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HOMOGENIZATION OF A DIFFUSION PROCESS IN A DIVERGENCE-FREE RANDOM FIELD

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We study the homogenization problem for a diffusion process in a divergence-free random drift field. In particular, in case of a small Gaussian field we derive an asymptotic expansion of the effective diffusion matrix in terms of its spectral density.

1. Introduction. In $\mathbb{R}^d$, $d \geq 2$, we study the asymptotic behaviour of a rescaled version of a diffusion process $X(t)$ solving

$$X(t) = \int_0^t \theta(X(s))\, ds + \sqrt{2}\,\beta(t). \tag{1.1}$$

$\beta(\cdot)$ is a standard Brownian motion in $\mathbb{R}^d$, and

$$\begin{aligned} &\theta(x) = (\theta_1(x), \dots, \theta_d(x)), \\ &\theta_k(x) = \sum_{j=1}^{d} \nabla_j H_{jk}(x), \qquad k \in \{1, \dots, d\}, \end{aligned} \tag{1.2}$$

for some stationary, ergodic, sufficiently smooth, zero-mean random field $H(\cdot) = H_{kj}(x)$, $k, j \in \{1, \dots, d\}$, $x \in \mathbb{R}^d$, with values in the space of skew symmetric matrices, i.e.,

$$H_{kj}(x) = -H_{jk}(x), \qquad x \in \mathbb{R}^d,\ k, j \in \{1, \dots, d\} \text{ a.s.}, \tag{1.3}$$

which is independent of $\beta(\cdot)$. By (1.3),

$$\operatorname{div} \theta(\cdot) \equiv \nabla \cdot \theta(\cdot) = 0 \quad \text{a.s.} \tag{1.4}$$

Therefore the process $X(\cdot)$, which has the generator

$$L = \theta \cdot \nabla + \Delta = \sum_{k, j=1}^{d} \nabla_k H_{kj}(\cdot) \nabla_j + \Delta, \tag{1.5}$$

is a Brownian motion in a "divergence-free" random drift field. We rescale $X(\cdot)$ by the usual diffusion scaling and study the processes $X_\delta(t) = \delta X(\delta^{-2} t)$ in the limit $\delta \to 0$.

In the case of bounded $H(\cdot)$, Papanicolaou and Varadhan [9] and Osada [8] proved that for almost all realizations of H the distributions of the processes $X_\delta(\cdot)$ converge weakly in the space of probability measures $\mathscr{P}(C([0, T]; \mathbb{R}^d))$ on

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path space $C([0, T]; \mathbb{R}^d)$ to the distribution of some nice diffusion process $X_0(t)$, $0 \le t \le T$, with drift 0 and a diffusion matrix $A = A_{kl}$, $k, l \in \{1, \dots, d\}$. A is given through the solution of a "resolvent equation" in some abstract Hilbert space. X_0 is called the "homogenization" of the processes X_δ with the effective diffusion matrix A.

Our starting point was to obtain a similar result for Gaussian fields H and to investigate how the effective diffusion matrix A can be expressed in terms of the correlation function of H or its spectral density. The answer to this question will be obtained in three successive steps.

As a first problem, we treat existence and uniqueness of the homogenization process $X_0(\cdot)$ for unbounded $H(\cdot)$.

FIRST RESULT. *Assume that $H(\cdot)$ and $\theta(\cdot)$ are square integrable and satisfy certain pathwise regularity and growth conditions. Then the processes $X_\delta(\cdot)$ converge in the limit as $\delta \to 0$ to some Gaussian diffusion with drift* 0 *and uniquely defined diffusion matrix $A = \mathbb{1} + D$ (cf. Theorem* 1).

Unfortunately, by the unboundedness of $H(\cdot)$ we cannot directly apply the results of [8] or [9]. To extend these results to our situation, we have to give a new proof of the unique existence of the solution of the above-mentioned "resolvent equation" (cf. Lemma 3.27). On the other hand, again the unboundedness of $H(\cdot)$ makes the use of Nash estimates or suitable generalizations (e.g., [8, Lemmas 1.1 and 1.2]) impossible, and therefore we have to be content with a weaker notion of convergence than that of weak convergence in $\mathscr{P}(C([0, T]; \mathbb{R}^d))$. For that reason we use a kind of Vaserstein-metric [cf. (2.7)] on the space of $\mathbb{R}^d$-valued random processes. Except for the solution of the resolvent equation and the use of the Vaserstein-metric, our proof of Theorem 1 essentially follows the arguments of Osada [9].

On a formal level we can describe the matrix A as follows. First we have by (1.1),

$$\delta X(\delta^{-2}t) = \delta \int_0^{\delta^{-2}t} \theta(X(s))\, ds + \delta W(\delta^{-2}t), \qquad W(t) = \sqrt{2}\,\beta(t).$$

Next let the random fields $\chi^k(\cdot)$ be defined as the solutions of

$$-L\chi^k = \theta_k = \sum_{r=1}^{d} \nabla_r H_{rk}. \tag{1.6}$$

Hence by Itô's formula,

$$\delta\chi^k(X(\delta^{-2}t)) = \delta\chi^k(0) - \delta\int_0^{\delta^{-2}t} \theta_k(X(s))\, ds + \delta\int_0^{\delta^{-2}t} \nabla\chi^k(X(s))\, dW(s).$$

If $\delta\chi(\cdot)$ is small enough, we obtain

$$\delta X^k(\delta^{-2}t) \approx \delta\int_0^{\delta^{-2}t} (1 + \nabla\chi^k(X(s)))\, dW(s),$$

i.e., $v \cdot \delta X(\delta^{-2}t)$, $v \in \mathbb{R}^d$, behaves like a martingale with quadratic variation

$$Q_\delta^v(t) = 2\delta^2 \int_0^{\delta^{-2}t} \left(v^2 + \sum_{l=1}^d \left(\sum_{k=1}^d v_k \nabla_l \chi^k(X(s)) \right)^2 + 2 \sum_{k,l=1}^d v_l v_k \nabla_l \chi^k(X(s)) \right) ds.$$

(1.3) implies that Lebesgue measure is invariant for the process $X(\cdot)$. Using this fact, the stationarity of $\nabla \chi^k(\cdot)$ and an ergodic theorem, one can conclude from the last formula that

$$\lim_{\delta \to 0} Q_\delta^v(t) = 2t \left(v^2 + \sum_{k,l=1}^d D_{kl} v_k v_l \right),$$

with

$$D_{kl} = E\left[\nabla \chi^k(0) \nabla \chi^l(0)\right].$$

(This last argument may be a weak point of this formal consideration. But, at least when H, and therefore $\nabla \chi$ too, is periodic, it seems plausible.)

A martingale convergence theorem now shows that $t \to \delta X(\delta^{-2}t)$ converges to a diffusion process with diffusion matrix $\mathbb{1} + D$. To obtain D in a more concise form, which is reminiscent of the starting point $H(\cdot)$, we apply $\nabla_l(-\Delta)^{-1}$ to (1.6). Hence we obtain for $G = G_{kl} = \nabla_k \chi^l$ and $\Gamma^f = -\nabla \otimes \nabla(-\Delta)^{-1}$ in operator notation $(\mathbb{1} + \Gamma^f H)(\mathbb{1} + G) = \mathbb{1}$, or $\mathbb{1} + G = (\mathbb{1} + \Gamma^f H)^{-1}$, i.e., since $E[G] = 0$,

$$\begin{aligned} \mathbb{1} + D &= E\left[(\mathbb{1} + G)(0)^T (\mathbb{1} + G)(0)\right] \\ &= E\left[(\mathbb{1} + \Gamma^f H)^{-1}(0)^T (\mathbb{1} + \Gamma^f H)^{-1}(0)\right]. \end{aligned} \tag{1.7}$$

The second step is the expansion of the uniquely determined diffusion matrix $A = \mathbb{1} + D$ for "small," but still not necessary bounded random fields $H(\cdot)$.

SECOND RESULT. *Assume additionally that H is L^p-integrable for any $p < \infty$. Then the effective diffusion matrix A^α corresponding to the field $\alpha H(\cdot)$ has an asymptotic expansion of the form*

$$A^\alpha = \mathbb{1} + D^\alpha = \mathbb{1} + \sum_{r=1}^M \alpha^{2r} D^{(2r)} + O(\alpha^{2M+2}), \quad \text{as } \alpha \to 0,\ M \in \mathbb{N} \tag{1.8}$$

(*cf. Theorem* 2).

(1.8) follows formally by expanding (1.7):

$$\begin{aligned} \mathbb{1} + D^\alpha &= \mathbb{1} + \sum_{l,m=1}^\infty (-\alpha)^{l+m} E\left[(\Gamma^f H)^l(0)^T (\Gamma^f H)^m(0)\right] \\ &\qquad\qquad \left(\text{since } E\left[(\Gamma^f H)^l(0)\right] = 0\right) \\ &= \mathbb{1} + \sum_{r=1}^\infty \alpha^{2r}(-1)^{r-1} E\left[(\Gamma^f H)^r(0)^T (\Gamma^f H)^r(0)\right]. \end{aligned}$$

This last equality follows from (1.3) and $\Gamma^f\Gamma^f = \Gamma^f$, which imply

$$E\left[(\Gamma^f H)^k(0)^T(\Gamma^f H)^l(0)\right] = \begin{cases} (-1)^{(k-l)/2} E\left[(\Gamma^f H)^{(k+l)/2}(0)^T(\Gamma^f H)^{(k+l)/2}(0)\right], & \text{if } k+l \text{ is even,} \\ 0, & \text{if } k+l \text{ is odd.} \end{cases}$$

Hence,

$$D^{(2r)} = (-1)^{r-1} E\left[(\Gamma^f H)^r(0)^T(\Gamma^f H)^r(0)\right]. \tag{1.9}$$

The L^p-assumption for $H(\cdot)$ is needed to assure that each matrix $D^{(2r)}$ is finite. For that we need an abstract version of the Caldéron–Zygmund theorem, which in its classical form states that each $\Gamma^f_{kl} = -\nabla_k\nabla_l(-\Delta)^{-1}$ is bounded on any $L^p(\mathbb{R}^d)$, $p < \infty$.

As final step we consider Gaussian random fields $H(\cdot)$.

THIRD RESULT. *Assume that $H(\cdot)$, and therefore $\theta(\cdot)$ too, is Gaussian. Then the matrices $D^{(2r)}$ can be evaluated in terms of the spectral density of $\theta(\cdot)$ (cf. Theorem* 3).

The organization of this paper is the following. Next we present a list of the notation which is used. Then we describe in Section 2 our results, whereas their proofs are deferred to Sections 3–5.

1.1. *Notation.* In this section we collect the notation which is used in this paper. By $L^p(\mathbb{R}^d)$, respectively, $L^p(\mathbb{R}^d; \mathbb{R}^d)$, we denote the usual spaces of $\mathbb{R}$-, respectively, $\mathbb{R}^d$-, valued functions on $\mathbb{R}^d$ with bounded $\|\cdot\|_p$-norm. Moreover,

$$H_1^2(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \colon \|\nabla f\|_2 < \infty \right\},$$

$$\overline{H}^2_{-1}(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \colon f = \nabla\cdot g \ (= \operatorname{div} g) \text{ for some } g \in L^2(\mathbb{R}^d; \mathbb{R}^d) \right\},$$

$C^k(\mathbb{R}^d)$ = space of real-valued functions on $\mathbb{R}^d$ with continuous partial derivatives of all orders $\leq k$, $k \in \{0, 1, \dots, \infty\}$,

$$C_c^k(\mathbb{R}^d) = \left\{ f \in C^k(\mathbb{R}^d) \colon f \text{ has compact support} \right\}, \qquad k \in \{0, 1, \dots, \infty\},$$

$$C^*(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \colon \tilde{f}(\cdot) \text{ has compact support} \right\},$$

where

$$\tilde{f}(\mu) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\mu x} f(x)\, dx$$

is the Fourier transform of f.

In Section 3 we need several spaces of functions on a probability space $(\boldsymbol{\Omega}, \mathscr{A}, \mathbf{P})$. These spaces are defined quite similarly as related spaces of functions on $\mathbb{R}^d$, with the operators $\hat{D}_k$ [cf. (3.6)] replacing the usual derivatives ∇_k. In particular, we define

$$H_1^2(\Omega) = \left\{ \hat{f} \in L^2(\Omega) \colon \hat{D}_k \hat{f} \in L^2(\Omega),\ k \in \{1, \dots, d\} \right\},$$

$$\overline{H}_{-1}^2(\Omega) = \left\{ \hat{f} \in L^2(\Omega) \colon \hat{f} = \sum_{k=1}^{d} \hat{D}_k \hat{g}_k = \hat{D} \cdot \hat{g} \text{ for some } \hat{g} \in L^2(\Omega; \mathbb{R}^d) \right\},$$

$$C_0^\infty(\Omega) = \left\{ \hat{f} \colon \Omega \to \mathbb{R} \colon \left(\hat{D}_1^{m_1} \cdots \hat{D}_d^{m_d} \right) \hat{f} \text{ exists and is bounded for any } m_1, \dots, m_d \in \{0, 1, 2, \dots\} \right\},$$

$H_{1,0}^{p,c}(\mathbb{R}^d \times \Omega) =$ space of those real-valued functions on $\mathbb{R}^d \times \Omega$ with $f(\cdot,\cdot)$, $\nabla_x f(\cdot,\cdot) \in L^p(\mathbb{R}^d \times \Omega)$, respectively, $L^p(\mathbb{R}^d \times \Omega; \mathbb{R}^d)$, and $\operatorname{supp}(f) = K \times \Omega$, where $K \subseteq \mathbb{R}^d$ is compact, $1 \le p \le \infty$. (∇_x denotes the "gradient" with respect to the spatial variable x.)

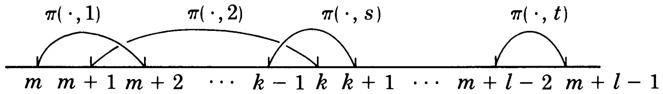
We denote by $\|\cdot\|_p$, respectively, $|||\cdot|||_p$, the norm in $L^p(\mathbb{R}^d)$, respectively, $L^p(\Omega)$. $\langle f, g \rangle = \int_{\mathbb{R}^d} f(x) g(x)\, dx$, respectively, $\langle\langle \hat{f}, \hat{g} \rangle\rangle = \int_\Omega \hat{f}(\omega) \hat{g}(\omega) \mathbf{P}(d\omega)$, whenever the right side is well defined.

$\mathscr{L}(X)$ denotes the distribution of some random variable or process X. By $C_k(a, b, \dots)$ we denote constants depending on parameters $a, b, \dots$.

To write down the coefficients of the expansion for the diffusion matrix of the limit process X_0, we need some additional notations.

Let $I(k) = \{1, 2, \dots, k\}$. For any interval $I = \{m, m+1, \dots, m+l-1\}$, ($l$ even) of $\mathbb{N}$, we denote by $\mathscr{P}_2^I$ the set of all exhaustive pair partitions of I. Any $\pi \in \mathscr{P}_2^I$ consists of the elements

$$\{\pi(1,1), \pi(2,1)\}, \{\pi(1,2), \pi(2,2)\}, \dots, \{\pi(1,r), \pi(2,r)\}, \dots,$$
$$\{\pi(1, l/2), \pi(2, l/2)\} \quad \text{with } \pi(1,r) < \pi(2,r),\ r \in \{1, 2, \dots, l/2\}.$$



The set of all "bonds" $\{m, m+1\}, \{m+1, m+2\}, \dots, \{m+l-2, m+l-1\}$ may be enumerated by $I' = I \setminus \{m+l-1\}$, i.e., the left endpoints. Let $\Lambda(r, \pi)$ be the set of pairs $\{\pi(1,n), \pi(2,n)\}$ "encircling" the bond $\{r, r+1\}$, i.e., $\Lambda(r, \pi) = \{n \le l/2 \colon \pi(1,n) \le r < \pi(2,n)\}$. By $H_1(\pi)$, respectively, $H_2(\pi)$, we denote the set of left, respectively, right, endpoints of pairs in π, i.e.,

$$H_1(\pi) = \{q \in I \colon q = \pi(1,r) \text{ for some } r \in \{1, \dots, l/2\}\},$$
$$H_2(\pi) = \{q \in I \colon q = \pi(2,r) \text{ for some } r \in \{1, \dots, l/2\}\}.$$

The sets $H_1(\pi)$ and $H_2(\pi)$ are disjoint with $H_1(\pi) \cup H_2(\pi) = I$. Moreover they generate a partition of I' into three sets,

$$B_0(\pi) = \{r \in I' \colon r \in H_1(\pi),\ r+1 \in H_2(\pi)\},$$
$$B_1(\pi) = \{r \in I' \colon r, r+1 \in H_1(\pi) \text{ or } r, r+1 \in H_2(\pi)\}$$

and

$$B_2(\pi) = \{r \in I' \colon r \in H_2(\pi),\, r+1 \in H_1(\pi)\}.$$

Obviously $|B_1(\pi)|$ is even, since

$$|B_1(\pi)| + 2|B_0(\pi)| = l. \tag{1.10}$$

For $q \in B_1(\pi)$, let

$$q'(\pi) = \begin{cases} q, & \text{if } q \in H_2(\pi), \\ q+1, & \text{if } q \in H_1(\pi). \end{cases}$$

Finally, let

$$\mathscr{P}_2^{I,e} = \{\pi \in \mathscr{P}_2^I \colon \Lambda(q,\pi) \neq \varnothing \text{ for all } q \in I'\},$$

i.e., $\mathscr{P}_2^{I,e}$ is the set of "elementary" pair partitions.

2. Formulation of the results. Instead of boundedness assumptions we assume in addition to (1.3) that the random field $x \to H(x) = H(x,\omega)$, $\omega \in \Omega$, which is defined on some separable probability space $(\Omega, \mathscr{A}, \mathbf{P})$ with expectation operator $\mathbf{E}$, satisfies

$$|||H_{kl}(x,\cdot)|||_2^2 = \mathbf{E}\big[|H_{kl}(x,\cdot)|^2\big] < \infty, \qquad k,l \in \{1,\dots,d\}, \tag{2.1}$$

$$x \to H_{kl}(x,\omega) \in C^2(\mathbb{R}^d), \qquad k,l \in \{1,\dots,d\}, \quad \mathbf{P}\text{-a.s.} \tag{2.2}$$

and

$$\sup_{x\in\mathbb{R}^d} \frac{1}{1+|x|} \left| \frac{\partial^{k_1+\cdots k_d}}{(\partial x_1)^{k_1}\cdots(\partial x_d)^{k_d}} H_{kl}(x,\omega) \right| < \infty, \tag{2.3}$$

$$k_1,\dots,k_d \in \{0,1,2\},\ k_1+\cdots+k_d \le 2,\ k,l \in \{1,\dots,d\}, \mathbf{P}\text{-a.s.}$$

Additionally, we shall use

$$|||\theta_k(x,\cdot)|||_2^2 = \mathbf{E}\big[|\theta_k(x,\cdot)|^2\big] < \infty, \qquad k \in \{1,\dots,d\}, \tag{2.4}$$

where $\theta_k(\cdot,\cdot)$ is defined in (1.2).

For fixed $\omega \in \Omega$, let $t \to X(t,\omega)$ be the diffusion process solving

$$X(t,\omega) = \int_0^t \theta(X(s,\omega),\omega)\,ds + W(t), \tag{2.5}$$

where $W(t) = \sqrt{2}\,\beta(t)$, $0 \le t < \infty$, for a standard Brownian motion $\beta(\cdot)$ in $\mathbb{R}^d$, which is stochastically independent of the random fields $H(\cdot)$ and $\theta(\cdot)$. Therefore we can assume that the random elements in our setting, i.e., the random fields $H(\cdot)$ and $\theta(\cdot)$ and the Brownian motion $W(\cdot)$, are defined on a probability space $(\underline{\Omega}, \underline{\mathscr{A}}, (\underline{\mathscr{A}}_t)_{t\ge 0}, \underline{\mathbf{P}})$, $\underline{\mathscr{A}}_t$ some filtration in $\underline{\mathscr{A}}$, such that H and θ are measurable with respect to $(\underline{\Omega}, \underline{\mathscr{A}}_0, \underline{\mathbf{P}}|_{A_0}) = (\Omega, \mathscr{A}, \mathbf{P})$, and $W(\cdot)$ is $(\underline{\mathscr{A}}_t)_{t\ge 0}$-adapted.

REMARK 2.6. In our results we shall consider the whole family $\{X(\cdot,\omega)\colon \omega \in \Omega\}$ as one $(\underline{\mathscr{A}}_t)_{t\ge 0}$-adapted random process $t \to X(t)$.

