The Annals of Probability
1988, Vol. 16, No. 3, 1034–1038

ON BROWNIAN PATHS CONNECTING BOUNDARY POINTS

By Krzysztof Burdzy

Polish Academy of Sciences

There exists a Greenian domain $D \subset \mathbb{R}^2$ such that for every set U of attainable minimal Martin boundary points which has null harmonic measure, there exist attainable minimal Martin boundary points $u, v \notin U$ which cannot be connected by an h-process in D starting from u and converging to v.

1. Introduction. Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a Greenian domain, let $\partial_a^M D$ denote the set of all attainable minimal Martin boundary points in D, let μ denote the harmonic measure on the Martin boundary of D and let Z be the set of all pairs (u, v) such that $u, v \in \partial_a^M D$, $u \neq v$, and there is no h-process in D starting from u and converging to v. Salisbury (1986) has proved that Z is nonempty in some domains [see Burdzy (1985) for a different proof of this theorem] and that $(\mu \times \mu)(Z) = 0$ in every domain. It will be shown that there exists a domain $D \subset \mathbb{R}^2$ with the property that for every set $U \subset \partial_a^M D$ which has null harmonic measure, there exist points $u, v \in \partial_a^M D \setminus U$ such that $(u, v) \in Z$. This implies that Z is not a subset of any set of the form $U \times U$, where $U \subset \partial_a^M D$, $\mu(U) = 0$.

The construction of the nonempty set Z given below is simpler than the ones mentioned above, although its main idea is not unrelated to that of Salisbury (1986).

Here is the main idea of the example.
Let $D = \mathbb{R}^2 \setminus W$, where

(i) W is a closed subset of a straight line M,
(ii) W is nowhere dense in M,
(iii) W has a strictly positive (1-dimensional) Lebesgue measure,
(iv) W is large enough so that its "typical" point z^1 corresponds to two minimal Martin boundary points u and v^1 on two sides of M [(iii) $\Rightarrow$ (iv)], and
(v) $M \setminus W$ is relatively equidistributed in M so that $P_u^{v^1}$ does not exist [(v) $\Rightarrow$ (ii)].

Another (unpublished yet) example due to Salisbury should be mentioned here. There exist a domain $D \subset \mathbb{R}^3$ and a point $u \in \partial_a^M D$ such that $(u, v) \in Z$ for all $v \in \partial_a^M D$, $v \neq u$.

2. Statement of the result. A short review of some fundamental concepts will be offered first. The monograph of Doob (1984) is the main source of the notation and results presented below. See also Salisbury (1986).

For each Greenian domain $D \subset \mathbb{R}^2$, fix a reference point $z^0 \in D$ and denote its Green function by G_D. Let

$$K_{z^0}^D(x, y) = G_D(x, y)/G_D(x, z^0),$$

Received July 1986; revised February 1987.
AMS 1980 *subject classifications*. Primary 60J50, 60J65.
Key words and phrases. Brownian motion, h-processes, Martin boundary.

for $x, y \in D$, $x \neq z^0$. There exists a unique up to a homeomorphism compactification D^M of D such that $K^D_{z^0}$ may be extended continuously to $(D^M \setminus \{z^0\}) \times D$ and $K^D_{z^0}(u, \cdot) \equiv K^D_{z^0}(v, \cdot)$ only if $u = v$. The set $\partial^M D \overset{\text{df}}{=} D^M \setminus D$ will be called the Martin boundary of D and the topology of D^M will be called the Martin topology. A point $u \in \partial^M D$ will be called minimal if every positive harmonic function in D majorized by $K^D_{z^0}(u, \cdot)$ is a constant multiple of $K^D_{z^0}(u, \cdot)$. The set of all minimal Martin boundary points will be denoted $\partial^M_1 D$. A set $A \subset D$ will be called minimal thin in D at $u \in \partial^M_1 D$ if u is an isolated point in the subset $\{u\} \cup A$ of D^M or there exists a measure ν such that

$$\int_D K^D_{z^0}(u, y)\nu(dy) < \liminf_{\substack{z \to u \\ z \neq u \\ z \in A}} \int_D K^D_{z^0}(z, y)\nu(dy).$$

This definition of minimal thinness is taken from Naim (1957); see Doob (1984), 1.XII.11, for an alternative definition. The minimal fine topology is defined by declaring that $u \in \partial^M_1 D$ is a minimal fine limit point of A if A is not minimal thin in D at u.

