e. (r, ω') the $(\mathscr{F}^0_{t+})$ stopping time $V^{r,\omega'}$ is in G and is announced by $(V_n^{r,\omega'})$ P^y a.s. on $B^{r,\omega'}$. But $G \setminus J$ is totally inaccessible under P^y, proving that $V_- \in J$ Q^y a.s. on B. □

6. Images of semi-Markov additive processes. Let $(M, X, \mathscr{R}(P^\cdot))$ be a regular p regenerative system as in Section 5 and let M_0 be as in Theorem 4.20. It follows from Theorem 5.5 that M_0 and X restricted to M_0 are determined by the MAP (S_t, E_t) (defined on an enlarged space). More precisely one has $M_0 = S_{\mathbb{R}_+}$, $X_{S_t} = E_t$ Q^x a.s. for each $x \in E^0$.

Conversely if $(S_t, E_t)_{t\geq 0}$ is a given MAP with state space E such that S is increasing and such that $S_s = S_t \Rightarrow E_s = E_t$ a.s. we can define $N_0 = S_{\mathbb{R}_+} \cap \mathbb{R}$, $N = N_0^-$ and for $s \in \mathbb{R}$,

$$\begin{aligned} H_s &= \sup\{t \leq s\colon t \in N_0\} = \sup\{t \leq s\colon t \in N\} \qquad (\sup\varnothing = -\infty),\\ &\qquad \Gamma(s) = \inf\{t\colon S_t \geq s\}, \qquad C_s = \inf\{t\colon S_t > s\}, \end{aligned}\tag{6.1}$$

$$Z_s\begin{cases} E_{C_s}, & \text{if } s \text{ is a right accumulation point of } N,\\ E_{\Gamma(H_s)}, & \text{otherwise, with } E_{-\infty} = x_0 \text{ (fixed in } E). \end{cases}$$

Then the process Z is right continuous and is constant on each N contiguous interval of the form $[a, b)$. It will follow from Theorem 6.5 that (N, Z) is then a regenerative system. Actually this result will be stated in the more general setting of the semi-Markov additive processes (SMAP), which was used by Weidenfeld [25] in the study of discontinuous time changes of Markov processes. We shall first fix some definitions and notations concerning the SMAP's, especially since they will be indexed by $\mathbb{R}$.

Let $(E, \mathscr{E})$ and E_δ be as before, but in this section W denotes the set of all right continuous functions from $\mathbb{R}$ into $(-\infty, +\infty] \times E$ which are increasing in the first component. In the example of Section 5 E_δ plays the role of E here. The process of the coordinates on W is denoted by $(U_t)_{t\in\mathbb{R}}$ and its natural filtration by $(\mathscr{M}_t^0)_{t\in\mathbb{R}}$. We set $\mathscr{M}^0 = \bigvee_t \mathscr{M}_t^0$. Denoting by S_t, E_t the two components of U_t, we also define the shifts (σ_t) and the set Σ by

$$U_s \circ \sigma_t = (S_{t+s^+} - S_t, E_{t+s^+}) \qquad (\pm\infty - \pm\infty = \pm\infty),$$

$$\Sigma = \{t: S_{t-} < S_t < \infty\} \cup \{t: S_{t+\varepsilon} > S_t, \forall \varepsilon > 0\}.$$

(6.2) Definition. Let $(\tilde{W}, \tilde{\mathscr{M}}, \tilde{\mathscr{M}}_t, \tilde{U}_t, \tilde{Q})$ be a stochastic process, where the filtration $(\tilde{\mathscr{M}}_t)_{t\in\mathbb{R}}$ is right continuous and complete and where the process $(\tilde{U}_t)_{t\in\mathbb{R}}$ is $(\tilde{\mathscr{M}}_t)$ adapted and satisfies $\tilde{U}_. \in W$ $\tilde{Q}$ a.s. Given an $\mathscr{E}$ measurable family (Q^x) of probability measures on $(W, \mathscr{M}^0)$ that satisfy $Q^x(S_0 = 0, E_0 = x) = 1$, the process $(\tilde{W}, \tilde{\mathscr{M}}, \tilde{\mathscr{M}}_t, \tilde{U}_t, \tilde{Q})$ is called a SMAP with respect to (Q^x) provided (with the notation $\tilde{Z} = Z \circ \tilde{U}_.$ for a function Z on W)

$$\tilde{Q}_{\tilde{\sigma}_T | \tilde{\mathscr{M}}_T} = Q^{\tilde{E}_T} \quad \text{on } \{T \in \tilde{\Sigma}\} \tag{6.3}$$

for each $(\tilde{\mathscr{M}}_t)$ stopping time T.

(6.4) Remark. If $\tilde{S}_t = \sup\{s \le t: s \in \tilde{M}\}$ ($\sup \varnothing = -\infty$), where $\tilde{M}$ is a random closed set, then $\tilde{M} = \{t: \tilde{S}_t = t\}$ is $(\tilde{\mathscr{M}}_t)$ progressive, and it follows from (6.3) that $(\tilde{W}, \tilde{\mathscr{M}}, \tilde{\mathscr{M}}_t, \tilde{M}, \tilde{E}_., \tilde{Q})$ is a strong $p_.$ regenerative system, since then $\tilde{\Sigma} = \underline{\tilde{M}}$ (the minimal right closed set with closure $\tilde{M}$, see Remark 2.17).

