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CHARACTERIZATION OF THE LAW OF THE ITERATED LOGARITHM IN BANACH SPACES

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In memory of Antoine Ehrhard

Using a Gaussian randomization technique, we prove that a random variable X with values in a Banach space B satisfies the (compact) law of the iterated logarithm if and only if (i) $E(\|X\|^2/LL\|X\|) < \infty$, (ii) $\{|\langle x^*, X\rangle|^2; x^* \in B^*, \|x^*\| \leq 1\}$ is uniformly integrable and (iii) $S_n(x)/a_n \to 0$ in probability. In particular, if B is of type 2, in order that X satisfy the law of the iterated logarithm it is necessary and sufficient that X have mean zero and satisfy (i) and (ii). The proof uses tools of the theory of Gaussian random vectors as well as by now classical arguments of probability in Banach spaces. It also sheds some light on the usual law of the iterated logarithm on the line.

1. Introduction. Let B be a Banach space with topological dual B^* and norm $\|\cdot\|$. By a random variable X with values in B we will always mean a measurable map from some probability space $(\Omega, \mathscr{F}, P)$ into B equipped with its Borel σ-field with separable range. Given a random variable X, we denote by $(X_n)_{n\in\mathbb{N}}$ a sequence of independent copies of X, and for each n we set $S_n(X) = X_1 + \cdots + X_n$. We write Lt to denote the function $\max(1, \log t)$ and LLt for the composition $L(Lt)$, $t \in \mathbb{R}_+$. We set further $a_n = (2nLLn)^{1/2}$, $n \in \mathbb{N}$.

Using a truncation argument to reduce to Kolmogorov's remarkable law of the iterated logarithm for independent but not necessarily identically distributed random variables [13], Hartman and Wintner [9], in the early 1940s, showed that for a real-valued random variable X such that $EX = 0$ and $\sigma^2 = EX^2 < \infty$,

$$\limsup_{n\to\infty} \frac{S_n(X)}{a_n} = -\liminf_{n\to\infty} \frac{S_n(X)}{a_n} = \sigma \quad \text{a.s.}$$

Twenty-five years later, Strassen [26] proved conversely that if the sequence $(S_n(X)/a_n)_{n\in\mathbb{N}}$ is almost surely bounded, then X has mean zero and $EX^2 < \infty$. He proved moreover [25] that the set of limit points of the sequence $(S_n(X)/a_n)_{n\in\mathbb{N}}$ is exactly the interval $[-\sigma, \sigma]$. (For a recent fairly elementary and self-contained proof of the Hartman–Wintner result in Strassen's formulation which does not use Kolmogorov's law of the iterated logarithm see, e.g., [2].) Actually, in this fundamental paper [25], Strassen obtained a functional law of the iterated logarithm as well as invariance principles which are in many

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respects at the origin of the study of the law of the iterated logarithm in the vector-valued setting. For example, LePage [20] showed then (as a particular case of a more general result) that for Brownian motion on $[0,1]$, considered as a random variable X with values in the space $C[0,1]$, almost surely the sequence $(S_n(X)/a_n)_{n\in\mathbb{N}}$ is relatively compact in $C[0,1]$ and

$$\lim_{n\to\infty} d\left(\frac{S_n(X)}{a_n}, K\right) = 0 \quad\text{and}\quad C\left(\frac{S_n(X)}{a_n}\right) = K,$$

where $d(x, A) = \inf\{\|x - y\|, y \in A\}$ stands for the distance of a point x from a set A and $C(x_n)$ for the set of limit points of the sequence (x_n), K being identified with the set of absolutely continuous functions x of $C[0,1]$ such that $x(0) = 0$ and $\int_0^1 \dot{x}(s)^2\,ds \le 1$, as described in [23].

These early results therefore motivated the study of the law of the iterated logarithm (in short LIL) for Banach space-valued random variables, a subject mostly developed through the impulse of Kuelbs. As a first definition, it is natural to say that a random variable X with values in a Banach space B satisfies the *bounded* LIL if the sequence $(S_n(X)/a_n)_{n\in\mathbb{N}}$ is almost surely bounded in B, or equivalently if the almost surely nonrandom limit

$$\Lambda(X) = \limsup_{n\to\infty} \frac{\|S_n(X)\|}{a_n} \tag{1.1}$$

is finite. Strassen's functional LIL also invites one to consider random variables X for which the sequence $(S_n(X)/a_n)_{n\in\mathbb{N}}$ is not only bounded almost surely but almost surely relatively compact in B. Kuelbs [14] (in a somewhat too restrictive setting as was pointed out later by Pisier [22]) showed that when $(S_n(X)/a_n)_{n\in\mathbb{N}}$ is relatively compact, there is a convex symmetric (and necessarily compact) set K in B such that

$$\lim_{n\to\infty} d\left(\frac{S_n(X)}{a_n}, K\right) = 0 \quad \text{a.s.} \tag{1.2}$$

and

$$C\left(\frac{S_n(X)}{a_n}\right) = K \quad \text{a.s.}, \tag{1.3}$$

and of course then $\Lambda(X) = \sup_{x\in K}\|x\|$. The limit set K is the unit ball of the reproducing kernel Hilbert space determined by the covariance structure of X and, for completeness, we would like to briefly describe it (cf. [14, 8] for more details). Assume that for each x^* in B^*, $E|\langle x^*, X\rangle|^2 < \infty$ (a set of necessary conditions for the LIL in view of the scalar case); consider the operator U: $B^* \to L_2(\Omega, \mathscr{F}, P)$ defined by $Ux^* = \langle x^*, X\rangle$. A closed graph argument easily shows that

$$\|U\| = \sup_{\|x^*\|\le 1} \left(E|\langle x^*, X\rangle|^2\right)^{1/2} = \sigma(X) < \infty.$$

Since X has separable range, the transpose U^* of U maps L_2 into B. The completion of the image of B^* by $S = U^*U$ with respect to the scalar product induced by L_2 on $S(B^*)$ is called the reproducing kernel Hilbert space associated to the covariance of X. Its unit ball K is convex and symmetric and $\sup_{x \in K} \|x\| = \sigma(X)$. K is compact if and only if S is a compact operator and this happens if and only if the family of real random variables $\{|\langle x^*, X\rangle|^2; x^* \in B^*, \|x^*\| \leq 1\}$ is uniformly integrable. As examples, if X is real-valued, K is trivially $[-\sigma, \sigma]$ where $\sigma = (EX^2)^{1/2}$; if X is Brownian motion on $[0,1]$, it is easily seen that K is the limit set of Strassen described previously.

According to Kuelbs' result, a second natural definition for the LIL is that a random variable X with values in a Banach space B satisfies the *compact* LIL whenever the sequence $(S_n(X)/a_n)_{n \in \mathbb{N}}$ is almost surely relatively compact in B, or equivalently therefore, that for some compact set K (1.2) and (1.3) hold. Of course bounded and compact LIL are equivalent in finite dimension but examples of Pisier [21] in c_0 disproved this equivalence in general. Actually this dichotomy appears even in Hilbert space [8].

The purpose of this paper is to investigate and establish necessary and sufficient conditions for a random variable with values in a Banach space to satisfy the bounded or compact LIL. The starting point is the Hartman–Wintner–Strassen equivalence which asserts that a real-valued random variable X satisfies the LIL if and only if $EX = 0$ and $EX^2 < \infty$. In this general question of finding necessary and sufficient conditions for a random variable to satisfy the LIL, we will give in Section 3 an elementary proof of this equivalence which thus makes our work complete in this regard. The preceding characterization of course easily extends to random variables with values in finite dimensional Banach spaces but there exist however almost surely *bounded* mean zero infinite dimensional random variables that do not satisfy the LIL. The first examples (concerning actually the central limit theorem) were constructed by Dudley and Strassen [5] in $C[0,1]$ and by Dudley [15] in L_p for $p < 2$; their relevance for the LIL was pointed out by Kuelbs [14, 15]. In spite of these negative observations, much has been done during the last few years in order to obtain necessary and sufficient conditions for a random variable with values in a Banach space to satisfy the LIL, a problem that we solve in this article. For simplicity we only discuss in greater detail the bounded LIL in the sequel of this introduction; the characterization of the compact LIL (which is announced in the abstract and actually follows from the bounded law) is given in Theorem 1.2 and discussed in the proofs in Section 3.

