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LARGE DEVIATIONS FOR NONINTERACTING INFINITE PARTICLE SYSTEMS

BY TZONG-YOW LEE

Princeton University

We consider noninteracting infinite particles, each of which follows a diffusion with generator $L \equiv (D^2 + D)/2$. The presence of many invariant distributions makes the situation radically different from the more familiar case where strong ergodicity assumptions are made. Explicit large deviation rates for the empirical density are obtained. The dependence of the rates on the initial distribution is strong and can be seen clearly. Some variational formulas for the scattering data associated with L are also obtained.

Introduction. Cox and Griffeath (1984) considered independent simple d-dimensional continuous time random walks with mean 1 holding times and distributed according to i.i.d. Poisson variables with intensity θ initially. A large deviation principle with explicit rates for the occupation times was proved. They also found that the large deviation probability is larger than exponential decay when each individual particle is recurrent and is the usual exponential decay when each particle is transient. In this paper we consider Brownian motion with constant drift instead of random walks on Z^d with $d \geq 3$ (both are transient) and prove a large deviation principle for the empirical density measure $D_{t,\omega}$ [see (E.D.) for definition] which can imply occupation time large deviations using the contraction principle [see, e.g., Varadhan (1984)]. Our rate functional also has explicit formulas and reveals an intimate connection between large deviations and potential theory. More precisely, $L = (D^2 + D)/2$ is the diffusion generator of a single particle in our noninteracting system. We now introduce some notation. Let Γ_x (E_x, resp.) be the probability distribution (expectation, resp.) of a single particle starting from x and let σ_l (σ_{-l}) be the hitting time of $x = l$ ($-l$, resp.). Throughout this paper, $X = \{x_i\}$ stands for a point configuration; $\omega = \{\omega_i(s)\}$ stands for trajectory of a configuration. We denote by $\mu_\lambda \equiv \mu_{\lambda(x)}$ the Poisson field with intensity $\lambda(x)\,dx$ and by $P_{a,b}$ (P_X, resp.) the probability distribution of our particle system with initial distribution $\mu_{a,b} \equiv \mu_{a+be^x}$ (δ_X, resp.). Our main result is a large deviation principle for the empirical density $D_{t,\omega}$ which is defined by

$$\text{(E.D.)}\qquad D_{t,\omega}(K) \equiv t^{-1}\int_0^t \sum_i \chi_K(\omega_i(s))\,ds \quad \text{for measurable} \quad K \subset R.$$

$D_{t,\omega}$ is regarded as an element in $M(R) \equiv \{$nonnegative σ-finite measures on $R\}$ on which we impose the weak topology $\mathscr{M}$ induced by continuous functions with compact support.

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To see that $P_{a,b}$ is stationary, it suffices to show that

$$E^{P_{a,b}}\Big\{\prod_i G(\omega_i(t))\Big\} = \exp\{\textstyle\int [G(x)-1](a+be^x)\,dx\} \quad \text{for } G(x) > 0$$

with $G(x) - 1 \in C_0(R)$. This can be computed from the fact that the left-hand side equals $E^{\mu_{a,b}}\{\prod_{x\in X} R(t,x)\}$ where $R(t,x) \equiv E_x\{G(\omega(t))\}$ satisfies $R_t = \frac{1}{2}(R'' + R')$ with $R(0,x) = G(x)$.

It is easy to check that $P_{a,b}$ is ergodic and that $D_{t,\omega}$ converges to $(a + be^x)\,dx$ in the $\mathscr{M}$-topology for a.e. ω. We are interested in the exponential decay rate of $P_{a,b}\{D_{t,\omega} \in B\}$, where B is a subset of $M(R)$. Let $k(\lambda)$ be $\lim_{x\to-\infty}\lambda(x)$. In our approach to lower bound estimates, $P_{a,b}$ is compared with another ergodic system Q_λ which is the noninteracting system with generator $L_\lambda \equiv [D^2 + (k(\lambda) + \lambda')/\lambda D]/2$ and initial distribution μ_λ [see, e.g., Donsker and Varadhan (1975)].

We shall study $P_{a,b}$ in Section 1 and P_X in Section 2.

1. Large deviation rates for $D_{t,\omega}$ under $P_{a,b}$. Our main result is an explicit formula for the rate function. Theorem 1.1 gives a lower bound. Theorem 1.2 estimates the cumulant generating function in terms of the scattering data [functionals of the potential $V(\cdot)$] and yields an upper bound. Theorem 1.3 proves the equivalence of upper and lower bounds. Theorem 1.4 characterizes the scattering data in the spirit of the classical variational formula of the maximal eigenvalue. Before giving the proofs we first introduce some notation and state the theorems. Some of the notation will be used in Section 2.

We define sets of functions on R:

$$F_{0,b} \equiv F'_{0,b} \equiv \Big\{\lambda(x)\colon \lambda \text{ is absolute continuous (a.c.), nonnegative,}$$
$$\lambda(x) \sim 0 \text{ as } x \to -\infty,\ \lambda(x) \sim be^x \text{ as } x \to \infty \text{ and}$$
$$\int\big[(\lambda' - \lambda)^2/8\lambda\big]\,dx < \infty\Big\},$$

$$F_{1,b} \equiv F_{0,b} \cup \Big\{\lambda(x)\colon \lambda \text{ is a.c., nonnegative, } \lambda(x) \sim k(\lambda) + be^x$$
$$\text{for some positive } k(\lambda) \text{ depending on } \lambda \text{ as } |x| \to \infty$$
$$\text{and } \int\big[(\lambda' + k(\lambda) - \lambda)^2/8\lambda\big]\,dx < \infty\Big\},$$

and for $a > 0$,

$$F'_{a,b} \equiv \Big\{\lambda(x)\colon \lambda \text{ is a.c., nonnegative, } \lambda(x) \sim a + be^x \text{ as } |x| \to \infty$$
$$\text{and } \int\big[(\lambda' + a - \lambda)^2/8\lambda\big]\,dx < \infty\Big\}.$$

Note that $F_{1,b} = F_{0,b} \cup \bigcup_{a>0} F'_{a,b}$.

Theorem 1.1. *If G is $\mathscr{M}$-open and $a > 0$, then*

$$\liminf_{t\to\infty} t^{-1} \log P_{0,b}\{D_{t,\omega} \in G\} \geq -\inf_{\lambda \in G\cap F_{0,b}} \int \left[(\lambda' - \lambda)^2/8\lambda\right] dx, \tag{1}$$

$$\begin{aligned}\liminf_{t\to\infty} t^{-1} \log P_{a,b}\{D_{t,\omega} \in G\} \geq -\inf_{\lambda\in G\cap F_{1,b}} \bigg\{ &\int \left[(\lambda' + k(\lambda) - \lambda)^2/8\lambda\right] dx \\ &+ \left[a + k(\lambda)\log(k(\lambda)/a) - k(\lambda)\right]/2 \bigg\}.\end{aligned} \tag{2}$$

Let $V(x)$ be continuously differentiable with compact support and

$$u(t,x) \equiv u(V,t,x) \equiv E_x\left\{\exp \int_0^t V(\omega(s))\, ds\right\}.$$

The Feynman–Kac formula asserts that $u(t,x)$ satisfies $u_t = (u_{xx} + u_x)/2 + Vu$ and $u(0,x) = 1$. The cumulant generating function of $D_{t,\omega}$ is a key ingredient in our upper estimate:

$$\begin{aligned}\log E^{P_{a,b}}\left\{\exp t\int V(x) D_{t,\omega}(dx)\right\} &= \log E^{\mu_{a,b}}\left\{\prod_{x\in X} E_x\left\{\exp \int_0^t V\right\}\right\} \\ &= \log E^{\mu_{a,b}}\left\{\prod_{x\in X} u(t,x)\right\} \\ &= \int [u(t,x) - 1](a + be^x)\, dx,\end{aligned} \tag{C.G.F.}$$

where the last equality is a well-known formula for the Poisson point distribution. When we study the linear growth of (C.G.F.) in Theorem 2, the following subjects arise naturally:

$$g(x) \equiv g(V,x) \equiv E_x\left\{\exp \int_0^\infty V(\omega(s))\, ds\right\} = \lim_{t\to\infty} u(t,x),$$

which exists when $V \in A$,

$A \equiv \{V \in C_0^1(R)$: There are positive $g(x)$ and two constants $\alpha(V)$ and $\beta(V)$ such that $\frac{1}{2}(g'' + g') + Vg = 0$, $g(x) = \alpha(V)$ for $x \leq \inf\{y: V(y) \neq 0\}$ and $g(x) = 1 + \beta(V)e^{-x}$ for $x \geq \sup\{y: V(y) \neq 0\}\}$.