Let Ω be the space of paths ω: $(0, \infty) \to \mathbb{R}^2 \cup \{\delta\}$, continuous on $(0, R)$ and equal to δ at their lifetime R and afterwards. Let X be the canonical process, i.e., $X_t(\omega) = \omega(t)$, and let $F = \sigma\{X_s,\ s > 0\}$. Let $P^D_t(x, dy)$ denote the transition probabilities of Brownian motion killed at the hitting time of $\mathbb{R}^2 \setminus D$. For each $x \in D$ and positive harmonic function h in D, there exists a measure P^x_h on (Ω, F) such that the process X starts from x and is strong Markov with the transition probabilities $P^D_t(x, dy)h(y)/h(x)$. If $h(\cdot) \equiv K^D_{z^0}(u, \cdot)$ for some $u \in \partial^M_1 D$, then the symbol P^x_u wil be used instead of P^x_h. $P^x_D \equiv P^x_1$, i.e., P^x_D will denote the distribution of Brownian motion in D.

The process with a distribution P^x_h will be called an h-process. For $u \in \partial^M_1 D$, P^u_h will denote the distribution of the h-process (if it exists) starting from u, i.e., $\lim_{t \to 0} X_t = u$, P^u_h-a.s. in the Martin topology. A point $u \in \partial^M_1 D$ will be called attainable if $P^x_u(R < \infty) = 1$ for some (and therefore for all) $x \in D$. The set of all attainable points will be denoted $\partial^M_a D$. The harmonic measure on $\partial^M D$ relative to $x \in D$ will be denoted μ_x.

THEOREM. *There exists an open, connected and Greenian set $D \subset \mathbb{R}^2$ such that for each set $U \subset \partial^M_a D$ with $\mu_x(U) = 0$, there exist points $u, v \in \partial^M_a D \setminus U$, $u \neq v$, for which P^u_v does not exist.*

3. Proof. The coordinates of a point $x \in \mathbb{R}^2$ will be denoted x_1 and x_2. Let

$$D_1 = \{x \in \mathbb{R}^2: x_2 > 0\}, \qquad D_2 = \{x \in \mathbb{R}^2: x_2 < 0\},$$
$$M = \{x \in \mathbb{R}^2: x_2 = 0\}, \qquad I = \{x \in M: 0 < x_1 < 1\},$$

and

$$T_A = \inf\Big\{t > 0: \lim_{s \to t^-} X_s \text{ exists and } \in A\Big\}.$$

∂A and A^c will denote the Euclidean boundary and the complement of a set $A \subset \mathbb{R}^2$. Consider annulae

$$A^k_j = \{x \in \mathbb{R}^2: a_k < |x - (ja_k(2 + 2^{-k}), 0)| < a_k(1 + 2^{-k})\}$$

and circles

$$B_j^k = \{x \in \mathbb{R}^2 \colon |x - (ja_k(2 + 2^{-k}), 0)| = a_k(1 + 2^{-k-1})\},$$

for integers k and j and reals $a_k > 0$ which will be chosen below. The Harnack principle implies that there exist constants $b_k > 0$ such that

$$h(x)/h(y) \geq b_k, \tag{3.1}$$

for all $x, y \in B_j^k$ and all positive harmonic functions h in A_j^k. The constants b_k do not depend on j or a_k, by the translation and scaling invariance of harmonic functions. Choose a_k's so small that

$$b_k/[a_k(1 + 2^{-k-1})] > k. \tag{3.2}$$

Denote $D = D_1 \cup D_2 \cup \bigcup_{k=2}^{\infty} \bigcup_{j=0}^{\infty} (A_j^k \cap I)$ and $J = \partial D \cap I$. It is easy to see that D is open, connected and Greenian. The measure of $\bigcup_{j=0}^{\infty}(A_j^k \cap I)$ is less or equal to 2^{-k} so the measure of $\bigcup_{k=2}^{\infty}\bigcup_{j=0}^{\infty}(A_j^k \cap I)$ does not exceed 2^{-1}. It follows that the measure of J is at least 2^{-1}.

