The Annals of Probability
1989, Vol. 17, No. 2, 700–722

SHARP RATES FOR INCREMENTS OF RENEWAL PROCESSES

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Let $\{N(t),\ t \geq 0\}$ be the renewal process associated to an i.i.d. sequence $X_1, X_2, \dots$ of nonnegative interarrival times having finite moment generating function near the origin. In this article we give strong and weak limiting laws for the maximal and minimal increments $\sup_{0 \leq t \leq T-K}(N(t+K) - N(t))$ and $\inf_{0 \leq t \leq T-k}(N(t+K) - N(t))$, where $K = K_T$ is a function of T such that $0 \leq K_T \leq T$.

1. Introduction. Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed nonnegative random variables satisfying the assumptions

(A) $\mu = E(X_1) > 0$;
(B) $P(X_1 = x) < 1$ for all x;
(C) $s_0 = \sup\{s: \phi(s) = E(\exp(sX_1)) < \infty\} > 0$.

Let $S_0 = 0$, $S_n = X_1 + \dots + X_n$ for $n \geq 1$, and consider the corresponding *renewal process* $\{N(t),\ t \geq 0\}$ defined by

$$N(t) = \max\{n \geq 0: S_n \leq t\} = -1 + \min\{n > 0: S_n > t\} \quad \text{for } t \leq 0. \tag{1.1}$$

The purpose of this article is to study the limiting behavior of the increments of $\{N(t),\ t \geq 0\}$. Throughout the sequel, we will consider a function $\{K_T\}_{T \geq 0}$ such that $0 \leq K_T \leq T$ for all $T \geq 0$ and define $\Delta_T^{\pm} = \Delta^{\pm}(T, K_T)$, where

$$\Delta^+(T, h) = \sup_{0 \leq t \leq T-h} (N(t+h) - N(t))$$

and

$$\Delta^-(T, h) = \inf_{0 \leq t \leq T-h} (N(t+h) - N(t)).$$

Several results are available concerning the limiting behavior of $\Delta^{\pm}(T, K_T)$ under various assumptions on K_T. Retka (1982) and Steinebach (1984, 1986) discuss strong laws for a general K_T, while Bacro, Deheuvels and Steinebach (1987) consider the "Erdös–Rényi case" where $K_T = C \log T$ for some $C > 0$.

In the following, we will show by a duality between events that it is possible to deduce the limiting behavior of $\Delta_T^{\pm}$ from the knowledge of the corresponding

Received December 1986; revised July 1988.

AMS 1980 *subject classifications*. Primary 60F15; secondary 60F05, 60F17, 60G55.

Key words and phrases. Renewal processes, laws of large numbers, weak laws, law of the iterated logarithm, invariance principles.

limiting behavior of the extremal increments of partial sums

$$U_n^+ = U^+(n, b_n) = \max_{0 \le k \le n - b_n} (S_{k+b_n} - S_k)$$

and

$$U_n^- = U^-(n, b_n) = \min_{0 \le k \le n - b_n} (S_{k+b_n} - S_k),$$

for suitable choices of $0 \le b_n \le n$.

Roughly speaking, our argument says that, for $T = S_n$, each time that $U_n^{\pm}$ exceeds (resp. becomes smaller than) K_T, then $\Delta_T^{\mp}$ becomes smaller than (resp. exceeds) b_n.

In the case where $K_T(\log\log T)^2/\log^3 T \to \infty$, the results we seek can be deduced from the limiting behavior of the increments of a Wiener process by using the new strong invariance principle due to Mason and van Zwet (1987) [see also Csörgő, Horváth and Steinebach (1987)].

In the range where $K_T = O(\log T)$ and $\log T = O(K_T)$ as $T \to \infty$, our results coincide with those of Bacro, Deheuvels and Steinebach (1987), which we will only cite for completeness.

From there, we see that the main situation uncovered by invariance principles or by the results of Bacro, Deheuvels and Steinebach (1987), corresponds to the case where

(a) $$K_T/\log T \to \infty \quad \text{as } T \to \infty$$

and

(b) $$K_T/\log^p T \to 0 \quad \text{as } T \to \infty \text{ for some } p > 1.$$

We will treat this case by our duality argument, used jointly with the results of Deheuvels and Steinebach (1987). It will become clear in the sequel that the methods of our proofs can be used to cover the case where K_T satisfies more general assumptions than (a) and (b), given the corresponding results for increments of partial sums.

The rest of this article is organized as follows. In Section 2, we consider the "*large-increment case*" corresponding namely to $K_T(\log\log T)^2/\log^3 T \to \infty$, which we treat by invariance principles. In Section 3, we establish asymptotic expansions related to the moment generating function $\phi(\cdot)$. In Section 4, we present our theorems for "*medium-size increments*," corresponding to (a) and (b). Finally, Section 5 contains the "*Erdös–Rényi case*," where $K_T = O(\log T)$. We conclude the article in Section 6, where we investigate how our results extend to "*generalized renewal processes*" obtained by relaxing the nonnegativity assumption on the X_i's.

2. Large increments. We will use throughout the notation $\sigma^2 = \mathrm{Var}(X_1)$. Notice that the assumptions (B) and (C) jointly imply that $0 < \sigma^2 < \infty$. The main results of this section are captured in Theorems 1 and 2 below.

