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ON THE CONVERGENCE RATE IN THE CENTRAL LIMIT THEOREM FOR ASSOCIATED PROCESSES

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We give uniform rates of convergence in the central limit theorem for associated processes with finite third moment. No stationarity is required. Using a coefficient $u(n)$ which describes the covariance structure of the process, we obtain a convergence rate $O(n^{-1/2}\log^2 n)$ if $u(n)$ exponentially decreases to 0. An example shows that such a rate can no longer be obtained if $u(n)$ decreases only as a power.

1. Introduction and notation. Let $\{X_j\colon j \in \mathbb{N}\}$ be a process of associated random variables, i.e., for every finite subcollection $X_{j(1)}, \dots, X_{j(m)}$ and every pair of coordinatewise nondecreasing functions $f, g\colon \mathbb{R}^m \to \mathbb{R}$ there holds

$$\operatorname{Cov}\big(f(X_{j(1)}, \dots, X_{j(m)}), g(X_{j(1)}, \dots, X_{j(m)})\big) \geq 0,$$

whenever the covariance is defined. Associated processes are of considerable use in physics and statistics and have been investigated in recent years to a great extent [see for example Newman (1984) and the references therein].

Assume in the following that $EX_j = 0$, $EX_j^2 < \infty$ and put $S_n = \sum_{j=1}^n X_j$, $\sigma_n^2 = ES_n^2$.

Several authors have shown that associated processes satisfy—under appropriate conditions—the central limit theorem, i.e.,

$$\Delta_n := \sup_{x \in \mathbb{R}} \left| P\{\sigma_n^{-1} S_n \leq x\} - \phi(x) \right| = o(1), \tag{1.1}$$

where $\phi(x) = (2\pi)^{-1/2}\int_{-\infty}^x \exp(-t^2/2)\,dt$ denotes the standard normal distribution function. Newman (1980) obtained (1.1) for strictly stationary associated processes, satisfying

$$0 < \sigma^2 = \operatorname{Cov}(X_1, X_1) + 2\sum_{j=2}^{\infty} \operatorname{Cov}(X_1, X_j) < \infty. \tag{1.2}$$

Using the coefficient

$$u(n) = \sup_{k \in \mathbb{N}} \sum_{j\colon |j-k| \geq n} \operatorname{Cov}(X_j, X_k), \qquad n \in \mathbb{N} \cup \{0\},$$

Cox and Grimmett (1984) weakened the assumption of stationarity. The conditions

$$u(n) = o(1), \qquad u(0) < \infty, \tag{1.3}$$

$$\inf_{j \in \mathbb{N}} EX_j^2 > 0, \tag{1.4}$$

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and

$$\sup_{j \in \mathbb{N}} E|X_j|^3 < \infty \tag{1.5}$$

imply the central limit theorem. Up to now, there is only one result which yields a convergence rate for Δ_n [see Wood (1983)]. The inequality given in the paper of Wood maximally leads to $\Delta_n = O(n^{-1/5})$, contrasting with his statement (see Remark 2.3). This convergence rate, however, is far from the optimal rate, the Berry–Esseen rate $O(n^{-1/2})$.

We prove that $\Delta_n = O(n^{-1/2}\log^2 n)$ if $u(n)$ exponentially decreases to 0, $\inf_{n \in \mathbb{N}} \sigma_n^2/n > 0$, and $\sup_{j \in \mathbb{N}} E|X_j|^3 < \infty$ (see Theorem 2.1). An example shows that we can no longer obtain this convergence rate for Δ_n if we only slightly weaken the assumption concerning $u(n)$ (see Example 2.2). If instead of $\sup_{j \in \mathbb{N}} E|X_j|^3 < \infty$, however, we assume $\sup_{j \in \mathbb{N}} E|X_j|^{3+\delta} < \infty$ for some $\delta > 0$, then even a convergence rate $O(n^{-1/2}\log n)$ can be obtained. We do not know whether the Berry–Esseen rate $O(n^{-1/2})$ is available. Let us remark that also for strongly mixing processes the convergence rate $O(n^{-1/2}\log^2 n)$ appears and it is still an open problem whether this is the optimal rate [see Tikhomirov (1980)].

In the next section we present the exact results, postponing some technical lemmas to Section 3.

2. The results.

Theorem 2.1. *Let $\{X_j\colon j \in \mathbb{N}\}$ be an associated process with $EX_j = 0$ satisfying*

$$u(n) = O(e^{-\lambda n}) \quad \textit{for some } \lambda > 0, \tag{2.1}$$

$$\inf_{n \in \mathbb{N}} \sigma_n^2/n > 0, \tag{2.2}$$

and

$$\sup_{j \in \mathbb{N}} E|X_j|^3 < \infty. \tag{2.3}$$

Then there exists a constant B not depending on n such that for all $n \in \mathbb{N}$

$$\Delta_n \le Bn^{-1/2}\log^2 n.$$

If instead of (2.3) *we assume*

$$\sup_{j \in \mathbb{N}} E|X_j|^{3+\delta} < \infty \quad \textit{for some } \delta > 0, \tag{2.3*}$$

then there exists B not depending on n such that for all $n \in \mathbb{N}$

$$\Delta_n \le Bn^{-1/2}\log n.$$

Proof. The theorem will be proved by modifying methods of Tikhomirov (1980) and Schneider (1981). We adopt their notation. Throughout the rest of the paper the symbols B, C, D with or without a subscript will denote a bounded quantity not depending on n. The symbol $\theta(t)$ with or without a subscript will denote a function such that $|\theta(t)| \le 1$, which may depend on n. Let $m = m(n) = [C \log n]$ and $k = k(n) = [D \log n]$, where $C, D > 0$ will be specified later, and let f_n be the characteristic function of $\sigma_n^{-1} S_n$.