Observe that

$$P_D^x(R = T_{\partial D} = T_M = T_J < \infty) > 0, \tag{3.3}$$

for $x \in D_1$, since the Brownian hitting distribution of a straight line is mutually absolutely continuous with the Lebesgue measure on this line and J has a positive measure. The distribution P_D^x is a mixture of the measures P_u^x for $u \in \partial_1^M D$ and the mixing measure is μ_x [see Doob (1984), 2.X.8]. This and (3.3) imply that there exists a set $W_1 \subset \partial_1^M D$ such that $\mu_x(W_1) > 0$ and

$$P_u^x(R = T_\partial D = T_M = T_j < \infty) > 0,$$

for all $u \in W_1$, $x \in D_1$. It follows that

$$P_u^x\Big(R < \infty,\ \lim_{t \to R^-} X_t \text{ exists and} \in J, \sup\{t > 0\colon X_t \in D_1^c\} < R\Big) > 0,$$

for $u \in W_1$, $x \in D_1$. The last event belongs to the tail σ-field, so it has probability 1 [see Doob (1984), 2.X.11(c1)] and, for a similar reason, $\lim_{t \to R^-} X_t = z^1$, P_u^x-a.s. for some point $z^1 = z^1(u) \in J$. Thus

$$P_u^x\Big(R < \infty,\ \lim_{t \to R^-} X_t = z^1(u) \in J, \sup\{t > 0\colon X_t \in D_1^c\} < R\Big) = 1, \tag{3.4}$$

for $u \in W_1$, $x \in D_1$. This means, in particular, that $W_1 \subset \partial_a^M D$. By symmetry with respect to M, for each $u \in W_1$ there exists $v^1 = v^1(u) \in \partial_a^M D$ such that

$$P_{v^1}^x\Big(R < \infty,\ \lim_{t \to R^-} X_t = z^1(u) \in J, \sup\{t > 0\colon X_t \in D_2^c\} < R\Big) = 1,$$

for $x \in D_2$.

Fix a point $u \in W_1$ and assume without loss of generality that the reference point z^0 is the same for D and D_1. The last exit time from D_1^c is P_u^x-a.s. strictly less than the lifetime R, by (3.4). Theorem 3.III.3 of Doob (1984) implies that $D \setminus D_1$ is minimal thin in D at u. It follows from Theorem 11 of Naim (1957) that

$$\liminf_{\substack{x \to u \\ x \in D_1}} G_D(x, z^0)/G_{D_1}(x, z^0) < \infty,$$

where $x \to u$ in the Martin topology.

Theorems 1.XII.14 and 1.XII.21 of Doob (1984) imply that

$$\lim_{\substack{x \to u \\ x \in D_1}} G_D(x, z^0)/G_{D_1}(x, z^0) = q = q(u) < \infty,$$

where x converges in the minimal fine topology and

$$\lim_{\substack{x \to z^1(u) \\ x_1 = z_1^1(u) \\ x \in D_1}} G_D(x, z^0)/G_{D_1}(x, z^0) = q, \tag{3.5}$$

where x converges in the Euclidean topology. The Green function is a monotone function of the set so $G_D(x, y)/G_{D_1}(x, y) \geq 1$. This and (3.5) yield

$$\begin{aligned} &\liminf_{\substack{x \to z^1(u) \\ x_1 = z_1^1(u) \\ x \in D_1}} K_{z^0}^D(x, y)/K_{z^0}^{D_1}(x, y) \\ &\qquad \geq \liminf_{\substack{x \to z^1(u) \\ x_1 = z_1^1(u) \\ x \in D_1}} \frac{G_D(x, y)}{G_D(x, z^0)} \frac{G_{D_1}(x, z^0)}{G_{D_1}(x, y)} \geq q^{-1} > 0, \end{aligned} \tag{3.6}$$

for $y \in D_1$. The Martin boundary of D_1 may be identified with the Euclidean boundary of D_1 near z^1 [Doob (1984), 1.XII.4]. The continuity of $K_{z^0}^D$ and $K_{z^0}^{D_1}$ in D^M and D_1^M combined with (3.6) gives $K_{z^0}^D(u, y)/K_{z^0}^{D_1}(z^1(u), y) \geq q^{-1}$ for $y \in D_1$.