By the unboundedness of $H(\cdot)$ and $\theta(\cdot)$, we cannot use estimates like those in [8, Lemma 1.1] to show the weak convergence of the distributions of the processes $t \to X_\delta(t) = \delta X(\delta^{-2}t)$ as $\delta \to 0$. For the weaker notion of convergence we have to be content with, we introduce on the space $S_T^\#$ of all $\mathbb{R}^d$-valued random processes $Y = Y(t)$, $0 \le t \le T$, the metric

$$p_T^\#(Y, Z) = \inf_{Q \in K(Y,Z)} E\left[\int_0^T 1 \wedge |Q_1(t) - Q_2(t)|\,dt\right], \tag{2.7}$$

where $K(Y, Z)$ is the set of all $\mathbb{R}^d \times \mathbb{R}^d$-valued processes $Q(\cdot) = (Q_1(\cdot), Q_2(\cdot)) = (Q_1(t), Q_2(t))$, $0 \le t \le T$, such that $\mathscr{L}(Q_1(\cdot)) = \mathscr{L}(Y(\cdot))$ and $\mathscr{L}(Q_2(\cdot)) = \mathscr{L}(Z(\cdot))$.

We can formulate now our first result.

THEOREM 1. *Assume that $H(\cdot)$ is a stationary ergodic zero-mean matrix-valued random field on $\mathbb{R}^d$, $d \ge 2$, satisfying* (1.3) *and* (2.1)–(2.3). *Furthermore let $\theta(\cdot)$, defined in* (1.2), *satisfy* (2.4). *Next let $X_\delta(t) = \delta X(\delta^{-2}t)$, where $X(t)$ is considered in the sense of Remark* 2.6 [*i.e.*, $(\Omega, \underline{\mathscr{A}}, (\underline{\mathscr{A}}_t)_{t \ge 0}, \underline{\mathbf{P}})$ *measurable*] *and $X_0(\cdot)$ the Gaussian diffusion with generator $\Delta + \sum_{k,l=1}^d D_{kl}\nabla_k\nabla_l$ and $X_0(0) = 0$, where the positive semidefinite matrix $D = D_{kl}$, $k, l \in \{1, \dots, d\}$, is defined in* (3.38) *and* (3.44). *Then for any fixed $T > 0$,*

$$\lim_{\delta \to 0} p_T^\#(X_\delta(\cdot), X_0(\cdot)) = 0.$$

Consequences of Theorem 1, respectively, its proof, which can be found in Section 3, are:

COROLLARY 2.8. *For $\omega \in \Omega$, $\beta, \delta > 0$ and $g \in L^2(\mathbb{R}^d)$, let $u_\delta(\cdot, \omega)$, respectively, $u_0(\cdot)$, be the solution of the resolvent equation*

$$\left(-\Delta - \sum_{k,l=1}^d \nabla_k H_{\delta,kl}(\cdot, \omega)\nabla_l + \beta\right) u_\delta(\cdot, \omega) = g(\cdot), \tag{2.9}$$

respectively,

$$\left(-\Delta - \sum_{k,l=1}^d D_{kl}\nabla_k\nabla_l + \beta\right) u_0(\cdot) = g(\cdot),$$

where $H_\delta(x, \omega) = H(x/\delta, \omega)$. Then

$$\lim_{\delta \to 0} \int_\Omega \mathbf{P}(d\omega)\|u_\delta(\cdot, \omega) - u_0(\cdot)\|_2^2 = 0. \tag{2.10}$$

If $g \in L^2(\mathbb{R}^d) \cap C^0(\mathbb{R}^d)$, then for any $x \in \mathbb{R}^d$,

$$\lim_{\delta \to 0} \int_\Omega \mathbf{P}(d\omega)|u_\delta(x, \omega) - u_0(x)|^2 = 0. \tag{2.11}$$

COROLLARY 2.12.

$$|D_{kl}| \le \left(\sum_{j=1}^d |||H_{jk}|||_2^2\right)^{1/2} \left(\sum_{j=1}^d |||H_{jl}|||_2^2\right)^{1/2} \le \sum_{m,n=1}^d |||H_{mn}|||_2^2.$$

Although the unique existence of an effective diffusion matrix $\mathbb{1} + D$ is deduced by Theorem 1, its representation through the solution of an abstract differential equation (3.38) and (3.44) is unsatisfactory. A problem that naturally arises is the calculation of D in terms of the random field $H(\cdot)$. One possibility is to derive an asymptotic expansion for small H based on modifying the classical scheme for L^∞ perturbation of the diffusion matrix [as in the work of Bergman, Milton, Golden and Papanicolaou (cf. [5] and the references therein)]. More precisely, for any $\alpha > 0$ we replace $H(\cdot)$ by $H^\alpha(\cdot) = \alpha H(\cdot)$. This random field $H^\alpha(\cdot)$ defines a process $X^\alpha(\cdot)$. By Theorem 1 the rescaled processes $X_\delta^\alpha(t) = \delta X^\alpha(\delta^{-2}t)$, $0 \le t \le T$, converge for any fixed $\alpha > 0$ as $\delta \to 0$ to some Gaussian diffusion $X_0^\alpha(\cdot)$ with infinitesimal generator $\Delta + \sum_{k,l=1}^d D_{kl}^\alpha \nabla_k \nabla_l$. We shall derive an asymptotic expansion for D_{kl}^α as $\alpha \to 0$.

THEOREM 2. *Assume* (1.3) *and* (2.2)–(2.4) *and, additionally,*

$$\||H_{kl}(x,\cdot)\||_p^p = \mathbf{E}\big[|H_{kl}(x,\cdot)|^p\big] < \infty, \qquad k,l \in \{1,\dots,d\},\ p \in [1,\infty). \tag{2.13}$$

Then for any $M \in \mathbb{N}$,

$$D_{kl}^\alpha = \sum_{r=1}^M \alpha^{2r} D_{kl}^{(2r)} + O(\alpha^{2M+2}) \quad \text{as } \alpha \to 0. \tag{2.14}$$

The precise definition of the coefficients $D_{kl}^{(2r)}$, *which are formally derived in* (1.9), *is given in* (4.15). *The matrices* $(-1)^{r+1}D^{(2r)}$, $r \in \{1,2,\dots\}$, *are positive semidefinite. Moreover the expansion* (2.14) *converges uniformly for* α *small enough if* $H(\cdot,\cdot)$ *is uniformly bounded.*

This theorem is proved in Section 4.

Probably an expansion like (2.14) is only of theoretical importance. In practice one would like to express the coefficients $D_{kl}^{(2r)}$ in terms of the correlation functions of the random field H. Then one has to replace the relation (4.15) defining $D_{kl}^{(2r)}$ in terms of H and the abstract operators Γ_{kl} in $L^p(\Omega)$ by a corresponding expression involving integrals over Euclidean spaces, correlation functions of H and concrete integral operators. Such a program may be quite tedious for general H.

For that reason, we assume now in addition to (1.3) that the random field $H(\cdot)$ is Gaussian with mean 0, since in this case all correlation functions are completely determined by the covariance functional

$$R_{kl,mn}^H(x) = \mathbf{E}\big[H_{kl}(x,\cdot)H_{mn}(0,\cdot)\big], \qquad k,l,m,n \in \{1,\dots,d\},\ x \in \mathbb{R}^d.$$

We assume that H has a spectral density $\sigma_{kl,mn}^H(\cdot)$, i.e., that $R_{kl,mn}^H(\cdot)$ can be written as

$$R_{kl,mn}^H(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \sigma_{kl,mn}^H(\mu) e^{-i\mu x}\, d\mu, \qquad k,l,m,n \in \{1,\dots,d\},\ x \in \mathbb{R}^d. \tag{2.15}$$

It is not hard to conclude from [1, Theorems 3.4.3, 6.5.3 and 6.9.4], that the assumptions of Theorem 2, in particular, the ergodicity and (2.2)–(2.4) and (2.13) are satisfied if

$$\sigma^H_{kl,mn}(\cdot) \in L^\infty(\mathbb{R}^d), \qquad k, l, m, n \in \{1, \ldots, d\}, \tag{2.16}$$

and

$$\int_{\mathbb{R}^d} \left|\sigma^H_{kl,mn}(\mu)\right|\left(1 + |\mu|^8\right) d\mu < \infty, \qquad k, l, m, n \in \{1, \ldots, d\}. \tag{2.17}$$

(1.2) and (2.17) imply that $\theta(\cdot)$ has a spectral density $\sigma^\theta_{k,l}(\cdot)$ too, namely,

$$\sigma^\theta_{k,l}(\mu) = \sum_{j,r=1}^{d} \mu_j \mu_r \sigma^H_{jk,rl}(\mu), \qquad k, l \in \{1, \ldots, d\},\ \mu \in \mathbb{R}^d. \tag{2.18}$$

To avoid the complications caused by the fact that the covariance functional and the spectral density of H are fourth order tensor fields on $\mathbb{R}^d$, we shall continue our calculations now in terms of the random field θ, respectively, its covariance functional and spectral density, which are tensor fields of order 2 only.

For the following result we use the definitions about pair partitions given in Section 1.1.

THEOREM 3. *Suppose that H, respectively, θ, is Gaussian. Moreover, assume $d \geq 3$ and* (1.3), (2.16) *and* (2.17). *Then the coefficients $D^{(2r)}_{kl}$ of* (2.14) *can be expressed in terms of the spectral density of the random field $\theta(\cdot)$. The explicit formulas are given by*

$$D^{(2r)}_{uv} = \sum_{\pi \in \mathscr{P}_2^{I(2r),e}} D^{I(2r),\pi}_{uv}, \tag{2.19}$$

with

$$\begin{aligned} D^{I(l),\pi}_{uv} = {} & (2\pi)^{-ld/4} \delta_{up_1} \delta_{vp_l} \int_{\mathbb{R}^d} \cdots \int_{R^d} \left(\prod_{r=1}^{l/2} \sigma^\theta_{p_{\pi(1,r)}, p_{\pi(2,r)}}(\mu^r) \right) \\ & \times \left(\prod_{q \in B_0(\pi)} \left| \sum_{s \in \Lambda(q,\pi)} \mu^s \right|^{-2} \right) \left(\prod_{q \in B_1(\pi)} \left(\sum_{s \in \Lambda(q,\pi)} \mu^s_{p_{q'(\pi)}} \right) \left| \sum_{s \in \Lambda(q,\pi)} \mu^s \right|^{-2} \right) \\ & \times \left(\prod_{q \in B_2(\pi)} \left(\sum_{s \in \Lambda(q,\pi)} \mu^s_{p_q} \right) \left(\sum_{s \in \Lambda(q,\pi)} \mu^s_{p_{q+1}} \right) \left| \sum_{s \in \Lambda(q,\pi)} \mu^s \right|^{-2} \right) \\ & \times (-1)^{l/2-1} d\mu^1 \cdots d\mu^{l/2}, \end{aligned} \tag{2.20}$$

for any $\pi \in \mathscr{P}_2^{I(l),e}$. In (2.20) *repeated indices are summed for $1, \ldots, d$. In particular,*

$$\begin{aligned} D^{(2)}_{kl} &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \sigma^\theta_{kl}(\mu) |\mu|^{-2} \, d\mu \\ &= \frac{\Gamma(d/2)}{(d-2)2\pi^{d/2}} \int_{\mathbb{R}^d} R^\theta_{kl}(x) |x|^{2-d} \, dx. \end{aligned} \tag{2.21}$$

The proof of Theorem 3 is given in Section 5.

REMARK 1. The proof of Theorem 3 shows that (2.21) remains true for non-Gaussian H, respectively, θ, as long as in addition to (1.3), (2.2)–(2.4) and (2.13) the assumptions on the spectral density, namely (2.16) and (2.17), are valid.

REMARK 2. The assumption $d \geq 3$ has technical reasons. For $d \geq 3$ certain integrals occurring during the proof of Theorem 3 are obviously convergent, whereas for $d = 2$ additional reasoning without further insight would be necessary to show that these integrals are well defined.

REMARK 3. For the proof of Theorem 3 we could use, of course, the representation of $D_{kl}^{(2r)}$ through (4.2) and (4.15). This would first require the translation of (4.15) into a more concrete expression involving integrals over $\mathbb{R}^d$, correlation function of H and integral operators in $\mathbb{R}^d$, and the subsequent evaluation of this new expression. We believe that this way has the conceptual disadvantage that the path from the concrete homogenization problem to its solution would follow a detour through a very abstract region, namely, the proofs of Theorems 1 and 2. This abstract formalism is not so helpful if one is mainly interested in a concrete representation of the matrices $D^{(2r)}$. For that reason we use in the proof of Theorem 3 a method which starts from a perturbation expansion of the solution of the resolvent equation (2.9) and which yields the desired expressions for $D_{kl}^{(2r)}$ as limits of certain "ordinary" integrals over $\mathbb{R}^d$ as $\delta \to 0$. In our opinion this way has the advantage that it uses only well known $L^2(\mathbb{R}^d)$-techniques and therefore is rather elementary, although far from being combinatorically trivial.

3. Homogenization of the process $X(\cdot)$.

3.1. *Proof of Theorem* 1. To define the random field $H(\cdot)$ in a suitable precise way, we can and do assume the following setting (cf. [8]): Let $(\Omega, \mathbf{d})$ be a separable metric space, $\mathscr{A}$ the Borel-σ-algebra and $\mathbf{P}$ a probability measure on $(\Omega, \mathscr{A})$. Next let $\hat{t} = \hat{t}_x$, $x \in \mathbb{R}^d$, a d-dimensional ergodic stochastically continuous flow on the probability space $(\Omega, \mathscr{A}, \mathbf{P})$, i.e.,

(3.1) $\hat{t}_x$ is a measure preserving transformation of Ω into Ω for any $x \in \mathbb{R}^d$,

$$\hat{t}_0 = id \quad \text{and} \quad \hat{t}_{x+y} = \hat{t}_x \circ \hat{t}_y, \qquad x, y \in \mathbb{R}^d, \tag{3.2}$$

(3.3) the only elements A of $\mathscr{A}$ with $\mathbf{P}[A \Delta_s \hat{t}_x^{-1} A] = 0$ for some $x \in \mathbb{R}^d \setminus \{0\}$, where Δ_s denotes the symmetric difference of sets, are those with $\mathbf{P}[A] = 0$ or $\mathbf{P}[A] = 1$

and

$$\lim_{h \to 0} \mathbf{P}\big[\omega \in \Omega \colon \mathbf{d}\big(\hat{t}_x \omega, \hat{t}_{x+h}\omega\big) > \delta\big] = 0 \qquad x \in \mathbb{R}^d,\ \delta > 0. \tag{3.4}$$

In this case the maps $\hat{T}_x$, $x \in \mathbb{R}^d$, defined by $(\hat{T}_x \hat{f})(\omega) = \hat{f}(\hat{t}_x \omega)$, $\hat{f} \in L^2(\Omega)$, on $L^2(\Omega) = L^2(\Omega, \mathscr{A}, \mathbf{P})$ constitute a strongly continuous unitary group $\hat{T}$, which has a "spectral representation"

$$\hat{T}_x = \int_{\mathbb{R}^d} e^{i\mu x} \hat{U}(d\mu), \tag{3.5}$$

where $\hat{U}(\cdot)$ is a "resolution of the identity."

Let $\hat{D}_k$, $k \in \{1, \dots, d\}$, be the infinitesimal generators of $\hat{T}$, i.e.,

$$(\hat{D}_k \hat{f})(\omega) = \lim_{h \to 0} \frac{1}{h}\left((\hat{T}_{he_k} \hat{f})(\omega) - \hat{f}(\omega)\right), \qquad \hat{f} \in \mathrm{dom}(\hat{D}), \tag{3.6}$$

where e_k is the unit vector in the kth direction of $\mathbb{R}^d$ and $\mathrm{dom}(\hat{D})$ the set of those $\hat{f} \in L^2(\Omega)$ for which the limit in (3.6) exists in $L^2(\Omega)$ for any $k \in \{1, \dots, d\}$. Note that the adjoint of $\hat{D}_k$ is $-\hat{D}_k$.

By stationarity we can assume that the random field H, respectively θ, is obtained from the fixed $\mathbb{R}^d \times \mathbb{R}^d$- ($\mathbb{R}^d$-) valued random variable $\hat{H}(\omega) = H(0, \omega)$, respectively, $\hat{\theta}(\omega) = \theta(0, \omega)$, as

$$H(x, \omega) = (\hat{T}_x \hat{H})(\omega), \tag{3.7}$$

respectively,

$$\theta(x, \omega) = (\hat{T}_x \hat{\theta})(\omega), \tag{3.8}$$

where (3.7) and (3.8) hold coordinatewise. Note that (1.2) implies

$$\hat{\theta}_k = \sum_{j=1}^{d} \hat{D}_j \hat{H}_{jk}. \tag{3.9}$$

By (2.2) and (2.3) the process $X(\cdot, \omega) = X(t, \omega)$, $0 \le t < \infty$, is for $\mathbf{P}$-almost all $\omega \in \Omega$ a well defined time homogenous diffusion process with a transition density $p(t, x, y, \omega)$, which by (1.4) satisfies

$$\int_{\mathbb{R}^d} p(t, x, y, \omega)\, dx = 1, \qquad t > 0,\ y \in \mathbb{R}^d,\ \mathbf{P}\text{-a.s.} \tag{3.10}$$

(cf. [8, Lemma 2.2]).

Moreover (2.3) implies that $\mathbf{P}$-a.s. for any $t > 0$ the distribution of $X(s, \omega)$, $0 \le s \le t$, is equivalent to that of Brownian motion (Girsanov–Cameron–Martin formula). Therefore,

$$\int_A p(t, x, y, \omega)\, dy > 0 \quad \text{for any } x \in \mathbb{R}^d,\ t > 0,\ A \subseteq \mathbb{R}^d \text{ open},\ \mathbf{P}\text{-a.s.} \tag{3.11}$$

(3.10) and (3.11) allow us to repeat the proof of [8, Proposition 2.1] word for word to obtain

LEMMA 3.12. *Let*

$$Z(\omega, s) = \hat{t}_{X(s, \omega)}\omega, \qquad Z(\omega, 0) = \omega. \tag{3.13}$$

For this Ω-valued process the probability measure $\mathbf{P}$ *is invariant and ergodic.*

The maps $\hat{V} = \hat{V}_t$, $0 \le t < \infty$, defined by

$$(\hat{V}_s \hat{f})(\omega) = E^1[\hat{f}(Z(\omega, s))], \qquad \hat{f} \in L^\infty(\Omega), \tag{3.14}$$

where $E^1[\cdots]$ in (3.14) means expectation with respect to the Brownian motion occurring in $X(\cdot, \omega)$, constitute a positive contraction semigroup in $L^\infty(\Omega)$, which by the definitions of $X(\cdot,\cdot)$, $\hat{H}(\cdot)$ and $\hat{D}_k$ has the infinitesimal generator

$$\hat{L} = \sum_{k=1}^d \hat{D}_k \hat{D}_k + \sum_{k,l=1}^d \hat{D}_k \hat{H}_{kl}(\cdot)\hat{D}_l = \hat{D}^2 + \sum_{k=1}^d \hat{\theta}_k(\cdot)\hat{D}_k,$$

defined on some suitable domain. Since the adjoint $\hat{L}^*$ of $\hat{L}$ equals $\hat{L}^* = \hat{D}^2 - \sum_{k,l=1}^d \hat{D}_k \hat{H}_{kl}(\cdot)\hat{D}_l$, we obtain for any positive $\hat{g} \in C_0^\infty(\Omega)$,

$$\begin{aligned}\langle\langle \hat{V}_t \hat{g}, 1\rangle\rangle &= \langle\langle \hat{g}, 1\rangle\rangle + \int_0^t \langle\langle \hat{L}\hat{V}_s \hat{g}, 1\rangle\rangle\, ds \\ &= \langle\langle \hat{g}, 1\rangle\rangle + \int_0^t \langle\langle \hat{V}_s \hat{g}, \hat{L}^* 1\rangle\rangle\, ds = \langle\langle \hat{g}, 1\rangle\rangle.\end{aligned}$$

We can conclude that $\hat{V}_t$, $t \ge 0$, is a contraction semigroup both on $L^\infty(\Omega)$ and on $L^1(\Omega)$ and hence by interpolation on any $L^p(\Omega)$, $1 \le p \le \infty$. Consequently, the resolvent $\hat{R}_\beta$, $\beta > 0$,

$$\hat{R}_\beta = \int_0^\infty e^{-\beta t}\hat{V}_t\, dt$$

is a well defined positive operator on any $L^p(\Omega)$ with

$$|||\hat{R}_\beta \hat{g}|||_p \le \frac{1}{\beta}|||\hat{g}|||_p, \qquad \hat{g} \in L^p(Z). \tag{3.15}$$

Note that by (2.4), the domain of $\hat{L}$ considered as the generator of $\hat{V}$ in $L^2(\Omega)$ contains $C_0^\infty(\Omega)$.

