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THE LAW OF THE ITERATED LOGARITHM FOR *B*-VALUED RANDOM VARIABLES WITH MULTIDIMENSIONAL INDICES[1]

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Given independent identically distributed random variables $\{X, X_{\bar{n}};\ \bar{n} \in \mathbb{N}^d\}$ indexed by d-tuples of positive integers and taking values in a separable Banach space B we approximate the rectangular sums $\{\sum_{\bar{k} \le \bar{n}} X_{\bar{k}};\ \bar{n} \in \mathbb{N}^d\}$ by a Brownian sheet and obtain necessary and sufficient conditions for X to satisfy, respectively, the bounded, compact and functional law of the iterated logarithm when $d \ge 2$. These results improve, in particular, the previous work by Morrow [17].

1. Introduction and results. Let $\mathbb{N}^d$ be the set of d-dimensional vectors $\bar{n} = (n_1, \dots, n_d)$ whose coordinates $n_1, \dots, n_d$ are natural numbers. The symbol $\le$ means coordinate-wise ordering in $\mathbb{N}^d$. For $\bar{n} \in \mathbb{N}^d$, we define

$$|\bar{n}| = \prod_{i=1}^{d} n_i$$

and

$$a_{\bar{n}} = (2d|\bar{n}|L_2|\bar{n}|)^{1/2}.$$

Here $Lx = \log \max(x, e)$, $L_2 x = L(Lx)$.

Let B be a real separable Banach space with dual space B^* and norm $\|\cdot\|$. Throughout $\{X, X_n, X_{\bar{n}};\ n \ge 1,\ \bar{n} \in \mathbb{N}^d\}$ are independent identically distributed (i.i.d.) B-valued random variables, $S_{\bar{n}} = \sum_{\bar{k} \le \bar{n}} X_{\bar{k}}$ and $S(n) = \sum_{i=1}^{n} X_i$ for $\bar{n} \in \mathbb{N}^d$ and $n \ge 1$. We say X satisfies the bounded $\mathrm{LIL}^{(d)}$ (and write $X \in \mathrm{BLIL}^{(d)}$) with respect to the normalizing constant $a_{\bar{n}}$ if

$$\limsup_{\bar{n}} \frac{\|S_{\bar{n}}\|}{a_{\bar{n}}} \triangleq \lim_{m \to +\infty} \sup_{|\bar{n}| \ge m} \frac{\|S_{\bar{n}}\|}{a_{\bar{n}}} < +\infty \quad \text{a.s.} \tag{1.1}$$

We say X satisfies the compact $\mathrm{LIL}^{(d)}$ (and write $X \in \mathrm{CLIL}^{(d)}$) with respect to the normalizing constant $a_{\bar{n}}$ if

$$P\left(\left\{\frac{S_{\bar{n}}}{a_{\bar{n}}};\ \bar{n} \in \mathbb{N}^d\right\} \text{ is conditionally compact in } B\right) = 1. \tag{1.2}$$

Obviously, $\mathrm{BLIL}^{(1)} \supset \mathrm{BLIL}^{(d)} \supset \mathrm{CLIL}^{(d)}$ and $\mathrm{CLIL}^{(1)} \supset \mathrm{CLIL}^{(d)}$. If $d = 1$, we write BLIL and CLIL instead of $\mathrm{BLIL}^{(1)}$ and $\mathrm{CLIL}^{(1)}$. We say X satisfies the central limit theorem (and write $X \in \mathrm{CLT}$) if there is a mean zero Gaussian

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random variable W with values in B such that

$$\mathscr{L}\big(S(n)/\sqrt{n}\big) \to \mathscr{L}(W). \tag{1.3}$$

In the case $d = 1$, under the assumption that $EX = 0$ and $E\|X\|^2 < +\infty$ Kuelbs [12] has shown that the compact (bounded) LIL is equivalent to the sequence of probability measures $\{\mathscr{L}(S(n)/\sqrt{2nL_2n});\ n \geq 1\}$ being uniformly tight on compact (bounded) sets on B. However, it is well known that the moment conditions $EX = 0$ and $E\|X\|^2 < +\infty$ are neither necessary nor sufficient for $X \in$ CLT or $X \in$ BLIL in the infinite-dimensional setting (see [11] and [20]). In addition, under the assumption that $X \in$ CLT, Goodman, Kuelbs and Zinn [4] and Heinkel [7] have shown that $X \in$ CLIL if and only if $E(\|X\|^2/L_2\|X\|) < +\infty$. Recently, Ledoux and Talagrand [15] have characterized the random variable satisfying the BLIL and CLIL; they showed that $X \in$ CLIL if and only if $E(\|X\|^2/L_2\|X\|) < +\infty$, $\{|x'(X)|^2;\ x' \in B^*,\ \|x'\| \leq 1\}$ is uniformly integrable and $S(n)/\sqrt{2nL_2n} \to_P 0$.

In the case $d \geq 2$, if $B = H$ (Hilbert space), Morrow [17] (the case $B = R$ is due to Wichura [22]) has shown that $X \in \mathrm{CLIL}^{(d)}$ if and only if $EX = 0$ and $E(\|X\|^2(L\|X\|)^{d-1}/L_2\|X\|) < +\infty$. If B is a general real separable Banach space and X is a B-valued random variable with $EX = 0$ and $E(\|X\|^2(L\|X\|)^{d-1} < +\infty$, Morrow [17] has shown that $X \in \mathrm{CLIL}^{(d)}$ if and only if $S(n)/\sqrt{2nL_2n} \to_P 0$.

In this article we improve Morrow's results and characterize the $\mathrm{BLIL}^{(d)}$ and $\mathrm{CLIL}^{(d)}$ for $d \geq 2$ in the following way.