For fixed n we will derive a differential equation for $f_n(t)$ in the region $0 \le t \le \sigma_n/(8b_m)$, where

$$b_m = \max_{1 \le p \le m} \sup_{l \in \mathbb{N} \cup \{0\}} \left(E|S_{p+l} - S_l|^3 \right)^{1/3}.$$

As in Tikhomirov (1980) and Schneider (1981) we obtain

$$\begin{aligned} f_n'(t) &= i\sigma_n^{-1} \sum_{j=1}^{n} E\left(X_j \exp\left(itS_j^{(1)} \right) \right) \\ &\quad + i\sigma_n^{-1} \sum_{r=2}^{k} \sum_{j=1}^{n} E\left(X_j \prod_{l=1}^{r-1} \xi_j^{(l)} \left(\exp\left(itS_j^{(r)} \right) - f_n(t) \right) \right) \\ &\quad + i\sigma_n^{-1} \sum_{r=3}^{k} \sum_{j=1}^{n} E\left(X_j \prod_{l=1}^{r-1} \xi_j^{(l)} \right) f_n(t) + i\sigma_n^{-1} \sum_{j=1}^{n} E\left(X_j \xi_j^{(1)} \right) f_n(t) \\ &\quad + i\sigma_n^{-1} \sum_{j=1}^{n} E\left(X_j \prod_{l=1}^{k} \xi_j^{(l)} \exp\left(itS_j^{(k)} \right) \right), \end{aligned} \tag{2.4}$$

where for $j = 1, \dots, n$ and $l = 1, \dots, k$,

$$S_j^{(0)} = S_{j,n}^{(0)} = \sigma_n^{-1} S_n,$$

$$S_j^{(l)} = S_{j,n}^{(l)} = \sigma_n^{-1} \sum_{\substack{1 \le \nu \le n \\ |\nu - j| > lm}} X_\nu,$$

$$\xi_j^{(l)} = \xi_{j,n}^{(l)}(t) = \exp\left(it\left(S_j^{(l-1)} - S_j^{(l)} \right) \right) - 1.$$

We will need some technical lemmas to estimate the summands in (2.4). These results, which are given in Section 3, are comparable to Lemmas 4.1–4.4 of Tikhomirov (1980), respectively, to the modified estimates of Schneider (1981).

Put $\delta(m) = \sum_{i=m+1}^{\infty} u(i)$. Assumption (2.1) implies $\delta(m) = O(e^{-\lambda m})$, and thus we can choose m and k such that

$$\delta(m)^{1/3} = O(n^{-2}),$$

$$k 4^k \delta(m)^{1/3} = O(1),$$

$$(1/2)^{k/2} = O(n^{-1})$$

[cf. the proof of Theorems 1–4 in Tikhomirov (1980)]. From (2.2) it follows that $n = O(\sigma_n^2)$. Using Hölder's and Minkowski's inequality, we get from (2.2) and (2.3) that $m^{1/2} = O(b_m)$, $b_m = O(m)$.

Lemmas 3.2–3.6 and a simple but tedious estimation of the summands in (2.4) lead to the following relation in the region $0 \le t \le \sigma_n/(8b_m)$:

$$\begin{aligned} f_n'(t) &= -tf_n(t) + B_1\theta_1(t) n^{-1/2} b_m^2 t^2 f_n(t) \\ &\quad + B_2\theta_2(t) n^{-1} m b_m t^2 + B_3\theta_3(t) n^{-3/2}. \end{aligned} \tag{2.5}$$

Since $b_m = O(m) = O(\log n)$, it follows from (2.5) as in Tikhomirov (1980) that

$$\left| f_n(t) - e^{-t^2/2} \right| \le B\left(n^{-1/2} \log^2 n\, t^3 e^{-t^2/4} + n^{-1} \log^2 n\, t + n^{-3/2} t \right),$$

which holds in the region $0 \le t \le \gamma n^{1/2} \log^{-2} n$. An application of Esseen's inequality now implies the first assertion of our theorem.

If (2.3*) is satisfied instead of (2.3), (2.1) and Theorem 1 of Birkel (1988) imply $b_m = O(m^{1/2}) = O(\log^{1/2} n)$. Then (2.5) leads to the estimate

$$\left| f_n(t) - e^{-t^2/2} \right| \le B\left(n^{-1/2} \log n\, t^3 e^{-t^2/4} + n^{-1} \log^{3/2} n\, t + n^{-3/2} t \right),$$

which holds in the region $0 \le t \le \gamma n^{1/2} \log^{-1} n$. According to Esseen's inequality, this implies the second assertion and completes the proof of our theorem. □

Let us remark that Theorem 2.1 provides convergence rates for Δ_n in the central limit theorem of Cox and Grimmett (1984) (note that $\sigma_n^2/n \ge \sum_{j=1}^n EX_j^2/n \ge \inf_{j \in \mathbb{N}} EX_j^2$, since the random variables are nonnegatively correlated). Using that for stationary associated processes

$$u(n) = 2 \sum_{j=n+1}^{\infty} \operatorname{Cov}(X_1, X_j), \qquad n \in \mathbb{N},$$

we also obtain convergence rates in the central limit theorem of Newman (1980).

We now present an example of associated processes satisfying the central limit theorem, for which lower bounds are obtained for Δ_n. It shows that a convergence rate $O(n^{-1/2}\log^2 n)$ can no longer be obtained for Δ_n if instead of (2.1) we assume that $u(n)$ decreases only as a power.