Next the second summand of $\hat{L}$, namely,

$$\hat{B} = \hat{\theta}(\cdot) \cdot \hat{D} = \sum_{k=1}^d \hat{\theta}_k(\cdot)\hat{D}_k = \sum_{k,l=1}^d \hat{D}_k \hat{H}_{kl}(\cdot)\hat{D}_l, \tag{3.16}$$

satisfies, by (1.3),

$$\langle\langle \hat{B}\hat{f}, \hat{f}\rangle\rangle = -\left\langle\!\!\left\langle \sum_{k,l=1}^d \hat{H}_{lk}(\cdot)(\hat{D}_k \hat{f}), (\hat{D}_l \hat{f})\right\rangle\!\!\right\rangle = 0,\ \hat{f} \in C_0^\infty(\Omega). \tag{3.17}$$

Since $\hat{L}$ is the infinitesimal generator of the semigroup $\hat{V}$, the function $\hat{f}_\beta = \hat{R}_\beta \hat{g}$ is the unique solution of

$$(-\hat{L} + \beta)\hat{f}_\beta = \hat{g}, \quad \text{i.e., } \hat{R}_\beta = (-\hat{L} + \beta)^{-1}; \tag{3.18}$$

cf. [3], where the meaning of (3.18) is made precise.

(3.18) can be written in the weak form

$$\langle\langle \hat{D}\hat{f}_\beta, \hat{D}\hat{h}\rangle\rangle + \sum_{k,l=1}^{d} \langle\langle \hat{H}_{lk}\hat{D}_k\hat{f}_\beta, \hat{D}_l\hat{h}\rangle\rangle + \beta\langle\langle \hat{f}_\beta, \hat{h}\rangle\rangle = \langle\langle \hat{g}, \hat{h}\rangle\rangle, \tag{3.19}$$
$$\hat{h} \in C_0^\infty(\Omega).$$

(3.17) implies

$$\begin{aligned}\langle\langle(-\hat{L}+\beta)\hat{f}, \hat{f}\rangle\rangle &= \langle\langle(-\hat{D}^2+\beta)\hat{f}, \hat{f}\rangle\rangle \\ &= |||\hat{D}\hat{f}|||_2^2 + \beta|||\hat{f}|||_2^2, \qquad \hat{f} \in C_0^\infty(\Omega).\end{aligned} \tag{3.20}$$

It is not hard to show

$$|||\hat{D}\hat{R}_\beta\hat{g}|||_2 < \infty, \qquad \hat{g} \in L^2(\Omega), \beta > 0. \tag{3.21}$$

(3.21) is obvious [by (1.3), (3.15) and (3.19)] for bounded $\hat{H}(\cdot)$ and then can be extended to the present situation by (2.1) and (2.3). Hence we can insert $\hat{f} = \hat{f}_\beta = \hat{R}_\beta\hat{g}$ into (3.20) to obtain, with (3.18),

$$\langle\langle \hat{g}, \hat{R}_\beta\hat{g}\rangle\rangle = |||\hat{D}\hat{R}_\beta\hat{g}|||_2^2 + \beta|||\hat{R}_\beta\hat{g}|||_2^2, \qquad \hat{g} \in L^2(\Omega).$$

This yields for any $\hat{g} \in \overline{H}^2_{-1}(\Omega)$, with $\hat{g} = \hat{D}\cdot\hat{w} = \sum_{k=1}^d \hat{D}_k\hat{w}_k$,

$$|||\hat{D}\hat{R}_\beta\hat{g}|||_2^2 + \beta|||\hat{R}_\beta\hat{g}|||_2^2 = -\langle\langle \hat{w}, \hat{D}\hat{R}_\beta\hat{g}\rangle\rangle \le \tfrac{1}{2}|||\hat{w}|||_2^2 + \tfrac{1}{2}|||\hat{D}\hat{R}_\beta\hat{g}|||_2^2 \tag{3.22}$$

and therefore,

$$|||\hat{D}\hat{R}_\beta\hat{g}|||_2 \le |||\hat{w}|||_2 \tag{3.23}$$

and

$$|||\hat{R}_\beta\hat{g}|||_2 \le (2\beta)^{-1/2}|||\hat{w}|||_2. \tag{3.24}$$

In particular we have

$$\lim_{\beta\to 0} \beta|||\hat{R}_\beta\hat{g}|||_2 = 0. \tag{3.25}$$

(3.23) shows that for any $\hat{g} \in \overline{H}^2_{-1}(\Omega)$, the set $\{\hat{D}\hat{R}_\beta\hat{g}\colon \beta > 0\}$ is bounded in $L^2(\Omega; \mathbb{R}^d)$ and therefore suggests that $\hat{D}\hat{R}_\beta\hat{g}$ is well defined even for $\beta = 0$. Moreover by (3.19) and (3.25), one expects that $\hat{F}_0 = \hat{D}\hat{R}_0\hat{g} = (\hat{F}_{0,1}, \dots, \hat{F}_{0,d})$ satisfies

$$\langle\langle \hat{F}_0, \hat{D}\hat{h}\rangle\rangle + \sum_{k,l=1}^{d} \langle\langle \hat{F}_{0,k}\hat{H}_{lk}, \hat{D}_l\hat{h}\rangle\rangle = \langle\langle \hat{g}, \hat{h}\rangle\rangle = -\langle\langle \hat{w}, \hat{D}\hat{h}\rangle\rangle, \tag{3.26}$$
$$\hat{h} \in C_0^\infty(\Omega),$$

where $\hat{w}$ solves $\hat{D}\cdot\hat{w} = \hat{g}$.

Indeed we have the following analogue of [9, Theorem 2], respectively [8, Proposition 3.1].

LEMMA 3.27. *For any* $\hat{g} \in \overline{H}^2_{-1}(\Omega)$, *there exists a unique function*

$$\hat{F}_0 = \left(\hat{F}_{0,1}, \dots, \hat{F}_{0,d}\right) \in L^2(\Omega; \mathbb{R}^d),$$

such that

$$\langle\langle \hat{F}_{0,k}, 1 \rangle\rangle = 0, \qquad k \in \{1, \dots, d\}, \tag{3.28}$$

and

$$\langle\langle \hat{F}_{0,k}, \hat{D}_j \hat{h} \rangle\rangle = \langle\langle \hat{F}_{0,j}, \hat{D}_k \hat{h} \rangle\rangle, \qquad \hat{h} \in H_1^2(\Omega),\ k, j \in \{1, \dots, d\}, \tag{3.29}$$

which solves (3.26). *Moreover we have*

$$|||\hat{F}_0|||_2 \le |||\hat{w}|||_2, \tag{3.30}$$

$$\lim_{\beta \to 0} |||\hat{D}\hat{R}_\beta \hat{g} - \hat{F}_0|||_2 = 0 \tag{3.31}$$

and

$$\lim_{\beta \to 0} \beta |||\hat{R}_\beta \hat{g}|||_2^2 = 0. \tag{3.32}$$

The proof of this lemma can be found in the next subsection.

(2.1), (2.4) and (3.9) imply

$$\hat{\theta}_k \in \overline{H}^2_{-1}(\Omega), \qquad k \in \{1, \dots, d\}. \tag{3.33}$$

Therefore estimate (3.23) is valid for $\hat{g} = \hat{\theta}_k$, respectively, $\hat{w} = (\hat{H}_{1k}, \dots, \hat{H}_{dk})$, and we obtain for $\hat{M}_k^\alpha = \hat{R}_\alpha \hat{\theta}_k$, $\alpha > 0$, $k \in \{1, \dots, d\}$,

$$\sup_{\alpha > 0} |||\hat{D}\hat{M}_k^\alpha|||_2 \le \left(\sum_{j=1}^d |||\hat{H}_{jk}|||_2^2 \right)^{1/2} < \infty, \qquad k \in \{1, \dots, d\}. \tag{3.34}$$

By Lemma 3.27,

$$\lim_{\alpha \to 0} \alpha^{1/2} |||\hat{M}_k^\alpha|||_2 = 0. \tag{3.35}$$

Furthermore there exist unique functions $\hat{G}^k = (\hat{G}_1^k, \dots, \hat{G}_d^k) \in L^2(\Omega; \mathbb{R}^d)$, $k \in \{1, \dots, d\}$, satisfying

$$\langle\langle \hat{G}_l^k, 1 \rangle\rangle = 0, \qquad k, l \in \{1, \dots, d\}, \tag{3.36}$$

$$\langle\langle \hat{G}_l^k, \hat{D}_j \hat{h} \rangle\rangle = \langle\langle \hat{G}_j^k, \hat{D}_l \hat{h} \rangle\rangle, \qquad \hat{h} \in H_1^2(\Omega),\ j, k, l \in \{1, \dots, d\}, \tag{3.37}$$

and

$$\begin{aligned} &\langle\langle \hat{G}^k, \hat{D}\hat{h} \rangle\rangle + \sum_{l, m=1}^d \langle\langle \hat{G}_l^k \hat{H}_{ml}, \hat{D}_m \hat{h} \rangle\rangle \\ &\qquad = \langle\langle \hat{\theta}_k, \hat{h} \rangle\rangle = -\sum_{l=1}^d \langle\langle \hat{H}_{lk}, \hat{D}_l \hat{h} \rangle\rangle, \\ &\qquad\qquad \hat{h} \in C_0^\infty(\Omega),\ k \in \{1, \dots, d\}. \end{aligned} \tag{3.38}$$

(3.31) implies

$$\lim_{\alpha \to 0} |||\hat{D}\hat{M}_k^\alpha - \hat{G}^k|||_2 = 0, \qquad k \in \{1, \dots, d\}. \tag{3.39}$$

Next we define for any $\alpha > 0$ and $k, l \in \{1, \dots, d\}$ the functions

$$M_k^\alpha(x, \omega) = \big(\hat{T}_x \hat{M}_k^\alpha\big)(\omega) \quad \text{and} \quad G_l^k(x, \omega) = \big(\hat{T}_x \hat{G}_l^k\big)(\omega), \quad \omega \in \Omega,\ x \in \mathbb{R}^d.$$

By (2.5) the rescaled process $X_\delta(t) = \delta X(\delta^{-2} t)$ satisfies

$$X_\delta(t, \omega) = \delta X(\delta^{-2} t, \omega) = \delta \int_0^{\delta^{-2} t} \theta(X(s, \omega), \omega)\, ds + \delta W(\delta^{-2} t),$$

i.e., if we use (3.8) and (3.13),

$$X_\delta(t, \omega) = \delta \int_0^{\delta^{-2} t} \hat{\theta}(Z(s, \omega))\, ds + \delta W(\delta^{-2} t). \tag{3.40}$$

The following computations which lead to (3.42) may seem to be a little bit formal. We shall show in Section 3.3 how they can be justified. Let $L(\omega) = \theta(\cdot, \omega) \cdot \nabla + \Delta = \sum_{k,l=1}^d \nabla_k H_{kl}(\cdot, \omega) \nabla_l + \Delta$. Then Itô's formula applied to the process $t \to \delta M_k^\alpha(X(\delta^{-2} t, \omega), \omega)$ yields for any $k \in \{1, \dots, d\}$, $\delta, \alpha > 0$ and $\mathbf{P}$-almost all $\omega \in \Omega$,

$$\begin{aligned}
\delta M_k^\alpha\big(X(\delta^{-2} t, \omega), \omega\big) &= \delta \hat{M}_k^\alpha\big(Z(\delta^{-2} t, \omega)\big) \\
&= \delta M_k^\alpha(0, \omega) + \delta \int_0^{\delta^{-2} t} \big(L(\omega) M_k^\alpha(\cdot, \omega)\big)(X(s, \omega))\, ds \\
&\quad + \delta \int_0^{\delta^{-2} t} \nabla M_k^\alpha(X(s, \omega), \omega)\, dW(s) \\
&= \delta \hat{M}_k^\alpha(\omega) + \delta \int_0^{\delta^{-2} t} \big((L(\omega) - \alpha) M_k^\alpha(\cdot, \omega)\big)(X(s, \omega))\, ds \\
&\quad + \delta \int_0^{\delta^{-2} t} \nabla M_k^\alpha(X(s, \omega), \omega)\, dW(s) \\
&\quad + \delta \int_0^{\delta^{-2} t} \alpha M_k^\alpha(X(s, \omega), \omega)\, ds \\
&= \delta \hat{M}_k^\alpha(\omega) - \delta \int_0^{\delta^{-2} t} \theta_k(X(s, \omega), \omega)\, ds \\
&\quad + \delta \int_0^{\delta^{-2} t} \nabla M_k^\alpha(X(s, \omega), \omega)\, dW(s) \\
&\quad + \delta \alpha \int_0^{\delta^{-2} t} M_k^\alpha(X(s, \omega), \omega)\, ds \\
&\qquad \big[\text{by the definition of } M_k^\alpha(\cdot, \cdot), \text{ which implies} \\
&\qquad (-L(\omega) + \alpha) M_k^\alpha(\cdot, \omega) = \theta_k(\cdot, \omega) \text{ for } \mathbf{P}\text{-almost all } \omega \in \Omega\big] \\
&= \delta \hat{M}_k^\alpha(\omega) - \delta \int_0^{\delta^{-2} t} \hat{\theta}_k(Z(s, \omega))\, ds \\
&\quad + \delta \int_0^{\delta^{-2} t} \hat{D} \hat{M}_k^\alpha(Z(s, \omega))\, dW(s) \\
&\quad + \delta \alpha \int_0^{\delta^{-2} t} \hat{M}_k^\alpha(Z(s, \omega))\, ds \qquad [\text{by (3.8) and (3.13)}].
\end{aligned} \tag{3.41}$$

Now we choose $\alpha = \alpha(\delta) = \delta^2$. Then we obtain by inserting (3.41) into (3.40) for any $k \in \{1, \dots, d\}$,

$$X_{\delta,k}(t,\cdot) = E_{\delta,k}(t,\cdot) + R_{\delta,k}(t,\cdot), \quad \mathbf{P}\text{-a.s.,} \tag{3.42}$$

where

$$E_{\delta,k}(t,\omega) = \delta W_k(\delta^{-2}t) + \delta \int_0^{\delta^{-2}t} \hat{D}\hat{M}_k^{\delta^2}(Z(s,\omega))\, dW(s)$$

and

$$R_{\delta,k}(t,\omega) = -\delta \hat{M}_k^{\delta^2}(Z(\delta^{-2}t,\omega)) + \delta \hat{M}_k^{\delta^2}(\omega) + \delta^3 \int_0^{\delta^{-2}t} \hat{M}_k^{\delta^2}(Z(s,\omega))\, ds.$$

From now on we consider all the processes $t \to X(t,\cdot)$, $t \to Z(t,\cdot)$, $t \to R_{\delta,k}(t,\cdot), \dots$ as $\underline{\mathscr{A}}_t$-adapted processes, i.e., in the sense of Remark 2.6. In particular $E_\delta(t,\cdot) = (E_{\delta,1}(t,\cdot), \dots E_{\delta,d}(t,\cdot))$ is an $\underline{\mathscr{A}}_t$-martingale. For any $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ the $\underline{\mathscr{A}}_t$-martingale $t \to v \cdot E_\delta(t,\cdot) = \sum_{k=1}^d v_k E_{\delta,k}(t,\cdot)$ has the quadratic variation

$$Q(\delta, v, t, \cdot) = 2\delta^2 \sum_{l=1}^d \int_0^{\delta^{-2}t} \left(v_l^2 + \left(\sum_{k=1}^d v_k \hat{D}_l \hat{M}_k^{\delta^2}(Z(s,\cdot)) \right)^2 \right.$$
$$\left. + 2 v_l \sum_{k=1}^d v_k \hat{D}_l \hat{M}_k^{\delta^2}(Z(s,\cdot)) \right) ds.$$

By Lemma 3.12 the distribution of the process $Z(t,\cdot)$, $t \geq 0$, $\mathscr{L}(Z(0,\cdot)) = \mathbf{P}$, on its path space $M([0,\infty); \Omega)$, i.e., the space of measurable functions from $[0,\infty)$ into $(\Omega, \mathscr{A})$, is invariant and ergodic with respect to the time shift. Therefore an application of Birkhoff's ergodic theorem (cf. [2]) and (3.36) and (3.39) yield

LEMMA 3.43. *For any $t \geq 0$,*

$$\lim_{\delta \to 0} Q(\delta, v, t, \cdot) = 2t \sum_{k,m=1}^d v_k v_m (\delta_{km} + D_{km}), \quad v \in \mathbb{R}^d \ (\text{in } \underline{\mathbf{P}}\text{-probability}),$$

with

$$D_{km} = \langle\langle \hat{G}^k, \hat{G}^m \rangle\rangle, \qquad k, m \in \{1, \dots, d\}. \tag{3.44}$$

An exact proof of this lemma follows in Section 3.4.

Lemma 3.43 and [6] imply

LEMMA 3.45. *For any $T > 0$ the distribution of the process $t \to E_\delta(t)$, $t \in [0,T]$, converges as $\delta \to 0$ weakly in $\mathscr{P}(C([0,T]; \mathbb{R}^d))$ to the distribution of $t \to X_0(t)$, $t \in [0,T]$, where $X_0(\cdot)$ is the Gaussian diffusion with diffusion matrix $\delta_{kl} + D_{kl}$, $k, l \in \{1, \dots, d\}$, i.e., infinitesimal generator $\Delta + \sum_{k,l=1}^d D_{kl} \nabla_k \nabla_l$.*

Next we can conclude by Lemma 3.12,

$$\underline{\mathbf{E}}\left[|\hat{M}_k^{\delta^2}(Z(t,\cdot))|\right] = |||\hat{M}_k^{\delta^2}|||_1 \leq |||\hat{M}_k^{\delta^2}|||_2, \qquad k \in \{1,\ldots,d\}, t \geq 0, \delta > 0,$$

where $\underline{\mathbf{E}}[\cdot]$ is the expectation operator in $(\underline{\Omega}, \underline{\mathscr{A}}, \underline{\mathbf{P}})$. Hence by (3.35),

$$\lim_{\delta \to 0} \sup_{t \in [0,T]} \underline{\mathbf{E}}\left[|R_{\delta,k}(t,\cdot)|\right] = 0, \qquad k \in \{1,\ldots,d\}. \tag{3.46}$$

The fact that we can prove only this convergence of $t \to R_{\delta,k}(t,\cdot)$ to the trivial process $t \to Y(t) \equiv 0$ and not the weak convergence of the distributions is the reason for introducing the norm $p_T^{\#}(\cdot,\cdot)$. By Skorokhod's theorem (cf. [7, page 9]) weak convergence in $\mathscr{P}(C([0,T];\mathbb{R}^d))$ implies convergence in $(S_T^{\#}, p_T^{\#})$. Hence Lemma 3.45 and (3.42) and (3.46) yield Theorem 1. □

3.2. *Proof of Lemma* 3.27. (a) Existence of a solution of (3.26). By (3.23) there exists for any sequence $\{\beta_n: n \in \mathbb{N}\}$ with $\lim_{n\to\infty}\beta_n = 0$, a subsequence $\{\beta_n': n \in \mathbb{N}\}$ such that the sequence $\{\hat{D}\hat{R}_{\beta_n'}\hat{g}: n \in \mathbb{N}\}$ is weakly convergent in $L^2(\Omega;\mathbb{R}^d)$. Let us denote the limit by $\hat{F}_0$. Obviously (3.28)–(3.30) hold for any $\hat{D}\hat{R}_\beta\hat{g}$, and so these properties hold for $\hat{F}_0$ too. Next (3.19) and (3.25) imply that $\hat{F}_0$ solves (3.26).