Theorem 1. *Let $\{X, X_n, X_{\bar{n}};\ n \geq 1,\ \bar{n} \in \mathbb{N}^d\}$ be i.i.d. B-valued random variables and $d \geq 2$. Then*

$$X \in BLIL^{(d)} \tag{1.4}$$

if and only if

$$\begin{cases} E\big(\|X\|^2(L\|X\|)^{d-1}/L_2\|X\|\big) < +\infty \\ \text{and the sequence } \big\{S(n)/\sqrt{2nL_2n}\,;\ n \geq 1\big\} \text{ is bounded in probability.} \end{cases} \tag{1.5}$$

Theorem 2. *Let $\{X, X_n, X_{\bar{n}};\ n \geq 1,\ \bar{n} \in \mathbb{N}^d\}$ be i.i.d. B-valued random variables and $d \geq 2$. Then*

$$X \in CLIL^{(d)} \tag{1.6}$$

if and only if

$$\begin{cases} E\big(\|X\|^2(L\|X\|)^{d-1}/L_2\|X\|\big) < +\infty \\ \text{and } S(n)/\sqrt{2nL_2n} \to_P 0. \end{cases} \tag{1.7}$$

We give the proofs of Theorems 1 and 2 in Sections 3 and 4, respectively. The methods of proof used in Theorems 1 and 2 are by now classical in probability in Banach spaces and rest on the ideas of [4], [7], [3] and [13]. Note in particular that since $d \geq 2$ the moment condition $E(\|X\|^2(L\|X\|)^{d-1}/L_2\|X\|) < +\infty$ implies that $E\|X\|^2 < +\infty$, which is not the case when $d = 1$. With respect to [15] the whole approach is simplified by this property and in particular no weak moments have to be studied. This cutoff between the cases $d = 1$ and $d \geq 2$ is already apparent on the real line as observed by Wichura [22]. By way of comparison, it is of interest to note that there is no such discontinuity with regard to d in the strong law of large numbers (see Mikosch and Norvaisa [16]).

The covariance function $T(\cdot,\cdot)$ of a B-valued random variable X is defined by

$$T(f,g) = E(f(X)g(X)), \qquad f,g \in B^*, \tag{1.8}$$

and X is said to be pre-Gaussian if its covariance structure is realized by some Gaussian measure on B.

Let $C_B([0,1]^d)$ denote the Banach space of B-valued continuous functions f on $[0,1]^d$ with

$$\|f\|_\infty = \sup_{\bar{t}\in[0,1]^d} \|f(\bar{t})\|. \tag{1.9}$$

For $\bar{n} \in \mathbb{N}^d$ and $\bar{t} \in [0,1]^d$ define $f_{\bar{n}} \in C_B([0,1]^d)$ by

$$f_{\bar{n}}(\bar{t}) = \begin{cases} a_{\bar{n}}^{-1} S_{\bar{m}}, & \text{for } t_i = \dfrac{m_i}{n_i},\ i = 1,\dots,d,\ \bar{m} \leq \bar{n}, \\ 0, & \text{if } t_i = 0 \text{ for some } i = 1,\dots,d, \\ \left\{\begin{array}{l} \text{Lagrange interpolation in } t_1,\dots,t_d \\ \text{over the cube } \{\bar{t} \in [0,1]^d; \\ (m_i - 1)/n_i \leq t_i \leq m_i/n_i,\ i = 1,\dots,d\}, \\ \bar{e} \leq \bar{m} \leq \bar{n}, \end{array}\right. \end{cases} \tag{1.10}$$

where $\bar{e} = (1,\dots,1)$. Let H_T be the reproducing kernel Hilbert space in B generated by the covariance function $T = T(\cdot,\cdot)$ and K be the closed unit ball of H_T (see Kuelbs [10]).

We say that X satisfies the functional LIL$^{(d)}$ (and write $X \in$ FLIL$^{(d)}$) if

$$\begin{cases} EX = 0, \qquad Ef^2(X) < +\infty, \qquad \forall f \in B^*, \\ K \text{ is a compact set on } B, \end{cases} \tag{1.11}$$

$$\lim_{\bar{n}} \inf_{f\in\mathscr{K}_T} \|f_{\bar{n}} - f\|_\infty = 0 \quad \text{a.s.} \tag{1.12}$$

and

$$P\Big(\Big\{f \in C_B\big([0,1]^d\big);\ f \text{ is a } \|\cdot\|_\infty\text{-limit point of } \{f_{\bar{n}};\ \bar{n} \in \mathbb{N}^d\}\Big\} = \mathscr{K}_T\Big) = 1, \tag{1.13}$$

where $\mathscr{K}_T$ is the set constructed on page 269 in [17].

Under the assumption that X is pre-Gaussian, $E(\|X\|^2(L\|X\|)^{d-1}) < +\infty$ and $S(n)/\sqrt{2nL_2n} \to_P 0$, Morrow [17] has shown that the rectangular sums $\{S_{\bar{n}};\ \bar{n} \in \mathbb{N}^d\}$ can be approximated by a Brownian sheet $\{W(\bar{t});\ \bar{t} \in [0, +\infty)^d\}$ in B (for the details see [17], page 266). Under the assumption that $E(\|X\|^2(L\|X\|)^{d-1}) < +\infty$ and $S(n)/\sqrt{2nL_2n} \to_P 0$, Morrow [17] has shown that $X \in \mathrm{FLIL}^{(d)}$ and conjectured "that the moment condition of this theorem can not be improved"; however, for $B = H$ he obtained an optimal moment condition ([17], Theorem 4).

We improve Morrow's results and obtain the following theorems.

THEOREM 3. *Let $\{X, X_{\bar{n}};\ \bar{n} \in \mathbb{N}^d\}$ be i.i.d. B-valued random variables. Then there is a Brownian sheet $\{W(\bar{t});\ \bar{t} \in [0, +\infty)^d\}$ in B with covariance function $T(\cdot, \cdot)$ determined by* (1.8) *such that*

$$\lim_{\bar{n}} \frac{\|S_{\bar{n}} - W(\bar{n})\|}{a_{\bar{n}}} = 0 \quad a.s. \tag{1.14}$$

if and only if

$$\begin{cases} X \in CLIL^{(d)}, \\ \text{and } X \text{ is pre-Gaussian.} \end{cases} \tag{1.15}$$

THEOREM 4. *Let X be a B-valued random variable. Then*

$$X \in FLIL^{(d)} \tag{1.16}$$

if and only if

$$X \in CLIL^{(d)}. \tag{1.17}$$

Since in type 2 spaces random variables X such that $EX = 0$ and $E\|X\|^2 < +\infty$ satisfy the CLT and are therefore necessarily pre-Gaussian (cf., e.g., [19]), the preceding theorems imply the following corollary.