From the probabilistic representation of $g(x)$, it is easy to see that A is convex and contains all nonpositive $V(x)$. The existence of positive $U(x) \in A$ will be shown in (38). The scattering data $\alpha(V)$ and $\beta(V)$ will appear in Theorem 1.2 and have some variational formula in Theorem 1.4.

We now state Theorems 1.2, 1.3 and 1.4.

THEOREM 1.2. *If $V \in A$ and $\theta \in R$, then*

$$\lim_{t\to\infty} t^{-1}\int[u^\theta(V,t,x)-1]\,dx = [\alpha^\theta(V)-1]/2, \tag{3}$$

$$\lim_{t\to\infty} t^{-1}\int[u^\theta(V,t,x)-1]e^x\,dx = \theta\beta(V)/2, \tag{4}$$

$$\begin{aligned}&\lim_{t\to\infty} t^{-1}\log E^{P_{a,b}}\left\{\exp\left[\int_0^t\sum_i V(\omega_i(s))\,ds\right]\right\}\\ &\quad= (a/2)[\alpha(V)-1]+(b/2)\beta(V),\end{aligned} \tag{5}$$

$$\limsup_{t\to\infty} t^{-1}\log P_{a,b}\{D_{t,\omega}\in C\} \le -\inf_{\tau\in C} I_{a,b}(\tau), \tag{6}$$

for all $\mathscr{M}$-closed subsets C, where

$$I_{a,b}(\tau) \equiv \sup_{V\in A}\left\{\int V\tau(dx) - [a(\alpha(V)-1)+b\beta(V)]/2\right\}.$$

THEOREM 1.3. *If $a > 0$ and $F_{0,b}$, $F_{1,b}$ are defined as in Theorem 1.1, then*

$$I_{0,b}(\tau) = \begin{cases}\int[(\lambda'-\lambda)^2/8\lambda]\,dx,\\ \qquad \text{for } \tau(dx)=\lambda(x)\,dx \text{ with } \lambda\in F_{0,b},\\ +\infty, \quad \text{otherwise},\end{cases} \tag{7}$$

$$I_{a,b}(\tau) = \begin{cases}\int[(\lambda'+k(\lambda)-\lambda)^2/8\lambda]\,dx\\ \qquad +[a+k(\lambda)\log(k(\lambda)/a)-k(\lambda)]/2,\\ \qquad \text{for } \tau(dx)=\lambda(x)\,dx \text{ with } \lambda\in F_{1,b}\\ +\infty, \quad \text{otherwise}.\end{cases} \tag{8}$$

THEOREM 1.4. *If $V \in A$, $a > 0$, then*

$$\begin{aligned}&\lim_{t\to\infty} t^{-1}\log E^{P_{0,b}}\left\{\exp\int_0^t\sum_i V(\omega_i(s))\,ds\right\}\\ &\quad= \sup_{\lambda\in F_{0,b}}\left\{\int V\lambda - [(\lambda'-\lambda)^2/8\lambda]\,dx\right\},\end{aligned} \tag{9}$$

$$\begin{aligned}&\lim_{t\to\infty} t^{-1}\log E^{P_{a,b}}\left\{\exp\int_0^t\sum_i V(\omega_i(s))\,ds\right\}\\ &\quad= \sup_{\lambda\in F_{1,b}}\left\{\int V\lambda - [(\lambda'+k(\lambda)-\lambda)^2/8\lambda]\,dx\right.\\ &\qquad\qquad \left. -[a+k(\lambda)\log(k(\lambda)/a)-k(\lambda)]/2\right\},\end{aligned} \tag{10}$$

and (9) *and* (10) *are equivalent to a result in differential equations*:

$$\lim_{t\to\infty} t^{-1}\int[u(V,t,x)-1](a+be^x)\,dx = \{a[\alpha(V)-1]+b\beta(V)\}/2 \tag{11}$$
$$= \sup_{\lambda\in F_{1,b}}\left\{\int V\lambda\,dx - I_{a,b}(\lambda)\right\}.$$

1.1. *Lower bound* (*Proof of Theorem* 1.1). Lemmas 1.5 and 1.6 are needed in the proof of Theorem 1.1.

LEMMA 1.5. *For* $a, b \geq 0$, *there exists a countable dense subset* $B_{a,b}$ *of* $M(R)$ *in the* $\mathscr{M}$ *topology, with properties*

$$\lambda\in B_{a,b}\Rightarrow(\lambda'+a)/\lambda\in L^\infty(R), \tag{12}$$
$$h(\mu_{a,b};\mu_\lambda)\equiv\int\lambda\log[\lambda/(a+be^x)]+(a+be^x-\lambda)\,dx<\infty,$$

$$\inf_{B_{a,b}\cap G}\int\left[(\lambda'+a-\lambda)^2/8\lambda\right]dx = \inf_{F'_{a,b}\cap G}\int\left[(\lambda'+a-\lambda)^2/8\lambda\right]dx \tag{13}$$

for all $\mathscr{M}$*-open subsets* G.

PROOF. Consider on $\Sigma_n=\{\lambda\colon \lambda\in C([-n,n],R^+),\ \lambda \text{ is a.c. and } \lambda'\in L^2\}$ the Sobolev H^1 norm which makes Σ_n separable. Denote B_n as a countable dense set of Σ_n and $\tilde{B}_n$ as

$$\tilde{B}_n\equiv\{\tilde\lambda\colon\lambda\in B_n\}\quad\text{where } \tilde\lambda(x) \text{ is defined as}$$
$$\tilde\lambda(x)=\begin{cases} a+[\lambda(-n)-a]e^{x+n}, & x\leq -n,\\ \lambda(x), & |x|\leq n,\\ e^x\left[b^{1/2}+\left(\lambda^{1/2}(n)e^{-n/2}-b^{1/2}\right)e^{-(x-n)}\right]^2, & x\geq n, \text{ when } b\neq 0,\\ a+[\lambda(n)-a]e^{-(x-n)}, & x\geq n, \text{ when } b=0.\end{cases}$$

Also define $B_{a,b}\equiv\bigcup_1^\infty\tilde{B}_n$. It can be checked that (12) holds and also that $\tilde{B}_n$ is dense in Σ_n. Because $\bigcup_1^\infty\Sigma_n$ is dense in $F'_{a,b}$, $B_{a,b}$ is now dense in $F'_{a,b}$ with respect to the H^1-norm. For (13), it suffices to prove, for an arbitrary $\xi(x)\in F'_{a,b}$ and $\varepsilon>0$, the existence of n and $\phi(x)\in\tilde{B}_n$ such that

$$\left|\int\left[(\phi'+a-\phi)^2/8\phi-(\xi'+a-\xi)^2/8\xi\right]dx\right|<\varepsilon. \tag{14}$$

To do this we first choose large n such that

$$\int_{|x|>n}(\xi'+a-\xi)^2/8\xi\,dx<\varepsilon/3 \tag{15}$$

and that

$$\int_{|x|>n} (\tilde{\xi}' + a - \tilde{\xi})^2/8\tilde{\xi}\, dx < \varepsilon/6, \tag{16}$$

where the relation between $\tilde{\xi}$ and ξ is given in the definition of $\tilde{B}_n$. Since $\tilde{B}_n$ is dense in Σ_n, there exists $\phi \in \tilde{B}_n$ such that

$$\left| \int_{|x|\le n} \left[(\xi' + a - \xi)^2/8\xi - (\phi' + a - \phi)^2/8\phi \right] dx \right| < \varepsilon/3. \tag{17}$$

Also it follows from (16) and the continuity of ϕ $(\pm n)$ with respect to the H^1 norm that we can pick ϕ such that

$$\int_{|x|>n} \left[(\phi' + a - \phi)^2/8\phi \right] dx < \varepsilon/3. \tag{18}$$

(15), (17) and (18) now imply (14). □

Some notation is needed to state Lemma 1.6.

We define $D_{t,\omega,1}$ and $D_{t,\omega,2}$ by

$$D_{t,\omega,1}(K) \equiv t^{-1} \int_0^t \sum_{\omega_i(0)\in[-t/2,0]} \chi_K(\omega_i(s))\, ds$$

and

$$D_{t,\omega,2}(K) \equiv t^{-1} \int_0^t \sum_{\omega_i(0)\in[-t/2,0]^c} \chi_K(\omega_i(s))\, ds.$$

Note that $D_{t,\omega} = D_{t,\omega,1} + D_{t,\omega,2}$. Also denote by F_t the Borel field generated by $\{\omega_i(s) | 0 \le s \le t\}$ and by G_t the Borel field generated by $\{\omega_i(s) | \omega_i(0) \in [-t/2, 0], 0 \le s \le t\}$.