Inspecting the necessary conditions for a random variable X with values in a Banach space B to satisfy the bounded LIL, it first turns out that the moment condition $E\|X\|^2 < \infty$ is unnecessarily restrictive and is actually necessary *only* in finite dimensional spaces [24]. A moment condition on the norm that is necessary and seems quite close to the preceding one but which induces a deep difference is

$$E\left(\|X\|^2/LL\|X\|\right) < \infty. \tag{1.4}$$

Indeed, if the sequence $(S_n(X)/a_n)_{n \in \mathbb{N}}$ is almost surely bounded, this is also the case for the sequence $(X_n/a_n)_{n \in \mathbb{N}}$ and thus, by the Borel–Cantelli lemma,

$\sum_n P\{\|X\| > Ma_n\} < \infty$ for some M, a condition which is clearly seen to be equivalent to (1.4). To this strong condition however, we can add a set of weak moment conditions since if X satisfies the bounded LIL, $\langle x^*, X\rangle$ also satisfies it for all x^* in B^*, and thus, by the scalar LIL,

$$\text{for each } x^* \text{ in } B^*, \quad E\langle x^*, X\rangle = 0 \quad \text{and} \quad E|\langle x^*, X\rangle|^2 < \infty. \tag{1.5}$$

As already mentioned, we note that (1.5) implies the finiteness of $\sigma(x) = \sup_{\|x^*\| \le 1}(E|\langle x^*, X\rangle|^2)^{1/2}$.

As we have seen, these necessary moment conditions (1.4) and (1.5) are not sufficient in general for a random variable X to satisfy the bounded LIL. They are however sufficient in certain spaces. Say that a Banach space B is of *type 2* if there is a constant C such that for all finite sequences $(x_1, \dots, x_n)$ in B,

$$E\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|^2 \le C\sum_{i=1}^n \|x_i\|^2,$$

where (ε_i) denotes a Rademacher sequence, i.e., a sequence of independent random variables taking the values $+1$ and -1 with probability $\frac{1}{2}$. For example, L_p-spaces for $2 \le p < \infty$ are of type 2. In an early result, Pisier [21] showed that a mean zero random variable X with values in a type 2 Banach space such that $E\|X\|^2 < \infty$ satisfies the LIL. The weaker (and necessary) conditions (1.4) and (1.5) are then shown to be sufficient in certain type 2 spaces (Hilbert spaces first [8], then L_p $(2 \le p < \infty)$ spaces [3] and 2-smooth spaces [18] and more generally uniformly convex spaces of type 2 [19]). We show in this paper how (1.4) and (1.5) characterize random variables satisfying the bounded LIL in any type 2 Banach space, but actually this result will appear as an immediate consequence of a much more general property valid in any Banach space that we describe now.

As moment conditions are insufficient to characterize the LIL in general, one has to seek an additional necessary condition which could also be sufficient together with (1.4) and (1.5). A trivial necessary condition for a random variable X with values in B to satisfy the bounded LIL is that

$$\text{the sequence } (S_n(X)/a_n)_{n\in\mathbb{N}} \text{ is bounded in probability.} \tag{1.6}$$

Of course, this necessary condition is quite different from the previous ones which only involve the law of the random variable X and not of the whole sequence $(S_n(X))_{n\in\mathbb{N}}$ as in (1.6). However characterizations of the LIL seem to require a condition like (1.6). In some spaces, like type 2 spaces, it is known [8] and will be proved later, that for a mean zero random variable X such that $E(\|X\|^2/LL\|X\|) < \infty$, $S_n(X)/a_n \to 0$ in probability and thus (1.6) holds in this case. This observation of course relates to the previous results in type 2 spaces in which the necessary condition (1.6) does not appear in the characterization of the LIL, justifying thus the interest in them. We note concerning (1.6) that it actually also contains the mean zero property (see Proposition 2.3) so that when (1.5) and (1.6) are involved together, centering may be omitted in (1.5). When we speak of mean zero random variables, it always concerns random variables X such that at least $E\|X\| < \infty$ so that the centering $EX = 0$ is unambiguous.

Strengthening Pisier's type 2 result [21], Kuelbs [16] showed that when $E\|X\|^2 < \infty$, X satisfies the bounded LIL if and only if (1.6) holds, a result at the basis of the present study. On the other hand, the relation between the central limit theorem and the LIL [22, 8 and 10] shows that when the sequence $(S_n(X)/\sqrt{n})_{n \in \mathbb{N}}$ is stochastically bounded, X satisfies the bounded LIL as soon as $E(\|X\|^2/LL\|X\|) < \infty$. However, neither $E\|X\|^2 < \infty$ nor the central limit theorem are necessary for a random variable X to satisfy the LIL, except in finite dimension (cf. [24]). The three necessary conditions (1.4)–(1.6) therefore appear as the optimal necessary conditions for the bounded LIL to hold and their sufficiency was established recently in uniformly convex spaces [19]. We prove here this equivalence, characterizing thus the random variables satisfying the LIL, *in any Banach space.*

THEOREM 1.1. *Let X be a random variable with values in a Banach space B. In order that X satisfy the bounded LIL it is necessary and sufficient that the following three conditions be fulfilled*:

$$E(\|X\|^2/LL\|X\|) < \infty; \tag{1.4}$$

$$\text{for each } x^* \text{ in } B^*, \quad E|\langle x^*, X\rangle|^2 < \infty; \tag{1.5'}$$

$$\text{the sequence } (S_n(X)/a_n)_{n \in \mathbb{N}} \text{ is bounded in probability.} \tag{1.6}$$

The compact LIL is characterized in a similar way (and actually follows from Theorem 1.1).

THEOREM 1.2. *Let X be a random variable with values in a Banach space B. In order that X satisfy the compact LIL it is necessary and sufficient that the following three conditions be fulfilled*:

$$E(\|X\|^2/LL\|X\|) < \infty; \tag{1.4}$$

$$\{|\langle x^*, X\rangle|^2;\ x^* \in B^*, \|x^*\| \le 1\} \quad \text{is uniformly integrable}; \tag{1.7}$$

$$S_n(X)/a_n \to 0 \quad \text{in probability.} \tag{1.8}$$

This result of course includes all the previous partial results giving sufficient or necessary and sufficient conditions for the LIL to hold as surveyed previously. In particular, let us mention the relation between the central limit theorem and the LIL [22, 8 and 10] since the proofs of Theorems 1.1 and 1.2 appear even simpler than the proof of this weaker result. Recall that a random variable X with values in a Banach space B *satisfies the central limit theorem* (CLT in short) if the sequence $(S_n(X)\sqrt{n})_{n \in \mathbb{N}}$ converges in law to a Gaussian random variable G. The CLT of course implies (1.7) since the Gaussian random variable G which has the same covariance structure as X has a strong second moment by the integrability properties of Gaussian vectors; it also clearly implies (1.8).

COROLLARY 1.3. *Let X be a random variable with values in a Banach space B. Assume that X satisfies the CLT; then X satisfies the compact LIL if and only if* $E(\|X\|^2/LL\|X\|) < \infty$.

As announced, Theorems 1.1 and 1.2 take simpler forms in type 2 spaces. The following result was actually obtained by the first-named author in an earlier version of the present paper.