On W we define $N_0 = S_{\mathbb{R}} \cap \mathbb{R}$ and N, H, Z as in (6.1). The process Z is right continuous, except possibly at $\alpha = \inf N_0$ if $-\infty < \alpha \notin N_0$.

(6.5) Theorem. *Let $(\tilde{W}, \tilde{\mathscr{M}}, \tilde{\mathscr{M}}_t, \tilde{U}_t, \tilde{Q})$ be a SMAP with respect to a family (Q^x) satisfying one of the conditions*

$$S_s = S_t \Rightarrow E_s = E_t \quad a.s.\ Q^x, \tag{6.6}$$

$$Q^x \bigcup_{t>0} \{S_{t-} < S_t, C_0 \circ \sigma_t > 0, S_{C_0} \circ \sigma_t = 0\} = 0. \tag{6.7}$$

Then with $C_t = \inf\{u: S_u > t\}$, $t \in \mathbb{R}$, the system $(\tilde{W}, \tilde{\mathscr{M}}, \tilde{\mathscr{M}}_{\tilde{C}_t}, \tilde{N}, \tilde{Z}_., \tilde{Q})$ is a strong regenerative system with regeneration laws $P^x = \varphi_0(N, Z_.)(Q^x)$ provided $\alpha \in N_0 \cup \{-\infty, +\infty\}$ a.s. Q^x for each x.

PROOF. By our assumptions (including the condition on α)

$$Q^x((N, Z_.) \in \Omega) = \tilde{Q}\big((\tilde{N}, \tilde{Z}_.) \in \Omega\big) = 1,$$

$$(D_t, X_{D_t}) \circ (N, Z_.) = U_{C_t} Q^x \quad \text{a.s. on } \{t < \sup N\},$$

$$(D_t, X_{D_t}) \circ (\tilde{N}, \tilde{Z}_.) = \tilde{U}_{\tilde{C}_t} \tilde{Q} \quad \text{a.s. on } \{t < \sup \tilde{N}\}.$$

One checks also that $X_{s \wedge D_t} \circ (\tilde{N}, \tilde{Z}_.)$ is $\tilde{\mathscr{M}}_{\tilde{C}_t}$ measurable for each s. Let T be an $(\tilde{\mathscr{M}}_{\tilde{C}_t})$ stopping time. Then $\tilde{C}_T$ is an $(\tilde{\mathscr{M}}_t)$ stopping time and $(\tilde{\mathscr{M}}_{\tilde{C}})_T \subset \tilde{\mathscr{M}}_{\tilde{C}_T}$. It follows from (6.3) that on $\{\tilde{C}_T \in \tilde{\Sigma}\} = \{T < \sup \tilde{N}\}$, one has

$$\tilde{Q}_{\varphi_0(N, Z_.) \circ \tilde{\sigma}_{\tilde{C}_T} | \tilde{\mathscr{M}}_{\tilde{C}_T}} = P^{\tilde{E}_{\tilde{C}_T}} \quad \text{on } \left\{\tilde{S}_{\tilde{C}_T} < \infty\right\}$$

or, equivalently, with the notation $\overline{F} = F \circ (\tilde{N}, \tilde{Z}_.)$ for a function F on Ω and with $S = \overline{D}_T = \tilde{S}_{\tilde{C}_T}$,

$$\tilde{Q}_{\overline{\varphi}_S | \tilde{M}_{\tilde{C}_T}} = P^{\overline{X}_S} \quad \text{on } \{S < \infty\}.$$

Finally note that $P^x(X_0 = x) = 1$, since $Q^x(S_0 = 0, E_0 = x) = 1$. □

(6.8) REMARKS. (1) Without the preceding assumption on α the result would still hold with a slight extension of the definition of a regenerative system: (2.3) should be required only on $\{\overline{D}_t \in \overline{M}'\}$, where $\overline{M}' = \overline{M}$ if $\inf \overline{M}$ is isolated in $\overline{M}$, $\overline{M} \setminus \{\inf \overline{M}\}$ otherwise.

(2) The assumption (6.7) amounts to requiring that every maximal interval of constancy of S which is bounded and preceded by a jump of S is also followed by a jump of S. It is automatically satisfied in the example of Remark 6.4, and the statement of Theorem 6.5 then reduces to the obvious fact that $(\tilde{W}, \tilde{\mathscr{M}}, \tilde{\mathscr{M}}_{\tilde{C}_t}, \tilde{H}, \tilde{E}, \tilde{Q})$ is a strong regenerative system, provided $(\tilde{W}, \tilde{\mathscr{M}}, \tilde{\mathscr{M}}_t, \tilde{H}, \tilde{E}, \tilde{Q})$ is a strong $p_.$ regenerative system (note that $\tilde{C}_t = \inf\{s > t \colon s \in \tilde{M}\}$). Note that each of the conditions (6.6), (6.7) of Theorem 6.5 can be replaced by the weaker condition

$$\text{(6.7}'\text{)} \quad Q^x \bigcup_{t>0} \left(\{S_{C_{t-}} < S_{C_t}\} \cap \sigma_{C_t}^{-1}\{0 < C_0 < \infty, S_{C_0} = 0, E_{C_0} \neq E_0\}\right) = 0.$$

(6.9) DEFINITION. The family (Q^x) is called a semi-Markov additive family (SMAF) provided $(W, \mathscr{M}, \mathscr{M}_t^x, U_t, Q^x)$ is a SMAP relative to (Q^x) for each x, where $(\mathscr{M}_t^x)$ denotes the completion of $(\mathscr{M}_{t+}^0)$ under Q^x.