THEOREM 1. *Assume that* $\{K_T\}_{T\geq 0}$ *satisfies* $0 \leq K_T \leq T$, $K_T/T \to 0$ *and* $K_T(\log\log T)^2/\log^3 T \to \infty$ *as* $T \to \infty$. *Then*

$$\lim_{T\to\infty} \frac{\pm(2\log(T/K_T))^{1/2}}{K_T^{1/2}\sigma\mu^{-3/2}\log\log(T/K_T)} \times \left\{\Delta_T^{\pm} - \mu^{-1}K_T \mp \sigma\mu^{-3/2}(2K_T\log(T/K_T))^{1/2}\right\} = \frac{1}{2} \quad \textit{in probability}. \tag{2.1}$$

THEOREM 2. *Assume that* $\{K_T\}_{T\geq 0}$ *satisfies* $0 \leq K_T \leq T$, $K_T/T \to 0$, $K_T \uparrow \infty$, $K_T(\log\log T)^2/\log^3 T \to \infty$ *and* $(\log\log(T/K_T))/\log\log T \to 1$ *as* $T \to \infty$. *Assume further that* K_T *has a continuous first derivative* K_T' *such that*

$$K_T'/K_T = O\big((T\log T)^{-1}\big) \quad \textit{as } T \to \infty. \tag{2.2}$$

Then

$$\limsup_{T\to\infty} \frac{\pm(2\log(T/K_T))^{1/2}}{K_T^{1/2}\sigma\mu^{-3/2}\log\log(T/K_T)} \times \left\{\Delta_T^{\pm} - \mu^{-1}K_T \mp \sigma\mu^{-3/2}(2K_T\log(T/K_T))^{1/2}\right\} = \frac{3}{2} \quad \textit{a.s.} \tag{2.3}$$

and

$$\liminf_{T\to\infty} \frac{\pm(2\log(T/K_T))^{1/2}}{K_T^{1/2}\sigma\mu^{-3/2}\log\log(T/K_T)} \times \left\{\Delta_T^{\pm} - \mu^{-1}K_T \mp \sigma\mu^{-3/2}(2K_T\log(T/K_T))^{1/2}\right\} = \frac{1}{2} \quad \textit{a.s.} \tag{2.4}$$

PROOF. The proof of both theorems relies on the result [see Mason and van Zwet (1987) and Csörgő, Horváth and Steinebach (1987)] that there exists a probability space on which sits a Wiener process $\{W(t),\ t \geq 0\}$, jointly with a sequence identical in distribution with $\{X_n,\ n \geq 1\}$ (which we will assume, without loss of generality, to coincide with $\{X_n,\ n \geq 1\}$), such that

$$N(t) = \mu^{-1}t + \sigma\mu^{-3/2}W(t) + O(\log t) \quad \text{a.s. as } t \to \infty. \tag{2.5}$$

By (2.5) and the observation that our assumptions imply that

$$\log T = o\left\{K_T^{1/2}(\log\log(T/K_T))(\log(T/K_T))^{-1/2}\right\} \quad \text{as } T \to \infty,$$

we see that in (2.1), (2.3) and (2.4) we may replace $\Delta_T^{\pm} - \mu^{-1}K_T$ by $\pm\sigma\mu^{-3/2}\sup_{0\leq t\leq T-K_T}(W(t+K_T) - W(t))$. The conclusion follows by applying the results of Révész (1982) and Ortega and Wschebor (1984), jointly with Theorem 1.5.5 in Csörgő and Révész (1981). We omit the details since this part of the proof is identical to the proof of Theorems 3 and 4 in Deheuvels and Steinebach (1987), corresponding to increments of partial sums. □

REMARK 1. Assume that $0 \le L_T \le T$ satisfies $L_T - K_T = O(\log T)$ as $T \to \infty$. Then, by (2.5), it can be seen that (2.3) and (2.4) remain valid with $\Delta^{\pm}(T, L_T)$ replacing $\Delta_T^{\pm} = \Delta^{\pm}(T, K_T)$. This enables us to relax partly the technical regularity conditions on K_T used in Theorem 2 through the use of an auxiliary function. It is naturally possible to deduce from (2.5) and the corresponding characterizations of the upper and lower classes for the increments of Wiener processes similar results for other choices of K_T [see, e.g., Csörgő and Révész (1981), Theorem 1.2.1, and Csáki and Révész (1979)]. Such evaluations are straightforward and will not be discussed here. We will limit ourselves to the following result whose proof follows directly from (2.5) and the results of Csörgő and Révész (1979).

THEOREM 3. *Assume that* $\{K_T\}_{T \ge 0}$ *satisfies* $0 \le K_T \le T$, $K_T \uparrow$, $K_T/T \downarrow$, $K_T/\log T \to \infty$ *and* $(\log(T/K_T))/\log\log T \to \infty$, *as* $T \uparrow \infty$. *Then*

$$\lim_{T\to\infty} \{\Delta_T^{\pm} - \mu^{-1}K_T\}/\left\{\sigma\mu^{-3/2}(2K_T \log(T/K_T))^{1/2}\right\} = \pm 1 \quad a.s. \tag{2.6}$$

REMARK 2. If, in Theorem 2, we replace the assumption that $K_T(\log\log T)^2/\log^3 T \to \infty$ by $K_T(\log\log T)^2/\log^3 T = O(1)$ as $T \to \infty$, the same arguments as above show that

$$\Delta_T^{\pm} - \mu^{-1}K_T \mp \sigma\mu^{-3/2}(2K_T \log(T/K_T))^{1/2} = O(\log T) \quad \text{a.s. as } T \to \infty. \tag{2.7}$$

The aim of making (2.7) precise is the main motivation for Sections 3 and 4 in the sequel.