COROLLARY 1.4. *A random variable X with values in a Banach space B of type 2 satisfies the bounded (resp., compact) LIL if and only if it has mean zero and* (1.4) *and* (1.5) [*resp.* (1.7)] *hold.*

The proofs we present of Theorems 1.1 and 1.2 are quite different from the previous approaches to characterization results, which use mainly the differentiability properties of the norm in the spaces under consideration. The method we follow relies on a simple Gaussian randomization technique, already used by Pisier in [21] (and more recently with great success by Giné and Zinn [7] in the study of limit properties of empirical processes), that is quite standard in probability in Banach spaces. It also relies on some of the powerful tools of the theory of Gaussian processes like Borell's inequality [1] and the comparison theorems based on Slepian's lemma. Some of the now classical arguments of probability in Banach spaces are also essential, especially Hoffmann-Jørgensen's inequality [11] and Yurinskii's idea [27].

The Gaussian randomization is presented in Section 2 after we have described some useful and well-known integrability properties. For the convenience of the reader and above all for the sake of completeness, and in order to make this work accessible to a large number of readers, we chose to give the proofs of some of these and other known results. As an illustration of the method, we present, in Section 3, an elementary proof of the LIL on the line (the sufficiency part reproducing Pisier's argument [21]) and then we establish our main result.

To conclude this introduction, let us mention that while the Gaussian randomization allows us to see in an elementary way that for a real nondegenerate random variable X, $0 < \Lambda(X) < \infty$ if and only if $EX = 0$ and $\sigma^2 = EX^2 < \infty$, it does not seem to be precise enough to check the equality $\Lambda(X) = \sigma$ which is crucial in the various questions and results concerning identification of the limit set of the sequence $(S_n(X)/a_n)_{n \in \mathbb{N}}$ and clustering phenomena in finite and infinite dimension like, for example, the equivalence between the compact LIL and (1.2) and (1.3). In particular, the study of the cluster set $C(S_n(X)/a_n)$ and the computation of the exact value of $\Lambda(X)$ (and not only a two-sided estimate) when X only satisfies the bounded LIL does not seem to be accessible by this approach. The results are in fact still very fragmentary in this area and some questions are described in [4]. de Acosta and Kuelbs [3] have shown that in some type 2 spaces, mainly L_p $(2 \le p < \infty)$ spaces, $\Lambda(X)$ is equal to $\sigma(X)$ (but this might not be the exact value in general — see [4]) when X satisfies the bounded LIL. They have also proved, in any Banach space, that if X satisfies (1.5) and

$S_n(X)/a_n \to 0$ in probability, then

$$C\left(\frac{S_n(X)}{a_n}\right) = K \quad \text{a.s.,} \tag{1.9}$$

but examples of Kuelbs [17] show that the cluster set $C(X_n(X)/a_n)$, while always contained in K, may be empty when $S_n(X)/a_n \nrightarrow 0$ in probability even if X is bounded and satisfies the bounded LIL. In type 2 spaces however, this result implies (by Proposition 3.7) that we have (1.9) whenever X satisfies the bounded LIL. Combining these various conclusions, de Acosta and Kuelbs moreover obtained a complete description of the LIL in Hilbert space by showing that when X satisfies the bounded LIL, almost surely,

$$\lim_{n\to\infty} d\left(\frac{S_n(X)}{a_n}, K\right) = 0 \quad \text{and} \quad C\left(\frac{S_n(X)}{a_n}\right) = K,$$

where K is the natural limit set, i.e., the unit ball of the reproducing kernel Hilbert space associated to the covariance of X; if (and only if) K is compact, X satisfies the compact LIL.

This aspect of the study of the LIL is not considered in this paper in which we wish to emphasize from a functional point of view the Gaussian randomization approach as a valuable and simple tool in the characterization of random variables satisfying the LIL. The characterization we obtain reduces, under necessary moment conditions, the almost surely statement of the LIL to a weak boundedness or convergence of the sequence $(S_n(X)/a_n)_{n\in\mathbb{N}}$ which in many respects is close to the CLT. It can therefore be hoped that ideas developed in the area of the CLT could also apply to yield better knowledge of this convergence in probability and therefore also of the LIL.

2. Gaussian randomization. This section is devoted to a discussion of the elementary Gaussian randomization that will be basic in the proof of our main result. We however first recall some well-known integrability properties (and prove some of them) that will be of help next. We start with the fundamental inequality of Hoffman-Jørgensen [11, page 164–165] (see also [12, page 16]).

PROPOSITION 2.1. *Let $Y_1, \dots, Y_n$ be independent symmetric random variables with values in a Banach space B; set $S_n = Y_1 + \cdots + Y_n$. Then, for all $t > 0$,*

$$P\{\|S_n\| > 3t\} \le 4\big(P\{\|S_n\| > t\}\big)^2 + P\Big\{\max_{i\le n}\|Y_i\| > t\Big\}. \tag{2.1}$$

It follows that if $p > 0$ and $E\|Y_i\|^p < \infty$ for all $i = 1, \dots, n$,

$$E\|S_n\|^p \le 2\cdot 3^p E\max_{i\le n}\|Y_i\|^p + 2\cdot(3t_0)^p, \tag{2.2}$$

where $t_0 = \inf\{t > 0\colon P\{\|S_n\| > t\} \le (8\cdot 3^p)^{-1}\}$.

Recall the usual conclusion drawn from that result [11]: Let (Y_n) be a sequence of independent random variables with values in a Banach space B and

(b_n) be an increasing sequence of nonnegative numbers; set $S_n = Y_1 + \cdots + Y_n$, $n \geq 1$. Then, if (S_n/b_n) is almost surely bounded, for all $0 < p < \infty$,

$$E \sup_n \|S_n/b_n\|^p < \infty \quad \text{iff} \quad E \sup_n \|Y_n/b_n\|^p < \infty.$$

This result has a first consequence in our framework. Let X be a random variable with values in B; set, for $0 < p < \infty$,

$$IL_p(X) = \left(E \sup_n \|S_n(X)/a_n\|^p \right)^{1/p}$$

(finite or not). It will be convenient for us, in some respects, to work with these IL_p norms in the sequel.

PROPOSITION 2.2 ([20]). *If X satisfies the bounded LIL, then $IL_p(X) < \infty$ for all $p < 2$.*

PROOF. By Hoffmann-Jørgensen's result recalled before, it is enough to show that

$$E \sup_n \|X_n/a_n\|^p < \infty.$$

But

$$\begin{aligned} E \sup_n \left\| \frac{X_n}{a_n} \right\|^p &\leq 1 + \int_1^\infty \sum_n P\{\|X\| > ta_n\}\, dt^p \\ &= 1 + E\left(\sum_n \int_1^\infty I_{\{ta_n < \|X\|\}}\, dt^p \right) \\ &\leq 1 + E\left(\|X\|^p \sum_n \frac{1}{a_n^p} I_{\{a_n < \|X\|\}} \right), \end{aligned}$$

which is finite under $E(\|X\|^2/LL\|X\|) < \infty$ whenever $p < 2$. Since we know indeed by the Borel–Cantelli lemma that this expectation is finite when X satisfies the bounded LIL, the proof is complete. □

We now state and prove the corresponding result for stochastic boundedness of the sequence $(S_n(X)/a_n)_{n \in \mathbb{N}}$.

PROPOSITION 2.3 ([20]). *If the sequence $(S_n(X)/a_n)_{n \in \mathbb{N}}$ is bounded in probability, i.e., if for each $\varepsilon > 0$ there is an M such that*

$$\sup_n P\{\|S_n(X)/a_n\| > M\} < \varepsilon,$$

then, for all $0 < p < 2$,

$$R_p(X) = \sup_n \left(E\|S_n(X)/a_n\|^p \right)^{1/p} < \infty.$$

Moreover $EX = 0$ and if $S_n(X)/a_n \to 0$ in probability, then $E\|S_n(X)/a_n\| \to 0$.