COROLLARY 1. *Let $d \geq 2$ and X be a random variable taking values in a space of type 2. Then the following statements are equivalent:*

$$EX = 0, \qquad E\left(\|X\|^2(L\|X\|)^{d-1}/L_2\|X\|\right) < +\infty. \tag{1.18}$$

$$X \in BLIL^{(d)}. \tag{1.19}$$

$$X \in CLIL^{(d)}. \tag{1.20}$$

$$X \in FLIL^{(d)}. \tag{1.21}$$

(1.22) *There is a Brownian sheet $\{W(\bar{t});\ \bar{t} \in [0, +\infty)^d\}$ in B with covariance function $T(\cdot, \cdot)$ determined by* (1.8) *such that*

$$\lim_{\bar{n}} \frac{\|S_{\bar{n}} - W(\bar{n})\|}{a_{\bar{n}}} = 0 \quad a.s.$$

Theorems 3 and 4 are proved in Sections 5 and 6, respectively. The proofs of Theorems 3 and 4 are obtained via an application of Thoerem 4 of Morrow [17] and our Theorem 2.

2. Preliminary lemmas. For the proofs of Theorems 1 and 2 we need the following lemmas.

LEMMA 1. *Let $\{Z_{\bar{n}};\ \bar{n} \in \mathbb{N}^d\}$ be a collection of B-valued independent symmetric random variables. Then for any $t \geq 0$,*

$$P\left(\max_{\bar{m} \leq \bar{n}} \left\| \sum_{\bar{k} \leq \bar{m}} Z_{\bar{k}} \right\| \geq t\right) \leq 2^d P\left(\left\| \sum_{\bar{k} \leq \bar{n}} Z_{\bar{k}} \right\| \geq t\right). \tag{2.1}$$

PROOF. This is a generalization of Lévy's inequality. For $d \geq 2$, it is easily found that

$$P\left(\max_{\bar{m} \leq \bar{n}} \left\| \sum_{\bar{k} \leq \bar{m}} Z_{\bar{k}} \right\| \geq t\right) \leq 2P\left(\max_{\substack{m_i \leq n_i \\ i=1,\dots,d-1}} \left\| \sum_{\substack{k_d \leq n_d \\ k_i \leq m_i \\ i=1,\dots,d-1}} Z_{k_1,\dots,k_d} \right\| \geq t\right), \tag{2.2}$$

where $\bar{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ and the conclusion follows by iterating this inequality. □

The following Lemma 2 is due to de Acosta [3], page 275. Further references for the B-valued case can be found in [12] and [23].

LEMMA 2. *Let $\{Z_k;\ k = 1, \dots, n\}$ be independent B-valued random variables such that $\|Z_k\| \leq b_n$ a.s., $k = 1, \dots, n$. Let q be a seminorm on B such that $q \leq \|\cdot\|$. Then for all $\lambda > 0$,*

$$E\exp\left\{\lambda\left(q\left(\sum_{k=1}^{n} Z_k\right) - Eq\left(\sum_{k=1}^{n} Z_k\right)\right)\right\} \leq \exp\left\{2\lambda^2 \sum_{k=1}^{n} Eq^2(Z_k) e^{2\lambda b_n}\right\}. \tag{2.3}$$

LEMMA 3 (Pyke [21]). *Let X be a B-valued random variable. Then*

$$\begin{aligned} &\sum_{\bar{n} \in \mathbb{N}^d} P(\|X\| \geq 2^d |\bar{n}|) \\ &\quad \leq \int_1^{+\infty} \cdots \int_1^{+\infty} P(\|X\| \geq x_1 \cdots x_d)\, dx_1 \cdots dx_d \\ &\quad = \sum_{j=1}^{d-1} (-1)^{d-j-1} E\left(\frac{\|X\| (L^+\|X\|)^j}{j!}\right) - (-1)^d E(\|X\| - 1)^+ \\ &\quad \leq \sum_{\bar{n} \in \mathbb{N}^d} P(\|X\| \geq |\bar{n}|), \end{aligned} \tag{2.4}$$

where $a^+ = \max(0, a)$, $L^+ x = \max(0, \log x)$.

LEMMA 4. *Let* $\{X, X_n;\ n \geq 1\}$ *be i.i.d. B-valued random variables and* $S(n) = X_1 + \cdots + X_n$, $n \geq 1$. *Then*

(i) $\{S(n)/\sqrt{2nL_2n};\ n \geq 1\}$ *is bounded in probability if and only if* $\sup_{n\geq 1} E\|S(n)/\sqrt{2nL_2n}\| < +\infty$; *and*

(ii) $S(n)/\sqrt{2nL_2n} \to_P 0$ *if and only if* $E\|S(n)/\sqrt{2nL_2n}\| \to 0$.

PROOF. This fact is due to Pisier (cf. [19], Proposition 2.1 with $\sqrt{2nL_2n}$ instead of $\sqrt{n}$). □

The next lemmas will take into account what happens above the level $\sqrt{|\bar{n}|/L_2|\bar{n}|}$.