It follows from Theorem 6.5 that, if (Q^x) is a SMAF satisfying (6.6) or (6.7), then the associated family (P^x) is regenerative [Definition 2.2(3)]. Therefore the process $\hat{E}_t = (S_{C_t} - t, E_{C_t}) = (R_t, X_{D_t}) \circ (N, Z_.)$ is Markovian under each Q^x. Actually this last property holds even in the general case, owing to Weidenfeld [25]. Here we shall produce a nice realization of the process $(\hat{E}_t)$ on the space W itself. The construction is similar to that used for the process (Y_t). Let (Q^x) be a general SMAF. On W let us consider the measures $Q^{r,x} = \tau_r(Q^x)$ for $r \in \mathbb{R}_+$, $x \in E$, where $\tau_r \colon W \to W$ is such that $U_t \circ \tau_r = (r + S_t, E_t)$ and $Q^{+\infty, \delta} = \varepsilon_{[(+\infty, \delta)]}$

(the constant function $[(+\infty, \delta)]$ is added to W). The completion of $(\mathscr{M}_{t+}^0)$ with respect to all measures $Q^{\hat{\nu}}$, where $\hat{\nu}$ is probability on $\hat{E} = \mathbb{R}_+ \times E \cup \{(+\infty, \delta)\}$, is denoted by $(\mathscr{M}_t)$. For $t \in \mathbb{R}$ we define $\eta_t: W \to W$ by

$$U_s \circ \eta_t = \left(S_{C_t+s^+} - t, E_{C_t+s^+}\right).$$

(6.10) THEOREM. *Let $(Q^x)_{x \in E}$ be a SMAF and let $\hat{E}_t = (S_{C_t} - t, E_{C_t})$. Then the collection $(W, \mathscr{M}, \mathscr{M}_{C_t}, \hat{E}_t, \eta_t, Q^{r,x}; t \in \mathbb{R}_+, (r,x) \in \hat{E})$ is a strong Markov process with shifts (η_t), i.e.,*

$$Q^{r,x}_{\eta_T | \mathscr{M}_{C_T}} = Q^{\hat{E}_T} \quad on\ \{T < \infty\}$$

for every $(\mathscr{M}_{C_t})$ stopping time T (in $\overline{\mathbb{R}}_+$).

The proof uses the fact that C_T is an $(\mathscr{M}_t)$ stopping time in Σ and it follows the lines of (1.17) of [20]. Note that the semigroup $(\hat{Q}_s)$ of $(\hat{E}_t)$ is given by

$$(6.11) \qquad \hat{Q}_s f(r,x) = Q^{r,x} f(\hat{E}_s) = \begin{cases} f(r-s, x), & \text{if } s < r, \\ Q^x f(\hat{E}_{s-r}), & \text{if } s \geq r. \end{cases}$$

It is Borel since (Q^x) is assumed to be measurable. Therefore, if we eliminate the branch points, $(W, \mathscr{M}, \mathscr{M}_{C_t}, \hat{E}_t, \eta_t, Q^{r,x})$ is a right process. If we assume only that (Q^x) is universally measurable, and if $(W, \mathscr{M}, \mathscr{M}_t, E_t, Q^x)$ is a right Markov process, then using the method of Glover [10] we can show that $(W, \mathscr{M}, \mathscr{M}_{C_t}, \hat{E}_t, \eta_t, Q^{r,x})$ is still a right process.

7. Stationary $p_.$ regenerative systems. Let $(\Omega, \mathscr{H}, \mathscr{F}_t, M, X, \varphi_t, \mathscr{R}(P^\cdot))$ be a regular $p_.$ regenerative system as defined in Definition 4.17 and let $(l, {}_*P, L)$ be its exit system as defined by Theorem 4.8. Then (4.10) and the results of Section 5 hold. In particular the process $(E_t = X_{S_t}, t \geq 0)$ is Markovian on the enlarged space $(W, \mathscr{G}, Q^x)$ with respect to the filtration $(\mathscr{M}_t)$ and the semigroup $(Q_s)_{s \geq 0}$ given by

$$(7.1) \qquad Q_s(x, A) = Q^x(E_s \in A), \qquad x \in E^0, A \in \mathscr{E}^0.$$

(7.2) THEOREM. *Each of the following formulas establishes a 1–1 correspondence between (P_s) invariant probabilities μ on $(U, \mathscr{U})$ such that $\mu(\{+\infty\} \times \tilde{\Omega}) = 0$ and (Q_s) invariant measures ν on $(E^0, \mathscr{E}^0)$ such that $\nu(l + {}_*P^\cdot(R)) = 1$ (in particular ν is finite):*

$$(7.3) \qquad \nu(f) = P^\mu \int_{(0,1]} f(X_s) L(ds), \qquad f \in \mathscr{E}^0_+,$$

$$(7.4) \qquad \mu(f) = \nu\left(lf(0, [\cdot]) + {}_*P^\cdot \int_0^R f(Y_a)\, da\right), \qquad f \in \mathscr{U}_+.$$

The following complementary result will be proved simultaneously. (A_t) denotes the age process $(t - G_t)$. The process (X_t^*) is defined a.s. on Ω [see

(4.16)] by

$$X_t^* = X_{t-}^D, \quad \text{if } t \in G \setminus J, \qquad (7.5)$$
$$= X_t, \quad \text{otherwise.}$$

Note that under the assumption of Remark 4.13(1) we would have $X_t^* = X_{t-}^D$ for all $t \in G$ a.s.