3. Expansions related to the moment generating function. In the following, we refer to Section 2 in Deheuvels, Devroye and Lynch (1986), Section 2 in Deheuvels and Steinebach (1987) and Section 1 in Bacro, Deheuvels and Steinebach (1987), where additional details are to be found.

Let $X = X_1$ satisfy conditions (A)–(C) of Section 1. Set $\phi(s) = E(e^{sX})$ and let $s_0 = \sup\{s: \phi(s) < \infty\}$. Clearly, $\phi(\cdot)$ is increasing and infinitely differentiable on $(-\infty, s_0)$. Introduce the following notation. Set $m(s) = \phi'(s)/\phi(s)$, and observe, using (B), that $m(\cdot)$ is increasing and such that $m(0) = \mu$. Define

$$\begin{gathered} 0 \le b = \lim_{s\downarrow -\infty} m(s) < \mu < a = \lim_{s\uparrow s_0} m(s) \le \infty, \\ A_0 = \begin{cases} 1/b & \text{if } b > 0, \\ \infty & \text{if } b = 0, \end{cases} \quad \text{and} \quad B_0 = \begin{cases} 1/a & \text{if } a < \infty, \\ 0 & \text{if } a = \infty. \end{cases} \end{gathered} \tag{3.1}$$

It can be verified that $0 \le B_0 < 1/\mu < A_0 \le \infty$, and that, for all $B_0 < \theta < A_0$, the equation in s: $\theta m(s) = 1$ has a unique solution $\hat{s} = \hat{s}(\theta)$. Moreover, $\hat{s}(\cdot)$ is decreasing on (B_0, A_0) and satisfies

$$\lim_{\theta \downarrow B_0} \hat{s}(\theta) = s_0, \qquad \hat{s}(1/\mu) = 0 \quad \text{and} \quad \lim_{\theta \uparrow A_0} \hat{s}(\theta) = -\infty. \tag{3.2}$$

For a fixed $\theta \in (B_0, A_0)$, the function $s - \theta \log \phi(s)$ of $s \in (-\infty, s_0)$ has first derivative $1 - \theta m(s)$ and strictly negative second derivative. Thus, it has a unique maximum on $(-\infty, s_0)$, reached for $s = \hat{s}(\theta)$, that is,

$$\Gamma(\theta) = \sup_s \{s - \theta \log \phi(s)\} = \hat{s}(\theta) - \theta \log \phi(\hat{s}(\theta)). \tag{3.3}$$

The function $\Gamma(\cdot)$ is differentiable on (B_0, A_0) and satisfies

$$\Gamma'(\theta) = -\log \phi(\hat{s}(\theta)) \quad \text{and} \quad \Gamma(\theta) - \theta\Gamma'(\theta) = \hat{s}(\theta). \tag{3.4}$$

$\Gamma(\cdot)$ decreases on $(B_0, 1/\mu]$ and increases on $[1/\mu, A_0)$. Moreover, $\Gamma(1/\mu) = 0$, and $\Gamma(\theta) > 0$ for all $B_0 < \theta (\neq 1/\mu) < A_0$. In addition [see, e.g., Bacro, Deheuvels and Steinebach (1987)], if c_0 and c_1 are defined by

$$c_1 = \lim_{\theta \uparrow A_0} (1/\Gamma(\theta)) \quad \text{and} \quad c_0 = \lim_{\theta \downarrow B_0} (1/\Gamma(\theta)), \tag{3.5}$$

then

$$c_0 = \begin{cases} -a/\log P(X = a) & \text{if } a = \operatorname{ess\,sup} X < \infty \text{ and } P(X = a) > 0, \\ a/\{as_0 - \log \phi(s_0)\} & \text{if } a < \operatorname{ess\,sup} X = \infty \text{ and } 0 < s_0 < \infty, \\ 1/s_0 & \text{otherwise (with the notation } 1/\infty = 0), \end{cases} \tag{3.6}$$

$$c_1 = \begin{cases} -b/\log P(X = b) & \text{if } b = \operatorname{ess\,inf} X > 0 \text{ and } P(X = b) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have the following lemma.

LEMMA 1. *For any $c_1 < C < \infty$, there exist a unique $A = A(C) \in (1/\mu, A_0)$ and a unique $s^{**} < 0$, solutions of the equations*

$$1/C = \sup_{s<0} \{s - A \log \phi(s)\} = s^{**} - A \log \theta(s^{**}). \tag{3.7}$$

Likewise, for any $c_0 < C < \infty$, there exist a unique $B = B(C) \in (B_0, 1/\mu)$ and a unique $s^* > 0$, solutions of the equations

$$1/C = \sup_{s>0} \{s - B \log \phi(s)\} = s^* - B \log \phi(s^*). \tag{3.8}$$

Observe that $\Gamma(A(C)) = \Gamma(B(C)) = 1/C$, and that $s^* = \hat{s}(B(C))$ and $s^{**} = \hat{s}(A(C))$. It follows from Lemma 1, that for all $\tau > 0$ sufficiently small [i.e., such that $1/\tau > \max(c_0, c_1)$], the equation

$$
(3.9)\qquad \begin{aligned}\tau &= \sup_s\left\{s - \left(\gamma + \frac{1}{\mu}\right)\log\phi(s)\right\}\\ &= \Gamma\left(\gamma + \frac{1}{\mu}\right) = \tilde{s}(\gamma) - \left(\gamma + \frac{1}{\mu}\right)\log\phi(\tilde{s}(\gamma)),\end{aligned}
$$

where we have used the notation $\tilde{s}(\gamma) = \hat{s}(\gamma + 1/\mu)$, has two solutions in γ which will be denoted by $\gamma_\tau^- < 0 < \gamma_\tau^+$.