Define $B(\lambda, V, \delta) \equiv \{\tau \in M(R): |\int V(x)[\tau(dx) - \lambda\, dx]| < \delta\}$ and note that $\{\bigcap_1^m B(\lambda, V_j, \delta): V_j \in A,\ \delta > 0,\ m \in N\}$ forms a basis of the open sets containing λ.

LEMMA 1.6. *Let $\lambda \in B_{a,0}$. We have*

$$\lim_{t\to\infty} t^{-1} \log dP_{k,0}/dP_{a,0}|_{G_t} = [a + k\log(k/a) - k]/2 \tag{19}$$

for a.e. ω with respect to Q_k and also in $L^1(Q_k)$ $[Q_k \equiv Q_\lambda$ with $\lambda(x) = k]$,

$$\lim_{t\to\infty} t^{-1} \log dQ_\lambda/dP_{k(\lambda),0}|_{F_t} = \int [\lambda' + k(\lambda) - \lambda]^2/8\lambda\, dx \tag{20}$$

for a.e. ω with respect to Q_λ and also in $L^1(Q_\lambda)$.

$$\textit{For any } \delta > 0, \quad \lim_{t\to\infty} P_{a,0}\left\{ D_{t,\omega,2} \in \bigcap_1^m B(0, V_j, \delta) \right\} = 1 \tag{21}$$

and

$$\text{(22)} \qquad \text{for any } \delta > 0, \qquad \lim_{t\to\infty} Q_\lambda\left\{D_{t,\omega,2} \in \bigcap_1^m B(0, V_j, \delta)\right\} = 1$$

and therefore

$$\lim_{t\to\infty} Q_\lambda\left\{D_{t,\omega,1} \in \bigcap_1^m B(\lambda, V_j, \delta)\right\} = 1.$$

PROOF. We first prove (19). Since $P_{k,0}$ and $P_{a,0}$ have the same evolution, it follows that

$$(dP_{k,0}/dP_{a,0})|_{G_t} = d\mu_{k,t}/d\mu_{a,t},$$

where $\mu_{k,t}$ ($\mu_{a,t}$, resp.) is the Poisson field on $[-t/2, 0]$ with intensity k (a, resp.).

The ergodicity of the Poisson field implies that

$$\begin{aligned}\lim_{t\to\infty} t^{-1} \log d\mu_{k,t}/d\mu_{a,t} &= \frac{1}{t}\int [\log d\mu_{k,t}/d\mu_{a,t}]\, d\mu_{k,t} \\ &= [a + k\log k/a - k]/2 \\ &\quad \text{for a.e. } X \text{ w.r.t. } \mu_k \text{ and in } L^1(\mu_k)\end{aligned}$$

[therefore a.e. ω w.r.t. Q_k and $L^1(Q_k)$], where the last equality is a straightforward computation. This proves (19). For (20), it follows also from the ergodicity of Q_λ that, for a.e. ω w.r.t. Q_λ and in $L^1(Q_\lambda)$,

$$\begin{aligned}\lim_{t\to\infty} t^{-1}\log dQ_\lambda/dP_{k(\lambda),0}|_{F_t} &= t^{-1}\int [\log dQ_\lambda/dP_{k(\lambda),0}]\, dQ_\lambda \\ &= \int (\lambda' + k(\lambda) - \lambda)^2/8\lambda\, dx,\end{aligned}$$

where we use the Cameron–Martin–Girsanov formula to establish the last equality.

Formula (21) amounts to showing that, for every $V \in C_0(-l, l)$ and every $\delta > 0$,

$$\text{(23)} \qquad \lim_{t\to\infty} P_{a,0}\{D_{t,\omega,2} \in B^c(0, V, \delta)\} = 0.$$

To show this let $m(x) \equiv E_x\{\int_0^t |V|(\omega(s))\, ds\}$ which is less than some constant N due to transience and divide, for $\varepsilon > 0$, $[-t/2, 0]^c$ into three parts:

$$\begin{aligned} A_t &\equiv (-\infty, -(l + \varepsilon t + t/2)], \\ B_t &\equiv (-(l + \varepsilon t + t/2), -t/2) \cup [0, \varepsilon t] \end{aligned}$$

and $(\varepsilon t, \infty)$.

We have that

$$
\begin{aligned}
&E^{P_{a,0}}\left\{t\int |V| D_{t,\omega,2}(dx)\right\}\\
&\quad = E^{P_{a,0}}\left\{\int_0^t \sum_{\omega_i(0)\in[-t/2,0]^c} |V|(\omega_i(s))\,ds\right\}\\
(24)\qquad &\quad = \int_{[-t/2,0]^c} m(x)\cdot a\,dx\\
&\quad \le \int_{A_t} N\cdot \Gamma_x\{\sigma_{-l} < t\}\cdot a\,dx + \int_{B_t} N\cdot a\,dx + \int_{\varepsilon t}^{\infty} N\Gamma_x\{\sigma_l < t\} a\,dx.
\end{aligned}
$$

The following inequalities are crucial and will be used several times in this paper:

$$
\begin{aligned}
(25)\qquad \Gamma_x\{\sigma_{-l} \le t\} &\le \operatorname{Prob}\left\{\max_{s<t} B(s) \ge -l - t/2 - x\right\}\\
&= 2\operatorname{Prob}\{B(t) \ge -l - t/2 - x\}\\
&\qquad\qquad \text{for } -l - t/2 - x > 0
\end{aligned}
$$

and

$$
\begin{aligned}
(26)\qquad \Gamma_x\{\sigma_l \le t\} &\le \operatorname{Prob}\left\{\max_{0\le s\le t} B(s) \ge x - l\right\}\\
&= 2\operatorname{Prob}\{B(t) \ge x - l\} \quad \text{for } x \ge l,
\end{aligned}
$$

where $B(s)$ is standard Brownian motion (B.M.). Both inequalities follow easily from path transformation and the equalities at the end are well known.

It now follows from (24), (25), (26) and the Chebychev inequality that

$$
\limsup_{t\to\infty} P_{a,0}\{D_{t,\omega,2} \in B^c(0,V,\delta)\} \le \lim_{t\to\infty} aN(l + 2\varepsilon t)/t\delta = 2aN\varepsilon/\delta.
$$

Letting ε tend to 0, we obtain (23).

To prove (22), we modify the argument used for (21). Let $t' \equiv (1 - \varepsilon/10)t$ and let $\Gamma'_x(E'_x, \sigma')$ stand for the probability distribution (expectation, stopping time, resp.) of a diffusion generated by $[D^2 + (\lambda + a)/\lambda]/2$; then we have

$$
\begin{aligned}
&E^{P_\lambda}\left\{t\int |V| D_{t,\omega,2}(dx)\right\}\\
&\quad = E^{P_\lambda}\left\{\int_{t'}^t \sum_{\omega_i(0)\in[-t/2,0]^c} |V(\omega_i(s))|\,ds\right\}\\
(27)\qquad &\qquad + E^{P_\lambda}\left\{\int_0^{t'} \sum_{\omega_i(0)\in[-t/2,0]^c} |V(\omega_i(s))|\,ds\right\}\\
&\quad = E^{P_\lambda}\left\{\int_{t'}^t \sum |V|\right\} + \int_{-\infty}^{-t/2-\varepsilon t} m(x)\lambda(x)\,dx + \int_{\varepsilon t}^{\infty} m(x)\lambda\,dx\\
&\qquad + \left[\int_{-t/2-\varepsilon t}^{-t/2} + \int_0^{\varepsilon t} m(x)\lambda\,dx\right],
\end{aligned}
$$

where $m(x) \equiv E'_x\{\int_0^\infty |V(\omega(s))|\,ds\} < N < \infty$ for some N. Since $-t/2 - \varepsilon t < -t'/2 - \varepsilon t' - (\varepsilon/20)t$ and $\varepsilon t = \varepsilon t' + (\varepsilon^2/10)t$, it follows that $m(x) < N\Gamma'_x\{\sigma'_{-(\varepsilon/20)t} < t'\}$ for $x < -t/2 - \varepsilon t$ and that $m(x) < N\Gamma'_x\{\sigma'_{(\varepsilon^2/10)t} < t'\}$ for $x > \varepsilon t$. Note that these estimates for $m(x)$ involve only $x \in (-\infty, -(\varepsilon/20)t) \cup ((\varepsilon^2/10)t, \infty)$, where $|(\lambda' + a)/\lambda - 1|$ can be made arbitrarily small by choosing large t. Thus, we can modify inequalities (25) and (26) and prove that the second and third terms in (27) are bounded for large t. From the fact that μ_λ is an invariant distribution, it follows that the first term in (27) is less than $(\varepsilon/10)t\int |V(x)|\lambda\,dx$. (22) can now be proved by the Chebychev inequality and by letting $\varepsilon \to 0$ as in the proof of (21). □