(7.6) THEOREM. *Assuming that μ and ν are associated by the correspondence of Theorem 7.2, let P be the element of $\mathscr{R}(P^\cdot)$ such that $\varphi_t(P) = P^\mu$ for each $t \in \mathbb{R}$ [one says that (M, X, P) is a stationary p regenerative system]. Then for all $t \in \mathbb{R}$ and positive appropriately measurable functions f on $E^0 \times \mathbb{R} \times \Omega$ or $E^0 \times \mathbb{R} \times U$*

$$P \sum_{s \in G} f(X_s^*, s, \varphi_s) = \int_{E^0 \times \mathbb{R}} \nu(dx)\, ds\, {}_*P^x f(x, s, \cdot), \qquad (7.7)$$

$$P(A_t < \infty) = 1 \quad \textit{and}$$

$$Pf(X_{G_t}^*, A_t, Y_t) = \int_{E^0} \nu(dx)\Big(l(x) f(x, 0, 0, [x]) + {}_*P^x \int_0^R f(x, a, Y_a)\, da \Big). \qquad (7.8)$$

PROOF. It will be convenient to denote by $\mathscr{A}$ and $\mathscr{B}$ the sets of measures on U and E^0 involved in Theorem 7.2. For $\mu \in \mathscr{A}$ we denote by $\alpha(\mu)$ the measure ν on E^0 given by (7.3), and for $\nu \in \mathscr{B}$ we denote by $\beta(\nu)$ the measure μ on U given by (7.4).

(1) Let $\mu \in \mathscr{A}$ and let P be the element of $\mathscr{R}(P^\cdot)$ such that $Y_t(P) = \mu$ [or $\varphi_t(P) = P^\mu$] for all $t \in \mathbb{R}$.

Since the process (X_{t-}^D) (with $X_{t-}^D = \delta$ when the limit does not exist in E^0) is $(\mathscr{H}_t^x)$ predictable for each x, it follows from Remark 4.21(1) and a monotone class argument that

$$P^x \sum_{s \in G} f(X_s^*, s, \varphi_s) = \int_\Omega P^x(d\omega) \int L(\omega, ds)_* P^{X_s(\omega)} f(X_s^*(\omega), s, \cdot).$$

In this equality one can replace X_s^* by X_s in the right-hand side, since L is a.s. carried by $\{s\colon X_s^* = X_s\}$, and one can replace P^x by P by noting that

$$P \sum_{s \in G} f(X_s^*, s, \varphi_s) = \lim_{t \downarrow -\infty} \int P(d\omega) P^{X_{D_t}(\omega)} \sum_{s \in G} f(X_s^*, D_t(\omega) + s, \varphi_s).$$

Therefore setting $g(x, s) = {}_*P^x f(x, s, \cdot)$ we get

$$P \sum_{s \in G} f(X_s^*, s, \varphi_s) = P \int g(X_s, s) L(ds). \qquad (7.9)$$

Let $\nu = \alpha(\mu)$. ν is finite [since $P^\mu(L(0,1]) \le e$]. By stationarity of (φ_t) under P and by homogeneity of L one has (using a classical argument)

$$P \int g(X_s, s) L(ds) = \int_{E^0 \times \mathbb{R}} \nu(dx)\, ds\, g(x, s), \qquad (7.10)$$

establishing (7.7). As a consequence of (7.7) one has for $f \in \mathscr{E}^0 \otimes \mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{U}_+$

$$P\Big(f(X^*_{S_t}, A_t, Y_t) I_{\{A_t > 0\}}\Big) = P\Big(\sum_{s \in G,\, s \le t} f(X^*_s, t-s, Y_{t-s}(\varphi_s)) I_{\{R(\varphi_s) > t-s\}}\Big)$$

$$(7.11) \qquad = \int_{E^0} \nu(dx) \int_{\mathbb{R}_+} da\, {}_*P^x(f(x, a, Y_a); R > a)$$

$$= \int_{E^0} \nu(dx)\, {}_*P^x \int_0^R f(x, a, Y_a)\, da.$$

Note that the property $P(A_t < \infty) = 1$ follows from the assumption that $P(R_t < \infty) = 1$ and from the stationarity of (M, X, P) [see (4.2) of [20]]. In order to prove (7.8), and thereby (7.4) and the equality $\nu(l + {}_*P^{\cdot}(R)) = 1$, it remains to show that

$$(7.12) \qquad P\Big(f(X^*_{G_t}, A_t, Y_t) I_{\{A_t = 0\}}\Big) = \int_{E^0} \nu(dx) l(x) f(x, 0, 0, [x]).$$

The left-hand side, equal to $P(f(X_t^*, 0, Y_t) I_{\{t \in M\}})$, does not depend on t. Hence it is equal to

$$P\int_{(0,1]} f(X_t^*, 0, Y_t) 1_M(t)\, dt = P\int_{(0,1]} f(X_t, 0, 0, [X_t]) l(X_t) L(dt),$$

proving (7.12).