Proof. Let X' be an independent copy of X and denote by $\tilde{X}$ the symmetric random variable $X - X'$. Then the sequence $(S_n(\tilde{X})/a_n)_{n\in\mathbb{N}}$ is clearly also stochastically bounded. Let m be fixed and set $S_0(\tilde{X}) = 0$,

$$Y_n^m = \frac{1}{a_m}\left[S_{mn}(\tilde{X}) - S_{m(n-1)}(\tilde{X})\right], \qquad n = 1, 2\dots,$$

which define independent identically distributed symmetric random variables. Note that $a_{mn} \le Ca_m a_n$ for some numerical constant C. Since

$$S_{mn}(\tilde{X}) = a_m \sum_{i=1}^{n} Y_i^m,$$

by symmetry, Lévy's inequality [12] implies that for all n and $t > 0$,

$$P\left\{\max_{i\le n}\|Y_i^m\| > Cta_n\right\} \le 2P\{\|S_{mn}(\tilde{X})\| > Cta_m a_n\}$$

$$\le 2P\{\|S_{mn}(\tilde{X})\| > ta_{mn}\}.$$

By hypothesis, there is an M such that, for all n,

$$P\{\|S_n(\tilde{X})\| > Ma_n\} \le \tfrac{1}{4}.$$

Thus

$$P\left\{\max_{i\le n}\|Y_i^m\| > CMa_n\right\} \le \tfrac{1}{2}$$

and, by independence and identical distribution, for all n,

$$P\{\|S_m(\tilde{X})/a_m\| > CMa_n\} \le 1 - \left(1 - \tfrac{1}{2}\right)^n \le 1/n,$$

from which it is easy to deduce that

$$E\|S_m(\tilde{X})/a_m\|^p \le D,$$

where D only depends on C, t and $p < 2$. Thus $R_p(\tilde{X}) < \infty$. In particular $E\|\tilde{X}\| = E\|X - X'\| < \infty$ and hence $E\|X\| < \infty$. By the strong law of large numbers,

$$S_n(X)/n \to EX \quad \text{a.s.}$$

and clearly we cannot have then $\|EX\| \ne 0$ when $(S_n(X)/a_n)_{n\in\mathbb{N}}$ is stochastically bounded. Therefore $EX = 0$ and Jensen's inequality then imply $R_p(X) < \infty$. The final conclusion is simply obtained by uniform integrability. □

In the following, we use the notation IL and R for IL_1 and R_1.

These preliminaries being described, we now discuss the Gaussian randomization. The following notations will be kept throughout the article. Whenever X is a Banach space random variable, g will always denote a standard normal random variable independent of X. The sequences $(X_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ of independent copies of X and g, respectively, will also therefore be understood to be independent. We will also need sometimes a Rademacher sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ both independent of $(X_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$. The idea behind the Gaussian randomization is simply that for every B-valued mean zero random variable X,

$$IL(X) \le CIL(gX), \tag{2.3}$$

where C denotes a numerical constant. Indeed, letting (X_n') be an independent copy of (X_n) also independent of (ε_n), we have, for all $N \ge 1$, by centering, Jensen's inequality and symmetry,

$$E\sup_{n\le N}\frac{1}{a_n}\|S_n(X)\| \le E\sup_{n\le N}\frac{1}{a_n}\left\|\sum_{i=1}^{n} X_i - X_i'\right\| \le 2E\sup_{n\le N}\frac{1}{a_n}\left\|\sum_{i=1}^{n}\varepsilon_i X\right\|.$$

Using independence and Jensen's inequality one more time,

$$E\sup_{n\le N}\frac{1}{a_n}\left\|\sum_{i=1}^{n}\varepsilon_i X_i\right\| \le (E|g|)^{-1}E\sup_{n\le N}\frac{1}{a_n}\left\|\sum_{i=1}^{n}\varepsilon_i|g_i|X_i\right\|$$
$$= (E|g|)^{-1}E\sup_{n\le N}\frac{1}{a_n}\left\|\sum_{i=1}^{n} g_i X_i\right\|,$$

where the last step follows from the fact that the sequences $(\varepsilon_i|g_i|)$ and (g_i) have the same distribution. Letting N go to infinity yields (2.3).

The IL norm actually satisfies classical contraction type inequalities (cf. [12, 11]); for example, as can be checked easily by independence, $IL(\alpha gX) \le IL(gX)$ for all X-measurable real random variables α such that $|\alpha| \le 1$ a.s. Let us also point out that many inequalities we will obtain concerning this norm or related quantities should actually be written first for a supremum over $n \le N < \infty$ (as before), letting then N tend to infinity. For simplicity, this will always be understood.

The important point to mention at this stage is that (2.3) is actually an equivalence.

Proposition 2.4. *There is a numerical constant C such that for all mean zero B-valued random variables X,*

$$C^{-1}IL(X) \le IL(gX) \le CIL(X). \tag{2.4}$$

Proof. The left-hand side was established previously and the right-hand portion follows from Theorem 4.1 of [19]; we reproduce here the argument and centering is not required in this part. Let ξ be the indicator function of some measurable set A independent of X. Checking Fourier transforms, it is easily seen that the sequences $(S_n(\xi X))_{n\in\mathbb{N}}$ and $(\sum_{i=1}^{S_n(\xi)}X_i)_{n\in\mathbb{N}}$ have the same distribution. Hence

$$IL(\xi X) = E\sup_{n}\frac{1}{a_n}\left\|\sum_{i=1}^{S_n(\xi)} X_i\right\|$$
$$= E\left(\sup_{n}\left(\frac{S_n(\xi)LLS_n(\xi)}{nLLn}\right)^{1/2}\frac{\|\sum_{i=1}^{S_n(\xi)}X_i\|}{a_{S_n(\xi)}}\right)$$

and therefore by independence of X and ξ,

$$IL(\xi X) \le E\left(\sup_n \left(\frac{S_n(\xi) LLS_n(\xi)}{nLLn}\right)^{1/2}\right) IL(X)$$

(2.5)

$$\le E\left(\sup\left(\frac{S_n(\xi)}{n}\right)^{1/2}\right) IL(X).$$

According to the strong law of large numbers (and its maximal inequality or the results and methods of [11]), there is a numerical constant C_1 such that for any real-valued random variable ξ,

$$E\left(\sup_n |S_n(\xi)/n|^{1/2}\right) \le C_1(E|\xi|)^{1/2}.$$

Therefore, if $\xi = I_A$, we have by (2.5)

$$IL(\xi X) \le C_1(P(A))^{1/2} IL(X). \tag{2.6}$$

For each $\varepsilon > 0$ let $g^\varepsilon = \sum_{k=1}^{\infty} \varepsilon I_{\{g > \varepsilon k\})}$ so that $\lim_{\varepsilon \to 0} g^\varepsilon = g^+$ a.s., where g^+ is the positive part of g. Clearly, for each $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \varepsilon(P\{g > \varepsilon k\})^{1/2} \le \int_0^\infty (P\{g > t\})^{1/2}\,dt \le 2$$

and therefore, by the triangle inequality and (2.6),

$$IL(g^\varepsilon X) \le \sum_{k=1}^{\infty} \varepsilon IL(I_{\{g > \varepsilon k\}} X) \le 2C_1 IL(X).$$

Letting ε go to 0 then implies $IL(g^+ X) \le 2C_1 IL(X)$ and hence the result by symmetry. □

Equivalence (2.4) thus rests on the fact that in spite of the type of an l_∞ norm (for which usually Gaussian and Rademacher averages are not equivalent), IL is constructed over a sequence of independent identically distributed random variables allowing such an equivalence. In the sequel, we actually make use of the right-hand side of (2.4) only for a proof of the LIL on the line, but still it clarifies things. However we use the corresponding result for stochastic boundedness which is proved entirely analogously and really goes back to [7, Lemma 2.9] in this form.