(b) Uniqueness of the solution of (3.26). We have to show that the only solution $\hat{G} = (\hat{G}_1,\ldots,\hat{G}_d) \in L^2(\Omega;\mathbb{R}^d)$ of the homogeneous equation,

$$\langle\langle \hat{G}, \hat{D}\hat{h}\rangle\rangle + \sum_{k,l=1}^{d} \langle\langle \hat{G}_k\hat{H}_{lk}, \hat{D}_l\hat{h}\rangle\rangle = 0, \qquad \hat{h} \in C_0^\infty(\Omega), \tag{3.47}$$

which satisfies

$$\langle\langle \hat{G}_k, 1\rangle\rangle = 0, \qquad k \in \{1,\ldots,d\}, \tag{3.48}$$

and

$$\langle\langle \hat{G}_k, \hat{D}_l\hat{h}\rangle\rangle = \langle\langle \hat{G}_l, \hat{D}_k\hat{h}\rangle\rangle, \qquad k,l \in \{1,\ldots,d\}, \hat{h} \in H_1^2(\Omega), \tag{3.49}$$

is the trivial one $\hat{G} = 0$. Let us fix any such solution. Since $\hat{G} \in L^2(\Omega;\mathbb{R}^d)$, the linear form $\hat{D}\hat{h} \to \langle\langle \hat{G}, \hat{D}\hat{h}\rangle\rangle$, $\hat{h} \in C_0^\infty(\Omega)$, and therefore the linear form $\hat{D}\hat{h} \to A_1(\hat{D}\hat{h}) = \sum_{k,l=1}^d \langle\langle \hat{G}_k\hat{H}_{lk}, \hat{D}_l\hat{h}\rangle\rangle$, $\hat{h} \in C_0^\infty(\Omega)$, too, can be extended continuously to the Hilbert space H^D, which is defined as the closure of the set $\{\hat{D}\hat{h}: \hat{h} \in C_0^\infty(\Omega)\}$ in $L^2(\Omega;\mathbb{R}^d)$.

We shall show now

$$\hat{G} \in H^D. \tag{3.50}$$

For that we define for any $\delta > 0$, $\hat{G}^\delta = (\hat{G}_1^\delta,\ldots,\hat{G}_d^\delta)$, $\hat{G}_k^\delta = \int_{\{|\mu|\geq\delta\}}\hat{U}(d\mu)\hat{G}_k$ and $\hat{\chi}^\delta = \sum_{j=1}^d \int_{\{|\mu|\geq\delta\}}((-i\mu_j)/|\mu|^2)\hat{U}(d\mu)\hat{G}_j$. Since the flow $\hat{t}$ is ergodic, 0 is a simple eigenvalue of the operators $\hat{D}_k$. Hence we obtain from (3.48)

$$\lim_{\delta\to 0} |||\hat{G} - \hat{G}^\delta|||_2 = 0. \tag{3.51}$$

Moreover we have for any $k \in \{1, \dots, d\}$ and $\hat{h} \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\langle\langle \hat{D}_k \hat{\chi}^\delta, \hat{h} \rangle\rangle &= \sum_{j=1}^{d} \left\langle\left\langle \int_{\{|\mu| \geq \delta\}} \frac{\mu_k \mu_j}{|\mu|^2} \hat{U}(d\mu) \hat{G}_j, \hat{h} \right\rangle\right\rangle \\
&= \sum_{j=1}^{d} \left\langle\left\langle \hat{G}_j, \int_{\{|\mu| \geq \delta\}} \frac{\mu_k \mu_j}{|\mu|^2} \hat{U}(d\mu) \hat{h} \right\rangle\right\rangle \\
&= \sum_{j=1}^{d} \left\langle\left\langle \hat{G}_j, \hat{D}_k \left(\int_{\{|\mu| \geq \delta\}} \frac{(-i\mu_j)}{|\mu|^2} \hat{U}(d\mu) \hat{h} \right) \right\rangle\right\rangle \\
&= \sum_{j=1}^{d} \left\langle\left\langle \hat{G}_k, \hat{D}_j \left(\int_{\{|\mu| \geq \delta\}} \frac{(-i\mu_j)}{|\mu|^2} \hat{U}(d\mu) \hat{h} \right) \right\rangle\right\rangle \quad \text{[by (3.49)]} \\
&= \sum_{j=1}^{d} \left\langle\left\langle \hat{G}_k, \int_{\{|\mu| \geq \delta\}} \frac{|\mu_j|^2}{|\mu|^2} \hat{U}(d\mu) \hat{h} \right\rangle\right\rangle \\
&= \left\langle\left\langle \hat{G}_k, \int_{\{|\mu| \geq \delta\}} \hat{U}(d\mu) \hat{h} \right\rangle\right\rangle = \langle\langle \hat{G}_k^\delta, \hat{h} \rangle\rangle.
\end{aligned}$$

Hence,

$$\hat{G}_k^\delta = \hat{D}_k \hat{\chi}^\delta. \tag{3.52}$$

(3.51) and (3.52) prove (3.50). Therefore $A_1(\hat{G})$ is well defined. Formally we obtain from (1.3),

$$A_1(\hat{G}) = \sum_{k,l=1}^{d} \langle\langle \hat{G}_k \hat{H}_{lk}, \hat{G}_l \rangle\rangle = \left\langle\left\langle \sum_{k,l=1}^{d} \hat{G}_k \hat{H}_{lk}, \hat{G}_l \right\rangle\right\rangle = 0. \tag{3.53}$$

(3.47) and (3.53) would imply

$$|||\hat{G}|||_2 = 0, \tag{3.54}$$

and the desired uniqueness would have been obtained. (3.54) is indeed true. However since $\hat{H}$ is not bounded, the second and third expressions in (3.53) are not well defined, and therefore (3.54) has to be proven in a different way. This can be done in just the same way as the subsequent derivation of (3.68), which is part of the proof of (3.31) and (3.32).

(c) Proofs of (3.31) and (3.32). We introduce the notation

$$g(x, \cdot) = (\hat{T}_x \hat{g})(\cdot), \qquad \hat{g} \in L^2(\Omega) \text{ or } L^2(\Omega; \mathbb{R}^d).$$

Then we obtain from (3.26) and the invariance of $\mathbf{P}$ with respect to $\hat{T}$,

$$\begin{aligned}
&\langle\langle F_0(x, \cdot), \hat{D}\hat{h}(\cdot) \rangle\rangle + \sum_{k,l=1}^{d} \langle\langle F_{0,k}(x, \cdot) H_{lk}(x, \cdot), \hat{D}_l \hat{h}(\cdot) \rangle\rangle \\
&\quad = -\langle\langle w(x, \cdot), \hat{D}\hat{h}(\cdot) \rangle\rangle, \qquad \hat{h} \in C_0^\infty(\Omega),\ x \in \mathbb{R}^d.
\end{aligned} \tag{3.55}$$

(3.55) implies (cf. [9, proof of Theorem 2])

$$\begin{aligned}\int_{\Omega}\int_{\mathbb{R}^d} F_0(x,\omega)\cdot\nabla h(x,\omega)\,dx\,\mathbf{P}(d\omega)&\\ +\sum_{k,l=1}^{d}\int_{\Omega}\int_{\mathbb{R}^d} F_{0,k}(x,\omega)H_{lk}(x,\omega)\nabla_l h(x,\omega)\,dx\,\mathbf{P}(d\omega)&\\ =-\int_{\Omega}\int_{\mathbb{R}^d} w(x,\omega)\cdot\nabla h(x,\omega)\,dx\,\mathbf{P}(d\omega),&\\ h\in H_{1,0}^{\infty,c}(\mathbb{R}^d\times\Omega).&\end{aligned}\tag{3.56}$$

Let us define now

$$\chi(x,\cdot)=\left(\sum_{j=1}^{d}\int_{\mathbb{R}^d}(e^{i\mu x}-1)\frac{(-i\mu_j)}{|\mu|^2}\hat{U}(d\mu)\hat{F}_{0,j}\right)(\cdot).$$

One can show (cf. [9, Theorem 2])

$$\nabla\chi(x,\cdot)=F_0(x,\cdot)\tag{3.57}$$

and

$$\lim_{\delta\to 0}\sup_{x\in K}\left\|\left|\delta\chi\left(\frac{x}{\delta},\cdot\right)\right\|\right|_2=0,\qquad K\subseteq\mathbb{R}^d\text{ compact.}\tag{3.58}$$

We insert now into (3.56) the function $h_{a,R}(x,\omega)=f_a(\chi(x,\omega))g_a(x)j_R(\omega)$, where for any $a>0$, $f_a(\cdot)\in C_c^\infty(\mathbb{R})$ and $g_a(\cdot)\in C_c^\infty(\mathbb{R}^d)$ such that $f_a(x)=x$ if $|x|\le a$, $f_a(x)=0$, if $|x|>3a$, $|f_a'(x)|\le 1$ and $|f_a(x)|\le 2a$ for all $x\in\mathbb{R}$, $g_a(x)=a^{-d}g(x/a)$ for some probability density g on $\mathbb{R}^d$ and $j_R(\omega)=\mathbb{1}_{M(R)}(\omega)$ for $M(R)=\{\omega\in\Omega:\ \sup_{x\in\mathbb{R}^d}1/(1+|x|)\sum_{k,l=1}^{d}|H_{kl}(x,\omega)|\le R\}$. Then we obtain by (3.56) and (3.57),

$$\begin{aligned}&\int_{\Omega}\int_{\mathbb{R}^d} F_0(x,\omega)^2 f_a'(\chi(x,\omega))g_a(x)j_R(\omega)\,dx\,\mathbf{P}(d\omega)\\ &\quad+\int_{\Omega}\int_{\mathbb{R}^d} F_0(x,\omega)\cdot\nabla g_a(x)f_a(\chi(x,\omega))j_R(\omega)\,dx\,\mathbf{P}(d\omega)\\ &\quad+\sum_{k,l=1}^{d}\int_{\Omega}\int_{\mathbb{R}^d} F_{0,k}(x,\omega)H_{lk}(x,\omega)F_{0,l}(x,\omega)\\ &\qquad\times f_a'(\chi(x,\omega))g_a(x)j_R(\omega)\,dx\,\mathbf{P}(d\omega)\\ &\quad+\sum_{k,l=1}^{d}\int_{\Omega}\int_{\mathbb{R}^d} F_{0,k}(x,\omega)H_{lk}(x,\omega)\nabla_l g_a(x,\omega)\\ &\qquad\times f_a(\chi(x,\omega))j_R(\omega)\,dx\,\mathbf{P}(d\omega)\\ &\quad=-\int_{\Omega}\int_{\mathbb{R}^d} w(x,\omega)\cdot F_0(x,\omega)f_a'(\chi(x,\omega))g_a(x)j_R(\omega)\,dx\,\mathbf{P}(d\omega)\\ &\qquad-\int_{\Omega}\int_{\mathbb{R}^d} w(x,\omega)\cdot\nabla g_a(x)f_a(\chi(x,\omega))j_R(\omega)\,dx\,\mathbf{P}(d\omega).\end{aligned}\tag{3.59}$$

Next we determine the asymptotic behaviour of the different terms in (3.59) as $R, a \to \infty$.

First we have by (1.3), (2.3) and since $\hat{F}_0(\cdot) \in L^2(\Omega; \mathbb{R}^d)$ for fixed $R, a < \infty$,

$$\sum_{k,l=1}^{d} \int_\Omega \int_{\mathbb{R}^d} F_{0,k}(x,\omega) H_{lk}(x,\omega) F_{0,l}(x,\omega) \times f_a'(\chi(x,\omega)) g_a(x) j_R(\omega)\, dx\, \mathbf{P}(d\omega) = 0. \tag{3.60}$$

Next

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^d} F_0(x,\omega)^2 f_a'(\chi(x,\omega)) g_a(x) j_R(\omega)\, dx\, \mathbf{P}(d\omega) \\
&\quad = \int_\Omega \int_{\mathbb{R}^d} F_0(ay,\omega)^2 f_a'(\chi(ay,\omega)) g(y) j_R(\omega)\, dy\, \mathbf{P}(d\omega) \\
&\quad = \int_\Omega \int_{\mathbb{R}^d} F_0(ay,\omega)^2 g(y) j_R(\omega)\, dy\, \mathbf{P}(d\omega) \\
&\qquad + \int_\Omega \int_{\mathbb{R}^d} F_0(ay,\omega)^2 (f_a'(\chi(ay,\omega)) - 1) g(y) j_R(\omega)\, dy\, \mathbf{P}(d\omega).
\end{aligned} \tag{3.61}$$

Since $g(\cdot)$ is a probability density and since $x \to F_0(x,\cdot) \in L^2(\Omega; \mathbb{R}^d)$ is stationary, we obtain by (2.3) and Lebesgue's bounded convergence theorem for any $a > 0$,

$$\begin{aligned}
&\lim_{R\to\infty} \int_\Omega \int_{\mathbb{R}^d} F_0(ay,\omega)^2 g(y) j_R(\omega)\, dy\, \mathbf{P}(d\omega) \\
&\quad = \int_\Omega \int_{\mathbb{R}^d} F_0(ay,\omega)^2 g(y)\, dy\, \mathbf{P}(d\omega) \\
&\quad = |||\hat{F}_0|||_2^2.
\end{aligned} \tag{3.62}$$

Moreover for any $R > 0$,

$$\begin{aligned}
&\left| \int_\Omega \int_{\mathbb{R}^d} F_0(ay,\omega)^2 (f_a'(\chi(ay,\omega)) - 1) g(y) j_R(\omega)\, dy\, \mathbf{P}(d\omega) \right| \\
&\quad \le 2 \int_\Omega \int_{\mathbb{R}^d} F_0(ay,\omega)^2 \left(\frac{1}{a} |\chi(ay,\omega)| \wedge 1 \right) g(y)\, dy\, \mathbf{P}(d\omega) \\
&\quad \le 2 \int_\Omega \int_{\mathbb{R}^d} |F_0(ay,\omega)| K \frac{1}{a} |\chi(ay,\omega)| g(y)\, dy\, \mathbf{P}(d\omega) \\
&\qquad + 2 \int_{\mathbb{R}^d} \left(\int_{\{\omega:\, |F_0(ay,\omega)| \ge K\}} F_0(ay,\omega)^2 \mathbf{P}(d\omega) \right) g(y)\, dy \\
&\quad \le C \left(K |||F_0|||_2 \left(\int_{\mathbb{R}^d} \left|\left|\left| \frac{1}{a} \chi(ay,\cdot) \right|\right|\right|_2^2 g(y)\, dy \right)^{1/2} \right. \\
&\qquad \left. + \mathbf{E}\left[F_0(\cdot)^2; |F_0(\cdot)| \ge K \right] \right).
\end{aligned} \tag{3.63}$$

The second term in (3.63) is less than any given $\varepsilon > 0$ if K is sufficiently large.

Next by (3.58) for any fixed K, the first term tends to 0 as $a \to \infty$. Hence (3.61)–(3.63) imply

$$\lim_{a\to\infty}\lim_{R\to\infty}\int_\Omega\int_{\mathbb{R}^d} F_0(x,\omega)^2 f_a'(\chi(x,\omega))g_a(x)j_R(\omega)\,dx\,\mathbf{P}(d\omega) = |||\hat{F}_0|||_2^2. \tag{3.64}$$

In quite the same way as (3.64), we obtain

$$\begin{aligned}&\lim_{a\to\infty}\lim_{R\to\infty}\int_\Omega\int_{\mathbb{R}^d} w(x,\omega)\cdot F_0(x,\omega)f_a'(\chi(x,\omega))g_a(x)j_R(\omega)\,dx\,\mathbf{P}(d\omega)\\ &\qquad = \langle\langle \hat{w}, \hat{F}_0\rangle\rangle.\end{aligned} \tag{3.65}$$

Next for any $k, l \in \{1,\dots,d\}$,

$$\begin{aligned}&\int_\Omega\int_{\mathbb{R}^d} F_{0,k}(x,\omega)H_{lk}(x,\omega)\nabla_l g_a(x)f_a(\chi(x,\omega))j_R(\omega)\,dx\,\mathbf{P}(d\omega)\\ &\quad = \int_\Omega\int_{\mathbb{R}^d} F_{0,k}(ay,\omega)H_{lk}(ay,\omega)\nabla_l g(y)\frac{1}{a}f_a(\chi(ay,\omega))j_R(\omega)\,dy\,\mathbf{P}(d\omega).\end{aligned}$$

The absolute value of this expression is less than

$$2\int_\Omega\int_{\mathbb{R}^d} |F_{0,k}(ay,\omega)|\,|H_{lk}(ay,\omega)|\left(\frac{1}{a}|\chi(ay,\omega)| \wedge 1\right)|\nabla_l g(y)|\,dy\,\mathbf{P}(d\omega).$$

By estimates similar to those that lead to (3.63), we may show now

$$\begin{aligned}&\lim_{a\to\infty}\lim_{R\to\infty}\int_\Omega\int_{\mathbb{R}^d} F_{0,k}(x,\omega)H_{lk}(x,\omega)\\ &\qquad\qquad \times\nabla_l g_a(x)f_a(\chi(x,\omega))j_R(\omega)\,dx\,\mathbf{P}(d\omega) = 0.\end{aligned} \tag{3.66}$$

Similarly, we get

$$\begin{aligned}&\lim_{a\to\infty}\lim_{R\to\infty}\left(\left|\int_\Omega\int_{\mathbb{R}^d} F_0(x,\omega)\cdot\nabla g_a(x)f_a(\chi(x,\omega))j_R(\omega)\,dx\,\mathbf{P}(d\omega)\right|\right.\\ &\qquad \left.+\left|\int_\Omega\int_{\mathbb{R}^d} w(x,\omega)\cdot\nabla g_a(x)f_a(\chi(x,\omega))j_R(\omega)\,dx\,\mathbf{P}(d\omega)\right|\right) = 0.\end{aligned} \tag{3.67}$$

(3.59), (3.60) and (3.64)–(3.67) imply

$$|||\hat{F}_0|||_2^2 = -\langle\langle \hat{w}, \hat{F}_0\rangle\rangle. \tag{3.68}$$

We know from (3.22) that

$$|||\hat{D}\hat{R}_\beta\hat{g}|||_2^2 + \beta|||\hat{R}_\beta\hat{g}|||_2^2 = -\langle\langle \hat{w}, \hat{D}\hat{R}_\beta\hat{g}\rangle\rangle. \tag{3.69}$$

On the other hand, the construction of $\hat{F}_0$ in (a) as the weak limit of $\hat{D}\hat{R}_{\beta_n}\hat{g}$ in $L^2(\Omega;\mathbb{R}^d)$ as $n\to\infty$ and its unique existence [cf. (b)] imply

$$\liminf_{\beta\to 0}|||\hat{D}\hat{R}_\beta\hat{g}|||_2^2 \geq |||\hat{F}_0|||_2^2$$

and

$$\lim_{\beta\to 0}\langle\langle \hat{w}, \hat{D}\hat{R}_\beta\hat{g}\rangle\rangle = \langle\langle \hat{w}, \hat{F}_0\rangle\rangle.$$

Hence,

$$|||\hat{F}_0|||_2^2 = -\lim_{\beta\to 0}\langle\langle \hat{w}, \hat{D}\hat{R}_\beta\hat{g}\rangle\rangle = \lim_{\beta\to 0}\left(|||\hat{D}\hat{R}_\beta\hat{g}|||_2^2 + \beta|||\hat{R}_\beta\hat{g}|||_2^2\right)$$
$$\geq \liminf_{\beta\to 0}|||\hat{D}\hat{R}_\beta\hat{g}|||_2^2 \geq |||\hat{F}_0|||_2^2.$$

This yields, first,

$$\lim_{\beta\to 0}\beta|||\hat{R}_\beta\hat{g}|||_2^2 = 0,$$

which proves (3.32) and, second,

$$\lim_{\beta\to 0}|||\hat{D}\hat{R}_\beta\hat{g}|||_2^2 = |||\hat{F}_0|||_2^2,$$

which together with the weak convergence of $\hat{D}\hat{R}_\beta\hat{g}$ to $\hat{F}_0$ proves (3.31). This completes the proof of Lemma 3.27. □

3.3. *Justification of* (3.42). Let us show that for any $\alpha > 0$, $k \in \{1,\dots,d\}$,

$$M_k^\alpha(\cdot,\omega) \in C^2(\mathbb{R}^d) \quad \mathbf{P}\text{-a.s.} \tag{3.70}$$

To prove (3.70), we fix some $\omega \in \Omega$ such that (2.2), (2.3) and

$$\int_A \left(|M_k^\alpha(x,\omega)|^2 + |\nabla M_k^\alpha(x,\omega)|^2\right)dx < \infty \quad \text{for any bounded } A \subseteq \mathbb{R}^d \tag{3.71}$$

hold for this fixed ω [cf. (3.34) and (3.35)]. Then we consider in any ball $B_R = \{x \in \mathbb{R}^d: |x| < R\}$ the Dirichlet problem

$$(-L(\omega) + \alpha)f = \theta_k(\cdot,\omega) \quad \text{in } B_R, \tag{3.72}$$

$$f\big|_{\partial B_R} = M_k^\alpha(\cdot,\omega)\big|_{\partial B_R}. \tag{3.73}$$

(1.2), (2.2), (2.3), (3.71) and [4, Theorem 8.3], imply that (3.72) and (3.73) have a unique solution $f \in H_1^2(B_R)$. It follows from [4, Theorem 8.24] that f is locally Hölder continuous in B_R. By the definition of $M_k^\alpha(\cdot,\cdot)$, we have $f = M_k^\alpha(\cdot,\omega)|_{B_R}$ **P**-a.s. and therefore [4, Theorem 6.13] implies $M_k^\alpha(\cdot,\omega)|_{B_R} \in C^2(B_R)$ **P**-a.s. This proves (3.70).