PROOF OF THEOREM 1.1. (1) is proved by a standard entropy argument. For $\lambda \in F_{0,b}$ with $\lambda'/\lambda \in L^\infty(R)$, the Cameron–Martin–Girsanov formula gives

$$\begin{aligned}\log dQ_\lambda/dP_{0,b}|_{F_t} &= \int_0^t \sum_i (\lambda'/2\lambda - \tfrac{1}{2})(\omega_i(s))[d\omega_i - \tfrac{1}{2}\,ds] \\ &\quad - \left[\int_0^t \sum_i (\lambda'/2\lambda - \tfrac{1}{2})^2(\omega_i(s))\,ds\right]/2 \\ &= \int_0^t \sum_i (\lambda' - \lambda)/2\lambda(\omega_i(s))[d\omega - \lambda'/2\lambda\,dx] \\ &\quad + \int_0^t \sum_i [(\lambda' - \lambda)/2\lambda]^2(\omega_i(s))\,ds/2.\end{aligned}$$

The ergodic theorem then implies

$$\lim_{t\to\infty} t^{-1}\log dQ_\lambda/dP_{0,b}|_{F_t} = \int (\lambda' - \lambda)^2/8\lambda\,dx, \tag{28}$$

a.e. ω with respect to Q_λ and also in $L^1(Q_\lambda)$.

In view of

$$\lim_{t\to\infty} Q_\lambda\left\{D_{t,\omega} \in \bigcap_1^k B(\lambda, V_j, \delta)\right\} = 1,$$

Jensen's inequality and (18) imply that

$$\min\inf_{t\to\infty} t^{-1}\log P_{0,b}\left\{D_{t,\omega} \in \bigcap_1^k B(\lambda, V_j, \delta)\right\} \geq -\int (\lambda' - \lambda)^2/8\lambda\,dx.$$

Since this holds for all $\lambda \in F_{0,b}$, $\lambda'/\lambda \in L^\infty(R)$, V_j and δ, a simple argument shows that, for all $\mathscr{M}$-open G,

$$\begin{aligned}&\liminf_{t\to\infty} t^{-1}\log P_{0,b}\{D_{t,\omega} \in G\} \\ &\qquad \geq - \inf_{\substack{\lambda\in F_{0,b}\cap G \\ \lambda'/\lambda \in L^\infty(R)}} \int (\lambda' - \lambda)^2/8\lambda\,dx.\end{aligned}$$

(1) now follows from Lemma 1.5.

We prove (2) only for the case when $b = 0$ because the argument extends easily to the other cases.

The reason for considering $D_{t,\omega,1}$ and $D_{t,\omega,2}$ is that $H(P_{a,0}; Q_\lambda) = \infty$ when $k(\lambda) \neq a$. The following modified entropy argument is made possible by Lemma 1.6:

$$\begin{aligned}
&\liminf_{t\to\infty} t^{-1}\log P_{a,0}\left\{D_{t,\omega} \in \bigcap_1^l B(\lambda, V_j, \delta)\right\}\\
&\quad\geq \liminf_{t\to\infty} t^{-1}P_{a,0}\left\{D_{t,\omega,1} \in \bigcap_1^l B(\lambda, V_j, \delta/2)\right\}\\
&\qquad+ \liminf_{t\to\infty} t^{-1}\log P_{a,0}\left\{D_{t,\omega,2} \in \bigcap_1^l B(0, V_j, \delta/2)\right\}\\
&\quad\geq \lim_{t\to\infty} -\int_{D_{t,\omega,1}\in\cap_1^l B(\lambda, V_j,\delta/2)} t^{-1}\big[\log dQ_\lambda/dP_{k(\lambda),0}\\
&\qquad\qquad+\log dP_{k(\lambda),0}/dP_{a,0}\big]_{G_t}\, dQ_\lambda + 0.
\end{aligned}$$

In view of Lemmas 1.5 and 1.6, (2) holds and Theorem 1.1 is proved. □

1.2. *Upper bound* (*Proof of Theorem* 1.2). First note that, for $V \in A$, $\lim_{t\to\infty} u(V, t, x) = g(V, x)$ and $\delta \leq u(V, t, x) \leq N$ for all x, t and some positive δ, N.

We divide R into four regions according to t, ε and $[-l, l]$, the support of V:

$$\begin{aligned}
A_t &\equiv \left(-\infty, -l - \left(\tfrac{1}{2} + \varepsilon\right)t\right],\\
B_t &\equiv \left(-l - \left(\tfrac{1}{2} + \varepsilon\right)t, -l - \left(\tfrac{1}{2} - \varepsilon\right)t\right],\\
C_t &\equiv \left(-l - \left(\tfrac{1}{2} - \varepsilon\right)t, -l\right] \quad\text{and}\quad (-l, \infty);
\end{aligned}$$

$$\begin{aligned}
t^{-1}\int_R \left[u^\theta(V, t, x) - 1\right] dx &= t^{-1}\left\{\int_{A_t}[\ ] + \int_{B_t}[\ ] + \int_{C_t}[\ ] + \int_{-l}^{\infty}[\ ]\right\}\\
&\equiv I_1(\varepsilon, t) + I_2(\varepsilon, t) + I_3(\varepsilon, t) + I_4(\varepsilon, t).
\end{aligned}$$

We shall estimate $\lim_{t\to\infty} I_j(\varepsilon, t)$, $j = 1, 2, 3, 4$, separately and let $\varepsilon \to 0$ in the end. We write σ_l, σ_{-l} as σ_+, σ_-.

A useful representation of u is

$$\begin{aligned}
u(t, x) &= E_x\left\{\exp\int_0^t V(\omega(s))\, ds;\ \sigma_- \leq t\right\} + \Gamma_x\{\sigma_- > t\}\\
&= E_x\left\{E_{-l}\left\{\exp\int_0^{t-\sigma_-} V;\ \sigma_- \leq t\right\}\right\} + \Gamma_x\{\sigma_- > t\} \qquad (29)\\
&= E_x\{u(t - \sigma_-, -l);\ \sigma_- \leq t\} + \Gamma_x\{\sigma_- > t\}.
\end{aligned}$$

Thus, we have $u(t,x) \le N\Gamma_x\{\sigma_- \le t\} + \Gamma_x\{\sigma_- > t\}$ and

$$(30)\quad u^\theta(t,x) - 1 \le \left[1 + (N-1)\Gamma_x\{\sigma_- \le t\}\right]^\theta - 1 \le \text{const.}\,\Gamma_x\{\sigma_- \le t\}.$$

It now follows from (25) and (30) that $\lim_{t\to\infty} I_1(\varepsilon, t) = 0$.

For $I_2(\varepsilon, t)$, it follows from the existence of N: $N \ge u(t,x) \ge N^{-1}$ that

$$2\varepsilon(N^{|\theta|} + 1) \ge \limsup_{t\to\infty} I_2 \ge \liminf_{t\to\infty} I_2 \ge -2\varepsilon(N^{|\theta|} + 1).$$

$I_3(\varepsilon, t)$ is the contributing part. We first rewrite (29) as

$$(31)\quad \begin{aligned} u(t,x) &= \Gamma_x\{\sigma_- \ge t\} + E_x\{u(t-\sigma_-, -l);\, (1-\varepsilon)t \le \sigma_- < t\} \\ &\quad + E_x\{u(t-\sigma_-, -l);\, \sigma_- < (1-\varepsilon)t\}. \end{aligned}$$

A crucial estimate is

$$(32)\quad \begin{aligned} \Gamma_x\{\sigma_- > (1-\varepsilon)t\} &\le \Gamma_x\{\omega((1-\varepsilon)t) \le -l\} \\ &\le \operatorname{Prob}\left\{B((1-\varepsilon)t) \le \frac{-\varepsilon}{2}t\right\} \to 0, \end{aligned}$$

as $t \to \infty$, where $B(s)$ is standard B.M. From this we see that the first two terms of (29) tend to 0 as $t \to \infty$ uniformly for $x \in C_t$.