PROPOSITION 2.5. *There is a numerical constant C such that for all mean zero B-valued random variables X*

$$C^{-1}R(X) \le R(gX) \le CR(X). \tag{2.7}$$

3. Proofs. Having described the Gaussian randomization in the preceding section and before turning to the proof of our main results, we would like to show how this Gaussian approach can be utilized to provide a simple proof of the characterization of the LIL on the line (but without identification of the lim sup) which almost reduces the LIL to the strong law of large numbers. We begin with an elementary and classical lemma concerning blocks which is also valid for vector-valued random variables. This argument is actually at the origin of the iterated logarithm in the LIL: Indeed, the LIL will be established by an exponential inequality (Gaussian in our context) which requires one logarithm for the convergence; the fact that, thanks to the regularity of the normalizing sequence (a_n), we are allowed to look at blocks of exponential size of the sequence $(S_n(X))$ shows that an iterated logarithm is enough. Let q be an integer greater than or equal to 2 and, for each k, $I(k)$ be the set of integers $\{q^k + 1, \dots, q^{k+1}\}$.

LEMMA 3.1. *For any symmetric random variable X,*

$$\frac{1}{6} IL(X) \le E \sup_k \frac{1}{a_{q^k}} \left\| \sum_{i \in I(k)} X_i \right\| \le (q+1) IL(X). \tag{3.1}$$

PROOF. Only the left-hand side requires a proof. Let $n \in I(k)$; we have

$$\begin{aligned}
\frac{1}{a_n} \|S_n(X)\| &\le \frac{1}{a_{q^k}} \|X_1\| + \frac{1}{a_{q^k}} \sum_{l=0}^{k-1} \left\| \sum_{i \in I(l)} X_i \right\| + \frac{1}{a_{q^k}} \left\| \sum_{i=q^k+1}^{n} X_i \right\| \\
&\le \frac{1}{a_{q^0}} \|X_1\| + \left(\sup_l \frac{1}{a_{q^l}} \left\| \sum_{i \in I(l)} X_i \right\| \right) \sum_{j=0}^{k-1} \frac{a_{q^j}}{a_{q^k}} \\
&\quad + \sup_l \sup_{m \in I(l)} \frac{1}{a_{q^l}} \left\| \sum_{i=q^l+1}^{m} X_i \right\|,
\end{aligned}$$

so that

$$IL(X) \le \left(1 + \frac{1}{\sqrt{q} - 1}\right) E \sup_k \frac{1}{a_{q^k}} \left\| \sum_{i \in I(k)} X_i \right\| + E \sup_{k \in n} \sup_{I(k)} \frac{1}{a_{q^k}} \left\| \sum_{i=q^k+1}^{n} X_i \right\|$$

and hence the lemma is proved since by Lévy's inequality and independence

$$E \sup_k \sup_{n \in I(k)} \frac{1}{a_{q^k}} \left\| \sum_{i=q^k+1} X_i \right\| \le 2 E \sup_k \frac{1}{a_{q^k}} \left\| \sum_{i \in I(k)} X_i \right\|. \qquad \square$$

THEOREM 3.2. *In order that a real random variable X satisfy the LIL, it is necessary and sufficient that $EX = 0$ and $EX^2 < \infty$.*

PROOF. Recall that for $q \ge 2$, we let $I(k) = \{q^k + 1, \dots, q^{k+1}\}$, $k \in \mathbb{N}$. Throughout the proof, $C_1, C_2, \dots$ will denote positive numerical constants independent of q. Assume first that $EX = 0$ and $EX^2 < \infty$. By (2.3) and Lemma 3.1

and independence and Gaussian distribution,

$$\begin{aligned} IL(X) &\le CIL(gX) \\ &\le 6CE \sup_k \frac{1}{a_{q^k}} \left| \sum_{i \in I(k)} g_i X_i \right| \\ &= 6CE\left(\sup_k \frac{|g_k|}{a_{q^k}} \left(\sum_{i \in I(k)} X_i^2 \right)^{1/2} \right) \\ &\le 6CE\left(\sup_k \frac{|g_k|}{\sqrt{2LL2^k}} \left(\frac{1}{q^k} \sum_{i \in I(k)} X_i^2 \right)^{1/2} \right). \end{aligned}$$

Since $E \sup_k(|g_k|/\sqrt{2LL2^k}) < \infty$ and, by the strong law of large numbers,

$$E\left(\sup_n \left(\frac{1}{n} \sum_{i=1}^n X_i^2 \right)^{1/2} \right) \le C_1 (EX^2)^{1/2},$$

we get

$$IL(X) \le C_2 \sqrt{q}\, (EX^2)^{1/2} < \infty$$

and this establishes the first part of the theorem. Conversely, let $\overline{X} = XI_{\{|X| \le t\}}$, $t > 0$. The contraction principle and (2.4) imply that

$$IL(g\overline{X}) \le IL(gX) \le CIL(X) < \infty.$$

Using the right-hand side inequality in (3.1) and arguing as before,

$$\begin{aligned} (q+1)IL(g\overline{X}) &\ge E\left(\sup_k \frac{|g_k|}{a_{q^k}} \left(\sum_{i \in I(k)} \overline{X}_i^2 \right)^{1/2} \right) \\ &\ge E\left(\sup_k \frac{|g_k|}{a_{q^k}} \left(\sum_{i=1}^{q^{k+1}} \overline{X}_i^2 \right)^{1/2} \right) - E\left(\sup_k \frac{|g_k|}{a_{q^k}} \left(\sum_{i=1}^{q^k} \overline{X}_i^2 \right)^{1/2} \right) \\ &\ge \sqrt{q}\, E\left(\limsup_{k \to \infty} \frac{|g_k|}{\sqrt{2LLq^k}} \left(\frac{1}{q^{k+1}} \sum_{i=1}^{q^{k+1}} \overline{X}_i^2 \right)^{1/2} \right) - C_3 (E\overline{X}^2)^{1/2} \\ &\ge \left(\sqrt{q} - C_3 \right) (E\overline{X}^2)^{1/2} \end{aligned}$$

since $\limsup_{k \to \infty} |g_k|/\sqrt{2LLq^k} = 1$ a.s. Letting q be large enough so that $\sqrt{q} > C_3$ and t go to infinity shows that $EX^2 < \infty$ whenever $IL(X) < \infty$ and therefore concludes the proof of the theorem since $EX = 0$ by the law of large numbers. □

We now turn to the proofs of our main results.

PROOF OF THEOREM 1.1. The necessity of conditions (1.4)–(1.6) has been discussed in the introduction; note that (1.5) also follows from Theorem 3.2. We turn to their sufficiency. Take $I(k) = \{q^k + 1, \dots, q^{k+1}\}$ for $q = 2$, for example, and for each k and i in $I(k)$ set

$$u_i = u_i(k) = X_i I_{\{\|X_i\| \le a_{2^k}\}}, \qquad v_i = v_i(k) = X_i - u_i.$$

As a consequence of (2.3) since $EX = 0$, and of (3.1) since gX is symmetric,

$$\begin{aligned} IL(X) &\le CIL(gX) \\ &\le 6CE \sup_k \frac{1}{a_{2^k}} \Big\| \sum_{i \in I(k)} g_i X_i \Big\| \\ &\le 6C\left(E \sup_k \frac{1}{a_{2^k}} \Big\| \sum_{i \in I(k)} g_i u_i \Big\| + E \sup_k \frac{1}{a_{2^k}} \Big\| \sum_{i \in I(k)} g_i v_i \Big\| \right). \end{aligned}$$

Now

$$\sum_k P\left\{ \Big\| \sum_{i \in I(k)} g_i v_i \Big\| > 0 \right\} \le \sum_k 2^k P\{\|X\| > a_{2^k}\}$$

and this last series is finite under (1.4) so that by the Borel–Cantelli lemma,

$$\sup_k \frac{1}{a_{2^k}} \Big\| \sum_{i \in I(k)} g_i v_i \Big\| < \infty \quad \text{a.s.}$$

Hoffmann-Jørgensen's integrability theorems as used in Proposition 2.2 then ensure, under (1.4), that

$$E \sup_k \frac{1}{a_{2^k}} \Big\| \sum_{i \in I(k)} g_i v_i \Big\| < \infty;$$

hence the proof of the theorem will be complete once we show that

$$E \sup_k \frac{1}{a_{2^k}} \Big\| \sum_{i \in I(k)} g_i u_i \Big\| < \infty. \tag{3.2}$$

In order to establish (3.2), we will make use of the next lemma which is an elementary consequence of the Gaussian isoperimetric inequality of Borell [1] and constitutes the main argument in the proof of our result. It describes the usual exponential inequality argument which is at the basis of any result on the LIL and this is its only appearance in this proof.