LEMMA 5. *Let* $\{X, X_{\bar{n}};\ \bar{n} \in \mathbb{N}^d\}$ *be i.i.d. B-valued random variables with* $E(\|X\|^2(L\|X\|)^{d-1}/L_2\|X\|) < +\infty$. *Let*

$$U_{\bar{n}} = X_{\bar{n}} I_{\{\|X_n\| \geq \sqrt{2|\bar{n}|L_2|\bar{n}|}\}}, \qquad \bar{n} \in \mathbb{N}^d. \tag{2.5}$$

Then

$$\lim_{\bar{n}} \sum_{\bar{k} \leq \bar{n}} U_{\bar{k}}/\sqrt{2|\bar{n}|L_2|\bar{n}|} = 0 \quad a.s. \tag{2.6}$$

PROOF. This follows since

$$\sum_{\bar{n}\in\mathbb{N}^d} P(U_{\bar{n}} \neq 0) = \sum_{\bar{n}\in\mathbb{N}^d} P\Big(\|X\| \geq \sqrt{2|\bar{n}|L_2|\bar{n}|}\Big) < +\infty \tag{2.7}$$

by Lemma 3 and $E(\|X\|^2(L\|X\|)^{d-1}/L_2\|X\|) < +\infty$. □

LEMMA 6. *Let* $\{X, X_n, X_{\bar{n}};\ n \geq 1,\ \bar{n} \in \mathbb{N}^d\}$ *be symmetric i.i.d. B-valued random variables with* $d \geq 2$. *Let*

$$V_{\bar{n}} = X_{\bar{n}} I_{\{\sqrt{|\bar{n}|/L_2|\bar{n}|} \leq \|X_{\bar{n}}\| \leq \sqrt{2|\bar{n}|L_2|\bar{n}|}\}} \tag{2.8}$$

and

$$T_{\bar{n}} = \sum_{\bar{k} \leq \bar{n}} V_{\bar{k}} \tag{2.9}$$

for $\bar{n} \in \mathbb{N}^d$. *Then we have*:

(I) *Under* (1.5),

$$\limsup_{\bar{n}} \|T_{\bar{n}}\|/\sqrt{2|\bar{n}|L_2|\bar{n}|} < +\infty \quad a.s. \tag{2.10}$$

(II) *Under* (1.7),

$$\lim_{\bar{n}} T_{\bar{n}}/\sqrt{2|\bar{n}|L_2|\bar{n}|} = 0 \quad a.s. \tag{2.11}$$

The idea of the proof used in Lemma 6 can be found in Goodman, Kuelbs and Zinn [4], Heinkel [7] and Kuelbs and Ledoux [13].

PROOF OF LEMMA 6. We only give the proof of (II), as the proof of (I) is analogous. For $\bar{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, let $I(\bar{n}) = \{(k_1, \dots, k_d);\ 2^{n_i - 1} \le k_i \le 2^{n_i} - 1,\ i = 1, \dots, d\}$, $\|\bar{n}\| = n_1 + \cdots + n_d$ and

$$\Lambda(\bar{n}) = \sum_{\bar{k} \in I(\bar{n})} E\left(\frac{\|V_{\bar{k}}\|^2}{2|\bar{k}| L_2 |\bar{k}|} \right). \tag{2.12}$$

It is easy to check

$$\begin{aligned} \Lambda(\bar{n}) &\le E\left\{ \|X\|^2 I_{\{\alpha_{\bar{n}} \le \|X\|^2 \le \beta_{\bar{n}}\}} \right\} / 2L_2 2^{\|\bar{n}\| - d} \\ &\le c_2 E\left\{ \frac{\|X\|^2}{L_2 \|X\|^2} I_{\{2^{\|\bar{n}\|} \le c_1 \|X\|^2 L_2 \|X\|^2\}} \right\}, \end{aligned} \tag{2.13}$$

where $\alpha_{\bar{n}} = 2^{\|\bar{n}\| - d} / L_2 2^{\|\bar{n}\|}$, $\beta_{\bar{n}} = 2 \cdot 2^{\|\bar{n}\|} L_2 2^{\|\bar{n}\|}$ and $c_1 > 0$, $c_2 > 0$ are constants such that for all $x \ge 0$, $L_2(x / 2^d L_2 x) \ge (2^d / c_1) L_2 x$ and $2L_2(x / 2^{d+1} L_2 x) \ge (1/c_2) L_2 x$. If X' is an independent copy of X, then

$$\begin{aligned} &\sum_{\bar{n} \in \mathbb{N}^d} \Lambda^2(\bar{n}) \\ &\quad \le 2c_2^2 E\left\{ \frac{\|X\|^2}{L_2 \|X\|^2} \frac{\|X'\|^2}{L_2 \|X'\|^2} \sum_{\bar{n} \in \mathbb{N}^d} I_{\{2^{\|\bar{n}\|} \le c_1 \|X\|^2 L_2 \|X\|^2,\ \|X\| \le \|X'\|\}} \right\} \\ &\quad \le 2c_2^2 E\left\{ \frac{\|X\|^2}{L_2 \|X\|} \frac{\|X'\|^2}{L_2 \|X'\|} c_3 \left(L\|X\| \right)^d I_{\{\|X\| \le \|X'\|\}} \right\} \\ &\quad \le 2c_2^2 c_3 E\left\{ \frac{\|X\|^2 \left(L\|X\| \right)^{d-1}}{L_2 \|X\|} \frac{\|X'\| \left(L\|X'\| \right)}{L_2 \|X'\|} \right\}, \end{aligned} \tag{2.14}$$

where $c_3 > 0$ is a constant such that $(2L(c_1 x^2 L_2 x^2))^d \le c_3 (Lx)^d$ for all $x \ge 0$. When $d \ge 2$, we get

$$\sum_{\bar{n} \in N^d} \Lambda^2(\bar{n}) \le 2c_2^2 c_3 \left(E\left(\frac{\|X\|^2 \left(L\|X\| \right)^{d-1}}{L_2 \|X\|} \right) \right)^2 < +\infty. \tag{2.15}$$

By standard methods and symmetry (2.11) is equivalent to

$$\lim_{\bar{n}} \sum_{\bar{k} \in I(\bar{n})} V_{\bar{k}} / \sqrt{2 \cdot 2^{\|\bar{n}\|} L_2 2^{\|\bar{n}\|}} = 0 \quad \text{a.s.} \tag{2.16}$$