Choose $k_0 = k_0(u)$ so large that for each $k \geq k_0$, $z^1(u)$ is inside the inner circle of an annulus A_j^k such that $A_j^k \cap M = A_j^k \cap I$. Easy geometry shows that for $k \geq k_0$ there exists a point $y^1 = y^1(u) \in B_j^k \cap D_1$ such that $y_1^1 = z_1^1(u)$ and $y_2^1 \leq a_k(1 + 2^{-k-1})$. One has $K_{z^0}^{D_1}(z^1, y) = c/y_2$ for some $c = c(z^0, z^1) > 0$ and all y such that $y_1 = z_1^1$ [see Doob (1984), 1.VIII.9]. Thus, by (3.2),

$$\begin{aligned} K_{z^0}^D(u, y^1) &\geq q^{-1} K_{z^0}^{D_1}(z^1(u), y^1) = q^{-1}c/y_2^1 \\ &\geq q^{-1}c/\left[a_k(1 + 2^{-k-1})\right] \geq q^{-1}ck/b_k. \end{aligned}$$

The function $K_{z^0}^D(u, \cdot)$ is harmonic in A_j^k so (3.1) implies that $K_{z^0}^D(u, y) \geq b_k q^{-1}ck/b_k = q^{-1}ck$ for all $y \in B_j^k$. The function $G_D(z^0, \cdot)$ is bounded in a neighborhood N of $z^1(u)$ by a constant $d = d(z^0, z^1) < \infty$, so

$$f(y) \stackrel{\text{df}}{=} K_{z^0}^D(u, y)/G_D(z^0, y) \geq q^{-1}ck/d,$$

for all large k and $y \in B_j^k \cap N$. It follows that $\limsup_{y \to z^1(u),\, y \in \Gamma} f(y) = \infty$ for every continuous path $\Gamma \subset D$ with endpoint z^1. Thus

$$P_{v^1}^{x}\Big(\limsup_{t \to R^-} f(X_t) = \infty\Big) = 1,$$

for $x \in D_2$. A theorem of Walsh, quoted by Salisbury (1986) as Theorem 2.3(c), implies that $P_u^{v^1}$ does not exist.

Let W_2 be the collection of all points $v^1 = v^1(u)$ for $u \in W_1$. Take any set $U \subset \partial_a^M D$ with $\mu_x(U) = 0$. The symmetry implies that $V \overset{\mathrm{df}}{=} \{u \in W_1 \colon v^1(u) \in U\}$ has null harmonic measure, since $\mu_x(W_2 \cap U) = 0$. Thus $\mu_x(W_1 \setminus (V \cup U)) > 0$. If $u \in W_1 \setminus (V \cup U)$, then $u, v^1 \in \partial_a^M D \setminus U$ and $P_u^{v^1}$ does not exist. □

Acknowledgments. I am grateful to Tom Salisbury for drawing my attention to the problem and for the long correspondence related to it. I would like to thank the referee for several helpful comments.

REFERENCES

BURDZY, K. (1985). On a theorem of Salisbury. *Probab. Math. Statist.* **6** 161–166.

DOOB, J. L. (1984). *Classical Potential Theory and Its Probabilistic Counterpart.* Springer, New York.

NAIM, L. (1957). Sur le rôle de la frontière de R. S. Martin dans le théorie du potential. *Ann. Inst. Fourier (Grenoble)* **7** 183–281.

SALISBURY, T. S. (1986). A Martin boundary in the plane. *Trans. Amer. Math. Soc.* **293** 623–642.

DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47907