Let us now check that $\nu Q_s = \nu$. As a particular case of Lemma 7.14 one has for $f \in b\mathscr{E}^0_+$

$$\nu(f) = Q^\mu \int_{(0,1]} f(X_t)\, dC_t$$

$$(7.13) \qquad = Q^\mu \int_{C_0}^{C_1} f(E_u)\, du,$$

$$\nu(Q_s f) = Q^\mu \int_{C_0}^{C_1} Q_s f(E_u)\, du.$$

By the Markov property of (E_t) relative to $(\mathscr{M}_t)$ under Q^μ (Theorem 5.5), one has

$$Q^\mu(Q_s f(E_u); C_0 < u \le C_1) = Q^\mu(f(E_{u+s}); C_0 < u \le C_1).$$

Therefore

$$\nu Q_s f = Q^\mu \int_{C_0 + s}^{C_1 + s} f(E_t)\, dt.$$

In order to prove that $\nu Q_s f = \nu f$, it remains to prove that $Q^\mu(F_1) = Q^\mu(F_0)$, with $F_t = \int_{C_t}^{C_t + s} f(E_u)\, du$. But F_t is $\mathscr{H} \otimes \mathscr{F}'$ measurable and satisfies $F_t = F_0(\psi_t)$ and the result follows from the stationarity of $(\psi_t)_{t \ge 0}$ under Q^μ (see Lemma 7.16).

(2) We saw that for each $\mu \in \mathscr{A}$ one has $\alpha(\mu) \in \mathscr{B}$ and $\beta(\alpha(\mu)) = \mu$. Let us now prove that for $\nu \in \mathscr{B}$, $\beta(\nu) \in \mathscr{A}$ and $\alpha(\beta(\nu)) = \nu$. Note first that ν is finite

since $1 = l + {}_*P \cdot (1 - e^{-R}) \le l + {}_*P \cdot (R)$ and that $\beta(\nu)$ is a probability on U since $\beta(\nu)(1) = \nu(l + {}_*P \cdot (R)) = 1$. We shall split the rest of the proof into three parts.

(i) $\beta(\nu)(f) = \mu'(f) \equiv Q^\nu \int_0^{S_1} f(Y_t)\,dt$ for $f \in b\mathscr{U}_+$. In fact, since $\sigma_1(Q^\nu) = Q^\nu$ due to (5.6),

$$Q^\nu \int_{S_1}^{D_{S_1}} f(Y_a)\,da = Q^\nu \int_0^R f(Y_a)\,da$$
$$= P^\nu \int_0^R f(Y_a)\,da$$
$$\le \nu({}_*P \cdot (R))\|f\| < \infty.$$

Therefore $\mu'(f) = Q^\nu \int_R^{D_{S_1}} f(Y_t)\,dt$ and by using Lemma 7.17, we obtain

$$\mu'(f) = Q^\nu \int_0^{S_1} f(Y_t) 1_M(t)\,dt + Q^\nu \sum_{s \in G,\, s \le S_1} \left(\int_0^R f(Y_a)\,da \right) \circ \varphi_s$$
$$= Q^\nu \int I_{\{0 < s \le S_1\}} g(X_s)\,dC_s \quad \left(\text{with } g = lf(0, [\cdot]) + P \int_0^R f(Y_a)\,da \right)$$
$$= Q^\nu \int_{C_0}^{C_{S_1}} g(E_u)\,du.$$

But $C_{S_1} = 1 + C_0 \circ \sigma_1$ and since $\sigma_1(Q^\nu) = Q^\nu$,

$$Q^\nu \int_1^{C_{S_1}} g(E_u)\,du = Q^\nu \int_0^{C_0} g(E_u)\,du$$
$$= Q^\nu\big(g(X_0)\lambda(X_0)U_0^0\big) \le \nu(g) < \infty.$$

Therefore $\mu'(f) = Q^\nu \int_0^1 g(E_u)\,du = \nu(g) = \beta(\nu)(f)$.

(ii) *The measure μ' is invariant for (P_s).* In fact let $f \in b\mathscr{U}_+$. Then by the Markov property of (Y_t) under Q^ν (see Lemma 7.15),

$$Q^\nu \int_0^{S_1} P_s f(Y_t)\,dt = Q^\nu \int_0^{S_1} f(Y_{t+s})\,dt$$
$$= Q^\nu \int_s^{S_1+s} f(Y_u)\,du$$
$$= Q^\nu \int_s^{S_1} f(Y_u)\,du + Q^\nu \int_{S_1}^{S_1+s} f(Y_u)\,du$$
$$= Q^\nu \int_s^{S_1} f(Y_u)\,du + Q^\nu \int_0^s f(Y_u)\,du$$
$$= Q^\nu \int_0^{S_1} f(Y_u)\,du,$$

where the fourth equality follows from

$$\int_{S_1}^{S_1+s} f(Y_u)\,du = \left(\int_0^s f(Y_u)\,du \right) \circ \sigma_1, \qquad \sigma_1(Q^\nu) = Q^\nu.$$

(iii) $\alpha(\mu') = \nu$. Since μ' is (P_s) invariant, $(\varphi_s)_{s\geq 0}$ is stationary under $P^{\mu'}$ and for $f \in b\mathscr{E}_+^0$

$$\alpha(\mu')(f) = P^{\mu'}\int_{(0,\infty)} e^{-s}f(X_s)L(ds)$$

$$= Q^{\mu'}\int_{(0,\infty)} e^{-s}f(X_s)\,dC_s = Q^{\mu'}(F),$$

with $F = \int_{(0,\infty)}e^{-s}f(X_s)\,dC_s$. By the definition of μ' and by the Markov property of (Y_t) under Q^ν one has

$$Q^{\mu'}(F) = Q^\nu\int_0^{S_1}Q^{Y_t}(F)\,dt = Q^\nu\int_0^{S_1}F\circ\psi_t\,dt$$