Example 2.2. For every $\beta > 0$ there exist an associated process $\{X_j: j \in \mathbb{N}\}$ with $EX_j = 0$ and a real number $\rho \in (0, 1/2)$ such that

$$u(n) = O(n^{-\beta}), \tag{2.1*}$$

(2.2) and (2.3) are satisfied, but

$$\limsup n^{\rho} \Delta_n = \infty$$

holds.

Proof. The following construction depends on an example of Tikhomirov (1980), but the details are quite different. For $\alpha > 0$ and $\delta \in (0, 1)$ let $\{\xi_i: i \in \mathbb{N}\}$ be a sequence of i.i.d. random variables satisfying

$$P\left\{ \xi_1 = \sum_{j=1}^{k} j^{\alpha} \right\} = P\left\{ \xi_1 = -\sum_{j=1}^{k} j^{\alpha} \right\} = Ck^{-1-(2+\delta)(1+\alpha)}, \qquad k \in \mathbb{N},$$

where $2C\sum_{k=1}^{\infty} k^{-1-(2+\delta)(1+\alpha)} = 1$.

For $j \in \mathbb{N}$ put

$$X_j = \sum_{i=1}^{j} \left(i^{\alpha} 1_{\{\xi_{j-i+1} > \sum_{l=1}^{i-1} l^{\alpha}\}} - i^{\alpha} 1_{\{\xi_{j-i+1} < -\sum_{l=1}^{i-1} l^{\alpha}\}} \right).$$

Then $EX_j = 0$, according to our construction.

Since

$$\sum_{i=1}^{j}\left(i^{\alpha}1_{\{t_{j-i+1}>\sum_{l=1}^{i-1}l^{\alpha}\}} - i^{\alpha}1_{\{t_{j-i+1}<-\sum_{l=1}^{i-1}l^{\alpha}\}}\right)$$

is nondecreasing in $t_1,\dots,t_j$, Theorem 2.1 and Property (P_4) of Esary, Proschan and Walkup (1967) imply that $\{X_j: j\in\mathbb{N}\}$ is an associated process.

We will prove the following relations:

$$\sup_{j\in\mathbb{N}} E|X_j|^3 < \infty \quad \text{if } \alpha < (1+\delta)/(1-\delta), \tag{2.6}$$

$$\inf_{n\in\mathbb{N}} \sigma_n^2/n \geq 1, \tag{2.7}$$

$$u(n) = O(n^{-\delta(1+\alpha)}), \tag{2.8}$$

$$\Delta_n \geq Bn^{-\delta/2}\log^{-2-\delta} n \quad \text{for some } B > 0. \tag{2.9}$$

It is easy to see that (2.6)–(2.9) lead to an example having the required properties. Therefore it remains to prove (2.6)–(2.9).

PROOF OF (2.6). By definition of X_j,

$$E|X_j|^3 \leq \sum_{\nu,\mu,\rho=1}^{j} \nu^{\alpha}\mu^{\alpha}\rho^{\alpha}P\left\{|\xi_{j-\nu+1}| > \sum_{l=1}^{\nu-1} l^{\alpha}, |\xi_{j-\mu+1}| > \sum_{l=1}^{\mu-1} l^{\alpha}, |\xi_{j-\rho+1}| > \sum_{l=1}^{\rho-1} l^{\alpha}\right\}$$
$$= T_1 + T_2 + T_3,$$

where T_1 means the sum over all equal indices, T_2 means the sum over all indices for which exactly one differs from the other two, and T_3 means the sum over all pairwise different indices.

As $\alpha < (1+\delta)/(1-\delta)$, we obtain

$$T_1 = \sum_{\nu=1}^{j} \nu^{3\alpha}P\left\{|\xi_1| > \sum_{l=1}^{\nu-1} l^{\alpha}\right\}$$
$$\leq C_1\sum_{\nu=1}^{\infty} \nu^{-\delta(1+\alpha)-2+\alpha} \leq C_2 < \infty.$$

Using the fact that the ξ_i are independent, T_2 and T_3 are estimated in a similar way. This proves (2.6).

PROOF OF (2.7). Our construction yields

$$\operatorname{Var}(X_j) \geq \operatorname{Var}\left(1_{\{\xi_j>0\}} - 1_{\{\xi_j<0\}}\right) = P\{|\xi_1| > 0\} = 1.$$

Since the X_j are associated, their covariances are nonnegative. Hence, $\sigma_n^2 \geq \sum_{j=1}^{n}\operatorname{Var}(X_j) \geq n$, which proves (2.7).

PROOF OF (2.8). Let $j, k \in \mathbb{N}$ be given. Then our construction yields

$$\begin{aligned}\operatorname{Cov}(X_j, X_{j+k}) &= 2\sum_{i=1}^{j} i^{\alpha}(i+k)^{\alpha} P\left\{\xi_1 > \sum_{l=1}^{i+k-1} l^{\alpha}\right\}\\ &\le C_3 \sum_{i=k+1}^{\infty} i^{-2-\delta(1+\alpha)}\\ &\le C_4 k^{-1-\delta(1+\alpha)}.\end{aligned}$$

According to the definition of $u(n)$, this immediately implies (2.8).