LEMMA 3.3. *There is a numerical constant C such that for every sequence* $(G_k)_{k \in \mathbb{N}}$ *of symmetric Gaussian vector-valued random variables*

$$E \sup_k \|G_k\| \le C\Big(\sup_k E\|G_k\| + E \sup_k |g_k| \sigma_k \Big), \tag{3.3}$$

where, for each k, $\sigma_k = \sigma(G_k)$. *Note that* (3.3) *is two-sided if the* G_k*'s are independent.*

PROOF. For each k, denote by m_k a median of $\|G_k\|$. Borell's inequality [1, Theorem 3.1] applied to the sets $\{x: \|x\| \ge m_k\}$ and $\{x: \|x\| \le m_k\}$ easily implies that for all $t > 0$ and integer k,

$$P\{|\|G_k\| - m_k| > t\} \le P\{|g_k|\sigma_k > t\} \tag{3.4}$$

since each point in the unit ball of the reproducing kernel Hilbert space associated to G_k has norm less than or equal to σ_k. Using that $m_k \le 2E\|G_k\|$, an integration by parts yields (3.3). □

REMARK. The deviation inequality (3.4) of a Gaussian random vector from its median or mean is actually a corollary of the more precise form of the isoperimetric inequality of [1] and can be proved directly in an elementary way as was shown recently by Maurey and Pisier (see [23]). Lemma 3.3 should therefore be considered as a basic but accessible fact about Gaussian random vectors.

We will use Lemma 3.3 conditionally and therefore denote by E_g and E_X expectations with respect to the independent sequences (g_i) and (X_i), respectively, when it will be necessary to distinguish them. By (3.3) and Fubini's theorem,

$$\begin{aligned} &E \sup_k \frac{1}{a_{2^k}} \Big\| \sum_{i \in I(k)} g_i u_i \Big\| \\ &\le C\Bigg(E_X\Bigg(\sup_k \frac{1}{a_{2^k}} E_g \Big\| \sum_{i \in I(k)} g_i u_i \Big\| \Bigg) \\ &\quad + E\Bigg(\sup_k \frac{|g_k|}{a_{2^k}} \sup_{\|x^*\| \le 1} \Big(\sum_{i \in I(k)} |\langle x^*, u_i\rangle|^2 \Big)^{1/2} \Bigg)\Bigg). \end{aligned} \tag{3.5}$$

In order to simplify the notation, let us denote by (I) and (II), respectively, the two terms inside the parentheses on the right-hand side of (3.5). We first study the quantity (I).

LEMMA 3.4. *Assume that* (1.4) *and* (1.6) *hold*; *then* (I) $< \infty$.

PROOF. Under (1.6), by Propositions 2.3 and 2.5, the contraction principle (cf. [11, 12]) and homogeneity, we may and do assume that

$$\text{for all } k, \quad E\Big\| \sum_{i \in I(k)} g_i u_i \Big\| \le a_{2^k}. \tag{3.6}$$

We first note that following Hoffmann-Jørgensen's argument to establish (2.1), we easily obtain that

$$P_X\Big\{ E_g \Big\| \sum_{i \in I(k)} g_i u_i \Big\| > 3s a_{2^k} \Big\} \le \Big(P_X\Big\{ E_g \Big\| \sum_{i \in I(k)} g_i u_i \Big\| > s a_{2^k} \Big\} \Big)^2$$

for all $s \geq 1$ since $\|u_i\| \leq a_{2^k}$, $i \in I(k)$, $k \in \mathbb{N}$. Here P_X denotes probability with respect to the sequence (X_i). It follows that

$$(\mathrm{I}) = 6\int_0^\infty P_X\left\{\sup_k \frac{1}{a_{2^k}} E_g\left\|\sum_{i\in I(k)} g_i u_i\right\| > 6t\right\} dt$$

$$\leq 6\left(1 + \sum_k \int_1^\infty P_X\left\{E_g\left\|\sum_{i\in I(k)} g_i u_i\right\| > 6ta_{2^k}\right\} dt\right)$$

$$\leq 6\left(1 + \sum_k \int_1^\infty \left(P_X\left\{E_g\left\|\sum_{i\in I(k)} g_i u_i\right\| > 2ta_{2^k}\right\}\right)^2 dt\right)$$

and thus, by (3.6),

$$(\mathrm{I}) \leq 6\left(1 + \sum_k \int_1^\infty \left(P_X\left\{E_g\left\|\sum_{i\in I(k)} g_i u_i\right\| - E\left\|\sum_{i\in I(k)} g_i u_i\right\| > ta_{2^k}\right\}\right)^2 dt\right). \tag{3.7}$$

We now apply to the averages $E_g\|\sum_{i\in I(k)} g_i u_i\|$ a technique due to Yurinskii [27], already exploited in the context of the LIL in [16]. For each k, let $\mathscr{F}_i$, $i \in I(k)$ denote the σ-field generated by the random variables $X_{2^k+1}, \dots, X_i$, and $\mathscr{F}_{2^k}$ denote the trivial field. They actually depend on k but we suppress this as well as in related quantities. Set, for all $i \in I(k)$,

$$d_i = E_X^{\mathscr{F}_i} E_g\left\|\sum_{j\in I(k)} g_j u_j\right\| - E_X^{\mathscr{F}_{i-1}} E_g\left\|\sum_{j\in I(k)} g_j u_j\right\|,$$

so that

$$\sum_{i\in I(k)} d_i = E_g\left\|\sum_{i\in I(k)} g_i u_i\right\| - E\left\|\sum_{i\in I(k)} g_i u_i\right\|.$$

Since by independence (cf. [27]),

$$E_X^{\mathscr{F}_i} E_g\left\|\sum_{\substack{j\in I(k)\\ j\neq i}} g_j u_j\right\| = E_X^{\mathscr{F}_{i-1}} E_g\left\|\sum_{\substack{j\in I(k)\\ j\neq i}} g_j u_j\right\|,$$

we see that

$$d_i = \left(E_X^{\mathscr{F}_i} - E_X^{\mathscr{F}_{i-1}}\right)(f_i), \tag{3.8}$$

where for each i in $I(k)$,

$$f_i = E_g\left\|\sum_{j\in I(k)} g_j u_j\right\| - E_g\left\|\sum_{\substack{j\in I(k)\\ j\neq i}} g_j u_j\right\|.$$

It is plain that

$$0 \le f_i \le \|u_i\|, \qquad i \in I(k). \tag{3.9}$$

The bound of f_i by $\|u_i\|$ is not enough to conclude. We need the following observation which together with the positivity of the f_i's that is proper to these averages will allow us to complete the proof.