$$= Q^\nu\int_0^{S_1}dt\int_{(t,\infty)}dC_u\,e^{-(u-t)}f(X_u)$$

$$= Q^\nu\int_{(0,\infty)}dC_u\,e^{-u}f(X_u)\int_0^{S_1\wedge u}e^t\,dt$$

$$= Q^\nu\int_{(0,S_1]}(1-e^{-u})f(X_u)\,dC_u + Q^\nu\int_{(S_1,\infty)}e^{-u}(e^{S_1}-1)f(X_u)\,dC_u$$

$$= Q^\nu\int_{(0,S_1]}f(X_u)\,dC_u - Q^\nu(F) + Q^\nu(F\circ\sigma_1)$$

$$= Q^\nu\int_{C_0}^{C_{S_1}}f(E_u)\,du = Q^\nu\int_0^1 f(E_u)\,du = \nu(f). \qquad \square$$

(7.14) Lemma. *For all $y \in U$ or $y \in E^0$ and all positive $\mathscr{H}$ measurable process Z on W one has*

$$P^y\int_{(0,\infty)}Z_tL(dt) = Q^y\int_{(0,\infty)}Z_tC(dt).$$

Proof. It suffices to prove this equality for the point parts of L, C. But

$$P^y\int_{(0,\infty)}Z_tL^d(dt) = Q^y\sum_{m,n\geq 1}Z_{S_n^m}\lambda(X_{S_n^m})$$

$$= \sum_{m,n\geq 1}Q^y\big(Z_{S_n^m}\lambda(X_{S_n^m})U_n^m\big)$$

$$= Q^y\int_{(0,\infty)}Z_tC(dt). \qquad \square$$

(7.15) Lemma. *Let $\mathscr{H}_t$ be defined like $\mathscr{G}_t$ with $\mathscr{H}_t^0$ instead of $\mathscr{F}_t^0$ in Theorem 5.4. Then for every $y \in U$ and every $(\mathscr{H}_t)$ stopping time T,*

$$Q^y(f(\psi_T)|\mathscr{H}_T) = Q^{Y_T} \quad \text{on } \{T<\infty\},\ f\in\mathscr{H}\otimes\mathscr{F}'_+.$$

PROOF. Same as for Theorem 5.4, by using (3.10) instead of Theorem 4.20(a) at the end of the proof. □

(7.16) LEMMA. *If μ is a probability measure on $(U, \mathscr{U})$ which is (P_s) invariant, then $(\psi_t)_{t\geq 0}$ is stationary under Q^μ.*

PROOF. By Lemma 7.15 one has $\psi_t(Q^\mu) = Q^\mu(Q^{Y_t}) = Q^\mu$. □

(7.17) LEMMA. *For every $y \in U$ or $y \in E^0$, every positive $(\mathscr{K}_t)$ predictable Z and every $f \in \mathscr{K}_+$ one has*

$$Q^y \sum_{0<s\in G} Z_s f \circ \varphi_s = Q^y \int_{(0,\infty)} Z_s {}_*P^{X_s}(f) C(ds).$$

PROOF. Owing to the Markov property of (Y_t) relative to $(\mathscr{K}_t)$ we only need to prove this equality for $Z_s = 1_{(0,u]}(s)$ with $u > 0$. Then the right-hand side equals (Lemma 7.14)

$$P^y \int_{(0,u]} {}_*P^{X_s}(f) L(ds) = Q^y \sum_{s\in G} Z_s f \circ \varphi_s. \qquad \square$$

(7.18) REMARK. If $g \in b\mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{B}_{\mathbb{R}_+}$, it follows from (7.8) that

$$Pg(A_t, R_t) = \int_{E^0} \nu(dx)\left(l(x) g(0,0) + {}_*P^x \int_0^R g(a, R-a)\, da \right).$$

If E^0 has only one point x_0, (S_s) is a subordinator with drift $\alpha = l(x_0)$ and Lévy measure $\lambda = R({}_*P^{x_0})$. Then $\nu\{x_0\} m = 1$, with $m = \alpha + \int_{(0,\infty)} r\lambda(dr)$, and we have as in [20]

$$(7.19)\qquad \begin{aligned} Pg(A_t, R_t) &= \frac{\alpha}{m} g(0,0) + \int_{(0,\infty)} \frac{1}{m} \lambda(dr) \int_0^r g(a, r-a)\, da, \\ P(A_t > u, R_t > v) &= \frac{1}{m} \int_{u+v}^\infty \lambda(a, \infty)\, da. \end{aligned}$$

If $\alpha = 0$ and λ is finite, λ, m can be replaced by $\nu = \lambda/(\lambda(0,\infty))$, $\bar\nu = \int_{(0,\infty)} x\nu(dx)$ in (7.19). This is the known formula for the joint distribution of the forward and backward recurrence times in renewal theory, ν being the distribution of the interrenewal epochs (see [4]).

We now turn to regenerative systems constructed from MAP's as in Example 2.13. This is a special case of what was done in Section 6 where one assumes the Markov property to hold on $\mathbb{R}_+$ and not on Σ only. Let $(Q^x)_{x\in E}$ be a MAF, by which we mean that (Q^x) is a SMAF (as defined in Remark 6.9) such that S_t is *finite* a.s. for each t and

$$Q^x_{\sigma_T|\mathscr{M}_T} = Q^{E_T} \quad \text{on } \{T < \infty\}$$

for any stopping time T of the filtration $(\mathscr{M}_t)_{t\geq 0}$. We shall use the notation

$(\hat{E}_t)$, $\hat{E}$, η_t, $(\hat{Q}_s), \ldots$ of Theorem 6.10, as well as the other notation $N, Z, \ldots$ of Section 6.