Using the Hoffmann-Jørgensen inequality [8], pages 164–165, in order to establish (2.16) it is enough to show that

$$\sum_{\bar{n} \in \mathbb{N}^d} P\left(\max_{\bar{k} \in I(\bar{n})} \|V_{\bar{k}}\| \ge \varepsilon \sqrt{2 \cdot 2^{\|\bar{n}\|} L_2 2^{\|\bar{n}\|}} \right) < +\infty, \qquad \forall\, \varepsilon > 0, \tag{2.17}$$

and

$$\sum_{\bar{n}\in\mathbb{N}^d}\left(P\left(\left\|\sum_{\bar{k}\in I(\bar{n})} V_{\bar{k}}\right\| \geq \varepsilon\sqrt{2\cdot 2^{\|\bar{n}\|}L_2 2^{\|\bar{n}\|}}\right)\right)^2 < +\infty, \qquad \forall\, \varepsilon > 0. \tag{2.18}$$

It is easily seen how $E(\|X\|^2(L\|X\|)^{d-1}/L_2\|X\|) < +\infty$ implies (2.17). Concerning (2.18), note first that, by symmetry and the contraction principle (cf. [8])

$$\begin{aligned} E&\left(\left\|\sum_{\bar{k}\in I(\bar{n})} V_{\bar{k}}\right\|\Big/\sqrt{2\cdot 2^{\|\bar{n}\|}L_2 2^{\|\bar{n}\|}}\right) \\ &\leq E\left(\left\|S(2^{\|\bar{n}\|-d})\right\|\Big/\sqrt{2\cdot 2^{\|\bar{n}\|}L_2 2^{\|\bar{n}\|}}\right). \end{aligned} \tag{2.19}$$

Hence, by Lemma 4 and (1.7), we have

$$\lim_{\bar{n}} E\left(\left\|\sum_{\bar{k}\in I(\bar{n})} V_{\bar{k}}\right\|\Big/\sqrt{2\cdot 2^{\|\bar{n}\|}L_2 2^{\|\bar{n}\|}}\right) = 0. \tag{2.20}$$

Now, (2.18) is equivalent to saying that for any $\varepsilon > 0$,

$$\sum_{\bar{n}\in\mathbb{N}^d}\left(P\left(\left\|\sum_{\bar{k}\in I(\bar{n})} V_{\bar{k}}\right\| - E\left\|\sum_{\bar{k}\in I(\bar{n})} V_{\bar{k}}\right\| \geq \varepsilon\sqrt{2\cdot 2^{\|\bar{n}\|}L_2 2^{\|\bar{n}\|}}\right)\right)^2 < +\infty. \tag{2.21}$$

The quadratic inequality of de Acosta [2], Theorem 2.1, shows that

$$\begin{aligned} P&\left(\left\|\sum_{\bar{k}\in I(\bar{n})} V_{\bar{k}}\right\| - E\left\|\sum_{\bar{k}\in I(\bar{n})} V_{\bar{k}}\right\| \geq \varepsilon\sqrt{2\cdot 2^{\|\bar{n}\|}L_2 2^{\|\bar{n}\|}}\right) \\ &\leq \frac{4}{2\cdot\varepsilon^2\cdot 2^{\|\bar{n}\|}L_2 2^{\|\bar{n}\|}}\sum_{\bar{k}\in I(\bar{n})} E\|V_{\bar{k}}\|^2, \end{aligned} \tag{2.22}$$

so that, by definition of $\Lambda^2(\bar{n})$, in order for (2.21) to hold, it suffices that $\sum_{\bar{n}\in\mathbb{N}^d}\Lambda^2(\bar{n}) < +\infty$. But this has been proved in (2.15); the proof of Lemma 6 is therefore complete. □

The following lemma is a generalization of a result of de Acosta [3], Lemma 3.2.

LEMMA 7. *Let $\{Z_{\bar{n}};\ \bar{n}\in\mathbb{N}^d\}$ be a collection of B-valued random variables. Assume*

$$P\left(\{Z_{\bar{n}};\ \bar{n}\in\mathbb{N}^d\} \text{ is bounded}\right) = 1 \tag{2.23}$$

and for every $\varepsilon > 0$ *there exists a finite-dimensional subspace* F *such that*

$$P\Big(\limsup_{\bar n} q_F(Z_{\bar n}) \le \varepsilon\Big) = 1, \tag{2.24}$$

where $q_F(x) = \inf_{y\in F}\|x - y\|$, $x \in B$. *Then*

$$P\big(\{Z_{\bar n};\ \bar n \in \mathbb{N}^d\} \text{ is relatively compact in } B\big) = 1. \tag{2.25}$$

Proof. Let A be a subset of B. It is easy to prove that A is relatively compact in B if and only if $\sup_{x\in A}\|x\| < +\infty$ and for every $\varepsilon > 0$ there exists a finite-dimensional subspace $F \subset B$ such that $\sup_{x\in A}\inf_{y\in F}\|x - y\| \le \varepsilon$. So the proof of the lemma is straightforward. □

3. Proof of Theorem 1. By a standard symmetrization procedure (cf., e.g., [1]), it suffices to prove the theorem under the assumption that X is symmetric, so we do this. Since $X \in \mathrm{BLIL}^{(d)}$ implies

$$\limsup_{\bar n} \frac{\|X_{\bar n}\|}{\sqrt{2|\bar n| L_2|\bar n|}} < +\infty \quad \text{a.s.} \tag{3.1}$$

by the Borel–Cantelli lemma there exists a constant $c_1 > 0$ such that

$$\begin{aligned} &\sum_{\bar n\in\mathbb{N}^d} P\Big(\|X_{\bar n}\| \ge c_1\sqrt{2|\bar n|L_2|\bar n|}\Big) \\ &\quad = \sum_{\bar n\in\mathbb{N}^d} P\Big(\|X\| \ge c_1\sqrt{2|\bar n|L_2|\bar n|}\Big) < +\infty; \end{aligned} \tag{3.2}$$