LEMMA 3.5. *Let $Y_1, \dots, Y_n$ be independent identically distributed B-valued random variables such that $E\|Y_i\| < \infty$, $i = 1, 2, \dots, n$. Set $S_i = Y_1 + \cdots + Y_i$ for $i \le n$. Then*

$$E\|S_n\| - E\|S_{n-1}\| \le \frac{1}{n} E\|S_n\|.$$

PROOF. We have

$$S_n = \frac{1}{n-1} \sum_{i=1}^{n} \left(\sum_{\substack{j=1 \\ j \ne i}}^{n} Y_j \right).$$

Then by the triangle inequality and identical distribution,

$$E\|S_n\| \le \frac{1}{n-1} \sum_{i=1}^{n} E \left\| \sum_{\substack{j=1 \\ j \ne i}}^{n} Y_j \right\| = \frac{n}{n-1} E\|S_{n-1}\|,$$

which is the result. Note that the lemma and its proof actually express a form of the reverse martingale property of (S_n/n). □

We can now conclude the proof of Lemma 3.4. Lemma 3.5 and (3.6) imply that, for all $i \in I(k)$,

$$\begin{aligned} Ef_i &= E\left\| \sum_{j \in I(k)} g_j u_j \right\| - E\left\| \sum_{\substack{j \in I(k) \\ j \ne i}} g_i u_i \right\| \\ &\le \frac{1}{2^k} E \left\| \sum_{j \in I(k)} g_j u_j \right\| \le \frac{a_{2^k}}{2^k}. \end{aligned}$$

Now, by (3.8), (3.9) and this estimate, for i in $I(k)$,

$$\begin{aligned} Ed_i^2 &\le Ef_i^2 \le E\left(\|u_i\|^{3/2} f_i^{1/2}\right) \\ &\le \left(E\|u_i\|^3\right)^{1/2} (Ef_i)^{1/2} \le \left(E\|u_i\|^3\right)^{1/2} \left(\frac{a_{2^k}}{2^k}\right)^{1/2}. \end{aligned}$$

Finally, if we turn back to (3.7) and apply the preceding for each k,

$$(\mathrm{I}) \le 6\left(1 + \sum_k \int_1^\infty \left(P_X\left\{\sum_{i\in I(k)} d_i > ta_{2^k}\right\}\right)^2 dt\right)$$

$$\le 6\left(1 + \sum_k \int_1^\infty \frac{dt}{t^4}\frac{1}{a_{2^k}^4}\left(\sum_{i\in I(k)} Ed_i^2\right)^2\right)$$

(by orthogonality of the martingale differences d_i)

$$\le 6\left(1 + \sum_k \frac{1}{a_{2^k}^4}\left[2^k\left(E\|u_{2^k+1}\|^3\right)^{1/2}\left(\frac{a_{2^k}}{2^k}\right)^{1/2}\right]^2\right)$$

$$= 6\left(1 + \sum_k \frac{1}{a_{2^k}^3} 2^k E\|u_{2^k+1}\|^3\right)$$

$$= 6\left(1 + \sum_k \frac{1}{\sqrt{2^k}\,(2LL2^k)^{3/2}} E\left(\|X\|^3 I_{\{\|X\|\le a_{2^k}\}}\right)\right)$$

$$\le 6\left(1 + E\left(\frac{\|X\|^3}{(LL\|X\|)^{3/2}}\sum_k \frac{1}{\sqrt{2^k}} I_{\{a_{2^k}\ge\|X\|\}}\right)\right),$$

which is finite under (1.4). This completes the proof of Lemma 3.4. □

We turn to the control of (II). We first simply observe that

$$(\mathrm{II}) \le E\left(\sup_k \frac{|g_k|}{a_{2^k}} \sup_{\|x^*\|\le 1} \left|\sum_{i\in I(k)} \left(|\langle x^*, u_i\rangle|^2 - E|\langle x^*, u_i\rangle|^2\right)\right|^{1/2}\right)$$

$$+\sigma(X) E \sup_k \frac{|g_k|}{\sqrt{2LL2^k}},$$

where $\sigma(X) = \sup_{\|x^*\|\le 1}(E|\langle x^*, X\rangle|^2)^{1/2} < \infty$ under (1.5) as already observed in the introduction. This is, by the way, the only place where the weak moments (1.5) are used. Let $\alpha \ge 1$. For simplicity we let in what follows $C = C(\alpha)$ be some constant possibly changing from line to line. Since $E|g|^{2\alpha} < \infty$ for all α,

$$(\mathrm{II}) \le C\left(\sigma(X) + \left(\sum_k \frac{1}{a_{2^k}^{2\alpha}} E\left(\sup_{\|x^*\|\le 1}\left|\sum_{i\in I(k)} \left(|\langle x^*, u_i\rangle|^2 - E|\langle x^*, u_i\rangle|^2\right)\right|^\alpha\right)\right)^{1/2\alpha}\right).$$

At this point the second Gaussian argument of this proof appears through an application of the Gaussian comparison properties (cf. [6]). For each k, by independence and Jensen's inequality, as in the proof of (2.3),

$$E\left(\sup_{\|x^*\|\le 1}\left|\sum_{i\in I(k)}\left(|\langle x^*,u_i\rangle|^2 - E|\langle x^*,u_i\rangle|^2\right)\right|^\alpha\right)$$

$$\le CE\left(\sup_{\|x^*\|\le 1}\left|\sum_{i\in I(k)}\varepsilon_i|\langle x^*,u_i\rangle|^2\right|^\alpha\right)$$

$$\le CE\left(\sup_{\|x^*\|\le 1}\left|\sum_{i\in I(k)}g_i|\langle x^*,u_i\rangle|^2\right|^\alpha\right).$$

We now look, conditionally on (u_i), at the Gaussian process

$$\left\{\sum_{i\in I(k)}g_i|\langle x^*,u_i\rangle|^2;\ x^*\in B^*,\ \|x^*\|\le 1\right\}.$$

Its corresponding metric (to the square) given by

$$\sum_{i\in I(k)}\left||\langle x^*,u_i\rangle|^2 - |\langle y^*,u_i\rangle|^2\right|^2$$

for x^*, y^* of norm ≤ 1 in B^* is easily bounded by

$$4\sum_{i\in I(k)}\|u_i\|^2|\langle x^*,u_i\rangle - \langle y^*,u_i\rangle|^2,$$

so that, by [6, Théorème 2.1.2],

$$E_g\left(\sum_{\|x^*\|\le 1}\left|\sum_{i\in I(k)}g_i|\langle x^*,u_i\rangle|^2\right|^\alpha\right)\le CE_g\left(\sup_{\|x^*\|\le 1}\left|\sum_{i\in I(k)}g_i\|u_i\|\langle x^*,u_i\rangle\right|^\alpha\right)$$

$$= CE_g\left\|\sum_{i\in I(k)}g_i\|u_i\|u_i\right\|^\alpha.$$

We therefore have obtained that

$$\text{(II)}\le C\left(\sigma(X)+\left(\sum_k\frac{1}{a_{2^k}^{2\alpha}}E\left\|\sum_{i\in I(k)}g_i\|u_i\|u_i\right\|^\alpha\right)^{1/2\alpha}\right).$$

Taking $\alpha = 3$, the conclusion will then follow from the next lemma. □

LEMMA 3.6. *If* (1.4) *and* (1.6) *hold, then*

$$\sum_k\frac{1}{a_{2^k}^6}E\left\|\sum_{i\in I(k)}g_i\|u_i\|u_i\right\|^3<\infty.$$