PROOF OF (2.9). As in (5.14) and (5.15) of Tikhomirov (1980) we obtain

$$\Delta_n \ge P\{S_n > \sigma_n \log n\} - o(n^{-1/2}). \tag{2.10}$$

From our construction it follows that

$$S_n = \sum_{i=1}^{n} \xi_i - \sum_{i=1}^{n} Y_i \quad \text{almost everywhere,}$$

where for $i = 1, \dots, n$,

$$Y_i = \left(\xi_i - \sum_{l=1}^{n+1-i} l^{\alpha}\right) 1_{\{\xi_i > \sum_{l=1}^{n+1-i} l^{\alpha}\}} + \left(\xi_i + \sum_{l=1}^{n+1-i} l^{\alpha}\right) 1_{\{\xi_i < -\sum_{l=1}^{n+1-i} l^{\alpha}\}}.$$

Hence, by (2.10),

$$\Delta_n \ge P\left\{\sum_{i=1}^{n} \xi_i \ge 2\sigma_n \log n\right\} - P\left\{\sum_{i=1}^{n} Y_i \ge \sigma_n \log n\right\} - o(n^{-1/2}). \tag{2.11}$$

Since the ξ_i are i.i.d. with symmetric distribution, it follows [cf. Petrov (1975), page 285] that for $t \in \mathbb{R}$,

$$P\left\{\sum_{i=1}^{n} \xi_i \ge t\right\} \ge (n/2) P\{\xi_1 \ge t\}\left(1 - 2(n-1)P\{\xi_1 \ge t\}\right).$$

Using this for $t = 2\sigma_n \log n$ and using $n \le \sigma_n^2 \le u(0)n$, we get

$$P\left\{\sum_{i=1}^{n} \xi_i \ge 2\sigma_n \log n\right\} \ge C_5 n^{-\delta/2} \log^{-2-\delta} n. \tag{2.12}$$

By construction we have

$$E\left|\sum_{i=1}^{n} Y_i\right| \le \sum_{i=1}^{n} E|\xi_1| 1_{\{|\xi_1| > \sum_{l=1}^{i} l^{\alpha}\}} \le C_6 \sum_{i=1}^{\infty} i^{-(1+\delta)(1+\alpha)} \le C_7 < \infty.$$

Hence Markov's inequality and (2.7) yield

$$P\left\{\sum_{i=1}^{n} Y_i \ge \sigma_n \log n\right\} = o(n^{-1/2}). \tag{2.13}$$

Relations (2.11)–(2.13) prove (2.9), which completes our example. □

We conclude this section with a remark concerning the inequality of Wood (1983). Under the conditions of Newman's (1980) theorem [i.e., $\{X_j\colon j \in \mathbb{N}\}$ is a strictly stationary process fulfilling (1.2)], he obtained the estimate for $n = mk$, $x \in \mathbb{R}$:

$$\begin{aligned} &\left|P\{n^{-1/2}S_n \le x\} - N(0,\sigma^2)((-\infty, x])\right| \\ &\quad \le \left[16\bar{\sigma}_k^4 m(\sigma^2 - \bar{\sigma}_k^2)/(9\pi\bar{\rho}_k^2)\right] + \left[3\bar{\rho}_k/(\bar{\sigma}_k^3 m^{1/2})\right], \end{aligned} \tag{2.14}$$

where $\bar{\sigma}_k^2 = \sigma_k^2/k$, $\bar{\rho}_k = E|S_k|^3/k^{3/2}$. But the convergence rate given by (2.14) is far from the optimal rate.

REMARK 2.3. Let $\{X_j\colon j \in \mathbb{N}\}$ be a strictly stationary associated process fulfilling $EX_1 = 0$, $E|X_1|^3 < \infty$ and (1.2). Assume that the random variables are not independent. Then (2.14) maximally leads to a convergence rate $O(n^{-1/5})$.

PROOF. Since uncorrelated associated random variables are independent [cf. Corollary 3 of Newman (1984)], we can choose $j_0 \ge 2$ such that $\mathrm{Cov}(X_1, X_{j_0}) > 0$. Hence, for $k \ge j_0$,

$$\begin{aligned} \sigma^2 - \bar{\sigma}_k^2 &= 2\sum_{j=k+1}^{\infty} \mathrm{Cov}(X_1, X_j) + (2/k)\sum_{j=2}^{k}(j-1)\mathrm{Cov}(X_1, X_j) \\ &\ge (2/k)(j_0 - 1)\mathrm{Cov}(X_1, X_{j_0}) \\ &= C_1/k, \qquad C_1 > 0. \end{aligned} \tag{2.15}$$

Note that the association of the process implies $\mathrm{Cov}(X_1, X_j) \ge 0$ for all j. W.l.g. we assume $E|S_n|^3 = O(n^{3/2})$. Using Hölder's inequality and $\bar{\sigma}_k^2 \ge \sum_{j=1}^{k} EX_j^2/k = EX_1^2 > 0$, we find positive constants C_i such that for all $k \in \mathbb{N}$

$$C_2 \le \bar{\sigma}_k^2 \le C_3, \qquad C_2 \le \bar{\rho}_k \le C_3. \tag{2.16}$$

From (2.15) and (2.16) we get for $n = mk$,

$$\begin{aligned} &\left[16\bar{\sigma}_k^4 m(\sigma^2 - \bar{\sigma}_k^2)/(9\pi\bar{\rho}_k^2)\right] + \left[3\bar{\rho}_k/(\bar{\sigma}_k^3 m^{1/2})\right] \\ &\quad \ge C_4(n/k^2 + k^{1/2}/n^{1/2}), \qquad C_4 > 0. \end{aligned}$$

Now it is easy to see that $k = k(n) = [n^{3/5}]$ yields the best possible convergence rate $O(n^{-1/5})$. □