Since, for any $\varepsilon > 0$,

$$\begin{aligned} \lim_{t\to\infty} \inf_{s>\varepsilon t} u(V, s, -l) &< E_x\{u(t-\sigma_-, -l);\, \sigma_- < (1-\varepsilon)t\} \\ &< \lim_{t\to\infty} \sup_{s>\varepsilon t} u(V, s, -l), \end{aligned}$$

it then follows from (31) and (32) that

$$\lim_{t\to\infty} I_3(\varepsilon, t) = \left[\alpha^\theta(V) - 1\right]\left(\tfrac{1}{2} - \varepsilon\right).$$

On (l, ∞), we have $\Gamma_x\{\sigma_+ < t\} \le \Gamma_x\{\sigma_+ < \infty\} = e^{-(x-l)}$ and

$$u(t,x) = \Gamma_x\{\sigma_+ \ge t\} + E_x\{u(t-\sigma_+, l); \sigma_+ < t\}.$$

The uniform integrability of $u^\theta - 1$ on $[-l, \infty)$ then follows from $u(t,x) < N$ and implies that

$$\lim_{t\to\infty} I_4(\varepsilon, t) = 0.$$

Adding the preceding four estimates and letting $\varepsilon \to 0$, (3) is proved. Now we prove (4).

Because $\lim_{t\to\infty} t^{-1}\int_{-\infty}^{y}[u^\theta(V,t,x) - 1]e^x\,dx = 0$ for any $y \in R$, we focus on integration of $[u^\theta - 1]e^x$ on $[y, \infty)$ with $y > l$.

The following lemma is crucial in proving (4).

Lemma 1.7. *If $V \in A$, then*

$$(33)\quad \lim_{t\to\infty} u_x(V, t, x) = g'(V, x)$$

and

$$\lim_{t\to\infty}\int_y^\infty u^{\theta-2}(V,t,x)u_x^2(V,t,x)e^x\,dx$$

(34)
$$=\int_y^\infty g^{\theta-2}(V,x)(g')^2 e^x\,dx$$

for all $y\in R$ *such that* $V(x)=0$ *for all* $x\geq y$.

PROOF. To derive (33), we use an integral representation,

(35)
$$u_x(V,t,x)=\int p_x(1,x,y)u(V,t-1,y)\,dy,$$

where $p(1,x,y)$ is a smooth kernel and $|p_x(1,x,y)|\leq c_1(x)e^{-c_2(x)y^2}$ for some positive $c_1(x)$, $c_2(x)$ because V is smooth and compactly supported.

Let t tend to ∞, (33) follows from (35) and the dominated convergence theorem.

Formula (34) amounts to showing that $u_x^2(V,t,x)e^x<f(x)$ for some $f\in L^1(y,\infty)$. Notice that $\zeta(t,x)\equiv u_x(t,x)$ satisfies

$$\zeta_t=(\zeta_{xx}+\zeta_x)/2,\qquad \zeta(0,x)=0,\quad \text{for}\quad x>y,\ t\geq 0;$$
$$\zeta(t,y)=u_x(t,y),$$

and therefore

(36)
$$u_x(t,x)=\Gamma_x\{\sigma_y\leq t\}\cdot u_x(t,y).$$

The lemma then follows from (33) and the fact that $\Gamma_x\{\sigma_y\leq t\}<\Gamma_x\{\sigma_y<\infty\}=e^{-(x-y)}$. □

We now prove (4). Notice, from (26) and (36), that $(u^\theta-1)e^x$ is integrable on $[y,\infty)$ and that $\lim_{x\to\infty}u_xe^x=0$. From the fact that $u_t(t,x)=E_x\{V(\omega(t))\int_0^tV(\omega(s))\,ds\}$ and from (26), it also follows that $\int_y^\infty u_te^x\,dx<\infty$. Thus,

$$\begin{aligned}\lim_{t\to\infty}t^{-1}\int_y^\infty[u^\theta-1]e^x\,dx &= \lim_{t\to\infty}\int_y^\infty\theta u^{\theta-1}u_te^x\,dx\\ &=\theta/2\lim_{t\to\infty}\int_y^\infty u^{\theta-1}(u_{xx}+u_x)e^x\,dx\\ &=\theta/2\lim_{t\to\infty}\int_y^\infty u^{\theta-1}(u_xe^x)_x\,dx\\ &=\theta/2\lim_{t\to\infty}u^{\theta-1}(V,t,y)\beta(V)\\ &\quad-(\theta-1)\int_y^\infty u^{\theta-2}u_x^2e^x\,dx,\end{aligned}$$

where the last equality is an integration by parts. Since y is arbitrary, we let y tend to ∞ and (4) is proved.

Recalling from (C.G.F.), we see that (5) follows from $\theta=1$ of (3) and (4).

By a standard method for upper bound estimates [e.g., see Donsker and Varadhan (1975)], (6) holds for compact subsets and also holds for closed subsets if $P_{a,b}$ has the property: There exist compact subsets K_N such that

$$\lim_{N\to\infty} \limsup_{t\to\infty} t^{-1}\log P_{a,b}\{D_{t,\omega} \in K_N^c\} = -\infty. \tag{37}$$

We shall establish (6) by proving (37). Take a nonnegative $U(x)$ such that $\alpha(U) < \infty$, $\beta(U) < \infty$. For example, computation shows, for $U = \chi_{[-1,0]}/8$, that $\alpha(U) = e^{1/2} \cdot \frac{4}{5}$ and $\beta(U) = \frac{1}{5}$. It then follows that

$$V(x) = \int U(x-z)(2\pi)^{-1/2}e^{-z^2/2}\,dz > 0 \tag{38}$$

and that

$$K_N \equiv \left\{\sigma\colon \int V(x)\sigma(dx) \le N\right\}$$

is $\mathscr{M}$-compact. Also from the convexity of $\alpha(\cdot), \beta(\cdot)$ and the fact that $\alpha(U(x-z)) = \alpha(U)$, $\beta(U(x-z)) = e^z\beta(U)$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} t^{-1}\log E^{P_{a,b}}\left\{\exp\int_0^t \sum_i V(\omega_i(s))\,ds\right\}\\
&\quad = \frac{a[\alpha(V)-1] + b\beta(V)}{2} \qquad (39)\\
&\quad \le \int (2\pi)^{-1/2}e^{-z^2/2}\,\frac{a[\alpha(U)-1] + be^z\beta(U)}{2}\,dz < \infty.
\end{aligned}$$

Since, by Chebychev's inequality

$$\begin{aligned}
&\limsup_{t\to\infty} t^{-1}\log P_{a,b}\{D_{t,\omega} \in K_N^c\}\\
&\qquad \le \limsup_{t\to\infty} t^{-1}\log E^{P_{a,b}}\left\{\exp\int_0^t \sum_i V(\omega_i(s))\,ds\right\} - N,
\end{aligned}$$

$P_{a,b}$ satisfies property (37). Therefore (6) holds. Theorem 1.2 is completely proved. □

1.3. *Equivalence of lower and upper bounds* (*Proof of Theorem* 1.3). We shall first consider the case when $\tau(dx)$ has some regularity properties. These restrictions will be removed at the end. Let us suppose that $\tau(dx)$ has a density $\lambda(\cdot)$ which is smooth and positive (strictly positive for $I_{a,b}$ with $a > 0$). Suppose $I_{0,b}(\lambda) \equiv I_{0,b}(\tau) < \infty$; we now start proving that $\lambda \in F_{0,b}$.

Using convexity we can find an $\varepsilon > 0$ and a positive V, with $V(x) \ge \varepsilon e^{-2x}$ for $x \ge 0$ such that $\beta(V) < \infty$. It then follows that $\int_0^\infty e^{-2x}\lambda\,dx < \infty$. Because $\beta(\chi_{[-c,0]}/8) = c/(c+4) < 1$ for $c > 0$, we also have that

$$\lambda \text{ must satisfy } \int_{-\infty}^0 \lambda\,dx < \infty. \tag{40}$$

If we define $h \equiv \log g(V, x)$ and integrate by parts, we have

$$\begin{aligned}
&\sup_V \left[\int V\lambda\, dx - b\beta(V)/2\right] \\
&\quad \geq 2^{-1} \sup_l \sup_{h' \in C_0^\infty(-l,l)} \int -\left[h'' + (h')^2 + h'\right]\lambda\, dx \\
(41)\qquad &\quad = 2^{-1} \sup_l \sup_{h' \in C_0^\infty(-l,l)} \int_{-l}^{l} \left[h'\lambda' - (h')^2\lambda - h'\lambda\right] dx \\
&\quad = 2^{-1} \sup_l \sup_{h \in C_0^\infty(-l,l)} \int_{-l}^{l} \left[-\lambda(h')^2 + (\lambda' - \lambda)l'\right] dx \\
&\quad = 2^{-1} \sup_l \int_{-l}^{l} (\lambda' - \lambda)^2/4\lambda\, dx,
\end{aligned}$$

where the second equality holds from the fact that the integrand contains no derivatives of h of order higher than 1. Therefore

$$\int_{-\infty}^{\infty} (\lambda' - \lambda)^2/8\lambda\, dx < \infty.$$

By Schwarz's inequality, it follows from $\int_{-\infty}^{0}\lambda < \infty$ and $\int_{-\infty}^{0}(\lambda' - \lambda)^2/8\lambda < \infty$, that $\lim_{x\to-\infty}\lambda(x) = 0$. The existence of $\lim_{x\to\infty}\lambda e^{-x}$ follows from $\int_0^\infty \lambda e^{-2x} < \infty$ and $\int_0^\infty(\lambda' - \lambda)^2/8\lambda < \infty$. To conclude that $\lambda \in F_{0,b}$ it remains to show that $\lambda(x) \sim be^x$.