(3.70) justifies the application of Itô's formula to the process $t \to M_k^\alpha(\chi(\delta^{-2}t,\omega),\omega)$. But first of all (3.40) and (3.41) are true only locally, i.e., we have to replace $\delta^{-2}t$ by the stopping time $s_R = s(\delta,\omega,R,t) = \inf\{v \geq 0: |X(\delta^{-2}v,\omega)| \geq R\} \wedge \delta^{-2}t$, where $R > 0$ is fixed. Then we obtain, instead of (3.42),

$$X_{\delta,k}(s_R,\omega) = E_{\delta,k}(s_R,\omega) + R_{\delta,k}(s_R,\omega). \tag{3.74}$$

To show that (3.40) and (3.41), and therefore (3.42) too, are true not only locally, i.e., that we can let $R \to \infty$ in (3.74), we need for any $\delta, t > 0$ and $k \in \{1,\dots,d\}$

the following estimates which follow from (2.4), (3.34), (3.35) and Lemma 3.12:

$$\underline{\mathbf{E}}\left[\delta\int_0^{\delta^{-2}t}|\hat{\theta}_k(Z(s,\cdot))|\,ds\right] = \delta^{-1}t\langle\langle|\hat{\theta}_k|,1\rangle\rangle < \infty,$$

$$\underline{\mathbf{E}}\left[\sup_{s\le t}\left|\delta\int_0^{\delta^{-2}s}\hat{D}\hat{M}_k^{\delta^2}(Z(v,\cdot))\,dW(v)\right|^2\right] \le 8\underline{\mathbf{E}}\left[\delta^2\int_0^{\delta^{-2}t}\left|\hat{D}\hat{M}_k^{\delta^2}(Z(v,\cdot))\right|^2 dv\right]$$
$$= 8t|||\hat{D}\hat{M}_k^{\delta^2}|||_2^2 < \infty$$

and

$$\underline{\mathbf{E}}\left[\delta^3\int_0^{\delta^{-2}t}|\hat{M}_k^{\delta^2}(Z(s,\cdot))|\,ds\right] = \delta t\langle\langle|\hat{M}_k^{\delta^2}|,1\rangle\rangle < \infty. \qquad \square$$

3.4. *Proof of Lemma* 3.43. Birkhoff's ergodic theorem and Lemma 3.12 imply

$$\begin{aligned}
&\lim_{\delta\to 0}\delta^2\sum_{l=1}^{d}\int_0^{\delta^{-2}t}\left(\sum_{k=1}^{d}v_k\hat{G}_l^k(Z(s,\cdot))\right)^2 ds\\
(3.75)\qquad &= t\sum_{l=1}^{d}\sum_{k,m=1}^{d}v_kv_m\langle\langle\hat{G}_l^k,\hat{G}_1^m\rangle\rangle\\
&= t\sum_{k,m=1}^{d}v_kv_m\langle\langle\hat{G}^k,\hat{G}^m\rangle\rangle \quad \text{(in } \mathbf{P}\text{-probability).}
\end{aligned}$$

By Lemma 3.12 and (3.39) we have

$$\begin{aligned}
&\lim_{\delta\to 0}\underline{\mathbf{E}}\left[\delta^2\sum_{l=1}^{d}\int_0^{\delta^{-2}t}\left|\left(\sum_{k=1}^{d}v_k\hat{D}_l\hat{M}_k^{\delta^2}(Z(s,\cdot))\right)^2 - \left(\sum_{k=1}^{d}v_k\hat{G}_l^k(Z(s,\cdot))\right)^2\right|ds\right]\\
(3.76)\qquad &= \lim_{\delta\to 0}\delta^2\sum_{l=1}^{d}\int_0^{\delta^{-2}t}\left\langle\left\langle\left|\left(\sum_{k=1}^{d}v_k\hat{D}_l\hat{M}_k^{\delta^2}\right)^2 - \left(\sum_{k=1}^{d}v_k\hat{G}_l^k\right)^2\right|,1\right\rangle\right\rangle ds\\
&\le \lim_{\delta\to 0}t\sum_{l,k,m=1}^{d}|v|^2|||\hat{D}_l\hat{M}_k^{\delta^2} - \hat{G}_l^k|||_2|||\hat{D}_l\hat{M}_m^{\delta^2} + \hat{G}_l^m|||_2 = 0.
\end{aligned}$$

Again by Birkhoff's ergodic theorem and (3.36),

$$\begin{aligned}
&\lim_{\delta\to 0}\delta^2\sum_{l,k=1}^{d}\int_0^{\delta^{-2}t}v_lv_k\hat{G}_l^k(Z(s,\cdot))\,ds\\
(3.77)\qquad &= t\sum_{l,k=1}^{d}v_lv_k\langle\langle\hat{G}_l^k,1\rangle\rangle = 0 \quad \text{(in } \underline{\mathbf{P}}\text{-probability).}
\end{aligned}$$

Finally, by (3.39),

$$\lim_{\delta\to 0}\underline{\mathbf{E}}\left[\delta^2\sum_{l,k=1}^{d}\int_0^{\delta^{-2}t}|v_l|\,|v_k|\,|\hat{D}_l\hat{M}_k^{\delta^2}(Z(s,\cdot)) - \hat{G}_l^k(Z(s,\cdot))|\,ds\right] \tag{3.78}$$

$$= \lim_{\delta\to 0} t\sum_{k,l=1}^{d}|v_k|\,|v_l|\langle\langle|\hat{D}_l\hat{M}_k^{\delta^2} - \hat{G}_l^k|,1\rangle\rangle = 0.$$

(3.75)–(3.78) imply for any finite set $R^{\#} \subseteq \mathbb{R}^d$ which spans $\mathbb{R}^d$ and for any $t \geq 0$,

$$\lim_{\delta\to 0} Q(\delta, v, t, \cdot) = t\sum_{k,l=1}^{d} v_k v_l\big(\delta_{kl} + \langle\langle\hat{G}^k, \hat{G}^l\rangle\rangle\big),$$

$$v \in R^{\#} \quad (\text{in } \underline{\mathbf{P}}\text{-probability}).$$

This suffices to finish the proof of Lemma 3.43. □

3.5. *Proof of Corollaries* 2.8 *and* 2.12. For $g \in C^0(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and fixed $\omega \in \Omega$ the unique solution of (2.9) is provided by

$$u_\delta(x,\omega) = E^1\left[\int_0^\infty \exp(-\beta t) g\big(X_{x,\delta}(t,\omega)\big)\,dt\right] = \big(R_{\delta,\beta}(\omega)g\big)(x) \tag{3.79}$$

(cf. [3]), where $X_{x,\delta}(\cdot,\omega)$ solves

$$X_{x,\delta}(t,\omega) = x + \int_0^t \delta^{-1}\theta\big(\delta^{-1}X_{x,\delta}(s,\omega),\omega\big)\,ds + W(t),$$

and $E^1[\cdots]$ denotes the expectation with respect to $W(\cdot)$. In a similar way to (3.15) we obtain

$$\|R_{\delta,\beta}(\omega)g\|_p \leq \frac{1}{\beta}\|g\|_p, \qquad p \in [1,\infty]. \tag{3.80}$$

We note that for fixed x, ω the random variable $X_{x,\delta}(t,\omega)$ has the same distribution as $\delta X_{x/\delta}(\delta^{-2}t,\omega)$, where $X_y(\cdot,\omega)$ is defined as the solution of

$$X_y(t,\omega) = y + \int_0^t \theta\big(X_y(s,\omega),\omega\big)\,ds + W(t).$$

Hence we have the same situation as studied in Theorem 1 [cf. (2.5)] and we obtain for any $T > 0$

$$\lim_{\delta\to 0} p_T^{\#}\big(X_{x,\delta}(\cdot), X_x(\cdot)\big) = 0, \tag{3.81}$$

where $X_x(\cdot)$ is the Gaussian diffusion with generator $\Delta + \sum_{k,l=1}^d D_{kl}\nabla_k\nabla_l$ and $X_x(0) = x$.

Similar to (3.79) we have the representation

$$u_0(x) = E^1\left[\int_0^\infty \exp(-\beta t) g\big(X_x(t)\big)\,dt\right]. \tag{3.82}$$

(2.11) follows from (3.79)–(3.82), and then (2.10) is obtained from (2.11) and (3.80).

Corollary 2.12 is an obvious consequence of (3.34), (3.39) and (3.44). □

4. Asymptotic expansion of the effective diffusion matrix: The general case.

4.1. *Proof of Theorem* 2. Theorem 1 and (3.68) imply that the matrix D^α is defined by

$$D^\alpha_{kl} = \langle\langle \hat{G}^{\alpha,k}, \hat{G}^{\alpha,l}\rangle\rangle = -\alpha \sum_{m=1}^{d} \langle\langle \hat{H}_{mk}, \hat{G}^{\alpha,l}_m\rangle\rangle, \tag{4.1}$$

where $\hat{G}^{\alpha,k} = (\hat{G}^{\alpha,k}_1,\dots,\hat{G}^{\alpha,k}_d) \in L^2(\Omega;\mathbb{R}^d)$ is the unique solution of (3.36)–(3.38) with $\alpha\hat{H}(\cdot)$ replacing $\hat{H}(\cdot)$.

Now for $n, l \in \{1,\dots,d\}$ we define the operators

$$\begin{aligned} -\hat{D}_n(-\hat{D}^2)^{-1}\hat{f} &= \int_{\mathbb{R}^d\setminus\{0\}} \frac{-i\mu_n}{|\mu|^2}\hat{U}(d\mu)\hat{f}, \quad \text{respectively,}\\ -\hat{D}_n\hat{D}_l(-\hat{D}^2)^{-1}\hat{f} &= \int_{\mathbb{R}^d\setminus\{0\}} \frac{\mu_n\mu_l}{|\mu|^2}\hat{U}(d\mu)\hat{f} = \Gamma_{nl}\hat{f}, \end{aligned} \tag{4.2}$$

where $\hat{U}(\cdot)$ is the resolution of the identity introduced in (3.5). It is easy to show that $-\hat{D}_n(-\hat{D}^2)^{-1}$ is well defined on a sufficiently large class of functions being dense in any $L^p(\Omega)$, $p \in [1,\infty)$. Moreover, Γ_{nl} is bounded not only in $L^2(\Omega)$, which is obvious from (4.2), but we even have

LEMMA 4.3. *Γ_{kl}, $k, l \in \{1,\dots,d\}$ can be extended to $\bigcup_{p\in(1,\infty)} L^p(\Omega)$ so as to have*

$$|||\Gamma_{kl}\hat{f}|||_p \le C_p|||\hat{f}|||_p, \qquad k,l \in \{1,\dots,d\},\ p\in(1,\infty), \tag{4.4}$$

where the constant C_p only depends on p.

We defer the proof of this lemma to the next subsection.

Now for fixed $n \in \{1,\dots,d\}$ we insert the function $\hat{h} = -\hat{D}_n(-\hat{D}^2)^{-1}\hat{f}$ into (3.38) to obtain for any $\alpha > 0$,

$$\begin{aligned} &-\sum_{l=1}^{d} \langle\langle \hat{G}^{\alpha,k}_l, \hat{D}_l\hat{D}_n(-\hat{D}^2)^{-1}\hat{f}\rangle\rangle\\ &\qquad -\alpha\sum_{m,l=1}^{d} \left\langle\left\langle \hat{G}^{\alpha,k}_l\hat{H}_{ml}, \hat{D}_m\hat{D}_n(-\hat{D}^2)^{-1}\hat{f}\right\rangle\right\rangle\\ &\qquad\qquad = \alpha\sum_{l=1}^{d} \langle\langle \hat{H}_{lk}, \hat{D}_l\hat{D}_n(-\hat{D}^2)^{-1}\hat{f}\rangle\rangle. \end{aligned} \tag{4.5}$$

By (3.37) the first term on the left side can be written as

$$\begin{aligned} &-\sum_{l=1}^{d} \langle\langle \hat{G}^{\alpha,k}_l, \hat{D}_l\hat{D}_n(-\hat{D}^2)^{-1}\hat{f}\rangle\rangle\\ &\qquad = -\sum_{l=1}^{d} \langle\langle \hat{G}^{\alpha,k}_n, \hat{D}_l\hat{D}_l(-\hat{D}^2)^{-1}\hat{f}\rangle\rangle = \langle\langle \hat{G}^{\alpha,k}_n, \hat{f}\rangle\rangle, \end{aligned}$$

i.e., by (4.5) after "integration by parts,"

$$\left\langle\left\langle \hat{G}_n^{\alpha,k} + \alpha \sum_{m,l=1}^{d} \Gamma_{nm}\hat{H}_{ml}\hat{G}_l^{\alpha,k}, \hat{f} \right\rangle\right\rangle = -\alpha \left\langle\left\langle \sum_{l=1}^{d} \Gamma_{nl}\hat{H}_{lk}, \hat{f} \right\rangle\right\rangle. \tag{4.6}$$

Using an obvious matrix notation and omitting multiplication with a test function and integration with respect to $\mathbf{P}(d\omega)$, one can write (4.6) as

$$(\mathbb{1} + \alpha\Gamma\hat{H})\hat{G}^{\alpha,k} = -\alpha\Gamma\hat{H}_{\cdot k}, \tag{4.7}$$

where $\hat{H}_{\cdot k} = (\hat{H}_{1k}, \dots, \hat{H}_{dk})$.

We define for any $m \in \mathbb{N}$,

$$\hat{G}^{\alpha,m,k} = \sum_{l=1}^{m} (-\alpha)^l (\Gamma\hat{H})^{l-1}\Gamma\hat{H}_{\cdot k}, \tag{4.8}$$

with $(\Gamma\hat{H})^l = \Gamma\hat{H}\Gamma\hat{H} \cdots \Gamma\hat{H}$ (l times).

By (4.7) we obtain

$$\begin{aligned}(\mathbb{1} + \alpha\Gamma\hat{H})&(\hat{G}^{\alpha,k} - \hat{G}^{\alpha,m,k}) \\ &= -\alpha\Gamma\hat{H}_{\cdot k} - \hat{G}^{\alpha,m,k} + \sum_{l=2}^{m+1} (-\alpha)^l (\Gamma\hat{H})^{l-1}\Gamma\hat{H}_{\cdot k} \\ &= (-\alpha)^{m+1}(\Gamma\hat{H})^m\Gamma\hat{H}_{\cdot k}.\end{aligned} \tag{4.9}$$

(3.36), (3.37) and (4.2) imply for $l, r \in \{1, \dots, d\}$,

$$\langle\langle \hat{G}_r^{\alpha,k} - \hat{G}_r^{\alpha,m,k}, 1 \rangle\rangle = 0, \tag{4.10}$$

$$\hat{D}_l\left(\hat{G}_r^{\alpha,k} - \hat{G}_r^{\alpha,m,k}\right) = \hat{D}_r\left(\hat{G}_l^{\alpha,k} - \hat{G}_l^{\alpha,m,k}\right). \tag{4.11}$$

After multiplying both sides of (4.9) with $\hat{D}\hat{h}$ and using (4.11), we obtain

$$\begin{aligned}\langle\langle \hat{G}^{\alpha,k} - \hat{G}^{\alpha,k,m}, \hat{D}\hat{h} \rangle\rangle &+ \sum_{l,n=1}^{d} \alpha \langle\langle \left(\hat{G}_l^{\alpha,k} - \hat{G}_l^{\alpha,m,k}\right)\hat{H}_{nl}, \hat{D}_n\hat{h} \rangle\rangle \\ &= (-\alpha)^{m+1} \langle\langle (\Gamma\hat{H})^m\Gamma\hat{H}_{\cdot k}, \hat{D}\hat{h} \rangle\rangle.\end{aligned} \tag{4.12}$$

(2.13) and Lemma 4.3 imply $(\Gamma\hat{H})^m\Gamma\hat{H}_{\cdot k} \in L^2(\Omega; \mathbb{R}^d)$, and therefore we can apply Lemma 3.27 to conclude from (4.12),

$$|||\hat{G}^{\alpha,k} - \hat{G}^{\alpha,m,k}|||_2 \le \alpha^{m+1}|||(\Gamma\hat{H})^m\Gamma\hat{H}_{\cdot k}|||_2 \le \alpha^{m+1}C(m), \tag{4.13}$$

where the constant $C(m)$ depends only on m and $\hat{H}(\cdot)$. Now we have for any $m \in \mathbb{N}$,

$$\begin{aligned}D_{kl}^{\alpha} &= -\alpha\langle\langle \hat{H}_{\cdot k}, \hat{G}^{\alpha,l} \rangle\rangle \quad [\text{by } (4.1)] \\ &= -\alpha\langle\langle \hat{H}_{\cdot k}, \hat{G}^{\alpha,m,l} \rangle\rangle - \alpha\langle\langle \hat{H}_{\cdot k}, \hat{G}^{\alpha,l} - \hat{G}^{\alpha,m,l} \rangle\rangle \\ &= \sum_{r=1}^{m} (-\alpha)^{r+1} \langle\langle \hat{H}_{\cdot k}, (\Gamma\hat{H})^{r-1}\Gamma\hat{H}_{\cdot l} \rangle\rangle + O(\alpha^{m+2}) \quad [\text{by } (4.13)].\end{aligned} \tag{4.14}$$

For any even r we have

$$\begin{aligned}\langle\langle \hat{H}_{\cdot k}, (\Gamma\hat{H})^{r-1}\Gamma\hat{H}_{\cdot l} \rangle\rangle &= (-1)^{r/2-1} \langle\langle (\Gamma\hat{H})^{r/2-1}\Gamma\hat{H}_{\cdot k}, \hat{H}(\Gamma\hat{H})^{r/2-1}\Gamma\hat{H}_{\cdot l} \rangle\rangle \\ &= A_{kl}^{\#}.\end{aligned}$$

Since $\hat{H}_{kl}(\cdot)$ is skew symmetric, the matrix $A_{kl}^{\#}$ is skew symmetric too. On the other hand, in an asymptotic expansion of the symmetric matrix D^α only symmetric terms can occur, i.e., $A^{\#}_{kl} = 0$, $k, l \in \{1, \dots, d\}$. For r odd we obtain

$$\langle\langle \hat{H}_{\cdot k}, (\Gamma\hat{H})^{r-1}\Gamma\hat{H}_{\cdot l}\rangle\rangle = (-1)^{(r-1)/2}\langle\langle(\hat{H}\Gamma)^{(r-1)/2}\hat{H}_{\cdot k}, \Gamma(\hat{H}\Gamma)^{(r-1)/2}\hat{H}_{\cdot l}\rangle\rangle.$$

Hence (2.14) is valid with

$$D^{(2r)}_{kl} = (-1)^{r-1}\langle\langle(\hat{H}\Gamma)^{r-1}\hat{H}_{\cdot k}, \Gamma(\hat{H}\Gamma)^{r-1}\hat{H}_{\cdot l}\rangle\rangle. \tag{4.15}$$

If we use $\Gamma^2 = \Gamma$, we obtain in analogy to (1.9), more formally,

$$D^{(2r)} = (-1)^{r-1}\langle\langle(\Gamma\hat{H})^r, (\Gamma\hat{H})^r\rangle\rangle = (-1)^{r-1}\mathbf{E}\big[((\Gamma\hat{H})^r)^T(\Gamma\hat{H})^r\big].$$

The positive semidefiniteness of the matrices $(-1)^{r+1}D^{(2r)}$ is obvious by (4.2) and (4.15).

Finally, we have for bounded $\hat{H}(\cdot)$, instead of (4.13),

$$|||\hat{G}^{\alpha,k} - \hat{G}^{\alpha,m,k}|||_2 \le \alpha^{m+1}\big(d\sup\{|||\hat{H}_{kl}(\cdot)|||_\infty : k, l \in \{1, \dots, d\}\}\big)^{m+1},$$

since the norm of Γ as operator in $L^2(\Omega; \mathbb{R}^d)$ is 1. This proves uniform convergence of the expansion (2.14) for α sufficiently small. □

4.2. *Proof of Lemma* 4.3. First we fix $p \in (1, 2]$ and some suitable $\mathbb{R}^d$-valued function $\hat{f} = (\hat{f}_1, \dots, \hat{f}_d)$ on Ω, where we can assume without restricting generality $\langle\langle \hat{f}, 1\rangle\rangle = 0$. Let $f(x, \cdot) = (\hat{T}_x\hat{f})(\cdot) = \int_{\mathbb{R}^d} e^{i\mu x}\hat{U}(d\mu)\hat{f}$.