Observe that our standardization $\sigma_n^{-1}S_n$ is different from the standardization $n^{-1/2}S_n$ used by Wood (1983). But in the stationary case this difference presents no difficulties: If $\{X_j\colon j \in \mathbb{N}\}$ is a stationary associated process fulfilling (1.2), we have

$$\begin{aligned} &\left|P\{n^{-1/2}S_n \le x\} - N(0,\sigma^2)((-\infty, x])\right| \\ &\quad \le \sup_{y\in\mathbb{R}}\left|P\{\sigma_n^{-1}S_n \le y\} - \phi(y)\right| + \left|\phi(n^{1/2}\sigma_n^{-1}x) - \phi(\sigma^{-1}x)\right|. \end{aligned}$$

As $\sigma^2 \geq \sigma_n^2/n \geq EX_1^2 > 0$, we obtain

$$\left|\phi\left(n^{1/2}\sigma_n^{-1}x\right) - \phi\left(\sigma^{-1}x\right)\right| \leq B_1\left(\sigma^2 - \sigma_n^2/n\right) \leq B_2 n^{-1},$$

according to (2.1). Hence Theorem 2.1 remains valid if we consider the standardization used by Wood (1983).

3. Auxiliary results. In this section we prove some results which we need for the proof of Theorem 2.1. The following lemma is the main tool for our estimates. We assume all occurring covariances to exist.

LEMMA 3.1. *Let A and B be finite sets and let X_j, $j \in A \cup B$, be associated random variables.*

(i) *If $f: \mathbb{R}^{\#A} \to \mathbb{R}$ and $g: \mathbb{R}^{\#B} \to \mathbb{R}$ are partially differentiable with bounded partial derivatives, then*

$$\left|\operatorname{Cov}\left(f\left((X_i)_{i\in A}\right), g\left((X_j)_{j\in B}\right)\right)\right| \leq \sum_{i\in A}\sum_{j\in B} \|\partial f/\partial t_i\|_\infty \|\partial g/\partial t_j\|_\infty \operatorname{Cov}(X_i, X_j).$$

(ii) *If $h: \mathbb{R} \to \mathbb{R}$ is a bounded differentiable function with bounded derivative, then*

$$\left|\operatorname{Cov}\left(\prod_{i\in A} h(X_i), \prod_{j\in B} h(X_j)\right)\right| \leq \|h\|_\infty^{\#A+\#B-2}\|h'\|_\infty^2 \sum_{i\in A}\sum_{j\in B}\operatorname{Cov}(X_i, X_j).$$

PROOF. Let $f_1: \mathbb{R}^{\#A} \to \mathbb{R}$ and $g_1: \mathbb{R}^{\#B} \to \mathbb{R}$ be defined by

$$f_1(\mathbf{s}) = \sum_{i\in A} \|\partial f/\partial t_i\|_\infty s_i,$$

$$g_1(\mathbf{s}) = \sum_{j\in B} \|\partial g/\partial t_j\|_\infty s_j.$$

Since $f_1 - f$, $f_1 + f$ and $g_1 - g$, $g_1 + g$ are coordinatewise nondecreasing, (i) follows from Proposition 15 of Newman (1984).

(ii) follows from (i), putting

$$f(\mathbf{s}) = \prod_{i\in A} h(s_i), \qquad g(\mathbf{s}) = \prod_{j\in B} h(s_j). \qquad \square$$

We now adopt the notation of Section 2.

LEMMA 3.2. *The inequality*

$$\begin{aligned} E\left|X_j\prod_{l=1}^{r-1}\xi_j^{(l)}\right| &\leq B(2tb_m/\sigma_n)^{r-1} + B(4tb_m/\sigma_n)^{(r-2)/2} r^{1/3}(t/\sigma_n)^{2/3}\delta(m)^{1/3} \\ &\quad + B2^r r^{2/3}(t/\sigma_n)^{4/3}\delta(m)^{2/3} \\ &= \Delta_1^{(r)} = \Delta_{1,n}^{(r)}(t) \end{aligned}$$

holds for all $j = 1, \dots, n$ and $r = 2, \dots, k+1$.

PROOF. Using Hölder's inequality and (2.3), we obtain

$$E\left|X_j \prod_{l=1}^{r-1} \xi_j^{(l)}\right| \le B_1\left(E\prod_{l=1}^{r-1}{}'|\xi_j^{(l)}|^3\right)^{1/3}\left(E\prod_{l=1}^{r-1}{}''|\xi_j^{(l)}|^3\right)^{1/3}, \tag{3.1}$$

where $\prod'$ indicates the product over all even indices and $\prod''$ the product over all odd indices. We have $|\xi_j^{(l)}|^3 = h(S_j^{(l-1)} - S_j^{(l)})$, where $h(x) = |\exp(itx) - 1|^3 = 2^{3/2}(1 - \cos(tx))^{3/2}$, $\|h\|_\infty = 8$, $\|h'\|_\infty \le 6t$. By Property (P_4) of Esary, Proschan and Walkup (1967), the random variables $S_j^{(l-1)} - S_j^{(l)}$, $l = 1,\dots, r-1$, are associated. Hence Lemma 3.1(ii) (put $A = \{2\}$, $B = \{2 < n \le r-1\colon n \text{ even}\}$) implies

$$E\prod_{l=1}^{r-1}{}'|\xi_j^{(l)}|^3 \le E|\xi_j^{(2)}|^3 E\prod_{\substack{l=1\\ l\ne 2}}^{r-1}{}'|\xi_j^{(l)}|^3 + 8^{r/2}(6t)^2 \sum_{\substack{l=3\\ l \text{ even}}}^{r-1} \operatorname{Cov}\left(S_j^{(1)} - S_j^{(2)}, S_j^{(l-1)} - S_j^{(l)}\right)$$

$$\le E|\xi_j^{(2)}|^3 E\prod_{\substack{l=1\\ l\ne 2}}^{r-1}{}'|\xi_j^{(l)}|^3 + B_2 8^{r/2}(t/\sigma_n)^2 \delta(m),$$

according to the definition of $u(m)$ and $\delta(m)$.