Let $M_r = E((X-\mu)^r)$ (resp. κ_r) stand for the rth centered moment (resp. cumulant) of X, with, in particular, $M_1 = 0$, $M_2 = \sigma^2$ and $\kappa_1 = \mu$. Our assumptions imply the existence of M_r and κ_r for all $r = 1, 2, \dots,$ together with the expansions when $s \to 0$,

$$
(3.10)\qquad \begin{aligned}\log\phi(s) &= \kappa_1 s + \frac{1}{2!}\kappa_2 s^2 + \frac{1}{3!}\kappa_3 s^3 + \cdots\\ &= \mu s + \frac{1}{2}\sigma^2 s^2 + \frac{1}{6}M_3 s^3 + \frac{1}{24}\left(M_4 - 3\sigma^4\right)s^4 + \cdots.\end{aligned}
$$

Noting that $\tilde{s}(\gamma)$ is the solution of $(d/ds)\log\phi(s) = (\gamma + \mu^{-1})^{-1}$, straightforward computations show that

$$
(3.11)\qquad \tilde{s}(\gamma) = -\mu^2\sigma^{-2}\gamma - \tfrac{1}{2}\mu^3\sigma^{-6}\left(M_3\mu - 2\sigma^4\right)\gamma^2 + O(\gamma^3) \quad \text{as } \gamma \to 0.
$$

Likewise, we see that $\tilde{s}(\gamma_\tau^\pm)$ is the solution of $s - ((d/ds)\log\phi(s))^{-1}\log\phi(s) = \tau$. It follows that

$$
(3.12)\qquad \begin{aligned}\tilde{s}(\gamma_\tau^\pm) &= \mp\left\{2\tau\mu/\sigma^2\right\}^{1/2}\\ &\quad - \left\{2\tau\mu/\sigma^2\right\}\left(\left(2M_3\mu - 3\sigma^4\right)/\left(6\mu\sigma^2\right)\right) + O(\tau^{3/2}) \quad \text{as } \tau \to 0,\end{aligned}
$$

which, jointly with (3.11), implies in turn that

$$
(3.13)\qquad \begin{aligned}\gamma_\tau^\pm &= \pm\left\{2\tau\sigma^2/\mu^3\right\}^{1/2}\\ &\quad - \left\{2\tau\sigma^2/\mu^3\right\}\left(\mu\left(M_3\mu - 3\sigma^4\right)/\left(6\sigma^4\right)\right) + O(\tau^{3/2}) \quad \text{as } \tau \to 0.\end{aligned}
$$

Consider now a function $\{K_T\}_{T \ge 0}$ satisfying $0 \le K_T \le T$, jointly with assumptions (a) and (b) of Section 1, which we repeat for convenience:

(a) $K_T/\log T \to \infty$ as $T \to \infty$;
(b) $K_T/\log^p T \to 0$ as $T \to \infty$ for some $p > 1$.

Let $\Lambda > 0$ be an arbitrary but fixed constant. By Lemma 1, there exists a $T_\Lambda < \infty$ such that $T \ge T_\Lambda$ implies the existence for any $|\lambda| \le \Lambda$ of a unique positive solution $a_T^+(\lambda)$ [resp. negative solution $a_T^-(\lambda)$] of the equation in γ,

$$
(3.14)\qquad K_T^{-1}\log\left(TK_T^{-1}\log^\lambda T\right) = \sup_s\left\{s - \left(\gamma + \frac{1}{\mu}\right)\log\phi(s)\right\} = \Gamma\left(\gamma + \frac{1}{\mu}\right).
$$

Let $a_T^\pm = a_T^\pm(0)$. We will make use of the following lemma.

LEMMA 2. *Let $\Lambda > 0$ be fixed, and let $a_T^{\pm}(\lambda)$, $a_T^{\pm}(\nu)$ and $a_T^{\pm} = a_T^{\pm}(0)$ be as in* (3.14) *for $|\lambda| \le \Lambda$, $|\nu| \le \Lambda$ and $T \ge T_\Lambda$. Then, as $T \to \infty$, we have uniformly over $|\lambda| \le \Lambda$ and $|\nu| \le \Lambda$ the asymptotic expansions*

$$
\begin{aligned}
K_T a_T^{\pm} = {} & \pm \sigma\mu^{-3/2}\{2K_T \log(T/K_T)\}^{1/2} \\
& -\tfrac{1}{3}(1+o(1))\sigma^{-2}\mu^{-2}(M_3\mu - 3\sigma^4)\log(T/K_T)
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
a_T^{\pm}(\lambda) - a_T^{\pm}(\nu) &= (1+o(1))(\lambda-\nu)\frac{\log\log T}{2\log T} a_T^{\pm} \\
&= \pm(1+o(1))(\lambda-\nu)\frac{\sigma\mu^{-3/2}\log\log T}{(2K_T\log T)^{1/2}}.
\end{aligned}
\tag{3.16}
$$

PROOF. (3.15) follows from (3.13) and the observation that $a_T^{\pm} = \gamma_\tau^{\pm}$ for the special choice of τ given by $\tau = K_T^{-1}\log(T/K_T)$. This, in turn, implies that $a_T^{\pm} \sim \pm\sigma\mu^{-3/2}K_T^{-1/2}(2\log T)^{1/2}$ and proves the second equality in (3.16). A similar argument shows that, uniformly over $|\theta| \le \Lambda$, $a_T^{\pm}(\theta) \sim a_T^{\pm}$ as $T \to \infty$.