Using $\lim_{x\to\infty} h'(x)e^x = -\beta(V) = c$, we have

$$\begin{aligned}
I_{0,b}(\lambda) &= \sup_h \int -\left[h'' + (h')^2 + h'\lambda\right] dx - b\beta(V) \\
&= \sup_{c\in R}\left\{c\left[b - \lim_{x\to\infty}\lambda(x)e^{-x}\right] + \sup_{h:\, h'e^x \to c} \int -\lambda(h')^2 + (\lambda' - \lambda)h'\right\}.
\end{aligned}$$

It is easy to check that the second term in { } equals $\int(\lambda' - \lambda)^2/4\lambda\, dx$. From the assumption that $I_{0,b}(\lambda) < \infty$, it then follows that $\lambda(x)e^{-x} \to b$.

We now prove the formula for $I_{0,b}$. It is already proved in (41) that

$$\sup_V \left[\int V\lambda\, dx - b\beta(V)/2\right] \geq \int(\lambda' - \lambda)^2/8\lambda\, dx.$$

The reverse inequality is just an easy consequence of integration by parts and the Schwarz inequality.

Before proving (8), we need the preliminary

LEMMA 1.8. *If $\lambda(x) > 0$ and a.c., then*

$$(42)\qquad \sup_{h \in C_0(-l,l)} \int -\lambda(h')^2 + (\lambda' - \lambda)h'\, dx = \inf \int_{m-l}^{l} (\lambda' + m - \lambda)^2/4\lambda\, dx,$$

$$(43)\qquad \sup_l \inf_m \int_{-l}^{l} (\lambda' + m - \lambda)^2/4\lambda\, dx = \inf_m \int_{-\infty}^{\infty} (\lambda' + m - \lambda)^2/4\lambda\, dx.$$

PROOF. From

$$\int_{-\infty}^{\infty}\left[-\lambda(h')^2+(\lambda'-\lambda)h'\right]dx=\int_{-l}^{l}\left[-\lambda(h')^2+(\lambda'+m-\lambda)h'\right]dx$$
$$\le\int_{-l}^{l}\frac{(\lambda'+m-\lambda)^2}{4\lambda}\,dx,$$

it follows that the left-hand side of (42) is smaller than or equal to the right-hand side. The minimum on the right-hand side will occur for m_* which satisfies

$$\int_{-l}^{l}(\lambda'+m_*(l)-\lambda)/2\lambda=0. \tag{44}$$

But now letting $h'=(\lambda'+m_*-\lambda)/2\lambda$, which is admissible due to (44), one sees that, in fact, (42) is an equality. One inequality in (43) is obvious and we prove the reverse inequality; we study the behavior of $m_*(l)$ as $l\to\infty$:

(1) If $\lim_{l\to\infty}|m_*(l)|=\infty$, then

$$\sup_l\inf_m\int_{-l}^{l}(\lambda'+m-\lambda)^2/4\lambda\,dx\ge\lim_{l\to\infty}\int_{-1}^{1}(\lambda'+m_*(l)-\lambda)^2/4\lambda\,dx=\infty.$$

(2) If there exists a subsequence, also named $m_*(l)$, converging to $m_*\neq\pm\infty$,

$$\sup_l\inf_m\int_{-l}^{l}(\lambda'+m-\lambda)^2/4\lambda\,dx$$
$$\ge\sup_{l_0}\lim_{l\to\infty}\int_{-l_0}^{l_0}[\lambda'+m_*(l)-\lambda]^2/4\lambda\,dx$$
$$=\int_{-\infty}^{\infty}(\lambda'+m_*-\lambda)^2/4\lambda.$$

The lemma is proved. □

Suppose $I_{1,0}(\lambda)<\infty$. Let us now prove that $\lambda\in F_{1,0}$: The existence of a nonnegative $V\in A$ with $V(x)\ge\varepsilon$ for $|x|\le 1$, implies that

$$\sup_{c\in R}\int_{c}^{c+1}\lambda\,dx<\infty$$

and, therefore, that there exist sequences $x_n\to-\infty$ and $y_n\to\infty$, such that

$$\lambda(x_n),\lambda(y_n)<N<\infty\quad\text{for some } N. \tag{45}$$

In view of (42) and (43), we have

$$\inf_m\int_{-\infty}^{\infty}(\lambda'+m-\lambda)^2/8\lambda\,dx<\infty,$$

which, from the strict positivity of λ, (45) and expanding $(\lambda'+m-\lambda)^2$, easily implies that

$$\int_{-\infty}^{\infty}(\lambda')^2/8\lambda+\int_{-\infty}^{\infty}(m_*-\lambda)^2/8\lambda<\infty. \tag{46}$$

Schwarz's inequality then implies the existence of $\lim_{x\to\pm\infty} m_* \log\lambda(x) - \lambda(x)$, and therefore of $\lim_{x\to\pm\infty}\lambda(x)$. But $\infty > \int_{-\infty}^{\infty}(m_* - \lambda)^2/8\lambda$ forces $\lim_{|x|\to\infty}\lambda(x) = m_*$. We denote m_* by $k(\lambda)$. The proof for $\lambda \in F_{1,0}$ is complete.

It can be proved by a mixture of reasoning used in the $I_{0,b}$ and $I_{a,0}$ situations that $\lambda \in F_{1,b}$ if $I_{a,b}(\lambda) < \infty$.

Let us now prove the formulas for $I_{a,b}(\lambda)$. We split the variational calculation into two parts as follows:

$$
\begin{aligned}
2I_{a,b}(\lambda) &= \sup_{V\in A} \int 2V\lambda\,dx - a[\alpha(V) - 1] - b\beta(V) \\
&\geq \sup_{h'\in C_0^\infty(R)} \int -\lambda(h')^2 + (\lambda' - \lambda)h' - a(e^{h(-\infty)} - 1) \\
&= \Big\{ \sup_{h(-\infty)=c\in R} [k(\lambda)c - a(e^c - 1)] \Big\} + \int [\lambda' + k(\lambda) - \lambda]^2/4\lambda\,dx,
\end{aligned}
$$

where the last equality follows by the technique used to prove (42). Therefore,

$$2I_{a,b}(\lambda) \geq \int [\lambda' + k(\lambda) - \lambda]^2/4\lambda\,dx + a + k(\lambda)\log[k(\lambda)/a] - k(\lambda).$$

The reverse inequality again follows easily from integration by parts.

We now remove the smoothness assumption.

LEMMA 1.9. *If $I_{a,b}(\tau) < \infty$, then $\tau(dx)$ must be of the form $\tau(dx) = \lambda(x)\,dx$ with λ a.c.*

PROOF. First choose a smooth nonnegative even test function $\phi(x)$ with $\int\phi(x)\,dx = 1$ and compact support. Let $C_\varepsilon = \log\int e^{\varepsilon x}\phi(x)\,dx$ and $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(\varepsilon^{-1}(x + C_\varepsilon))$. $\{\phi_\varepsilon(x)\colon \varepsilon > 0\}$ then satisfies

(i) $\int e^x\phi_\varepsilon(x)\,dx = 1$,
(ii) ϕ_ε tends to δ_0 weakly because $\lim_{\varepsilon\to 0}\varepsilon^{-1}C_\varepsilon = 0$,
(iii) $V \in A \Rightarrow V * \phi_\varepsilon \in A$.