Then we define the functions

$$\hat{W} = \sum_{k=1}^{d}\int_{\mathbb{R}^d}\frac{i\mu_k}{|\mu|^2}\hat{U}(d\mu)\hat{f}_k, \quad \text{respectively,}$$

$$W(x, \cdot) = (\hat{T}_x\hat{W})(\cdot) = \sum_{k=1}^{d}\int_{\mathbb{R}^d} e^{i\mu x}\frac{i\mu_k}{|\mu|^2}\hat{U}(d\mu)\hat{f}_k,$$

which solve

$$-\hat{D}^2\hat{W} = \hat{D}\cdot\hat{f} = \sum_{k=1}^{d}\hat{D}_k\hat{f}_k, \quad \text{respectively,} \tag{4.16}$$

$$-\Delta W(\cdot, \omega) = (\nabla\cdot f)(\cdot, \omega). \tag{4.17}$$

For later computational simplicity we assume that

$$\begin{aligned}&\hat{f},\ \hat{D}\cdot\hat{f},\ \hat{W} \text{ and } \hat{D}\hat{W}, \text{ and therefore } f(\cdot,\cdot), \nabla\cdot f(\cdot,\cdot),\\ &W(\cdot,\cdot) \text{ and } \nabla W(\cdot,\cdot) \text{ too, are uniformly bounded.}\end{aligned} \tag{4.18}$$

It is easy to prove that functions $\hat{f}$ satisfying (4.18) are dense in any space $\{\hat{g} \in L^q(\Omega; \mathbb{R}^d): \langle\langle \hat{g}, 1\rangle\rangle = 0\}$, $q \in (1, \infty)$.

Next we choose a sequence $r_N(\cdot) \in C_c^\infty(\mathbb{R}^d)$, $N \in \mathbb{N}$, satisfying

$$r_N(x) = \begin{cases} 1, & \text{if } |x| \le N,\\ 0, & \text{if } |x| \ge N + 10\end{cases}$$

and

$$\sup_{N\in\mathbb{N}}\ \sup_{m\in\{1,\dots,d\}}\ \max\{\|r_N\|_\infty, \|\nabla_m r_N\|_\infty\} \le 1.$$

For $\omega \in \Omega$, $\beta \in (0,1)$, let $W_{N,\beta}(\cdot,\omega)$ be the solution of the resolvent equation

$$(-\Delta + \beta) W_{N,\beta}(\cdot,\omega) = \nabla \cdot \big(f(\cdot,\omega) r_N(\cdot) \big). \tag{4.19}$$

The Caldéron–Zygmund theorem [10, Chapter 2], applied to the kernels $K_{\beta,lk}(x) = \nabla_l \nabla_k (-\Delta + \beta)^{-1}(x)$ implies

$$\|\nabla W_{N,\beta}(\cdot,\omega)\|_p^p \le C_1(p) \| f(\cdot,\omega) r_N(\cdot)\|_p^p,$$

where the constant $C_1(p)$ only depends on p and d, in particular, not on β. Since the random field $x \to f(x,\cdot)$ is stationary, we obtain after integration with respect to $\mathbf{P}(d\omega)$,

$$\begin{aligned}
&\int_\Omega \mathbf{P}(d\omega) \|\nabla W_{N,\beta}(\cdot,\omega)\|_p^p \\
&\quad \le C_1(p) \int_\Omega \mathbf{P}(d\omega) \int_{\mathbb{R}^d} dx |f(x,\omega) r_N(x)|^p \\
&\quad = C_1(p) |||\hat f|||_p^p \|r_N\|_p^p \le C_1(p) |||\hat f|||_p^p |B_{N+10}| \\
&\quad \le C_2(p) |||\hat f|||_p^p N^d,
\end{aligned} \tag{4.20}$$

where $|B_R|$ is the volume of the sphere $B_R = \{x \in \mathbb{R}^d \colon |x| < R\}$. (4.17) and (4.19) imply for $\mathbf{P}$-almost all $\omega \in \Omega$,

$$\begin{aligned}
&-\Delta\big(W(\cdot,\omega) r_N(\cdot) - W_{N,\beta}(\cdot,\omega)\big) \\
&\quad = -\Delta W(\cdot,\omega) r_N(\cdot) - 2\nabla W(\cdot,\omega) \cdot \nabla r_N(\cdot) - W(\cdot,\omega) \Delta r_N(\cdot) \\
&\qquad -\nabla \cdot \big(f(\cdot,\omega) r_N(\cdot)\big) + \beta W_{N,\beta}(\cdot,\omega) \\
&\quad = -\nabla \cdot \big(W(\cdot,\omega) \nabla r_N(\cdot)\big) - \big(\nabla W(\cdot,\omega) + f(\cdot,\omega)\big) \cdot \nabla r_N(\cdot) \\
&\qquad + \beta W_{N,\beta}(\cdot,\omega).
\end{aligned} \tag{4.21}$$

Multiplying both sides by $W(\cdot,\omega) r_N(\cdot) - W_{N,\beta}(\cdot,\omega)$ yields

$$\begin{aligned}
&\|\nabla\big(W(\cdot,\omega) r_N(\cdot) - W_{N,\beta}(\cdot,\omega)\big)\|_2^2 \\
&\quad = \langle \nabla\big(W(\cdot,\omega) r_N(\cdot) - W_{N,\beta}(\cdot,\omega)\big), W(\cdot,\omega) \nabla r_N(\cdot)\rangle \\
&\qquad - \langle W(\cdot,\omega) r_N(\cdot) - W_{N,\beta}(\cdot,\omega), \big(\nabla W(\cdot,\omega) + f(\cdot,\omega)\big) \cdot \nabla r_N(\cdot)\rangle \\
&\qquad + \beta \langle W(\cdot,\omega) r_N(\cdot) - W_{N,\beta}(\cdot,\omega), W_{N,\beta}(\cdot,\omega)\rangle \\
&\quad \le \tfrac{1}{2} \|\nabla\big(W(\cdot,\omega) r_N(\cdot) - W_{N,\beta}(\cdot,\omega)\big)\|_2^2 \\
&\qquad + C_3(\hat f)\big(\|\nabla r_N\|_2^2 + (1+\beta^{-1})\|\nabla r_N\|_1 + \beta^{1/2}\|r_N\|_2^2\big) - \beta\|W_{N,\beta}(\cdot,\omega)\|_2^2.
\end{aligned} \tag{4.22}$$

Here we have used (4.18) and the inequalities

$$\|W_{N,\beta}(\cdot,\omega)\|_\infty \le \frac{1}{\beta} \|\nabla \cdot \big(f(\cdot,\omega) r_N(\cdot)\big)\|_\infty, \quad \text{respectively,}$$

$$\beta \|W_{N,\beta}(\cdot,\omega)\|_2^2 \le \frac{1}{2} \| f(\cdot,\omega) r_N(\cdot)\|_2^2,$$

which are easily derived from (4.19).

After integration with respect to $\mathbf{P}(d\omega)$, we obtain from (4.22)

$$\begin{aligned}\int_{\Omega}&\mathbf{P}(d\omega)\|\nabla\big(W(\cdot,\omega)r_N(\cdot)-W_{N,\beta}(\cdot,\omega)\big)\|_2^2\\&\le C_4(\hat{f})\big((1+\beta^{-1})\|\nabla r_N\|_1+\beta^{1/2}|B_{N+10}|\big)\\&\le C_5(\hat{f})\big((1+\beta^{-1})N^{d-1}+\beta^{1/2}N^d\big),\end{aligned}$$

and therefore

$$\begin{aligned}&\int_{\Omega}\mathbf{P}(d\omega)\|\nabla\big(W(\cdot,\omega)r_N(\cdot)-W_{N,\beta}(\cdot,\omega)\big)\|_p^p\\&\quad\le C_6(p)\int_{\Omega}\mathbf{P}(d\omega)\Big(\|\nabla\big(W(\cdot,\omega)r_N(\cdot)-W_{N,\beta}(\cdot,\omega)\big)\mathbb{1}_{B_{N+20}}\|_p^p\\&\qquad\qquad+\|\nabla W_{N,\beta}(\cdot,\omega)\mathbb{1}_{B_{N+20}^c}\|_p^p\Big)\\&\quad\le C_7(\hat{f},p)\Bigg(\bigg(\int_{\Omega}\mathbf{P}(d\omega)\|\nabla\big(W(\cdot,\omega)r_N(\cdot)-W_{N,\beta}(\cdot,\omega)\big)\|_2^2\bigg)^{p/2}\|\mathbb{1}_{B_{N+20}}\|_{2/(2-p)}\\&\qquad+\int_{\Omega}\mathbf{P}(d\omega)\int_{B_{N+20}^c}dx\bigg|\int_{\mathbb{R}^d}dy\,\nabla\cdot\big(f(y,\omega)r_N(y)\big)\\&\qquad\qquad\times\nabla(-\Delta+\beta)^{-1}(x-y)\bigg|^p\Bigg)\\&\quad\le C_8(\hat{f},p)\Bigg(\big((1+\beta^{-1})N^{d-1}+\beta^{1/2}N^d\big)^{p/2}N^{(d(2-p))/2}\\&\qquad+\int_{B_{N+20}^c}dx\bigg(\int_{B_{N+10}}dy\big|\nabla(-\Delta+\beta)^{-1}(x-y)\big|\bigg)^p\Bigg).\end{aligned}$$

The relation $(-\Delta+\beta)^{-1}(x)=\int_0^\infty(4\pi t)^{-d/2}\exp(-x^2/4t-\beta t)\,dt$ implies for $|x|>1$,

$$|\nabla(-\Delta+\beta)^{-1}(x)|\le C_9\exp\big(-\tfrac{1}{2}\sqrt{\beta}|x|\big),$$

and so we obtain

$$\int_{B_{N+20}^c}dx\bigg(\int_{B_{N+10}}dy\big|\nabla(-\Delta+\beta)^{-1}(x-y)\big|\bigg)^p\le C_{10}(p)N^{d-1}\beta^{-3d/2}.$$

Hence for $\beta\in(0,1)$, $p\in(1,2]$,

$$\begin{aligned}\int_{\Omega}&\mathbf{P}(d\omega)\|\nabla\big(W(\cdot,\omega)r_N(\cdot)-W_{N,\beta}(\cdot,\omega)\big)\|_p^p\\&\le C_{11}(\hat{f},p)\Big(\big((1+\beta^{-1})N^{d-1}+\beta^{1/2}N^d\big)^{p/2}N^{(d(2-p))/2}\\&\qquad+N^{d-1}\beta^{-3d/2}\Big)\\&\le C_{12}(\hat{f},p)\big((1+\beta^{-3d/2})N^{d-1/2}+\beta^{1/4}N^d\big).\end{aligned}\tag{4.23}$$

Therefore,

$$
\begin{aligned}
|||\hat{D}\hat{W}|||_p^p &\le |B_N|^{-1}\|r_N\|_p^p|||\hat{D}\hat{W}|||_p^p \\
&= |B_N|^{-1}\int_\Omega \mathbf{P}(d\omega)\|\nabla W(\cdot,\omega)r_N(\cdot)\|_p^p \\
&\le C_{13}(p)|B_N|^{-1}\bigg(\int_\Omega \mathbf{P}(d\omega)\|\nabla(W(\cdot,\omega)r_N(\cdot))\|_p^p \\
&\qquad + \int_\Omega \mathbf{P}(d\omega)\|W(\cdot,\omega)\nabla r_N(\cdot)\|_p^p\bigg) \\
&\le C_{14}(p)|B_N|^{-1}\bigg(\int_\Omega \mathbf{P}(d\omega)\|\nabla\big(W(\cdot,\omega)r_N(\cdot) - W_{N,\beta}(\cdot,\omega)\big)\|_p^p \\
&\qquad + \int_\Omega \mathbf{P}(d\omega)\|\nabla W_{N,\beta}(\cdot,\omega)\|_p^p + \int_\Omega \mathbf{P}(d\omega)\|W(\cdot,\omega)\nabla r_N(\cdot)\|_p^p\bigg) \\
&\le \frac{1}{|B_N|}\Big(C_{15}(\hat{f},p)\big((1+\beta^{-3d/2})N^{d-1/2} + \beta^{1/4}N^d\big) + C_{16}(p)|||\hat{f}|||_p^p N^d\Big) \\
&\qquad\qquad [\text{by } (4.18), (4.20) \text{ and } (4.23)].
\end{aligned}
\tag{4.24}
$$

Therefore we obtain with $\beta = \beta(N) = N^{-1/6d}$ in the limit as $N \to \infty$,

$$
\left|\left|\left| \sum_{k=1}^{d} \Gamma_{nk}\hat{f} \right|\right|\right|_p^p \le |||\hat{D}\hat{W}|||_p^p \le C_{16}(p)|||f|||_p^p.
$$

Since the arguments leading to this inequality can be repeated for a sufficiently large class of functions $\hat{f}$, we have proved Lemma 4.3 for $p \in (1,2]$.

Finally, we have for $\hat{f}, \hat{g} \in L^p(\Omega)$, $p > 2$, by the self-adjointness of Γ_{nk} in $L^2(\Omega)$,

$$
|\langle\langle \hat{g}, \Gamma_{nk}\hat{f}\rangle\rangle| = |\langle\langle \Gamma_{nk}\hat{g}, \hat{f}\rangle\rangle| \le C_q|||\hat{f}|||_p|||\hat{g}|||_q,
$$

where $q \in (1,2]$ is defined by $1/p + 1/q = 1$. This proves Lemma 4.3 for $p \in (2,\infty)$. □

5. Asymptotic expansion of the diffusion matrix: The Gaussian case.

5.1. *Proof of Theorem* 3. To prove Theorem 3, we write down for fixed $\omega \in \Omega$ a perturbation expansion for the solution $f = R^{\alpha}_{\delta,\beta}(\omega)g$ of the resolvent equation

$$
(-L^{\alpha}_{\delta}(\omega) + \beta)f = g, \tag{5.1}
$$

where

$$
L^{\alpha}_{\delta}(\omega)f = \left(\Delta + \sum_{k,l=1}^{d} \nabla_k H^{\alpha}_{\delta,kl}(\cdot,\omega)\nabla_l\right)f = (\Delta + \theta^{\alpha}_{\delta}(\cdot,\omega)\cdot\nabla)f,
$$

$$
H^{\alpha}_{\delta,kl}(x,\omega) = H^{\alpha}_{kl}\left(\frac{x}{\delta},\omega\right), \qquad \theta^{\alpha}_{\delta,l}(x,\omega) = \sum_{k=1}^{d}\nabla_k H^{\alpha}_{\delta,kl}(x,\omega) = \frac{1}{\delta}\theta^{\alpha}_l\left(\frac{x}{\delta},\omega\right),
$$

and then analyze the behaviour of the different terms of this expansion as $\delta \to 0$.

Let us define for $g \in C^*(\mathbb{R}^d)$ and $M \in \{0,1,\dots\}$,

$$g_M^\alpha = g - \left(B_\delta^\alpha(\omega)(-\Delta+\beta)^{-1}\right)^{M+1} g$$

and

$$f_M^\alpha = \sum_{k=0}^{M} (-\Delta+\beta)^{-1}\left(B_\delta^\alpha(\omega)(-\Delta+\beta)^{-1}\right)^k g,$$

where $B_\delta^\alpha(\omega) = \sum_{k,l=1}^d \nabla_k H_{\delta,kl}^\alpha(\cdot,\omega)\nabla_l$. Since $g \in C^*(\mathbb{R}^d)$, g and all its partial derivatives are exponentially decreasing at infinity. Hence by (2.2) and (2.3) $f_M^\alpha \in L^2(\mathbb{R}^d) \cap C^3(\mathbb{R}^d)$ and $g_M^\alpha \in L^2(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$.

A trivial computation yields

$$(5.2)\qquad (-L_\delta^\alpha(\omega)+\beta)f_M^\alpha = g_M^\alpha, \quad \text{i.e., } f_M^\alpha = R_{\delta,\beta}^\alpha(\omega)g_M^\alpha.$$

Corollary 2.8 implies for $h, g \in C^*(\mathbb{R}^d)$ and $R_\beta^\alpha = (-\Delta - \sum_{k,l=1}^d D_{kl}^\alpha \nabla_k\nabla_l + \beta)^{-1}$,

$$(5.3)\qquad \begin{aligned} \langle h, R_\beta^\alpha g\rangle &= \lim_{\delta\to 0} \mathbf{E}\left[\langle h, R_{\delta,\beta}^\alpha(\cdot)g\rangle\right] \\ &= \lim_{\delta\to 0} \mathbf{E}\left[\langle h, R_{\delta,\beta}^\alpha(\cdot)g_M^\alpha\rangle\right] + \lim_{\delta\to 0} \mathbf{E}\left[\langle h, R_{\delta,\beta}^\alpha(\cdot)(g-g_M^\alpha)\rangle\right] \\ &= \lim_{\delta\to 0} \sum_{k=0}^{M} \mathbf{E}\left[\left\langle h, (-\Delta+\beta)^{-1}\left(B_\delta^\alpha(\cdot)(-\Delta+\beta)^{-1}\right)^k g\right\rangle\right] \\ &\quad + \lim_{\delta\to 0} \mathbf{E}\left[\langle h, R_{\delta,\beta}^\alpha(\cdot)(g-g_M^\alpha)\rangle\right]. \end{aligned}$$

To estimate the second summand on the right side of (5.3), we use the following analogue of (3.23) and (3.24).

Lemma 5.4. *Suppose there exists some $F \in H_1^2(\mathbb{R}^d;\mathbb{R}^d)$ such that $f = \nabla\cdot F$. Then for any $\omega\in\Omega$, $\alpha,\delta,\beta > 0$,*

$$(5.5)\qquad \|\nabla R_{\delta,\beta}^\alpha(\omega)f\|_2 \le \|F\|_2$$

and

$$(5.6)\qquad \|R_{\delta,\beta}^\alpha(\omega)f\|_2 \le (2\beta)^{-1/2}\|F\|_2.$$

For the proof of this lemma we can repeat the arguments leading to (3.23) and (3.24), where $\hat{D}_k$ has to be replaced by ∇_k. Therefore, we have

$$(5.7)\qquad \begin{aligned} &\left|\langle h, R_{\delta,\beta}^\alpha(\omega)(g-g_M^\alpha)\rangle\right| \\ &\quad = \left|\langle h, R_{\delta,\beta}^\alpha(\omega)\nabla\cdot\nabla(-\Delta)^{-1}(g-g_M^\alpha)\rangle\right| \\ &\quad \le (2\beta)^{-1/2}\|h\|_2\|\nabla(-\Delta)^{-1}(g-g_M^\alpha)\|_2 \\ &\quad = (2\beta)^{-1/2}\|h\|_2\|(-\Delta)^{-1/2}\left(B_\delta^\alpha(\omega)(-\Delta+\beta)^{-1}\right)^{M+1}g\|_2 \\ &\quad = (2\beta)^{-1/2}\|h\|_2\Big\langle \left(B_\delta^\alpha(\omega)(-\Delta+\beta)^{-1}\right)^{M+1}g, \\ &\qquad\qquad (-\Delta)^{-1}\left(B_\delta^\alpha(\omega)(-\Delta+\beta)^{-1}\right)^{M+1}g\Big\rangle^{1/2} \\ &\quad = (2\beta)^{-1/2}\|h\|_2\Big|\Big\langle g, \left((-\Delta+\beta)^{-1}B_\delta^\alpha(\omega)\right)^{M+1} \\ &\qquad\qquad \times(-\Delta)^{-1}\left(B_\delta^\alpha(\omega)(-\Delta+\beta)^{-1}\right)^{M+1}g\Big\rangle\Big|^{1/2}. \end{aligned}$$

Here we used the fact that $B_\delta^\alpha(\omega)$ is a skew symmetric operator in $L^2(\mathbb{R}^d)$.

Hence both the first $M + 1$ summands and the upper bound for the last term in (5.3) have essentially the same structure. The asymptotic behaviour of such expressions can be computed in a straightforward way since H and θ are Gaussian with mean 0, and therefore are completely determined by their covariance functionals.

For the proof of Lemma 5.8, see Section 5.2. Note that in the rest of this section we use the convention that repeated indices are summed for $1, \dots, d$.