hence $E(\|X\|^2(L\|X\|)^{d-1}/L_2\|X\|) < +\infty$ by using Lemma 3. It remains to show that (1.5) implies (1.4). Now, following [3], we truncate as follows:

$$\begin{aligned} Y_{\bar n}(\tau) &= X_{\bar n} I_{\{\|X_{\bar n}\| \le \tau\sqrt{|\bar n|/L_2|\bar n|}\}}, \\ W_{\bar n}(\tau) &= \sum_{\bar k \le \bar n} Y_{\bar k}(\tau), \end{aligned} \qquad \bar n \in \mathbb{N}^d, \tag{3.3}$$

where $\tau > 0$ is a parameter. By Lemmas 4, 5 and 6, we have

$$\limsup_{\bar n} \frac{\|S_{\bar n} - W_{\bar n}(\tau)\|}{\sqrt{2|\bar n|L_2|\bar n|}} < +\infty \quad \text{a.s.} \tag{3.4}$$

and

$$\sup_{\bar n\in\mathbb{N}^d} E\frac{\|W_{\bar n}(\tau)\|}{\sqrt{2|\bar n|L_2|\bar n|}} \le \sup_{n\ge 1} E\frac{\|S(n)\|}{\sqrt{2nL_2 n}} \triangleq \gamma < +\infty. \tag{3.5}$$

Let $I(\bar n)$, $\|\bar n\|$ be as in Lemma 6; then there exists a constant $c_2 > 0$ such that for all $\bar n \in \mathbb{N}^d$ and all $\bar k \in I(\bar n)$, $\sqrt{2|\bar k|L_2|\bar k|} \ge c_2\sqrt{2\cdot 2^{\|\bar n\|}L_2 2^{\|\bar n\|}}$. By applying Lemma 1 and Lemma 2 with $\lambda = (2a^2)^{-1}tL_2 2^{\|\bar n\|}$ (where $a^2 = E\|X\|^2$ is finite under the necessary integrability condition and $d \geq 2$) we have

$$P\left(\sup_{\bar{k}\in I(\bar{n})} \frac{\|W_{\bar{k}}(\tau)\|}{\sqrt{2|\bar{k}|L_2|\bar{k}|}} > \frac{t+\gamma}{c_2}\right)$$

$$\leq 2^d P\left(\left(\|W_{2^{\bar{n}}}(\tau)\| - E\|W_{2^{\bar{n}}}(\tau)\|\right)/\sqrt{2\cdot 2^{\|\bar{n}\|}L_2 2^{\|\bar{n}\|}} > t\right)$$

$$\leq 2^d e^{-\lambda t} E\left(\exp\left\{\lambda\left(\|W_{2^{\bar{n}}}(\tau)\| - E\|W_{2^{\bar{n}}}(\tau)\|\right)/\sqrt{2\cdot 2^{\|\bar{n}\|}L_2 2^{\|\bar{n}\|}}\right\}\right) \tag{3.6}$$

$$\leq 2^d e^{-\lambda t}\exp\left\{2\lambda^2 \sum_{\bar{k}\leq 2^{\bar{n}}} \left(E\|\bar{Y}_{\bar{k}}\|^2/2\cdot 2^{\|\bar{n}\|}L_2 2^{\|\bar{n}\|}\right)e^{2\lambda b_{2^{\bar{n}}}}\right\}$$

$$\leq 2^d \exp\left\{-\left(\frac{t}{2a}\right)^2\left(2-\exp\left(t\tau/\sqrt{2}\,a^2\right)\right)L_2 2^{\|\bar{n}\|}\right\},$$

where $\bar{n} = (n_1,\dots,n_d) \in \mathbb{N}^d$, $2^{\bar{n}} = (2^{n_1},\dots,2^{n_d}) \in \mathbb{N}^d$ and $b_{2^{\bar{n}}} = \tau/\sqrt{2}\,L_2 2^{\|\bar{n}\|}$. Take $t = 2(d+1)a$ and $\tau > 0$ such that $2 - \exp(2(d+1)\tau/a) > \frac{1}{2}$. Then

$$P\left(\sup_{\bar{k}\in I(\bar{n})} \frac{\|W_{\bar{k}}(\tau)\|}{\sqrt{2|\bar{k}|L_2|\bar{k}|}} > \frac{2(d+1)a+\gamma}{c}\right)$$

$$\leq 2^d \exp\left\{-\frac{(d+1)^2}{2}L_2 2^{\|\bar{n}\|}\right\} \tag{3.7}$$

$$\leq 2^d(\log 2)^{-(d+1)}(\|\bar{n}\|)^{-(d+1)}.$$

Since

$$\sum_{\bar{n}\in\mathbb{N}^d} (\|\bar{n}\|)^{-(d+1)} < +\infty, \tag{3.8}$$

we get

$$\limsup_{\bar{n}} \frac{\|W_{\bar{n}}(\tau)\|}{\sqrt{2|\bar{n}|L_2|\bar{n}|}} < +\infty \quad \text{a.s.}, \tag{3.9}$$

which together with (3.4) implies that $X \in \mathrm{BLIL}^{(d)}$. The theorem is proved. □

4. Proof of Theorem 2. That (1.6) implies (1.7) follows easily from Kuelbs' (compact) LIL ([12], Theorem 4.1) and (3.1). By [1] it suffices to prove that (1.7) implies (1.6) under the assumption that X is symmetric. Let $Y_{\bar{n}}(\tau), W_{\bar{n}}(\tau)$, $\bar{n} \in \mathbb{N}^d$ be as in the proof of Theorem 1. Since (1.7) and $d \geq 2$ imply $E\|X\|^2 < +\infty$, we have

$$\lim_{\bar{n}} E\frac{\|W_{\bar{n}}(\tau)\|}{\sqrt{2|\bar{n}|L_2|\bar{n}|}} = 0 \tag{4.1}$$

and

$$\lim_{\bar{n}} \frac{S_{\bar{n}} - W_{\bar{n}}(\tau)}{\sqrt{2|\bar{n}|L_2|\bar{n}|}} = 0 \quad \text{a.s.} \tag{4.2}$$

by applying Lemmas 4, 5, and 6. Let $q(\cdot)$ be a seminorm on B such that $q(\cdot) \leq \|\cdot\|$; then we have

$$\lim_{\bar{n}} Eq\left(\frac{W_{\bar{n}}(\tau)}{\sqrt{2|\bar{n}|L_2|\bar{n}|}} \right) = 0. \tag{4.3}$$