It follows from convexity that

$$\alpha(V * \phi_\varepsilon) \leq \alpha(V), \qquad \beta(V * \phi_\varepsilon) \leq \beta(V).$$

Therefore, by duality, we have

$$I_{a,b}(\tau) \geq I_{a,b}(\phi_\varepsilon(-\,\cdot\,) * \tau).$$

If we write $\lambda_\varepsilon(x)\,dx$ for $\phi_\varepsilon(-\,\cdot\,) * \tau$, this implies that

$$\limsup_{\varepsilon\to 0} I_{a,b}(\lambda_\varepsilon) < \infty. \tag{47}$$

For $a \neq 0$, (47) implies that $\limsup_{\varepsilon\to 0}\int(\lambda'_\varepsilon + k(\lambda_\varepsilon) - \lambda_\varepsilon)^2/\lambda_\varepsilon\,dx < \infty$. By the same technique used to derive (46), it follows that $\limsup_{t\to 0}\int((\lambda'_\varepsilon)^2/\lambda_\varepsilon)\,dx < \infty$. This then implies that $\tau = \text{weak-lim}_{\varepsilon\to 0}\lambda_\varepsilon(x)\,dx$ must be of the form $\lambda(x)\,dx$ with a.c. $\lambda(x)$ and in fact with $\int((\lambda)'^2/\lambda)\,dx < \infty$. For τ such that $I_{0,b}(\tau) < \infty$, we note that absolute continuity is after all a local property and we therefore concentrate on finite intervals. By the same technique used to prove (40) and the fact that ϕ has compact support, it follows that $\tau(K) < N|K|$ for all $K \subseteq [0,1]$ for some constant N and $\lambda_\varepsilon(K) < N|K|$. (47) then implies that $\limsup_{\varepsilon \to 0} \int_0^1 ((\lambda'_\varepsilon)^2/\lambda_\varepsilon)\,dx < \infty$ and therefore that τ is of the form $\lambda(x)\,dx$ with a.c. $\lambda(x)$. Note that $[0,1]$ can be replaced by an arbitrary finite interval. The proof is complete. □

We now remove the positiveness assumption: Because $I_{0,b}(be^x) = 0$, convexity and lower semicontinuity imply that

$$\begin{aligned} I_{0,b}(\lambda) &= \lim_{\varepsilon \to 0} I_{0,b}(\varepsilon b e^x + (1-\varepsilon)\lambda) \\ &= \lim_{\varepsilon \to 0} (1-\varepsilon)^2 \int (\lambda' - \lambda)^2 / 8[\varepsilon b e^x + (1-\varepsilon)\lambda] \\ &= \int (\lambda' - \lambda)^2 / 8\lambda \, dx \end{aligned}$$

for $a \neq 0$; the expression becomes more involved but the same argument works nicely. The proof of Theorem 1.3 is complete. □

1.4. *Variational formulas of scattering data* (*Proof of Theorem* 1.4). Recall from Theorem 1.2 that $V = -[h'' + (h')^2 + h']/2$ and $\int -h''\lambda = b\beta(V) + \int \lambda' h'$. We need only show that

$$\sup_{\lambda \in F_{0,b}} -\int \lambda \{h' - [(\lambda' - \lambda)/2\lambda]\}^2 \, dx = 0.$$

By solving $(\lambda' - \lambda)/2\lambda = h'$, we see that 0 is attained by $\lambda(x) = bg^2(V, x)e^x$. (9) is proved.

Similar reasoning also reduces (10) to

$$\begin{aligned} \sup_{\lambda \in F_{1,b}} \int &- \lambda\{h' - [(\lambda' + k(\lambda) - \lambda)/2\lambda]^2 \, dx \\ &+ k(\lambda)\{\log[a\alpha(V)/k(\lambda)] + 1 - [a\alpha(V)/k(\lambda)] = 0. \end{aligned}$$

Since the equations

$$[\lambda' + a\alpha(V) - \lambda]/2 = h',$$
$$k(\lambda) = a\alpha(V) \quad \text{and} \quad \lambda \in F_{1,b}$$

have the solution

$$\lambda(x) = bg^2(x)e^x + a\alpha(V)g^2(x)\int_x^\infty e^{-(y-x)} g^{-2}(y)\,dy,$$

we see that 0 is attained and (10) is proved. □

2. Large deviation rate function of $D_{t,\omega}$ under P_X. As to large deviation rates of $D_{t,\omega}$ under P_X, it is easy to see that the rates are not too sensitive to the specific configuration X which we start with.

Our main result in Section 2 is that the rate function equals some $I'_{a,b}(\cdot)$ (defined in Theorem 2.2) for a.e. X with respect to $\mu_{a,b}$ and that, if $a \neq 0$ and $\lambda \in F_{1,b} - F'_{a,b}$, $I'_{a,b}(\lambda) = \infty > I_{a,b}(\lambda)$.

Theorems of Section 2 differ from Section 1 only in that "$P_{a,b}$" is always replaced by "P_X for a.e. X with respect to $\mu_{a,b}$." The proofs, therefore, proceed very similarly. We shall concentrate only on this relatively new aspect, i.e., "P_X for a.e. X w.r.t. $\mu_{a,b}$." Again we first introduce some notation and state the theorems. Let $\Omega_{a,b,V}$ be

$$\left\{X:\ \lim_{t\to\infty} t^{-1}\log E^{P_X}\left\{\exp\bigcap_0^t \sum_i V(\omega_i(s))\,ds\right\} = [a\log\alpha(V) + b\beta(V)]/2\right\}$$

and let $\Omega_{a,b}$ be $\bigcap_{V\in A}\Omega_{a,b,V}$.

Theorems are labeled to note the analogy between Sections 1 and 2.

THEOREM 2.1. *For each $a, b \geq 0$, we have*

$$\mu_{a,b}\left\{X: \liminf_{t\to\infty} t^{-1}\log P_X\{D_{t,\omega}\varepsilon G\} \geq \inf_{\lambda\in F'_{a,b}\cap G}\int(\lambda' + a - \lambda)^2/8\lambda \right.$$
$$\left.\text{for all } \mathscr{M}\text{-open subsets } G\right\} = 1.$$

THEOREM 2.2. *We have*

$$\mu_{a,b}\{\Omega_{a,b}\} = 1. \tag{48}$$

If C is $\mathscr{M}$-closed, $a, b \geq 0$, and $X \in \Omega_{a,b}$, then

$$\limsup_{t\to\infty} t^{-1}\log P_X\{D_{t,\omega}\in C\} \leq -\inf_{\tau\in C} I'_{a,b}(\tau), \tag{49}$$

where $I'_{a,b}(\tau)$, for $\tau \in M(R)$, is defined as

$$\sup_{V\in A}\int V\tau(dx) - [a\log\alpha(V) + b\beta(V)]/2.$$

THEOREM 2.3. *For $a, b \geq 0$,*

$$I'_{a,b}(\tau) = \begin{cases}\int(\lambda' + a - \lambda)^2/8\lambda\,dx, & \text{for } \tau = \lambda(x)\,dx \text{ with } \lambda\in F'_{a,b},\\ +\infty, & \text{otherwise}.\end{cases} \tag{50}$$

Note that, when $a > 0$ and $\lambda \in F_{1,b} - F'_{a,b}$, $I'_{a,b}(\lambda) > I_{a,b}(\lambda)$.

THEOREM 2.4. *If $V \in A$ and $X \in \Omega_{a,b}$, then*

$$\lim_{t\to\infty} t^{-1}\log E^{P_X}\left\{\exp\int_0^t \sum_i V(\omega_i(s))\,ds\right\} = \sup_{\lambda\in F'_{a,b}} \int\left[V\lambda - (\lambda' + a - \lambda)^2/8\lambda\right]dx. \tag{51}$$

This is equivalent to a result in differential equations:

$$\lim_{t\to\infty} t^{-1}\sum_{x\in X}\log u(V,t,x) = [a\log\alpha(V) + b\beta(V)]/2 = \sup_{\lambda\in F'_{a,b}}\int V\lambda\,dx - I'_{a,b}(\lambda), \tag{52}$$

for a.e. X with respect to $\mu_{a,b}$.

Before going into detail, it is instructive to investigate why $I'_{a,0} = \infty > I_{a,0}(\lambda)$ for $\lambda \in F_{1,0} - F'_{a,0}$: The cause of this inequality is that

$$P_{a,0}\{|1/t(\text{ Number of } \omega_i(0) \text{ such that } \omega_i(0) \in [-t/2,0]) - k| < \varepsilon\}$$

is asymptotically greater than $\exp[-t/2(a + k\log k/a - k)]$ in contrast with $P_X\{|\ \ | < \varepsilon\}$ is zero when $|k - a| > \varepsilon$ and t large, for a.e. X w.r.t. $\mu_{a,0}$.

2.1. *Lower bound* (*Proof of Theorem* 2.1). We first show that

$$\mu_\lambda\left\{X: \liminf_{t\to\infty} t^{-1}\log P_X\left\{D_{t,\omega} \in \bigcap_1^m B(\lambda, V_j, \delta)\right\} \geq -\int(\lambda' + a - \lambda)^2/8\lambda\right\} = 1 \tag{53}$$

when $\lambda \in F'_{a,b}$ and $h(\mu_{a,b}; \mu_\lambda) < \infty$.