Applying Lemma 3.1(ii) consecutively, we obtain (note that $|\xi_j^{(l)}|^3 \le 8$)

$$E\prod_{l=1}^{r-1}{}'|\xi_j^{(l)}|^3 \le \prod_{l=1}^{r-1}{}'E|\xi_j^{(l)}|^3 + B_3(r-1)8^{r/2}(t/\sigma_n)^2 \delta(m). \tag{3.2}$$

In the same way we get

$$E\prod_{l=1}^{r-1}{}''|\xi_j^{(l)}|^3 \le \prod_{l=1}^{r-1}{}''E|\xi_j^{(l)}|^3 + B_3(r-1)8^{r/2}(t/\sigma_n)^2 \delta(m). \tag{3.3}$$

As in Tikhomirov [(1980), cf. (3.2)–(3.4)] it is not hard to show that for $l = 1,\dots, r-1$

$$E|\xi_j^{(l)}|^3 \le (2tb_m/\sigma_n)^3. \tag{3.4}$$

Since $2tb_m/\sigma_n < 1$, (3.1)–(3.4) lead to the required estimate. □

LEMMA 3.3. *The inequality*

$$\left|\operatorname{Cov}\left(X_j \prod_{l=1}^{r-1} \xi_j^{(l)}, \exp\left(itS_j^{(r)}\right)\right)\right| \le B4^r(t/\sigma_n)u(m+1) + B4^r(t/\sigma_n)^{4/3}\delta(m)^{2/3}$$

$$= \Delta_2^{(r)} = \Delta_{2,n}^{(r)}(t)$$

holds for all $j = 1,\dots, n$ *and* $r = 2,\dots, k$ [*here* $\operatorname{Cov}(\xi,\eta) = E(\xi\eta) - E(\xi)E(\eta)$].

PROOF. Let $\psi_0\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying $\psi_0(x) = x$ for $|x| \le N/2$, $0 \le \psi_0'(x) \le 1$ for $x \in \mathbb{R}$ and $\|\psi_0\|_\infty = N$. The constant $N > 0$ will be specified later. For $z \in \mathbb{C}$ put $h_1(z) = \mathrm{Re}(z)$, $h_2(z) = \mathrm{Im}(z)$. Then we have

$$\begin{aligned}
&\left|\mathrm{Cov}\left(X_j \prod_{l=1}^{r-1} \xi_j^{(l)}, \exp\left(itS_j^{(r)}\right)\right)\right| \\
&\quad\le \sum_{\nu\in\{1,2\}^{r-1}} \left|\mathrm{Cov}\left(\psi_0(X_j)\prod_{l=1}^{r-1} h_{\nu_l}\left(\xi_j^{(l)}\right), \cos\left(tS_j^{(r)}\right)\right)\right| \\
&\qquad + \sum_{\nu\in\{1,2\}^{r-1}} \left|\mathrm{Cov}\left(\psi_0(X_j)\prod_{l=1}^{r-1} h_{\nu_l}\left(\xi_j^{(l)}\right), \sin\left(tS_j^{(r)}\right)\right)\right| \qquad (3.5)\\
&\qquad + \left|\mathrm{Cov}\left(\left(X_j - \psi_0(X_j)\right)\prod_{l=1}^{r-1} \xi_j^{(l)}, \exp\left(itS_j^{(r)}\right)\right)\right| \\
&\quad= T_1 + T_2 + T_3.
\end{aligned}$$

Put $f_1(x) = \cos(tx) - 1$, $f_2(x) = \sin(tx)$. Then

$$h_\nu\left(\xi_j^{(l)}\right) = f_\nu\left(S_j^{(l-1)} - S_j^{(l)}\right), \qquad \nu = 1,2.$$

For fixed $\nu \in \{1,2\}^{r-1}$ we now apply Lemma 3.1(i) with

$$f(\mathbf{s}) = \psi_0(s_0)\prod_{l=1}^{r-1} f_{\nu_l}(s_l), \qquad \mathbf{s} = (s_0,\dots,s_{r-1}) \in \mathbb{R}^r,$$

and $g(s) = \cos(ts)$, $s \in \mathbb{R}$. Since the random variables X_j, $S_j^{(l-1)} - S_j^{(l)}$ and $S_j^{(r)}$ are associated by Property (P_4) of Esary, Proschan and Walkup (1967), and since $\|\partial f/\partial s_0\|_\infty \le 2^{r-1}$, $\|\partial f/\partial s_l\|_\infty \le Nt2^{r-2}$, $1 \le l \le r-1$, and $\|\partial g/\partial s\|_\infty \le t$, we obtain

$$\begin{aligned}
&\left|\mathrm{Cov}\left(\psi_0(X_j)\prod_{l=1}^{r-1} h_{\nu_l}\left(\xi_j^{(l)}\right), \cos\left(tS_j^{(r)}\right)\right)\right| \\
&\quad\le 2^{r-1}t\,\mathrm{Cov}\left(X_j, S_j^{(r)}\right) + \sum_{l=1}^{r-1} Nt^2 2^{r-2}\mathrm{Cov}\left(S_j^{(l-1)} - S_j^{(l)}, S_j^{(r)}\right) \\
&\quad\le 2^{r-1}(t/\sigma_n)u(m+1) + 2^{r-1}N(t/\sigma_n)^2\,\delta(m).
\end{aligned}$$