It was shown by Çinlar [3] that $S_t = K_t + S_t^d$, where K is a continuous additive functional of (E_t) and S^d is a pure jump process. If we assume that the process (S_t) is quasi left continuous with respect to $(\mathscr{M}_t)$, then it follows from [2] (see also [19]) that there exists a continuous additive functional B of (E_t) and a kernel N from $(E, \mathscr{E}^*)$ into $(\mathbb{R}_+ \times E, \mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{E})$ such that for all $f \in \mathscr{E} \otimes \mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{E}_+$

$$\begin{aligned} &Q^x \sum_{s \le t} I_{\{S_{s-} \neq S_s\}} f(E_{s-}, S_s - S_{s-}, E_s) \\ &\quad = Q^x \int_{(0,t]} dB_s \int_{\mathbb{R}_+ \times E} N(E_s, dr\, dy) f(E_s, r, y). \end{aligned} \tag{7.20}$$

(7.21) THEOREM. *Let ν be a finite invariant measure for the semigroup (Q_s) of (E_s) and consider the Revuz measure of K (resp., B),*

$$\nu_K(\nu_B) = Q^\nu \int_{(0,1]} I_{\{E_s \in \cdot\}}\, dK_s(dB_s).$$

Then the measure $\hat{\mu}$ defined on $\hat{E}$ by

$$\begin{aligned} \hat{\mu}(f) &= \int_E \nu_K(dx) f(0, x) \\ &\quad + \int_E \nu_B(dx) \int_{\mathbb{R}_+ \times E} N(x, dr\, dy) \int_0^r f(r - a, y)\, da \end{aligned} \tag{7.22}$$

is invariant for the semigroup $(\hat{Q}_s)$ provided it is σ-finite. If one of the conditions (6.6), (6.7) *is satisfied, then the process $(Y_s)_{s \ge 0}$ is a stationary Markov process under $P = \varphi_0(N, Z_\cdot)(Q^{\hat{\mu}})$.*

(7.23) REMARKS. (1) Although $\hat{\mu}$ is not necessarily a probability measure, the system $(\Omega, \mathscr{H}, \mathscr{H}_t, M, X, \varphi_t, t \ge 0; P)$ deserves to be called a stationary regenerative system (indexed by $\mathbb{R}_+$), with regeneration laws $P^x = \varphi_0(N, Z_\cdot)(Q^x)$. Note that the process Z could be modified in this statement as long as the equality $(X_0, \hat{X}_\cdot)(N, Z) = (E_0, \hat{E}_\cdot)$, with $\hat{X}_t = (R_t, X_{D_t})$, remains satisfied a.s. Q^x. In fact it is easy to verify that, given a regenerative family (P^x) and a measurable map $\Phi: (\Omega, \mathscr{H}) \to (\Omega, \mathscr{H})$ such that $(X_0, \hat{X}_\cdot) \circ \Phi = (X_0, \hat{X}_\cdot)$ a.s. P^x, then $(\Phi(P^x))$ is still a regenerative family. Note that in the theorem of Section 6 of [13] we have implicitly identified two such regenerative families (P^x) and $(\Phi(P^x))$ [Ω in that paper was restricted to the points of the form $\omega = (\omega^0, \omega^1)$, where $\omega^0 \in \Omega^0$ and $\omega^1 \in \Omega^1$ is constant on each interval $[\alpha, \beta)$ contiguous to ω^0, but despite this restriction the statement of the theorem remains incorrect].

(2) In the conditions of Theorem 7.2, it follows from this theorem that, given a finite (Q_s) invariant measure ν on E^0 such that $\nu(l + {}_*P^\cdot(R)) < \infty$, then the measure $\hat{\mu}$ defined on $\mathbb{R}_+ \times E$ by

$$\mu(f) = \nu\left(lf(0, \cdot) + {}_*P^\cdot \int_0^R f(R - a, X_R)\, da\right)$$

is invariant for the semigroup $(\hat{P}_s)$ of the process $(\hat{X}_t)$. This formula can be understood as a special case of (7.22) with $K_t = \int_0^t l(E_s)\,ds$, $B_t = t$, $N(x, dr\,dy) = {}_*P^x(R \in dr, X_R \in dy)$.

(3) Note that

$$\int_{\mathbb{R}_+ \times E} N(x, dr\,dy) \int_0^r f(r-a, y)\,da = \int_{\mathbb{R}_+} dt \int_E N(x, (t,\infty) \times dy) f(t, y).$$

In particular in the case of Remark 7.23(2) one has

$$\hat{\mu}(f) = \nu\left(lf(0,\cdot) + \int_{\mathbb{R}_+} dt \int_E {}_*P^{\cdot}(R > t, X_R \in dy) f(t, y) \right).$$

When $E^0 = \{x_0\}$, the measure $\hat{\mu}$, considered as a measure on $\mathbb{R}_+$, is of the form $\alpha\varepsilon_0 + \lambda(t,\infty)\,dt$ that appears in [8], [20], [24].