PROOF. Since in the Hoffmann-Jørgensen inequality (2.2), $t_0\le 8.3^pE\|S_n\|$, applying this inequality for $p = 3$, it is enough to show that

$$\sum_k\frac{1}{a_{2^k}^6}E\left(\max_{i\in I(k)}|g_i|^3|u_i|^6\right)<\infty \tag{3.10}$$

and

$$\sum_k \frac{1}{a_{2^k}^6}\left(E\left\|\sum_{i\in I(k)} g_i\|u_i\|u_i\right\|\right)^3 < \infty. \tag{3.11}$$

Now

$$\sum_k \frac{1}{a_{2^k}^6} E\left(\max_{i\in I(k)} |g_i|^3\|u_i\|^6\right) \le E|g|^3 \sum_k \frac{1}{a_{2^k}^6} \sum_{i\in I(k)} E\|u_i\|^6$$

and this last series is finite under the integrability condition $E(\|X\|^2/LL\|X\|) < \infty$ as in the very final step of the proof of Lemma 3.4. To prove (3.11), since we have (1.6), we may assume from Propositions 2.3 and 2.5 that for all k,

$$E\left\|\sum_{i\in I(k)} g_i u_i\right\|^{3/2} \le a_{2^k}^{3/2}.$$

Independence, the contraction principle and Hölder's inequality then indicate that

$$\begin{aligned} E\left\|\sum_{i\in I(k)} g_i\|u_i\|u_i\right\| &\le E\left(\max_{i\in I(k)}\|u_i\|\left\|\sum_{i\in I(k)} g_i u_i\right\|\right) \\ &\le \left(E\max_{i\in I(k)}\|u_i\|^3\right)^{1/3}\left(E\left\|\sum_{i\in I(k)} g_i u_i\right\|^{3/2}\right)^{2/3} \\ &\le \left(\sum_{i\in I(k)} E\|u_i\|^3\right)^{1/3} a_{2^k}, \end{aligned}$$

and hence

$$\sum_k \frac{1}{a_{2^k}^6}\left(E\left\|\sum_{i\in I(k)} g_i\|u_i\|u_i\right\|\right)^3 \le \sum_k \frac{1}{a_{2^k}^3} \sum_{i\in I(k)} E\|u_i\|^3,$$

which is finite under (1.4). This finishes the proof of Lemma 3.6 and therefore that of Theorem 1.1. □

PROOF OF THEOREM 1.2. We first prove the necessity of (1.7); that (1.8) is necessary follows from the fact that if X satisfies the compact LIL, the sequence of laws of the $S_n(X)/a_n$ is tight with only 0 as limit point. Assume that $\{|\langle x^*, X\rangle|^2,\ x^* \in B,\ \|x^*\| \le 1\}$ is not uniformly integrable; then there is an $\varepsilon > 0$ such that

$$\limsup_{c\to\infty} \sup_{\|x^*\|\le 1} \int_{\{|\langle x^*, X\rangle| > c\}} |\langle x^*, X\rangle|^2\, dP > \varepsilon.$$

Therefore, one can construct a sequence (c_k) of positive numbers increasing to infinity and a family (x_k^*) of linear functionals of norm less than or equal to 1, such that, for each k,

$$\int_{\{\|X\| > c_k\}} |\langle x^*, X\rangle|^2 \, dP > \int_{\{|\langle x_k^*, X\rangle| > c_k\}} |\langle x^*, X\rangle|^2 \, dP > \varepsilon. \tag{3.12}$$

Let $(x_{k'}^*)$ be a weakly convergent subsequence of (x_k^*) converging to some x^*. By compactness,

$$\lim_{k' \to \infty} \sup_n \left| \left\langle x_{k'}^* - x^*, \frac{S_n(X)}{a_n} \right\rangle \right| = 0 \quad \text{a.s.}$$

The scalar LIL, for example in the form of Theorem 3.2, then implies that

$$\lim_{k' \to \infty} E|\langle x_{k'}^* - x^*, X\rangle|^2 = 0,$$

which contradicts (3.12) and thus establishes the necessity of (1.7).

We turn to sufficiency. Let $\Phi(t) = t^2/LLt$, $t \geq 0$, and define

$$\|X\|_\Phi = \inf\{a > 0\colon E\Phi(\|X\|/a) \leq 1\}.$$

It is a norm on the Orlicz space of all B-valued random variables satisfying $E(\|X\|^2/LL\|X\|) < \infty$. By the closed graph theorem, Theorem 1.1 implies the existence of a constant C such that for all random variables X with values in B,

$$IL(X) \leq C(\|X\|_\Phi + \sigma(X) + R(X)). \tag{3.13}$$

Given X satisfying (1.4), (1.7) and (1.8), consider an increasing family $(\mathscr{F}_N)_{N \in \mathbb{N}}$ of finite σ-fields generating the σ-field of X (remember that we assume that X has separable range) and let $X^N = E^{\mathscr{F}_N}X$. It is easily seen that

$$\lim_{N \to \infty} \left(\|X - X^N\|_\Phi + \sigma(X - X^N) + R(X - X^N)\right) = 0,$$

indeed, that $\lim_{N \to \infty}\|X - X^N\|_\Phi$ follows from the martingale convergence theorem as well as $\lim_{N \to \infty}\sigma(X - X^N) = 0$ together with uniform integrability. Since $E\|S_n(X)/a_n\| \to 0$ by Proposition 2.3, and since, by Jensen's inequality, for each N,

$$E\|S_n(X - X^N)/a_n\| \leq 2E\|S_n(X)/a_n\|,$$

we also have that $\lim_{N \to \infty}R(X - X^N) = 0$. Applying (3.13) to $X - X^N$ for each N yields

$$\lim_{N \to \infty} IL(X - X^N) = 0$$

and this approximation property by finite-valued random variables (satisfying thus the compact LIL) easily establishes the almost sure relative compactness of $(S_n(X)/a_n)_{n \in \mathbb{N}}$. □

Proof of Corollary 1.4. The corollary is an immediate consequence of the following proposition [8, Proposition 7.2].

Proposition 3.7. *Let X be a mean zero random variable with values in a Banach space B of type 2 satisfying $E(\|X\|^2/LL\|X\|) < \infty$; then $S_n(X)/a_n \to 0$ in probability.*

PROOF. Let X' be an independent copy of X and $\tilde{X} = X - X'$. Since $E(\|\tilde{X}\|^2/LL\|\tilde{X}\|) < \infty$, clearly

$$\lim_{n \to \infty} nP\{\|\tilde{X}\| > a_n\} = 0.$$

For each n, define the symmetric independent random variables

$$Y_i = \tilde{X}_i I_{\{\|\tilde{X}_i\| \le a_n\}}, \qquad i = 1, \dots, n.$$

For each $\varepsilon > 0$,

$$P\{\|S_n(\tilde{X})\| > \varepsilon a_n\} \le P\left\{\left\|\sum_{i=1}^n Y_i\right\| > \varepsilon a_n\right\} + nP\{\|\tilde{X}\| > a_n\}.$$

Now, since B is of type 2,

$$\begin{aligned} P\left\{\left\|\sum_{i=1}^n Y_i\right\| > \varepsilon a_n\right\} &\le \frac{C}{\varepsilon^2 a_n^2} \sum_{i=1}^n E\|Y_i\|^2 \\ &= \frac{C}{2\varepsilon^2 LLn} E\left(\|\tilde{X}\|^2 I_{\{\|\tilde{X}\| \le a_n\}}\right) \\ &\le \frac{Ct^2}{2\varepsilon^2 LLn} + \frac{C}{\varepsilon^2} E\left(\frac{\|\tilde{X}\|^2}{LL\|\tilde{X}\|} I_{\{\|\tilde{X}\| > t\}}\right) \end{aligned}$$

for each $t > 0$. Letting n and then t go to infinity yields that $S_n(\tilde{X})/a_n \to 0$ in probability. By Proposition 2.3, $E\|S_n(\tilde{X})\|/a_n \to 0$ and since $EX = 0$, Jensen's inequality furnishes the conclusion. □

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