Next, by (3.14), $\Gamma(a_T^{\pm}(\theta) + 1/\mu) = K_T^{-1}\log(TK_T^{-1}\log^{\theta} T)$. By (3.4), $\Gamma'(\gamma + 1/\mu) = -\log\phi(\tilde{s}(\gamma))$, so that the mean value theorem implies that, for some $0 < \rho < 1$,

$$
\Gamma\left(a_T^{\pm}(\nu) + \frac{1}{\mu} + h\right) = \Gamma\left(a_T^{\pm}(\nu) + \frac{1}{\mu}\right) - h\log\phi\left(\tilde{s}\left(a_T^{\pm}(\nu) + \rho h\right)\right).
\tag{3.17}
$$

The fact that $a_T^{\pm}(\nu) \sim a_T^{\pm}(\lambda) \sim a_T^{\pm}$ implies that, in (3.17), the choice of h given by $h = a_T^{\pm}(\lambda) - a_T^{\pm}(\nu)$ satisfies $h = o(a_T^{\pm})$ as $T \to \infty$. Since by (a) and (b) and the second equality in (3.16) $a_T^{\pm} \to 0$, by (3.11), $\tilde{s}(a_T^{\pm}(\nu) + \rho h) = \tilde{s}((1+o(1))a_T^{\pm}) \sim -\mu^2\sigma^{-2}a_T^{\pm}$ as $T \to \infty$. This, jointly with (3.10), implies in turn that, for the above choice of h,

$$
\log\phi\left(\tilde{s}\left(a_T^{\pm}(\nu) + \rho h\right)\right) = -(1+o(1))\mu^3\sigma^{-2}a_T^{\pm} \quad \text{as } T \to \infty.
\tag{3.18}
$$

It follows from (3.14), (3.17) and (3.18) that

$$
\begin{aligned}
\Gamma\left(a_T^{\pm}(\lambda) + \frac{1}{\mu}\right) - \Gamma\left(a_T^{\pm}(\nu) + \frac{1}{\mu}\right) &= (\lambda-\nu)K_T^{-1}\log\log T \\
&\sim \left(a_T^{\pm}(\lambda) - a_T^{\pm}(\nu)\right)\mu^3\sigma^{-2}a_T^{\pm} \quad \text{as } T \to \infty,
\end{aligned}
$$

which, in view of (3.15), completes the proof of (3.16). □

In the sequel, $R > 0$ will denote a fixed constant, the value of which will be precised later on. We will define two integer sequences $\{b_n^+(\lambda)\}$ and $\{b_n^-(\lambda)\}$ by

$$
b_n^{\pm}(\lambda) = \left[K_T\left(a_T^{\pm}(\lambda) + \frac{1}{\mu}\right)\right] \quad \text{for } T = Rn,\ n \ge 1 + T_\Lambda/R, \text{ and } |\lambda| \le \Lambda,
\tag{3.19}
$$

where here and in the sequel, $[u] \le u < [u] + 1$ denotes the integer part of u.

Note that (a), (b) and Lemma 2 jointly imply that $b_n = b_n^{\pm}(\lambda) \sim K_{Rn}/\mu$ as $n \to \infty$, and that we have

$$\text{(i)} \qquad b_n/\log n \to \infty \quad \text{as } n \to \infty$$

and

$$\text{(ii)} \qquad b_n/\log^p n \to 0 \quad \text{as } n \to \infty \text{ for some } p > 1.$$

LEMMA 3. *Let $|\lambda| \le \Lambda$ be fixed and assume that $\{K_T\}_{T \ge 0}$ satisfies, in addition to (a) and (b),*

(c) *$K_T/\log T$ is nondecreasing in the upper tail;*
(d) *K_T has a continuous first derivative K_T' such that*

$$K_T'/K_T = O\big((T \log T)^{-1}\big) \quad \text{as } T \to \infty. \tag{3.20}$$

Let $b_n = b_n^{\pm}(\lambda)$ be as in (3.19). *Then $\{b_n\}$ is nondecreasing in the upper tail and there exists a real-valued sequence $\{\tilde{b}_n\}$ such that*

$$\text{(iii)} \qquad b_n - \tilde{b}_n = O(b_n/\log n) \quad \text{as } n \to \infty;$$

$$\text{(iv)} \qquad \tilde{b}_{n+1} - \tilde{b}_n = O\big(\tilde{b}_n/(n \log n)\big) \quad \text{as } n \to \infty.$$

PROOF. Observe that (iii) and (iv) coincide with conditions (i) and (ii) in Theorem 2, page 382, in Deheuvels and Steinebach (1987) (see, e.g., Theorem A in the sequel).

We will show that (iii) and (iv) are satisfied with the choice of $\tilde{b}_n$ given by $\tilde{b}_n = k(Rn)$, where $k(T) = K_T(a_T^{\pm}(\lambda) + 1/\mu)$. Since $|b_n - \tilde{b}_n| \le 1$ by (3.19), (iii) is a straightforward consequence of (i), and we need only prove that, as $T \to \infty$,

$$0 < k'(T)/k(T) = O\big((T \log T)^{-1}\big). \tag{3.21}$$

Here, we have used the fact that our choice of $\tilde{b}_n$ ensures that

$$\big(\tilde{b}_{n+1} - \tilde{b}_n\big)/\tilde{b}_n \sim \log\big(\tilde{b}_{n+1}/\tilde{b}_n\big) = \int_{Rn}^{Rn+R} (k'(t)/k(t))\, dt.$$