Using $\#\{1,2\}^{r-1} = 2^{r-1}$, we get

$$T_1 \le 2^{2r-2}(t/\sigma_n)u(m+1) + 2^{2r-2}N(t/\sigma_n)^2\,\delta(m), \tag{3.6}$$

and analogously

$$T_2 \le 2^{2r-2}(t/\sigma_n)u(m+1) + 2^{2r-2}N(t/\sigma_n)^2\,\delta(m). \tag{3.7}$$

The properties of ψ_0 imply $|X_j - \psi_0(X_j)| \le 4N^{-2}|X_j|^3$. Since $|\xi_j^{(l)}| \le 2$ and $\sup_{j\in\mathbb{N}} E|X_j|^3 < \infty$, it is now easy to see that

$$T_3 \le B_1 N^{-2} 2^r. \tag{3.8}$$

W.l.g. we assume $t > 0$. As uncorrelated associated random variables are independent, the condition $\delta(m) = 0$ implies that the process $\{X_j\colon j \in \mathbb{N}\}$ is m-dependent and thus Theorem 2.1 follows from Tikhomirov (1980) and Schneider (1981). Hence we also assume $\delta(m) > 0$. Putting $N = \delta(m)^{-1/3}(t/\sigma_n)^{-2/3}$, (3.5)–(3.8) imply the assertion. □

LEMMA 3.4. *The inequality*

$$\left| \sum_{j=1}^{n} E\left(X_j \prod_{l=1}^{r-1} \xi_j^{(l)} \left(\exp\left(itS_j^{(r)} \right) - f_n(t) \right) \right) \right|$$
$$\le n\Delta_2^{(r)} + Bn\Delta_1^{(r)} r^{1/2} m^{1/2} (t/\sigma_n) |f_n(t)| + Bn^{1/2} \Delta_1^{(r)} rm(t/\sigma_n)$$

holds for all $r = 2, \dots, k$.

PROOF. Elementary estimates yield

$$\begin{aligned} &\left| \sum_{j=1}^{n} E\left(X_j \prod_{l=1}^{r-1} \xi_j^{(l)} \left(\exp\left(itS_j^{(r)} \right) - f_n(t) \right) \right) \right| \\ &\quad \le \sum_{j=1}^{n} \left| \operatorname{Cov}\left(X_j \prod_{l=1}^{r-1} \xi_j^{(l)}, \exp\left(itS_j^{(r)} \right) \right) \right| \\ &\qquad + \left| \sum_{j=1}^{n} a_j^{(r)} \left(E \exp\left(itS_j^{(0)} \right) - E \exp\left(itS_j^{(r)} \right) \right) \right| \\ &\qquad\qquad \left(\text{where } a_j^{(r)} = a_{j,n}^{(r)}(t) = EX_j \prod_{l=1}^{r-1} \xi_j^{(l)} \right) \\ &\quad \le n\Delta_2^{(r)} + \left| f_n(t) \sum_{j=1}^{n} a_j^{(r)} E\eta_j^{(r)} \right| \\ &\qquad + \left| E\left(\exp\left(it\sigma_n^{-1} S_n \right) \sum_{j=1}^{n} a_j^{(r)} \left(\eta_j^{(r)} - E\eta_j^{(r)} \right) \right) \right|, \end{aligned} \tag{3.9}$$

according to Lemma 3.3, where

$$\eta_j^{(r)} = \eta_{j,n}^{(r)}(t) = 1 - \exp\left(-it\left(S_j^{(0)} - S_j^{(r)} \right) \right).$$

Using Hölder's inequality and

$$E\left(\left(\sum_{\nu=a+1}^{b} X_\nu \right) \left(\sum_{\mu=c+1}^{d} X_\mu \right) \right) \le u(0) \min\{ b - a, d - c \} \qquad \text{for } a < b,\ c < d, \tag{3.10}$$

we get

$$\begin{aligned}\left|E\eta_j^{(r)}\right| &\le tE\left|S_j^{(0)} - S_j^{(r)}\right| \\ &\le (t/\sigma_n)\left(E\left(\sum_{\substack{1\le\nu\le n\\ |\nu-j|\le rm}} X_\nu\right)^2\right)^{1/2} \\ &\le B_1 r^{1/2} m^{1/2}(t/\sigma_n).\end{aligned}$$

Hence, by Lemma 3.2,

$$\left|f_n(t)\sum_{j=1}^n a_j^{(r)} E\eta_j^{(r)}\right| \le B_1 n\Delta_1^{(r)} r^{1/2} m^{1/2}(t/\sigma_n)|f_n(t)|. \tag{3.11}$$

We now derive an estimate for the third summand in (3.9). Since $|\exp(it\sigma_n^{-1}S_n)| = 1$, we get

$$\begin{aligned}&\left|E\left(\exp(it\sigma_n^{-1}S_n)\sum_{j=1}^n a_j^{(r)}\left(\eta_j^{(r)} - E\eta_j^{(r)}\right)\right)\right| \\ &\quad\le \left(E\left|\sum_{j=1}^n a_j^{(r)}\left(\eta_j^{(r)} - E\eta_j^{(r)}\right)\right|^2\right)^{1/2}.\end{aligned} \tag{3.12}$$