LEMMA 5.8. *Assume $d \geq 3$ and* (2.16) *and* (2.17). *Then*:

(i) *For any odd $k \in \mathbb{N}$, $\alpha, \beta > 0$ and $h, g \in C^*(\mathbb{R}^d)$*,

$$\lim_{\delta \to 0} \mathbf{E}\left[\left\langle h, (-\Delta + \beta)^{-1}\left(B_\delta^\alpha(\cdot)(-\Delta + \beta)^{-1}\right)^k g\right\rangle^2\right] = 0. \tag{5.9}$$

(ii) *For any even $k \geq 2$, $\alpha, \beta > 0$ and $h, g \in C^*(\mathbb{R}^d)$*,

$$\begin{aligned}\lim_{\delta \to 0} \mathbf{E}\Bigg[\Bigg(&\left\langle h, (-\Delta + \beta)^{-1}\left(B_\delta^\alpha(\cdot)(-\Delta + \beta)^{-1}\right)^k g\right\rangle \\ &- \alpha^k \sum_{v=1}^{k/2} \sum_{\substack{l_1, \dots, l_v \geq 1 \\ l_1 + \cdots + l_v = k/2}} \\ &\times \left\langle h, (-\Delta + \beta)^{-1}\left(\prod_{q=1}^{v}\left(D_{p_{2q-1}p_{2q}}^{(2l_q)} \nabla_{p_{2q-1}} \nabla_{p_{2q}}(-\Delta + \beta)^{-1}\right)\right) g\right\rangle\Bigg)^2\Bigg] = 0.\end{aligned} \tag{5.10}$$

(iii) *For any even $M \geq 2$, $\alpha, \beta > 0$ and $g \in C^*(\mathbb{R}^d)$*,

$$\begin{aligned}\lim_{\delta \to 0} \mathbf{E}\Bigg[\Bigg(&\left\langle g, \left((-\Delta + \beta)^{-1} B_\delta^\alpha(\cdot)\right)^{M+1}(-\Delta)^{-1}\left(B_\delta^\alpha(\cdot)(-\Delta + \beta)^{-1}\right)^{M+1} g\right\rangle \\ &- \alpha^{2M+2} \sum_{v=1}^{M+1} \sum_{\substack{l_1, \dots, l_v \geq 1 \\ l_1 + \cdots + l_v = M+1}} \\ &\times \left\langle g, (-\Delta + \beta)^{-1}\left(\prod_{q=1}^{v}\left(D_{p_{2q-1}p_{2q}}^{(2l_q)} \nabla_{p_{2q-1}} \nabla_{p_{2q}}(-\Delta + \beta)^{-1}\right)\right) g\right\rangle\Bigg)^2\Bigg] = 0.\end{aligned} \tag{5.11}$$

Let us define now for M even,

$$D_{uv}^{[\alpha, M]} = \sum_{k=1}^{M/2} \alpha^{2k} D_{uv}^{(2k)}, \qquad u, v \in \{1, \dots, d\}. \tag{5.12}$$

Obviously any matrix $D^{(2l)} = D^{(2l)}_{uv}$, $u, v \in \{1, \dots, d\}$ is symmetric. Therefore the matrix $\mathbb{1} + D^{[\alpha, M]}$ is symmetric too and, at least for α small enough, positive. Hence for α sufficiently small the resolvent

$$R^{[\alpha, M]}_{\beta} = \left(-\Delta - \sum_{u,v=1}^{d} D^{[\alpha, M]}_{uv} \nabla_u \nabla_v + \beta \right)^{-1}$$

is for any $\beta > 0$ a well defined bounded operator on $L^2(\mathbb{R}^d)$. The representation (5.12) implies that similar to (5.3) for any $\alpha, \beta > 0$, α sufficiently small, and $h, g \in C^*(\mathbb{R}^d)$,

$$\begin{aligned}
\langle h, R^{[\alpha, M]}_{\beta} g \rangle &= \sum_{v=0}^{M/2} \left\langle h, (-\Delta + \beta)^{-1} \left(\sum_{m,n=1}^{d} D^{[\alpha, M]}_{mn} \nabla_m \nabla_n (-\Delta + \beta)^{-1} \right)^{v} g \right\rangle \\
&\quad + C_1(h, g, \alpha, \beta, M) \\
&= \langle h, (-\Delta + \beta)^{-1} g \rangle + \sum_{k=1}^{M/2} \alpha^{2k} \sum_{w=1}^{k} \sum_{\substack{l_1, \dots, l_w \geq 1 \\ l_1 + \cdots + l_w = k}} \\
&\quad \times \left\langle h, (-\Delta + \beta)^{-1} \left(\prod_{q=1}^{w} \left(D^{(2l_q)}_{p_{2q-1} p_{2q}} \nabla_{p_{2q-1}} \nabla_{p_{2q}} (-\Delta + \beta)^{-1} \right) \right) g \right\rangle \\
&\quad + C_2(h, g, \alpha, \beta, M),
\end{aligned} \tag{5.13}$$

where

$$|C_2(h, g, \alpha, \beta, M)| \leq C_3(\beta, M) \|h\|_2 \|g\|_2 \alpha^{M+2}. \tag{5.14}$$

Next (5.3), (5.7) and (5.9)–(5.11) imply

$$\begin{aligned}
&\Bigg| \langle h, R^{\alpha}_{\beta} g \rangle - \langle h, (-\Delta + \beta)^{-1} g \rangle - \sum_{k=1}^{M/2} \alpha^{2k} \sum_{w=1}^{k} \sum_{\substack{l_1, \dots, l_w \geq 1 \\ l_1 + \cdots + l_w = k}} \\
&\quad \times \left\langle h, (-\Delta + \beta)^{-1} \left(\prod_{q=1}^{w} \left(D^{(2l_q)}_{p_{2q-1} p_{2q}} \nabla_{p_{2q-1}} \nabla_{p_{2q}} (-\Delta + \beta)^{-1} \right) \right) g \right\rangle \Bigg| \\
&\quad \leq C_4(\beta, M) \|h\|_2 \|g\|_2 \alpha^{M+2}.
\end{aligned} \tag{5.15}$$

(5.13)–(5.15) imply

$$\begin{aligned}
&\left| \langle h, R^{\alpha}_{\beta} g \rangle - \langle h, R^{[\alpha, M]}_{\beta} g \rangle \right| \\
&\quad = \Bigg| \int_{\mathbb{R}^d} \overline{\tilde{h}(\tau)} \Bigg(\left(\beta + |\tau|^2 + \sum_{u,v=1}^{d} D^{\alpha}_{uv} \tau_u \tau_v \right)^{-1} \\
&\qquad - \left(\beta + |\tau|^2 + \sum_{u,v=1}^{d} D^{[\alpha, M]}_{uv} \tau_u \tau_v \right)^{-1} \Bigg) \tilde{g}(\tau) \, d\tau \Bigg| \\
&\quad \leq C_5(\beta, M) \|h\|_2 \|g\|_2 \alpha^{M+2}.
\end{aligned} \tag{5.16}$$

Corollary 2.12 implies

$$|D_{kl}^{\alpha}| \le C_6 \alpha^2, \qquad k, l \in \{1, \dots, d\}, \alpha > 0. \tag{5.17}$$

Hence we obtain from (5.16) for

$$\tilde{h}(\tau) = \left(v_d r^d\right)^{-1/2} \mathbb{1}_{B(a,r)}(\tau)$$

and

$$\tilde{g}(\tau) = \tilde{h}(\tau)\left(\beta + |\tau|^2 + \sum_{u,v=1}^{d} D_{uv}^{\alpha} \tau_u \tau_v\right),$$

where v_d is the volume of the unit sphere in $\mathbb{R}^d$ and $B(a, r)$ is the sphere with center a and radius r, the estimate

$$\left|\left(v_d r^d\right)^{-1} \int_{B(a,r)} \frac{\beta + |\tau|^2 + \sum_{u,v=1}^{d} D_{uv}^{\alpha} \tau_u \tau_v}{\beta + |\tau|^2 + \sum_{u,v=1}^{d} D_{uv}^{[\alpha, M]} \tau_u \tau_v} \, d\tau - 1\right| \alpha^{-M-2} \le C_7(\beta, M),$$

uniformly in $|a|$, $r \in (0,1)$ and α sufficiently small.

Hence,

$$\left|\frac{\beta + |a|^2 + \sum_{u,v=1}^{d} D_{uv}^{\alpha} a_u a_v}{\beta + |a|^2 + \sum_{u,v=1}^{d} D_{uv}^{[\alpha, M]} a_u a_v} - 1\right| \alpha^{-M-2} \le C_7(\beta, M),$$

uniformly in $|a| < 1$ and for α sufficiently small, and therefore by (5.17) for such a, α,

$$\left|\sum_{u,v=1}^{d} \left(D_{uv}^{\alpha} - D_{uv}^{[\alpha, M]}\right) a_u a_v\right| \le C_8(\beta, M) \alpha^{M+2}.$$

Since the matrices of D^{α} and $D^{[\alpha, M]}$ are symmetric, Theorem 3 follows. □

5.2. *Proof of Lemma* 5.8. In this section we use the convention that repeated indices are summed for $1, \dots, d$.

For the proof of (5.10) we obviously have to show for any even $k \ge 2$,

$$\begin{aligned} &\lim_{\delta \to 0} \mathbf{E}\left[\langle h, (-\Delta + \beta)^{-1}\left(B_{\delta}^{\alpha}(\cdot)(-\Delta + \beta)^{-1}\right)^k g\rangle^b\right] \\ &\quad = \left(\alpha^k \sum_{v=1}^{k/2} \sum_{\substack{l_1, \dots, l_v \ge 1 \\ l_1 + \cdots l_v = k/2}} \right. \\ &\qquad \left. \times \left\langle h, (-\Delta + \beta)^{-1}\left(\prod_{q=1}^{v} \left(D_{p_{2q-1} p_{2q}}^{(2l_q)} \nabla_{p_{2q-1}} \nabla_{p_{2q}} (-\Delta + \beta)^{-1}\right)\right) g\right\rangle\right)^b \end{aligned} \tag{5.18}$$

for $b = 1, 2$.

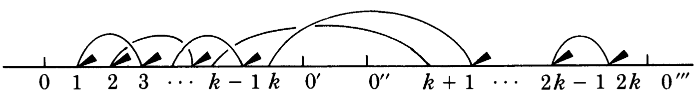
Let us begin with $b = 2$. Since the random field $x \to \theta_\delta(\cdot,\cdot) = (1/\delta)\theta(x/\delta,\cdot)$ is Gaussian, we obtain

$$
\begin{aligned}
&A(k,\beta,\delta,h,g,\alpha)\\
&\quad = \mathbf{E}\left[\left\langle h, (-\Delta+\beta)^{-1}\left(B_\delta^\alpha(\cdot)(-\Delta+\beta)^{-1}\right)^k g\right\rangle^2\right]\\
&\quad = \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\Bigg(h(x_0)(-\Delta+\beta)^{-1}(x_0-x_1)\\
&\qquad\qquad \times\left(\prod_{q=1}^{k-1}\nabla_{p_q}(-\Delta+\beta)^{-1}(x_q-x_{q+1})\right)\\
&\qquad\qquad \times\nabla_{p_k}(-\Delta+\beta)^{-1}(x_k-x_0')g(x_0')\Bigg)\\
&\qquad\qquad \times\Bigg(h(x_0'')(-\Delta+\beta)^{-1}(x_0''-x_{k+1})\\
&\qquad\qquad \times\left(\prod_{q=k+1}^{2k-1}\nabla_{p_q}(-\Delta+\beta)^{-1}(x_q-x_{q+1})\right)\\
&\qquad\qquad \times\nabla_{p_{2k}}(-\Delta+\beta)^{-1}(x_{2k}-x_0''')g(x_0''')\Bigg)\\
&\qquad\qquad \times\left(\sum_{\pi\in\mathscr{P}_2^{I(2k)}}\left(\prod_{r=1}^{k}R^\theta_{\delta,p_{\pi(1,r)},p_{\pi(2,r)}}\left(x_{\pi(1,r)}-x_{\pi(2,r)}\right)\right)\alpha^{2k}\right)\\
&\qquad\qquad \times dx_0\,dx_0'\,dx_0''\,dx_0'''\,dx_1\cdots dx_{2k}\\
&\quad = \alpha^{2k}\sum_{\pi\in\mathscr{P}_2^{I(2k)}}\hat{A}(k,\beta,\delta,h,g,\pi),
\end{aligned}
\tag{5.19}
$$

where $R^\theta_{\delta,k,l}(x) = \mathbf{E}[\theta_{\delta,k}(x)\theta_{\delta,l}(0)]$ is the correlation function of the random field $\theta_\delta(\cdot)$.

Now we fix some $\pi\in\mathscr{P}_2^{I(2k)}$ and use (1.4) and integration by parts to arrange $\hat{A}(k,\beta,\delta,h,g,\pi)$ in a more easily understandable way. To make this procedure more transparent we represent π by "bridges" connecting $\pi(1,r)$ and $\pi(2,r)$, $r = 1,\dots,k$, e.g.,

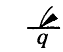
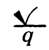
(5.20)



The sign " $\overline{q}$ " represents the gradient operator, which in (5.19) always acts on the "right"of each position q, i.e., on $(-\Delta+\beta)^{-1}(x_q-x_{q+1})\cdots$. In (5.19) and (5.20) the gradients occur in two situations:

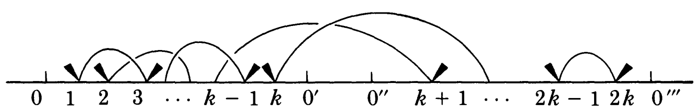
(i) "∇ is covered by the bridge" $\underset{q}{\_}$, i.e., $q\in H_1(\pi)$.

(ii) "∇ is not covered by the bridge" $\underset{q}{\_}$, i.e., $q\in H_2(\pi)$.

It is well known, that gradients do not like being covered by bridges, and therefore we use integration by parts to help the gradients  to come to the sunny side of the bridge, i.e.,  $\to$  . By (1.4) the gradients have no difficulties coming around the bridge. They only have to change their sign, i.e.,

$$\sum_{s=1}^{d} \int_{\mathbb{R}^d} G(x_q) R^{\theta}_{\delta, s, t}(x_q - x_{q'}) \nabla_s F(x_q)\, dx_q = - \sum_{s=1}^{d} \int_{\mathbb{R}^d} \nabla_s G(x_q) R^{\theta}_{\delta, s, t}(x_q - x_{q'}) F(x_q)\, dx_q.$$

After performing integration by parts for each gradient being covered by the bridge, we obtain from (5.20),



or from (5.19),

$$\begin{aligned} &\hat{A}(k, \beta, \delta, h, g, \pi) \\ &\quad = - \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \Big(\nabla_{p_1} h(x_0)(-\Delta + \beta)^{-1}(x_0 - x_1) \underline{\mathbf{R}}^{(1, \pi)}_{\beta, p_k}(x_k - x_0') g(x_0') \Big) \\ &\qquad \times \Big(h(x_0'') \underline{\mathbf{R}}^{(2, \pi)}_{\beta, p_{k+1}}(x_0'' - x_{k+1})(-\Delta + \beta)^{-1}(x_{2k} - x_0''') \nabla_{p_{2k}} g(x_0''') \Big) \\ &\qquad \times \left(\prod_{\substack{q \in B_0(\pi) \\ q \neq k}} (-\Delta + \beta)^{-1}(x_q - x_{q+1}) \right) \\ &\qquad \times \left(\prod_{\substack{q \in B_1(\pi) \\ q \neq k}} \nabla_{p_{q'(\pi)}} (-\Delta + \beta)^{-1}(x_q - x_{q+1}) \right) \\ &\qquad \times \left(\prod_{\substack{q \in B_2(\pi) \\ q \neq k}} \nabla_{p_q} \nabla_{p_{q+1}} (-\Delta + \beta)^{-1}(x_q - x_{q+1}) \right) \\ &\qquad \times \left(\prod_{r=1}^{k} R^{\theta}_{\delta, p_{\pi(1,r)}, p_{\pi(2,r)}}(x_{\pi(1,r)} - x_{\pi(2,r)}) \right) dx_0\, dx_0'\, dx_0''\, dx_0'''\, dx_1 \cdots dx_{2k}. \end{aligned} \tag{5.21}$$

Here we have used the notation

$$\underline{\mathbf{R}}^{(1, \pi)}_{\beta, p}(x) = \begin{cases} (-\Delta + \beta)^{-1}(x), & \text{if } k \in H_1(\pi), \\ \nabla_p (-\Delta + \beta)^{-1}(x), & \text{if } k \in H_2(\pi) \end{cases}$$

and

$$\underline{\mathbf{R}}^{(2, \pi)}_{\beta, p}(x) = \begin{cases} (-\Delta + \beta)^{-1}(x), & \text{if } k + 1 \in H_2(\pi), \\ \nabla_p (-\Delta + \beta)^{-1}(x), & \text{if } k + 1 \in H_1(\pi). \end{cases}$$

The covariance functional $R^{\theta}_{\delta,k,l}(\cdot)$ of $\theta_\delta(\cdot)$ satisfies

$$\begin{aligned} R^{\theta}_{\delta,k,l}(x) &= \mathbf{E}\big[\theta_{\delta,k}(x,\cdot)\theta_{\delta,l}(0,\cdot)\big] = \delta^{-2}R^{\theta}_{k,l}(x/\delta) \\ &= \delta^{-2}(2\pi)^{-d/2}\int_{\mathbb{R}^d}\exp(-i(\mu/\delta)x)\sigma^{\theta}_{k,l}(\mu)\,d\mu, \end{aligned} \tag{5.22}$$

where $\sigma^{\theta}_{\cdot,\cdot}(\mu)$ is the spectral density of the covariance functional of $\theta(\cdot)$ [cf. (2.18)]. Therefore we can write

$$\begin{aligned} &R^{\theta}_{\delta,p_{\pi(1,r)},p_{\pi(2,r)}}(x_{\pi(1,r)} - x_{\pi(2,r)}) \\ &= \delta^{-2}(2\pi)^{-d/2}\int_{\mathbb{R}^d}\exp\left(-i\frac{\mu^r}{\delta}(x_{\pi(1,r)} - x_{\pi(2,r)})\right) \\ &\qquad\qquad \times \sigma^{\theta}_{p_{\pi(1,r)},p_{\pi(2,r)}}(\mu^r)\,d\mu^r \\ &= \begin{cases} \delta^{-2}(2\pi)^{-d/2}\displaystyle\int_{\mathbb{R}^d}\exp\left(-i\frac{\mu^r}{\delta}\left(\sum_{s=\pi(1,r)}^{\pi(2,r)-1}(x_s - x_{s+1})\right)\right) \\ \qquad\qquad \times\sigma^{\theta}_{p_{\pi(1,r)},p_{\pi(2,r)}}(\mu^r)\,d\mu^r, \\ \qquad\qquad\qquad \text{if } \pi(1,r) > k \text{ or } \pi(2,r) \le k, \\ \delta^{-2}(2\pi)^{-d/2}\displaystyle\int_{\mathbb{R}^d}\exp\left(-i\frac{\mu^r}{\delta}\left(\sum_{s=\pi(1,r)}^{k-1}(x_s - x_{s+1})\right.\right. \\ \qquad\qquad + (x_k - x_0') + x_0' - x_0'' + (x_0'' - x_{k+1}) \\ \qquad\qquad \left.\left. + \displaystyle\sum_{s=k+1}^{\pi(2,r)-1}(x_s - x_{s+1})\right)\right)\sigma^{\theta}_{p_{\pi(1,r)},p_{\pi(2,r)}}(\mu^r)\,d\mu^r, \\ \qquad\qquad\qquad \text{if } \pi(1,r) \le k < \pi(2,r). \end{cases} \end{aligned} \tag{5.23}$$