For the proof of (3.21), we see that joint use of (3.14) and (b) yields

$$\begin{aligned}\frac{d}{dT}\big(K_T^{-1} \log\big(TK_T^{-1} \log^{\lambda} T\big)\big) &= \frac{d}{dT}\Gamma\left(a_T^{\pm}(\lambda) + \frac{1}{\mu}\right)\\ &= (1 + o(1))(TK_T)^{-1} - (1 + o(1))(\log T)K_T'/K_T^2,\end{aligned}$$

which by (3.20) is $O((TK_T)^{-1})$. Moreover, by (3.4), (3.10), (3.11) and (3.15),

$$\begin{aligned}\frac{d}{dT}\Gamma\left(a_T^{\pm}(\lambda) + \frac{1}{\mu}\right) &= \big\{-\log \phi\big(\tilde{s}\big(a_T^{\pm}(\lambda)\big)\big)\big\} \frac{d}{dT} a_T^{\pm}(\lambda)\\ &\sim \pm\sigma^{-1}\mu^{3/2}\big(2K_T^{-1} \log T\big)^{1/2} \frac{d}{dT} a_T^{\pm}(\lambda).\end{aligned}$$

Hence

$$\frac{d}{dT}a_T^{\pm}(\lambda) = O\big(T^{-1}(K_T \log T)^{-1/2}\big),$$

and by (a)–(d),

$$\begin{aligned} k'(T)/k(T) &= K_T'/K_T + (1 + o(1))\mu \frac{d}{dT}a_T^{\pm}(\lambda) \\ &= O\big((T \log T)^{-1}\big) \quad \text{as } T \to \infty, \end{aligned}$$

which proves the right-hand side of (3.21). In addition, (c) implies that, in the upper tail, $K_T'/K_T \geq (T \log T)^{-1}$. It follows that $k'(T)/k(T) \sim K_T'/K_T > 0$ as desired. This proves the second half of (3.21), jointly with the fact that $k(T)\uparrow$, which in turn implies that $b_n = [k(Rn)]$ is nondecreasing in the upper tail. The proof of Lemma 3 is now complete. □

Consider now an arbitrary integer sequence $\{b_n\}$, which will be assumed in the sequel to satisfy (i) and (ii), that is,

(i) $$b_n/\log n \to \infty \quad \text{as } n \to \infty;$$

(ii) $$b_n/\log^p n \to 0 \quad \text{as } n \to \infty \text{ for some } p > 1.$$

By the same arguments as used in (3.14) [see Deheuvels and Steinebach (1987), (2.4), page 372], it can be shown that, if $\Lambda > 0$ is any fixed constant, there exists an n_Λ such that $n \geq n_\Lambda$ implies the existence for any $|\lambda| \leq \Lambda$ of a unique positive solution $\delta_n^+(\lambda)$ [resp. negative solution $\delta_n^-(\lambda)$] of the equation in δ,

$$\begin{aligned} b_n^{-1} \log\big(n b_n^{-1} \log^\lambda n\big) &= \sup_s \left\{ s\left(\delta + \frac{1}{\mu}\right)^{-1} - \log \phi(s) \right\} \\ &= \left(\delta + \frac{1}{\mu}\right)^{-1} \Gamma\left(\delta + \frac{1}{\mu}\right). \end{aligned} \tag{3.22}$$

Assume, from now on, that $b_n = b_n^{\pm}(\lambda)$, and define accordingly $\alpha_n^{\pm}(\lambda)$ by $\alpha_n^{\pm}(\lambda) = \delta_n^{\pm}(\lambda)$. Because of the close relationship between (3.14) and (3.22), it may be expected that $\alpha_n^{\pm}(\lambda) - a_{Rn}^{\pm}(\lambda)$ should be small. This is formulated precisely in the following lemma.

LEMMA 4. *Let $|\lambda| \leq \Lambda$ be fixed, and let $a_T^{\pm}(\lambda)$ and $\alpha_T^{\pm}(\lambda)$ be defined via* (3.3), (3.14), (3.19) *and* (3.22). *Then*

$$\alpha_n^{\pm}(\lambda) - a_T^{\pm}(\lambda) = -(1 + o(1))(\log(R/\mu))(2 \log T)^{-1} a_T^{\pm} \quad \text{as } T = Rn \to \infty. \tag{3.23}$$

PROOF. Throughout the proof, we use the notation $T = Rn$. By (3.19), (a) and (b), we have, as $T \to \infty$,

$$b_n^{-1} \log\left(nb_n^{-1} \log^\lambda n\right) = \left\{K_T^{-1} \log\left(TK_T^{-1} \log^\lambda T\right)\right\} \times \left(a_T^\pm(\lambda) + \frac{1}{\mu}\right)^{-1} \left(1 - (1 + o(1)) \frac{\log(R/\mu)}{\log T}\right). \tag{3.24}$$

Now define ν by the equation

$$b_n^{-1} \log\left(nb_n^{-1} \log^\lambda n\right) = \left\{K_T^{-1} \log\left(TK_T^{-1} \log^\nu T\right)\right\} \left(a_T^\pm(\nu) + \frac{1}{\mu}\right)^{-1}. \tag{3.25}$$

Recall by (3.14) that

$$K_T^{-1} \log\left(TK_T^{-1} \log^\nu T\right) = \Gamma\left(a_T^\pm(\nu) + \frac{1}{\mu}\right).$$