Let Q_X (Q_λ, resp.) denote the particle system generated by $[D^2 + (\lambda' + a)/\lambda D]/2$ and starting from X (μ_λ, resp.); F_t is as in Section 1.

First for Q_λ, the ergodic theorem implies that

$$\mu_\lambda\left\{X: \lim_{t\to\infty} Q_X\left\{D_{t,\omega} \in \bigcap_1^m B(\lambda, V_j, \delta)\right\}\right\} = 1. \tag{54}$$

Again the Cameron–Martin–Girsanov formula implies that

$$\mu_\lambda\left\{X: \lim_{t\to\infty} t^{-1}\log dQ_X/dP_X|_{F_t} = \int(\lambda' + a - \lambda)^2/8\lambda \text{ for a.e. } \omega \text{ w.r.t. } Q_X \text{ and in } L^1(Q_X)\right\} = 1. \tag{55}$$

Formula (53) then follows from a standard entropy argument.

To complete the proof, we have to replace, in (53), μ_λ by $\mu_{a,b}$, and also $\bigcap_1^m B(\lambda, V_j, \delta)$ by a general $\mathscr{M}$-open G. This is easy using Lemma 1.5. □

2.2. *Upper bound* (*Proof of Theorem* 2.2). Three steps of observations will reduce this theorem to Theorem 1.2.

(1) Since, due to the noninteracting nature of the evolution,

$$\log E^{P_X}\left\{\exp\left[\int_0^t \sum_i V(\omega_i(s))\,ds\right]\right\} = \sum_{x\in X} \log u(V,t,x)$$

is additive in X, we only need to show that

$$\mu_{1,0}\{\Omega_{1,0}\} = 1, \qquad \mu_{0,1}\{\Omega_{0,1}\} = 1. \tag{56}$$

(2) It is elementary to construct $\mathscr{D} = \{V_k\colon V_k \in A,\ k = 1,2,3,\dots\}$ so that $\Omega_{1,0} = \bigcap_{k=1}^\infty \Omega_{1,0,V_k}$ and $\Omega_{0,1} = \bigcap_{k=1}^\infty \Omega_{0,1,V_k}$. It therefore suffices to show that, for $V \in A$, $\mu_{1,0}\{\Omega_{1,0,V}\} = 1$ and $\mu_{0,1}\{\Omega_{0,1,V}\} = 1$, i.e.,

$$\mu_{1,0}\left\{X\colon \lim_{t\to\infty} t^{-1} \sum_{x\in X} \log u(V,t,x) = \log \alpha(V)/2\right\} = 1 \tag{57}$$

and

$$\mu_{0,1}\left\{X\colon \lim_{t\to\infty} t^{-1} \sum_{x\in X} \log u(V,t,x) = \beta(V)/2\right\} = 1.$$

(3) Formula (57) follows easily from the result: For $V \in A$ and $\theta \in R$ we have

$$\lim_{t\to\infty} t^{-1} \log E^{\mu_{1,0}}\left\{\exp\left[\theta \sum_{x\in X} \log u(V,t,x)\right]\right\} = \left[\alpha^\theta(V) - 1\right]/2$$

and

$$\lim_{t\to\infty} t^{-1} \log E^{\mu_{0,1}}\left\{\exp\left[\theta \sum_{x\in X} \log u(V,t,x)\right]\right\} = \theta\beta(V)/2.$$

In view of (C.G.F.), this is exactly (3) and (4) in Theorem 1.2.

For (49) the proof of (6) works nicely after replacing $P_{a,b}$, $\alpha(V) - 1$ and $I_{a,b}$ by P_X, $\log \alpha(V)$ and $I'_{a,b}$. □

2.3. *The equivalence of lower and upper bounds* (*Proof of Theorem* 2.3). The $I'_{0,b}$ formula is the same as $I_{0,b}$ since they are equal by definition.

Since

$$\begin{aligned} 2I'_{a,b}(\lambda) &\ge \sup_l \sup_{h\in C_0^\infty(-l,l)} \int -[h'' + h'^2 + h']\lambda + ah'\,dx \\ &= \sup_l \left[\sup_{h\in C_0^\infty(-l,l)} \int_{-l}^{l} -\lambda\left[h' - ((\lambda' + a - \lambda)/2\lambda)\right]^2 dx \right. \\ &\qquad \left. + \int_{-l}^{l} (\lambda' + a - \lambda)^2/4\lambda\,dx\right] \\ &= \int_{-\infty}^{\infty} (\lambda' + a - \lambda)^2/4\lambda\,dx \end{aligned} \tag{58}$$

it follows from the same reasoning as in the proof of Theorem 1.3 that

$$\lambda \sim a + be^x \quad \text{as} \quad x \to \pm\infty \quad \text{if} \quad I'_{a,b}(\lambda) < \infty.$$

For the $I'_{a,b}$ formula, we have

$$2I'_{a,b}(\lambda) \ge \int_{-\infty}^{\infty} (\lambda' + a - \lambda)^2/4\lambda \, dx$$

in (58).

The reverse inequality again holds because the difference equals $\int -\lambda\{h' - [(\lambda' + a - \lambda)/2\lambda]\}^2 \, dx$, which is nonpositive. □

2.4. *Variational formulas of scattering data* (*Proof of Theorem* 2.4). We only need to show that

$$\sup_{\lambda} -\int \lambda\{h' - [(\lambda' + a - \lambda)/2\lambda]\}^2 \, dx = 0.$$

Again, 0 is attained by

$$\lambda = be^x g^2(x) + ag^2(x) \int_x^{\infty} e^{-|y-x|} g^{-2}(y) \, dy,$$

which is the solution of $(\lambda' + a - \lambda)/2\lambda = h'$ and $\lambda \in F'_{a,b}$. □

In Donsker and Varadhan (1987), a large deviation principle for empirical density is also proved. We find it instructive to contrast the model studied here to that in Donsker and Varadhan (1987). Some remarks are:

(1) It affects technicality for proofs rather than the qualitative large deviation results whether each Markovian particle is a diffusion or a chain with countable states [as in Donsker and Varadhan (1987)]. For example, a dependence of the rates on the initial distribution is a common feature for both models.
(2) Worth noting is the difference between (3)–(4) in this paper and (2.23) in Donsker and Varadhan (1987). This difference results in the inequality $I_{a,b}(\cdot) > I'_{a,b}(\cdot)$ (as functionals) when $a \neq 0$ which does not occur in the model considered in Donsker and Varadhan (1987). Also this difference can be explained by the fact that our model has $\Gamma_x\{\sigma_{-l} < \infty\} = 1$ for all $x \le -l$ which violates an assumption made in Donsker and Varadhan (1987).
(3) An interesting model is the independent B.M. particles on R^d ($d \ge 3$) which "essentially" (regardless of countable states assumption) satisfies all assumptions made in Donsker and Varadhan (1987). $\{\mu_a\colon a > 0\}$ is the invariant distribution and therefore only one functional, instead of $\alpha(\cdot)$ and $\beta(\cdot)$, appears in the limit of the cumulant generating function [namely capacity $C(V)$]

$$C(V) \equiv \lim_{|x| \to \infty} [g(x) - 1]|x|^{d-2}$$

where $g(x)$ satisfies $(\Delta/2 + V)g = 0$ with $g(x) > 0$ and $\lim_{|x| \to \infty} g(x) = 1$.

Following the approach used here and in Donsker and Varadhan (1987), the rate can be computed as $I_a(\lambda) \equiv \int(|\nabla\lambda|^2/8\lambda)\,dx$ for $\lambda \sim a$ as $|x| \to \infty$ and "nice" λ. A variational formula for $C(V)$ can then be proved:

$$k_d C(V) = \sup_{\lambda \sim 1} \int \left[V\lambda - \frac{|\nabla\lambda|^2}{8\lambda} \right] dx,$$

where k_d depends on dimension only. This should be contrasted with a classical variational formula

$$\text{the maximum eigenvalue of } \left(\frac{\Delta}{2} + W \right) = \sup_{f>0,\, \|f\|_1 = 1} \int \left[Wf - \frac{|\nabla f|^2}{8f} \right] dx.$$

(4) For models with recurrent particles, e.g., random walk or B.M. in one or two dimensions, Cox and Griffeath (1984) and Lee (1988) prove that the large deviation probabilities decay more slowly than exponentially.

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Department of Mathematics
Princeton University
Princeton, New Jersey 08544