Let $I(\bar{n})$, $\|\bar{n}\|$ and the constant $c_1 > 0$ be as in the proof of Theorem 1. Proceeding as in that proof, for every $\varepsilon > 0$ we get

$$\begin{aligned} &P\left(\sup_{\bar{k} \in I(\bar{n})} q\left(W_{\bar{k}}(\tau)/\sqrt{2|\bar{k}|L_2|\bar{k}|} \right) > (2(d+1)a + \varepsilon)/c_1 \right) \\ &\quad \leq 2^d/(\log 2)^{d+1}(\|\bar{n}\|)^{d+1}, \qquad |\bar{n}| \gg 1, \end{aligned} \tag{4.4}$$

by using Lemmas 1 and 2, where $a^2 = Eq^2(X)$ and $\tau > 0$ is a constant such that $2 - \exp(\sqrt{2}(d+1)\tau/a) > \frac{1}{2}$. Therefore by using (4.2), we get

$$P\left(\limsup_{\bar{n}} q\left(S_{\bar{n}}/\sqrt{2|\bar{n}|L_2|\bar{n}|} \right) \leq \frac{2(d+1)\left(Eq^2(X) \right)^{1/2}}{c_1} \right) = 1. \tag{4.5}$$

In particular,

$$P\left(\limsup_{\bar{n}} \frac{\|S_{\bar{n}}\|}{\sqrt{2|\bar{n}|L_2|\bar{n}|}} \leq \frac{2(d+1)\left(E\|X\|^2 \right)^{1/2}}{c_1} \right) = 1. \tag{4.6}$$

Given $\varepsilon > 0$, choose a finite-dimensional subspace F such that $E(q_F^2(X)) \leq \varepsilon^2 c_1^2/4(d+1)^2$; hence

$$P\left(\limsup_{\bar{n}} q_F\left(S_{\bar{n}}/\sqrt{2|\bar{n}|L_2|\bar{n}|} \right) \leq \varepsilon \right) = 1. \tag{4.7}$$

By applying Lemma 7, we have $X \in \mathrm{CLIL}^{(d)}$, so the proof is complete. □

REMARK. Let $\{X, X_{\bar{n}};\ \bar{n} \in \mathbb{N}^d\}$ be i.i.d. B-valued random variables, $d \geq 2$, and $X \in \mathrm{CLIL}^{(d)}$. Then

$$\lim_{\bar{n}} \inf_{x \in K} \left\| \frac{S_{\bar{n}}}{a_{\bar{n}}} - x \right\| = 0 \quad \text{a.s.}, \tag{4.8}$$

$$P\left(C\left\{ \frac{S_{\bar{n}}}{a_{\bar{n}}};\ \bar{n} \in \mathbb{N}^d \right\} = K \right) = 1 \tag{4.9}$$

and

$$\limsup_{\bar{n}} \frac{\|S_{\bar{n}}\|}{\sqrt{2|\bar{n}|L_2|\bar{n}|}} = \sqrt{d} \sup_{x \in K} \|x\| \quad \text{a.s.}, \tag{4.10}$$

where $a_{\bar{n}} = \sqrt{2d|\bar{n}|L_2|\bar{n}|}$, K is the closed unit ball of the reproducing kernel Hilbert space H_T and $T(f,g) = Ef(X)g(X)$, $(f, g \in B^*)$, and $C\{x_{\bar{n}};\ \bar{n} \in \mathbb{N}^d\}$ denotes the all limit points of the sequence $\{x_{\bar{n}};\ \bar{n} \in \mathbb{N}^d\}$. Indeed, under the assumption that $X \in \mathrm{CLIL}^{(d)}$ Wichura's LIL implies that

$$\limsup_{\bar{n}} f\left(\frac{S_{\bar{n}}}{a_{\bar{n}}}\right) = \left(Ef^2(X)\right)^{1/2} \quad \text{a.s.}, \qquad \forall f \in B^*. \tag{4.11}$$

The remark then follows from the argument provided in [10] for $d = 1$.

5. Proof of Theorem 3. That (1.14) implies $X \in \mathrm{CLIL}^{(d)}$ is clear and for any $f \in B^*$, we get

$$\limsup_{\bar{n}} f\left(\frac{S_{\bar{n}}}{a_{\bar{n}}}\right) = \limsup_{\bar{n}} f\left(\frac{W(\bar{n})}{a_{\bar{n}}}\right) \quad \text{a.s.} \tag{5.1}$$

By applying the Hartman–Wintner LIL [5] and Wichura's LIL [22] we have

$$\left(Ef^2(X)\right)^{1/2} = \left(Ef^2(W(\bar{e}))\right)^{1/2}, \tag{5.2}$$

where $\bar{e} = (1,\dots,1) \in \mathbb{N}^d$, so for any $f, g \in B^*$,

$$Ef(X)g(X) = Ef(W(\bar{e}))g(W(\bar{e})), \tag{5.3}$$

that is, X is pre-Gaussian.