We insert (5.23) into (5.21) and then collect for any term $x_0, x_0 - x_1, \dots, x_{k-1} - x_k$, $x_k - x_0'$, x_0', x_0'', $x_0'' - x_{k+1}, \dots, x_{2k} - x_0'''$ and x_0''', all factors of the integrand of the resulting expression which contain this term. We observe, e.g., that for any $q \in \{1,\dots,2k\}\setminus\{k\}$, the product

$$\prod_{s\in\Lambda(q,\pi)}\exp(-(i/\delta)\mu^s(x_q - x_{q+1})) = \exp\left(-i/\delta\left(\sum_{s\in\Lambda(q,\pi)}\mu^s\right)(x_q - x_{q+1})\right)$$

multiplied by $(-\Delta + \beta)^{-1}(x_q - x_{q+1})$ [respectively, some derivative of $(-\Delta + \beta)^{-1}(x_q - x_{q+1})$] gives a contribution. Next we obtain for $x_k - x_0'$ the contribution $\underline{\mathbf{R}}^{(1,\pi)}_{\beta,p_k}(x_k - x_0')\exp(-i/\delta(\sum_{s\in\Lambda(k,\pi)}\mu^s)(x_k - x_0'))$. The terms x_0', x_0'' and $x_0'' - x_{k+1}$ are contained in similar products, such that finally (5.21) can be written as

$$
\begin{aligned}
&\hat{A}(k,\beta,\delta,h,g,\pi)\\
&\quad = -\delta^{-2k}(2\pi)^{-kd/2}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\left(\prod_{r=1}^{k}\sigma^{\theta}_{p_{\pi(1,r)},\,p_{\pi(2,r)}}(\mu^r)\right)\\
&\qquad\times\Bigg(\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\nabla_{p_1}h(x_0)(-\Delta+\beta)^{-1}(x_0-x_1)\\
&\qquad\quad\times\left(\underline{\mathbf{R}}^{(1,\pi)}_{\beta,\,p_k}(x_k-x_0')\exp\left(-\frac{i}{\delta}\left(\sum_{s\in\Lambda(k,\pi)}\mu^s\right)(x_k-x_0')\right)\right)\\
&\qquad\quad\times\left(g(x_0')\exp\left(-\frac{i}{\delta}\left(\sum_{s\in\Lambda(k,\pi)}\mu^s\right)x_0'\right)\right)\\
&\qquad\quad\times\left(h(x_0'')\exp\left(\frac{i}{\delta}\left(\sum_{s\in\Lambda(k,\pi)}\mu^s\right)x_0''\right)\right)\\
&\qquad\quad\times\left(\underline{\mathbf{R}}^{(2,\pi)}_{\beta,\,p_{k+1}}(x_0''-x_{k+1})\exp\left(-\frac{i}{\delta}\left(\sum_{s\in\Lambda(k,\pi)}\mu^s\right)(x_0''-x_{k+1})\right)\right)\\
&\qquad\quad\times(-\Delta+\beta)^{-1}(x_{2k}-x_0''')\nabla_{p_{2k}}g(x_0''')\\
&\qquad\quad\times\Bigg(\prod_{\substack{q\in B_0(\pi)\\ q\neq k}}(-\Delta+\beta)^{-1}(x_q-x_{q+1})\\
&\qquad\qquad\qquad\times\exp\left(-\frac{i}{\delta}\left(\sum_{s\in\Lambda(q,\pi)}\mu^s\right)(x_q-x_{q+1})\right)\Bigg)\\
&\qquad\quad\times\Bigg(\prod_{\substack{q\in B_1(\pi)\\ q\neq k}}\nabla_{p_{q'(\pi)}}(-\Delta+\beta)^{-1}(x_q-x_{q+1})\\
&\qquad\qquad\qquad\times\exp\left(-\frac{i}{\delta}\left(\sum_{s\in\Lambda(q,\pi)}\mu^s\right)(x_q-x_{q+1})\right)\Bigg)\\
&\qquad\quad\times\Bigg(\prod_{\substack{q\in B_2(\pi)\\ q\neq k}}\nabla_{p_q}\nabla_{p_{q+1}}(-\Delta+\beta)^{-1}(x_q-x_{q+1})\\
&\qquad\qquad\qquad\times\exp\left(-\frac{i}{\delta}\left(\sum_{s\in\Lambda(q,\pi)}\mu^s\right)(x_q-x_{q+1})\right)\Bigg)\\
&\qquad\quad\times dx_0\,dx_0'\,dx_0''\,dx_0'''\,dx_1\cdots dx_{2k}\Bigg)\,d\mu^1\cdots d\mu^k.
\end{aligned}
$$

We observe that the integrand with respect to $dx_0\,dx_0'\,dx_1\,\cdots\,dx_k$ has the convolution form $f_0(x_0)f_1(x_0-x_1)\cdots f_k(x_k-x_0')f_{k+1}(x_0')$. It will be convenient to use Parseval's theorem ($\int_{\mathbb{R}^d}\bar{f}(x)g(x)\,dx=\int_{\mathbb{R}^d}\bar{\tilde{f}}(\tau)\tilde{g}(\tau)\,d\tau$) to express $dx_0\,dx_0'\,dx_1\,\cdots\,dx_k$ integration in Fourier space. Similarly we transform $dx_0''\,dx_0'''\,dx_{k+1}\,\cdots\,dx_{2k}$ integration, such that we obtain

$$
\begin{aligned}
&\hat{A}(k,\beta,\delta,h,g,\pi)\\
&\quad=-\delta^{-2k}(2\pi)^{d-kd/2}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\left(\prod_{r=1}^{k}\sigma^{\theta}_{p_{\pi(1,r)},\,p_{\pi(2,r)}}(\mu^r)\right)\\
&\quad\times\Bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\overline{\widetilde{\nabla_{p_1}h}}(\tau)(\tau^2+\beta)^{-1}\overline{\widetilde{\underline{\mathbf{R}}^{(1,\pi)}_{\beta,p_k}}}\left(\tau-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\\
&\quad\times\tilde{g}\left(\tau-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\\
&\quad\times\bar{\tilde{h}}\left(\tau_1-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\widetilde{\underline{\mathbf{R}}^{(2,\pi)}_{\beta,p_{k+1}}}\left(\tau_1-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\\
&\quad\times(\tau_1^2+\beta)^{-1}\widetilde{\nabla_{p_{2k}}g}(\tau_1)\\
&\quad\times\left(\prod_{\substack{q\in B_0(\pi)\\ q\neq k}}\left(\left(\tau(q)-\frac{1}{\delta}\sum_{s\in\Lambda(q,\pi)}\mu^s\right)^2+\beta\right)^{-1}\right)\\
&\quad\times\left(\prod_{\substack{q\in B_1(\pi)\\ q\neq k}}(-i)\left(\tau(q)_{p_{q'(\pi)}}-\frac{1}{\delta}\sum_{s\in\Lambda(q,\pi)}\mu^s_{p_{q'(\pi)}}\right)\right.\\
&\quad\times\left.\left(\left(\tau(q)-\frac{1}{\delta}\sum_{s\in\Lambda(q,\pi)}\mu^s\right)^2+\beta\right)^{-1}\right)\\
&\quad\times\left(\prod_{\substack{q\in B_2(\pi)\\ q\neq k}}(-1)\left(\tau(q)_{p_q}-\frac{1}{\delta}\sum_{s\in\Lambda(q,\pi)}\mu^s_{p_q}\right)\right.\\
&\quad\times\left(\tau(q)_{p_{q+1}}-\frac{1}{\delta}\sum_{s\in\Lambda(q,\pi)}\mu^s_{p_{q+1}}\right)\\
&\quad\times\left.\left(\left(\tau(q)-\frac{1}{\delta}\sum_{s\in\Lambda(q,\pi)}\mu^s\right)^2+\beta\right)^{-1}\right)d\tau\,d\tau_1\Bigg)\,d\mu^1\cdots d\mu^k,
\end{aligned}
\tag{5.24}
$$

with

$$
\tau(q)=\begin{cases}\tau, & \text{if } q\le k,\\ \tau_1, & \text{if } q\ge k+1.\end{cases}
$$

The inequality

$$\left((a+\tau)^2+\beta\right)^{-1} \le 4\left(1+\frac{\tau^2}{\beta}\right)|a|^{-2}$$

shows that the absolute value of the integrand in (5.24) (with respect to the measure $d\tau\, d\tau_1\, d\mu^1 \cdots d\mu^k$ on $\mathbb{R}^{(k+2)d}$) is less than

$$\begin{aligned}
&C_9(k,\beta)\delta^{-2k}\left(\prod_{r=1}^{k}|\sigma^{\theta}(\mu^r)|\right)\left|\widetilde{\nabla h}(\tau)\right|\left|\widetilde{\underline{\mathbf{R}}_{\beta}^{(1,\pi)}}\left(\tau-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\right|\\
&\quad\times\left|\tilde{g}\left(\tau-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\right|\left|\tilde{h}\left(\tau_1-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\right|\\
&\quad\times\left|\widetilde{\underline{\mathbf{R}}_{\beta}^{(2,\pi)}}\left(\tau_1-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\right|\left|\widetilde{\nabla g}(\tau_1)\right|\left(\prod_{\substack{q\in B_0(\pi)\\ q\ne k}}\left(\delta^2\left|\sum_{s\in\Lambda(q,\pi)}\mu^s\right|^{-2}\right)\right) \qquad (5.25)\\
&\quad\times\left(\prod_{\substack{q\in B_1(\pi)\\ q\ne k}}\left(\delta\left|\sum_{s\in\Lambda(q,\pi)}\mu^s\right|^{-1}\right)\right)\left(1+|\tau|^2+|\tau_1|^2\right)^k\\
&\quad= B\left(k,\beta,\delta,h,g,\pi,\tau,\tau_1,\mu^1,\ldots,\mu^k\right).
\end{aligned}$$

We can estimate

$$\begin{aligned}
&\left|\widetilde{\underline{\mathbf{R}}_{\beta}^{(1,\pi)}}\left(\tau-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\right|\left|\widetilde{\underline{\mathbf{R}}_{\beta}^{(2,\pi)}}\left(\tau_1-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\right|\\
&\quad\le C_{10}(\beta)\delta^{t(\pi)}\left|\sum_{s\in\Lambda(k,\pi)}\mu^s\right|^{-t(\pi)}(1+\tau^2)(1+\tau_1^2),
\end{aligned}$$

where $t(\pi)=2-j$ if $k\in B_j(\pi)$. Then by (1.10) the δ's in the numerator and denominator of the resulting upper bound for $B(k,\beta,\delta,h,g,\pi,\tau,\tau_1,\mu^1,\ldots,\mu^k)$ cancel, and we obtain

$$\begin{aligned}
&B\left(k,\beta,\delta,h,g,\pi,\tau,\tau_1,\mu^1,\ldots,\mu^k\right)\\
&\quad\le C_{11}(k,\beta)\left(\prod_{r=1}^{k}|\sigma^{\theta}(\mu^r)|\right)\left(\prod_{q\in B_0(\pi)}\left|\sum_{s\in\Lambda(q,\pi)}\mu^s\right|^{-2}\right)\\
&\quad\times\left(\prod_{q\in B_1(\pi)}\left|\sum_{s\in\Lambda(q,\pi)}\mu^s\right|^{-1}\right)\left|\widetilde{\nabla h}(\tau)\right|\left|\widetilde{\nabla g}(\tau_1)\right|\left|\tilde{g}\left(\tau-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\right| \qquad (5.26)\\
&\quad\times\left|\tilde{h}\left(\tau_1-\frac{1}{\delta}\sum_{s\in\Lambda(k,\pi)}\mu^s\right)\right|\left(1+\tau_{h,g}^2\right)^{k+2},
\end{aligned}$$

with

$$\tau_{h,g}=\sup\{|\tau|:\ \tau\in\operatorname{supp}(g)\cup\operatorname{supp}(h)\}.$$

The construction of the sets $B_a(\pi)$, $a = 0,1,2$ (cf. Section 1.1), implies that for any $\Lambda = \{s_1,\dots,s_t\} \subseteq \{1,\dots,k\}$ there exists at most one $q \in B_0(\pi)$ or two $q_1, q_2 \in B_1(\pi)$ with $\Lambda(q,\pi) = \Lambda$ [respectively, $\Lambda(q_1,\pi) = \Lambda(q_2,\pi) = \Lambda$]. Hence the singularities in (5.26) with respect to $\mu^1,\dots,\mu^k$ integration have order less than 2. Such singularities are integrable in $\mathbb{R}^d$, $d \geq 3$. Using this fact we can prove by induction on k that the right side of (5.26) is integrable with respect to $d\mu^1 \cdots d\mu^k$. Hence we can apply Lebesgue's bounded convergence theorem to conclude from (2.16)–(2.18), (5.24) and (5.26),

$$\lim_{\delta\to 0}\hat{A}(k,\beta,\delta,h,g,\pi) = 0 \quad \text{if } \Lambda(k,\pi) \neq \varnothing. \tag{5.27}$$

Since for k odd any $\pi \in \mathscr{P}_2^{I(2k)}$ satisfies $\Lambda(k,\pi) \neq \varnothing$, we obtain for any $\alpha > 0$,

$$\lim_{\delta\to 0} A(k,\beta,\delta,h,g,\alpha) = 0 \quad \text{if } k \text{ is odd,} \tag{5.28}$$

and (5.9) is proved.

If k is even and if $\Lambda(k,\pi) = \varnothing$, we have $k \in H_2(\pi)$ and $k+1 \in H_1(\pi)$, and so we can write in (5.21)

$$\underline{\mathbf{R}}_{\beta,p}^{(1,\pi)}(x) = \underline{\mathbf{R}}_{\beta,p}^{(2,\pi)}(x) = \nabla_p(-\Delta+\beta)^{-1}(x).$$

Therefore we obtain from (5.24),

$$\hat{A}(k,\beta,\delta,h,g,\pi) = A_1(k,\beta,\delta,h,g,\zeta)A_1(k,\beta,\delta,h,g,\eta), \tag{5.29}$$

where ζ, respectively η is the restriction of π to the set $\{1,\dots,k\}$, respectively, $\{k+1,\dots,2k\}$, and

$$\begin{aligned}
&A_1(k,\beta,\delta,h,g,\zeta)\\
&\quad= -\delta^{-k}(2\pi)^{-kd/4}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\left(\prod_{r=1}^{k/2}\sigma^{\theta}_{p_{\zeta(1,r)},\,p_{\zeta(2,r)}}(\mu^r)\right)\\
&\qquad\times\left(\int_{\mathbb{R}^d}\left(\overline{\widetilde{\nabla_{p_1}h}}\right)(\tau)(\tau^2+\beta)^{-1}\left(\prod_{q\in B_0(\zeta)}\left(\left(\tau-\frac{1}{\delta}\sum_{s\in\Lambda(q,\zeta)}\mu^s\right)^2+\beta\right)^{-1}\right)\right.\\
&\qquad\times\left(\prod_{q\in B_1(\zeta)}\left(\tau_{p_{q'(\zeta)}}-\frac{1}{\delta}\sum_{s\in\Lambda(q,\zeta)}\mu^s_{p_{q'(\zeta)}}\right)\left(\left(\tau-\frac{1}{\delta}\sum_{s\in\Lambda(q,\zeta)}\mu^s\right)^2+\beta\right)^{-1}\right)\\
&\qquad\times\left(\prod_{q\in B_2(\zeta)}\left(\tau_{p_q}-\frac{1}{\delta}\sum_{s\in\Lambda(q,\zeta)}\mu^s_{p_q}\right)\left(\tau_{p_{q+1}}-\frac{1}{\delta}\sum_{s\in\Lambda(q,\zeta)}\mu^s_{p_{q+1}}\right)\right.\\
&\qquad\qquad\left.\times\left(\left(\tau-\frac{1}{\delta}\sum_{s\in\Lambda(q,\zeta)}\mu^s\right)^2+\beta\right)^{-1}\right)\\
&\qquad\left.\times(-i)\tau_{p_k}(\tau^2+\beta)^{-1}\tilde{g}(\tau)\,d\tau\right)(-1)^{k/2-1}\,d\mu^1\dots d\mu^{k/2}.
\end{aligned}\tag{5.30}$$

An estimate like (5.26) justifies the application of Lebesgue's bounded convergence theorem to (5.30). Therefore, and by (1.10),

$$
\begin{aligned}
&\lim_{\delta\to 0} A_1(k,\beta,\delta,h,g,\zeta)\\
&\quad = -(2\pi)^{-kd/4}\Biggl(\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\Biggl(\prod_{r=1}^{k/2}\sigma^{\theta}_{p_{\zeta(1,r)},\,p_{\zeta(2,r)}}(\mu^r)\Biggr)(-1)^{k/2-1}\\
&\qquad\times\Biggl(\prod_{q\in B_0(\zeta)}\Bigl|\sum_{s\in\Lambda(q,\zeta)}\mu^s\Bigr|^{-2}\Biggr)\Biggl(\prod_{q\in B_1(\zeta)}\Bigl(\sum_{s\in\Lambda(q,\zeta)}\mu^s_{p_{q'(\zeta)}}\Bigr)\Bigl|\sum_{s\in\Lambda(q,\zeta)}\mu^s\Bigr|^{-2}\Biggr)\\
&\qquad\times\Biggl(\prod_{\substack{q\in B_2(\zeta)\\ \Lambda(q,\zeta)\neq\varnothing}}\Bigl(\sum_{s\in\Lambda(q,\zeta)}\mu^s_{p_q}\Bigr)\Bigl(\sum_{s\in\Lambda(q,\zeta)}\mu^s_{p_{q+1}}\Bigr)\Bigl|\sum_{s\in\Lambda(q,\zeta)}\mu^s\Bigr|^{-2}\Biggr)\,d\mu^1\dots d\mu^{k/2}\Biggr)\\
&\qquad\times\Biggl(\int_{\mathbb{R}^d}\Bigl(\widetilde{\nabla_{p_1}h}\Bigr)(\tau)(\tau^2+\beta)^{-1}\Biggl(\prod_{\substack{q\in B_2(\zeta)\\ \Lambda(q,\zeta)=\varnothing}}\tau_{p_q}\tau_{p_{q+1}}(\tau^2+\beta)^{-1}\Biggr)\\
&\qquad\qquad\times(-i)\tau_{p_k}(\tau^2+\beta)^{-1}\tilde{g}(\tau)\,d\tau\Biggr).
\end{aligned}\tag{5.31}
$$

Obviously any $\zeta\in\mathscr{P}_2^{I(k)}$ can be considered a "product" $\zeta=\zeta_1\circ\zeta_2\cdots\circ\zeta_{M(\zeta)}$, where $\zeta_1,\dots,\zeta_{M(\zeta)}$ are "maximal connected components of ζ," i.e., $M(\zeta)=|\{q\in\{1,\dots,k-1\}\colon \Lambda(q,\zeta)=\varnothing\}|+1$ and ζ_j is the restriction of ζ to some subinterval

$$
\begin{gathered}
I_{j,\zeta}=\{q_{j,\zeta},q_{j,\zeta}+1,\dots,q_{j+1,\zeta}-1=q_{j,\zeta}+l_{j,\zeta}-1\},\ l_{j,\zeta}\text{ even},\\
1=q_{1,\zeta}<\cdots<q_{M(\zeta)+1,\zeta}=k+1,\ \zeta_j\in\mathscr{P}_2^{I_{j,\zeta},e},\ j=1,\dots,M(\zeta).
\end{gathered}
$$

By (2.20) and (5.31) we can write

$$
\begin{aligned}
&\lim_{\delta\to 0} A_1(k,\beta,\delta,h,g,\zeta)\\
&\quad = -\Biggl(\int_{\mathbb{R}^d}\Bigl(\widetilde{\nabla_{p_1}h}\Bigr)(\tau)(\tau^2+\beta)^{-1}\Biggl(\prod_{\substack{q\in B_2(\zeta)\\ \Lambda(q,\zeta)=\varnothing}}(-1)\tau_{p_q}\tau_{p_{q+1}}(\tau^2+\beta)^{-1}\Biggr)\\
&\qquad\qquad\times\Biggl(\prod_{j=1}^{M(\zeta)}D^{I_{j,\zeta},\zeta_j}_{p_{q_{j,\zeta}}p_{q_{j+1,\zeta}-1}}\Biggr)(-i)\tau_{p_k}(\tau^2+\beta)^{-1}\tilde{g}(\tau)\Biggr)\,d\tau
\end{aligned}\tag{5.32}
$$

$$
\begin{aligned}
&= \Biggl\langle h,(-\Delta+\beta)^{-1}\Biggl(\prod_{j=1}^{M(\zeta)}\Bigl(D^{I_{j,\zeta},\zeta_j}_{p_{q_{j,\zeta}}p_{q_{j+1,\zeta}-1}}\nabla_{p_{q_{j,\zeta}}}\\
&\qquad\qquad\times\nabla_{p_{q_{j+1,\zeta}-1}}(-\Delta+\beta)^{-1}\Bigr)\Biggr)g\Biggr\rangle.
\end{aligned}\tag{5.33}
$$

Note that (5.32) contains a summation over the indices

$$
\{p_{q_{j,\zeta}-x}\colon x=0,1;\ j=2,\dots,M(\zeta)\}\cup\{p_1,p_k\},
$$

since each of these indices occurs twice in (5.32). Summation of (5.33) over $\zeta \in \mathscr{P}_2^{I(k)}$, (5.27) and (5.29) prove (5.18) for both $b = 1$ and $b = 2$.

The proof of (5.11) is completely analogous to the proof of (5.10). We only have to note that M even, i.e., $M + 1$ odd, implies that for any $\zeta \in \mathscr{P}_2^{I(2M+2)}$ $\Lambda(M + 1, \zeta) \neq \varnothing$. Therefore the operator $(-\Delta)^{-1}$ in the first term on the left side of (5.11) contributes in exactly the same way to the asymptotics as $(-\Delta + \beta)^{-1}$ in the same place would do. This finishes the proof of Lemma 5.8. □

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