Thus, we see from (3.22) and (3.25) that $a_T^\pm(\nu) = \alpha_n^\pm(\lambda)$. By (3.16), (a) and (b),

$$\left(a_T^\pm(\nu) + \frac{1}{\mu}\right)^{-1} = \left(a_T^\pm(\lambda) + \frac{1}{\mu}\right)^{-1} \left(1 + o\left((\nu - \lambda) \frac{\log\log T}{\log T}\right)\right),$$

so that a simple expansion in (3.25) yields

$$\begin{aligned} b_n^{-1} \log\left(nb_n^{-1} \log n\right) &= \Big\{K_T^{-1} \log\left(TK_T^{-1} \log^\lambda T\right) \\ &\qquad + (\nu - \lambda) K_T^{-1} \log\log T\Big\} \left(a_T^\pm(\nu) + \frac{1}{\mu}\right)^{-1} \\ &= \left\{K_T^{-1} \log\left(TK_T^{-1} \log^\lambda T\right)\right\} \left(a_T^\pm(\lambda) + \frac{1}{\mu}\right)^{-1} \\ &\qquad \times \left(1 + (1 + o(1))(\nu - \lambda) \frac{\log\log T}{\log T}\right). \end{aligned} \tag{3.26}$$

A comparison of (3.24) and (3.26) shows that $(\nu - \lambda) \sim (-\log(R/\mu))/\log\log T$, so that, using again (3.16), we have, as $T \to \infty$,

$$\begin{aligned} \alpha_n^\pm(\lambda) - a_T^\pm(\lambda) &= a_T^\pm(\nu) - a_T^\pm(\lambda) \\ &\sim (\nu - \lambda) \frac{\log\log T}{2 \log T} a_T^\pm \sim \frac{-\log(R/\mu)}{2 \log T} a_T^\pm, \quad \text{as sought.} \qquad \square \end{aligned}$$

We now have in our hands all the technical tools necessary for the proofs of our main theorems which may now be presented.

4. Medium-size increments. In the sequel, we shall assume at times that assumptions (a)–(d) of Sections 1–3 hold. We repeat these assumptions for convenience.

(a) $K_T/\log T \to \infty$ as $T \to \infty$;

(b) $K_T/\log^p T \to 0$ as $T \to \infty$ for some $p > 1$;

(c) $K_T/\log T$ is ultimately nondecreasing;
(d) $K_T'/K_T = O((T\log T)^{-1})$ as $T \to \infty$.

We shall denote by a_T^+ (resp. a_T^-) the positive (resp. negative) solution of the equation in γ,

$$
\begin{aligned}
K_T^{-1}\log\left(TK_T^{-1}\right) &= \sup_s\left\{s - \left(\gamma + \frac{1}{\mu}\right)\log\phi(s)\right\} \\
&= \tilde{s}(\gamma) - \left(\gamma + \frac{1}{\mu}\right)\log\phi(\tilde{s}(\gamma)).
\end{aligned} \tag{4.1}
$$

Note (see Section 3) that $a_T^\pm$ exists for all T sufficiently large.
Our main results are listed in Theorems 4 and 5 below.

THEOREM 4. *Assume that* $\{K_T\}_{T\geq 0}$ *satisfies* $0 \leq K_T \leq T$, $K_T/\log T \to \infty$, *and, for some* $p > 1$, $K_T/\log^p T \to 0$ *as* $T \to \infty$. *Let* $a_T^\pm$ *be as in* (4.1) *and* $\Delta_T^\pm$ *be as in Section* 1. *Then*

$$
\begin{aligned}
&\lim_{T\to\infty}\frac{\pm(2\log(T/K_T))^{1/2}}{K_T^{1/2}\sigma\mu^{-3/2}\log\log(T/K_T)}\left\{\Delta_T^\pm - \mu^{-1}K_T - K_Ta_T^\pm\right\} \\
&\quad = \frac{1}{2} \quad \textit{in probability}.
\end{aligned} \tag{4.2}
$$

THEOREM 5. *Assume that* $\{K_T\}_{T\geq 0}$ *satisfies* $0 \leq K_T \leq T$, $K_T/\log T \uparrow \infty$, *and, for some* $p > 1$, $K_T/\log^p T \to 0$ *as* $T \to \infty$. *Assume further that* K_T *has a continuous first derivative* K_T' *such that*

$$
K_T'/K_T = O\left((T\log T)^{-1}\right) \quad \textit{as } T \to \infty. \tag{4.3}
$$

Let $a_T^\pm$ *be as in* (4.1) *and* $\Delta_T^\pm$ *be as in Section* 1. *Then*

$$
\limsup_{T\to\infty}\frac{\pm(2\log(T/K_T))^{1/2}}{K_T^{1/2}\sigma\mu^{-3/2}\log\log(T/K_T)}\left\{\Delta_T^\pm - \mu^{-1}K_T - K_Ta_T^\pm\right\} = \frac{3}{2} \quad a.s. \tag{4.4}
$$

and

$$
\liminf_{T\to\infty}\frac{\pm(2\log(T/K_T))^{1/2}}{K_T^{1/2}\sigma\mu^{-3/2}\log\log(T/K_T)}\left\{\Delta_T^\pm - \mu^{-1}K_T - K_Ta_T^\pm\right\} = \frac{1}{2} \quad a.s. \tag{4.5}
$$