Now we prove that (1.15) implies (1.14). Let $T(f,g) = Ef(X)g(X)$, $f, g \in B^*$, $T = T(\cdot,\cdot)$ and $\{\varphi_v^*;\ v \geq 1\}$ be a sequence of bounded linear functionals on B with the property that the points $\varphi_v = \int_B \xi \varphi_v^*(\xi) P(X \in d\xi)$, $v \geq 1$, constitute a C.O.N.S. $\{\varphi_v;\ v \geq 1\}$ in H_T and $\xi = \sum_{v=1}^{\infty} \varphi_v^*(\xi)\varphi_v$, for $\xi \in H_T$ (see, e.g., [10], Lemma 2.1). The inner product $(\cdot,\cdot)$ in H_T is given by $(\varphi_u, \varphi_v) = \int_B \varphi_u^*(\xi)\varphi_v^*(\xi) P(X \in d\xi)$. We first prove that for each $\theta > 0$, there is a Brownian sheet $\{W_\theta(\bar{t});\ \bar{t} \in [0, +\infty)^d\}$ in B with covariance function $T(\cdot,\cdot)$ defined by $T(f,g) = Ef(X)g(X)$, $f, g \in B^*$, such that

$$\limsup_{\bar{n}} \frac{\|S_{\bar{n}} - W_\theta(\bar{n})\|}{a_{\bar{n}}} \leq \theta \quad \text{a.s.} \tag{5.4}$$

For this we employ the maps Π_N associated to the covariance function $T(\cdot,\cdot)$ of X, $\Pi_N(\xi) = \sum_{v=1}^{N} \varphi_v^*(\xi)\varphi_v$, $\xi \in B$. Let $Q_N = I - \Pi_N$; as shown in Theorem 3.1 of [10], given $\theta > 0$ there exists N_θ with

$$\sup_{\xi \in K} \left\| Q_{N_\theta}(\xi) \right\| \leq \frac{\theta}{2}. \tag{5.5}$$

Hence $X \in \mathrm{CLIL}^{(d)}$ implies

$$\lim_{\bar{n}} \inf_{\xi \in K} \left\| a_{\bar{n}}^{-1} Q_{N_\theta}(S_{\bar{n}}) - Q_{N_\theta}(\xi) \right\| = 0 \quad \text{a.s.} \tag{5.6}$$

and

$$\limsup_{\bar{n}} a_{\bar{n}}^{-1} \left\| S_{\bar{n}} - \Pi_{N_\theta}(S_{\bar{n}}) \right\| \le \frac{\theta}{2} \quad \text{a.s.} \tag{5.7}$$

Let $p = \min(N_\theta, \dim H_T)$. Then $\Pi_{N_\theta}(B)$ is the Euclidean space $\mathbb{R}^p$ equipped with the norm $|\cdot| = \|\cdot\|_T$ induced by B-norm on $H_T \subset B$. We define $\hat{X} = \Pi_{N_\theta}(X)$, $\hat{S}_{\bar{n}} = \sum_{\bar{k} \le \bar{n}} \hat{X}_{\bar{k}}$; then there exists a Brownian sheet $\{W_\theta(\bar{t});\ \bar{t} \in [0, +\infty)^d\}$ in B with covariance function $T = T(\cdot, \cdot)$ of X such that

$$\lim_{\bar{n}} a_{\bar{n}}^{-1} \left\| \hat{S}_{\bar{n}} - \Pi_{N_\theta}(W_\theta(\bar{n})) \right\| = 0 \quad \text{a.s.} \tag{5.8}$$

by applying Theorem 4 of [17]. Therefore

$$\limsup_{\bar{n}} \frac{\|S_{\bar{n}} - W_\theta(\bar{n})\|}{a_{\bar{n}}} \le \theta \quad \text{a.s.} \tag{5.9}$$

The rest is similar to [17], pages 282–283. □

REMARK. Let $d \ge 2$ and $\{X, X_n, X_{\bar{n}};\ n \ge 1,\ \bar{n} \in \mathbb{N}^d\}$ be i.i.d. B-valued random variables. Then, as a corollary to Theorems 2 and 3, there is a Brownian sheet $\{W(\bar{t});\ \bar{t} \in [0, +\infty)^d\}$ in B with covariance function $T(\cdot, \cdot)$ of X such that

$$\lim_{\bar{n}} \frac{\|S_{\bar{n}} - W(\bar{n})\|}{a_{\bar{n}}} = 0 \quad \text{a.s.} \tag{5.10}$$

if and only if

$$\begin{cases} E\left(\dfrac{\|X\|^2 (L\|X\|)^{d-1}}{L_2\|X\|} \right) < +\infty, \\ S(n)/\sqrt{2nL_2 n} \to_P 0 \\ \text{and } X \text{ is pre-Gaussian,} \end{cases} \tag{5.11}$$

where $S(n) = X_1 + \cdots + X_n$, $n \ge 1$.

6. Proof of Theorem 4. Since $X \in \mathrm{FLIL}^{(d)}$ implies $X \in \mathrm{CLIL}^{(d)}$ is clear, we only need to prove that $X \in \mathrm{CLIL}^{(d)}$ implies $X \in \mathrm{FLIL}^{(d)}$. Let Π_N be as in the proof of Theorem 3, $N \ge 1$. Since $\Pi_N(X) \in \mathrm{CLIL}^{(d)}$ and $\Pi_N(X)$ is pre-Gaussian, it follows that

$$\lim_{\bar{n}} \inf_{f \in \mathscr{K}_T} \left\| \Pi_N(f_{\bar{n}} - f) \right\|_\infty = 0 \quad \text{a.s.} \tag{6.1}$$

and

$$P\Big(\Big\{ f \in C_B\big([0,1]^d\big);\ f \text{ is a } \|\cdot\|_\infty\text{-limit point of } \big\{\Pi_N(f_{\bar n});\ \bar n \in \mathbb{N}^d\big\}\Big\} = \Pi_N(\mathscr{K}_T)\Big) = 1 \tag{6.2}$$

by using Theorem 3 and Morrow's Theorem 2 in [17]. Furthermore,

$$\lim_{N \to +\infty} \limsup_{\bar n} a_{\bar n}^{-1} \|S_{\bar n} - \Pi_N(S_{\bar n})\| = 0 \quad \text{a.s.} \tag{6.3}$$

yields $X \in \mathrm{FLIL}^{(d)}$ and the theorem is proved. □

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