PROOF. By the definition of ν_K, ν_B and by (7.20) one has (with $\Delta S_t = S_t - S_{t-}$)

$$\begin{aligned}\hat{\mu}(f) &= Q^\nu\left(\int_{(0,1]} f(0, E_t) I_{\{\Delta S_t = 0\}}\,dS_t + \sum_{0<t\le 1} I_{\{\Delta S_t > 0\}} \int_0^{\Delta S_t} f(\Delta S_t - a, E_t)\,da \right)\\ &= Q^\nu\left(\int_0^{S_1} f(\hat{E}_s) I_{\{\Delta S(C_s) = 0\}}\,ds + \int_0^{S_1} f(\hat{E}_s) I_{\{\Delta S(C_s) > 0\}}\,ds \right)\\ &= Q^\nu \int_0^{S_1} f(\hat{E}_s)\,ds.\end{aligned}$$

Replacing Y_t, P_s and σ_t by $\hat{E}_t$, $\hat{Q}_s$ and η_t in part (2ii) of the proof of Theorem (7.2), we obtain the $(\hat{Q}_s)$ invariance of $\hat{\mu}$. The second assertion of the theorem is clear by Theorem 6.5. □

REFERENCES

[1] CANETTI, J. and EL-KAROUI, N. (1977). Processus de Markov associé à une fonctionelle additive gauche. Unpublished.

[2] ÇINLAR, E. (1972). Markov additive processes. II. *Z. Wahrsch. verw. Gebiete* **24** 95–121.

[3] ÇINLAR, E. (1974). Additive processes from the point of view of regeneration. Unpublished.

[4] ÇINLAR, E. (1975). *Introduction to Stochastic Processes*. Prentice Hall, Englewood Cliffs, N.J.

[5] ÇINLAR, E. and KASPI, H. (1983). Regenerative systems and Markov additive processes. In *Seminar on Stochastic Processes, 1982* (E. Çinlar, K. L. Chung and R. K. Getoor, eds.) 123–137. Birkhäuser, Boston.

[6] DELLACHERIE, C. (1972). *Capacités et Processus Stochastiques*. Springer, Berlin.

[7] EL KAROUI, N. and WEIDENFELD, G. (1977). Theorie générale et changement de temps. *Séminaire de Probabilités XI. Lecture Notes in Math.* **581** 79–108. Springer, Berlin.

[8] GEMAN, D. and HOROWITZ, J. (1973). Remarks on Palm measures. *Ann. Inst. H. Poincaré Probab. Statist.* **9** 215–232.

[9] GETOOR, R. K. (1975). *Markov Processes: Ray Processes and Right Processes. Lecture Notes in Math.* **440**. Springer, Berlin.

[10] GLOVER, J. (1983). Discontinuous time changes of semi-regenerative processes and balayage theorems. *Z. Wahrsch. verw. Gebiete* **65** 145–160.

[11] KASPI, H. (1983). Excursions of Markov processes: An approach via Markov additive processes. *Z. Wahrsch. verw. Gebiete* **64** 251–268.

[12] KASPI, H. (1984). On invariant measures and dual excursions of Markov processes. *Z. Wahrsch. verw. Gebiete* **66** 185–204.

[13] KASPI, H. and MAISONNEUVE, B. (1986). Stationary regenerative systems. In *Semi-Markov Models: Theory and Applications* (J. Janssen, ed.) 13–22. Plenum, New York.

[14] KASPI, H. and MAISONNEUVE, B. (1986). Predictable local times and exit systems. *Séminaire de Probabilités XX. Lecture Notes in Math.* **1204**. Springer, Berlin.

[15] LE JAN, Y. (1979). Martingales et changements de temps. *Séminaire de Probabilités XIII. Lecture Notes in Math.* **721** 385–399. Springer, Berlin.

[16] MAISONNEUVE, B. (1974). *Systèmes Régénératifs. Astérisque* **15**. Soc. Math. France, Paris.

[17] MAISONNEUVE, B. (1975). Entrance–exit results for semi-regenerative processes. *Z. Wahrsch. verw. Gebiete* **32** 81–94.

[18] MAISONNEUVE, B. (1975). Exit systems. *Ann. Probab.* **3** 399–411.

[19] MAISONNEUVE, B. (1977). Changement de temps d'un processus markovien additif. *Séminaire de Probabilités XI. Lecture Notes in Math.* **581** 529–537. Springer, Berlin.

[20] MAISONNEUVE, B. (1983). Ensembles régénératifs de la droite. *Z. Wahrsch. verw. Gebiete* **63** 501–510.

[21] MAISONNEUVE, B. and MEYER, P. A. (1974). Ensembles aléatoires markoviens homogènes. III. *Séminaire de Probabilités VIII. Lecture Notes in Math.* **381** 212–226. Springer, Berlin.

[22] MAISONNEUVE, B. and MEYER, P. A. (1974). Ensembles aléatoires markoviens homogènes. IV. *Séminaire de Probabilités VIII. Lecture Notes in Math.* **381** 227–241. Springer, Berlin.

[23] SHARPE, M. J. (1987). *General Theory of Markov Processes*. To appear.

[24] TAKSAR, M. I. (1980). Regenerative sets on the real line. *Séminaire de Probabilités XIV. Lecture Notes in Math.* **784** 437–474. Springer, Berlin.

[25] WEIDENFELD, G. (1980). Changements de temps de processus de Markov. *Z. Wahrsch. verw. Gebiete* **53** 123–136.

FACULTY OF INDUSTRIAL ENGINEERING
AND MANAGEMENT
TECHNION
HAIFA 32000
ISRAEL

IMSS
47X-38040 GRENOBLE CÉDEX
FRANCE