Splitting the terms into real and imaginary parts and again applying Lemma 3.2, we obtain

$$\begin{aligned}E&\left|\sum_{j=1}^n a_j^{(r)}\left(\eta_j^{(r)} - E\eta_j^{(r)}\right)\right|^2 \\ &\le 2\left(\Delta_1^{(r)}\right)^2\Bigg\{\sum_{1\le i,\,j\le n}\left|\operatorname{Cov}\left(\operatorname{Re}\left(\eta_i^{(r)}\right), \operatorname{Re}\left(\eta_j^{(r)}\right)\right)\right| \\ &\qquad +2\sum_{1\le i,\,j\le n}\left|\operatorname{Cov}\left(\operatorname{Re}\left(\eta_i^{(r)}\right), \operatorname{Im}\left(\eta_j^{(r)}\right)\right)\right| \\ &\qquad +\sum_{1\le i,\,j\le n}\left|\operatorname{Cov}\left(\operatorname{Im}\left(\eta_i^{(r)}\right), \operatorname{Im}\left(\eta_j^{(r)}\right)\right)\right|\Bigg\} \\ &= 2\left(\Delta_1^{(r)}\right)^2\Bigg\{\sum_{1\le i,\,j\le n}\left|\operatorname{Cov}\left(\cos\left(-t\left(S_i^{(0)} - S_i^{(r)}\right)\right), \cos\left(-t\left(S_j^{(0)} - S_j^{(r)}\right)\right)\right)\right| \\ &\qquad +2\sum_{1\le i,\,j\le n}\left|\operatorname{Cov}\left(\cos\left(-t\left(S_i^{(0)} - S_i^{(r)}\right)\right), \sin\left(-t\left(S_j^{(0)} - S_j^{(r)}\right)\right)\right)\right| \\ &\qquad +\sum_{1\le i,\,j\le n}\left|\operatorname{Cov}\left(\sin\left(-t\left(S_i^{(0)} - S_i^{(r)}\right)\right), \sin\left(-t\left(S_j^{(0)} - S_j^{(r)}\right)\right)\right)\right|\Bigg\}.\end{aligned} \tag{3.13}$$

According to Property (P_4) of Esary, Proschan and Walkup (1967), the random variables $S_i^{(0)} - S_i^{(r)}$ and $S_j^{(0)} - S_j^{(r)}$, $1 \le i,\ j \le n$, are associated. Hence Lemma

3.1(i) [put $f(x) = g(x) = \cos(-tx)$] implies

$$\sum_{1\le i,\, j\le n} \left|\operatorname{Cov}\left(\cos\left(-t\left(S_i^{(0)} - S_i^{(r)}\right)\right), \cos\left(-t\left(S_j^{(0)} - S_j^{(r)}\right)\right)\right)\right|$$

$$(3.14)\qquad \begin{aligned} &\le (t/\sigma_n)^2 \sum_{\substack{1\le i,\, j\le n \\ |i-j|\le 3rm}} \operatorname{Cov}\left(\sum_{\substack{1\le \nu\le n \\ |\nu - i|\le rm}} X_\nu, \sum_{\substack{1\le \mu\le n \\ |\mu-j|\le rm}} X_\mu\right) \\ &\quad + (t/\sigma_n)^2 \sum_{\substack{1\le i,\, j\le n \\ |i-j|> 3rm}} \operatorname{Cov}\left(\sum_{\substack{1\le \nu\le n \\ |\nu - i|\le rm}} X_\nu, \sum_{\substack{1\le \mu\le n \\ |\mu-j|\le rm}} X_\mu\right). \end{aligned}$$

Using (3.10), the first summand is bounded by $B_2 nr^2m^2(t/\sigma_n)^2$. The second summand is bounded by

$$\begin{aligned} &(t/\sigma_n)^2 \sum_{i=1}^{n} \operatorname{Cov}\left(\sum_{\substack{1\le \nu\le n \\ |\nu - i|\le rm}} X_\nu, \sum_{\substack{1\le j\le n \\ |i-j|> 3rm}} \sum_{\substack{1\le \mu\le n \\ |\mu-j|\le rm}} X_\mu\right) \\ &\quad \le B_3(t/\sigma_n)^2 nrm\delta(rm) \\ &\quad \le B_4 nr^2m^2(t/\sigma_n)^2. \end{aligned}$$

Hence (3.14) yields

$$\sum_{1\le i,\, j\le n} \left|\operatorname{Cov}\left(\cos\left(-t\left(S_i^{(0)} - S_i^{(r)}\right)\right), \cos\left(-t\left(S_j^{(0)} - S_j^{(r)}\right)\right)\right)\right| \le B_5 nr^2m^2(t/\sigma_n)^2.$$

The other summands in (3.13) are estimated in a similar way. Combining (3.9) and (3.11)–(3.13), we get the required inequality. □

LEMMA 3.5. *The following inequality holds*:

$$i\sigma_n^{-1} \sum_{j=1}^{n} EX_j\xi_j^{(1)} = -t + \theta_1(t)n\left(t/\sigma_n^2\right)u(m+1) + B\theta_2(t)nb_m^2\left(t^2/\sigma_n^3\right).$$

PROOF. Using the definition of $u(n)$, the proof follows easily from the proof of Lemma 3.4 of Tikhomirov (1980). □

Finally we need

LEMMA 3.6. *The following inequality holds*:

$$i\sigma_n^{-1} \sum_{j=1}^{n} E\left(X_j \exp\left(itS_j^{(1)}\right)\right) = B\theta(t)n\left(t/\sigma_n^2\right)u(m+1).$$

PROOF. Using the decomposition $\exp(itS_j^{(1)}) = \cos(tS_j^{(1)}) + i\sin(tS_j^{(1)})$, the proof follows easily from Lemma 3.1(i). □

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