REMARK 3. Let $\mathscr{A}_T^\pm = a_T^\pm\mathscr{K}_T$, where $\mathscr{K}_T = [K_{[T]}]$, and set $d_T^\pm = \mu^{-1}\mathscr{K}_T^{1/2} + \mathscr{A}_T^\pm$. Denote by $\hat{\rho}(\alpha) = \inf_s\{\hat{\phi}(s)e^{-\alpha s}\}$, with $\hat{\phi}(s) = E(\exp(-(X_1 - \mu)s/\sigma)) = \phi(-s/\sigma)e^{s\mu/\sigma}$. Routine computations show that $\mathscr{A}_T^+$ (resp. $\mathscr{A}_T^-$) is the unique positive (resp. negative) solution of the system of equations in $\mathscr{A}$,

$$
\hat{\rho}^{d\mathscr{K}^{1/2}}\left(\frac{\mu\mathscr{A}}{\sigma d}\right) = T^{-1}\mathscr{K}, \qquad d = \mu^{-1}\mathscr{K}^{1/2} + \mathscr{A}, \qquad \mathscr{K} = \mathscr{K}_T. \tag{4.6}
$$

Observe that (4.6) coincides with (viii) in Theorem D of Steinebach (1986), page 549, so that with our notation, the above-mentioned Theorem D can be shown to be equivalent to

$$\Delta_T^+ - \mu^{-1}K_T - K_T a_T^+ = o\big(K_T^{1/2}\big) \quad \text{a.s. as } T \to \infty. \tag{4.7}$$

It is clear that (4.4) and (4.5) improve upon (4.7) by replacing the $o(K_T^{1/2})$ term by $O(K_T^{1/2}(\log T)^{-1} \log\log T)$. Moreover, our results cover also the case of Δ_T^-.

It is noteworthy that some regularity assumptions on K_T are necessary in the proofs. In the above-mentioned Theorem D of Steinebach (1986), it is assumed that $K_T/T \downarrow$, $K_T/\log T \uparrow \infty$ and $K_T/\log^p T \to 0$ as $T \to \infty$. It is obvious that these conditions are implied by those of Theorem 5 [in particular, (4.3) $\Rightarrow$ $K_T/T \downarrow$], which are therefore slightly more restrictive.

Observe also that, in the intersections of the ranges covered by Theorems 2 and 5, the conditions imposed on K_T in each of these theorems are equivalent [with the only exception of $K_T(\log\log T)^2/\log^3 T \to \infty$].

The question whether these conditions [and in particular (2.2) and (4.3)] may be relaxed is, to our best knowledge, open, even in the case of Wiener processes. It is possible though [see, e.g., Deheuvels and Steinebach (1987), Remark 7] to prove that the result holds under slightly weaker but more cumbersome assumptions, and we limit ourselves to the relatively simple case discussed above, for the sake of conciseness.

REMARK 4. Theorems 4 and 5 are in agreement with Theorems 1 and 2 in the range where $K_T(\log\log T)^2/\log^3 T \to \infty$ and $K_T/\log^p T \to 0$ as $T \to \infty$, for some fixed $p > 3$. In general, (3.15) implies that, for $M_3\mu - 3\sigma^4 \neq 0$ [note here that $\log(T/K_T) \sim \log T$],

$$\begin{aligned} &\frac{(2\log T)^{1/2}}{K_T^{1/2}\sigma\mu^{-3/2}\log\log T}\left\{K_T a_T^{\pm} \mp \sigma\mu^{-3/2}\big(2K_T\log(T/K_T)\big)^{1/2}\right\} \\ &\quad = -(1+o(1))2^{1/2}3^{-1}\sigma^{-3}\mu^{-1/2}\big(M_3\mu - 3\sigma^4\big)\frac{\log^{3/2} T}{K_T^{1/2}\log\log T} \\ &\qquad\qquad \text{as } T \to \infty. \end{aligned} \tag{4.8}$$

Thus, we see that, whenever $M_3\mu - 3\sigma^4 \neq 0$, the results of Theorems 1 and 2 are not valid if

$$K_T(\log\log T)^2/\log^3 T \nrightarrow \infty \quad \text{as } T \to \infty. \tag{4.9}$$

On the other hand, if $M_3\mu - 3\sigma^4 = 0$, this may still be the case. An interesting example is given by the *inverse Gaussian* distribution. In standard form corresponding to expectation μ and standard deviation $\sigma = \mu^{3/2}$, this distribution has density [see, e.g., Johnson and Kotz (1970), Chapter 15]

$$f(x) = x^{-3/2}(2\pi)^{-1/2}\exp\big(-(x-\mu)^2/(2\mu^2 x)\big) \quad \text{for } x > 0, \tag{4.10}$$

and cumulant generating function

$$\log \phi(s) = \mu^{-1}\left(1 - (1 - 2\mu^2 s)^{1/2}\right) \quad \text{for } s < s_0 = 1/(2\mu^2). \tag{4.11}$$

Straightforward computations show that in this case $\Gamma(\gamma + 1/\mu) = \frac{1}{2}\gamma^2$, so that we have *exactly* (recall that here $\sigma\mu^{-3/2} = 1$)

$$a_T^{\mp} = \pm\left(2K_T^{-1}\log(T/K_T)\right)^{1/2}. \tag{4.12}$$

In view of (4.12) and Theorems 4 and 5, we see that, whenever $X_1, X_2, \dots$ have a common inverse Gaussian distribution, the results of Theorems 1 and 2 can be extended to the case where $K_T/\log T \to \infty$ a $T \to \infty$. It is obvious that $\Gamma(\gamma + 1/\mu) = \frac{1}{2}\gamma^2$ characterizes the inverse Gaussian distribution given in (4.10). In general, the same arguments as used in Section 2 show that if a distribution has the same cumulants as the inverse Gaussian up to order r, then the leading term in (4.8) will be (for $r \geq 2$) $K_T^{-(r-1)/2}(\log T)^{(r-1)/2}(\log\log T)^{-1}$. In this case, the results of Theorems 1 and 2 will fail to be true in the range $K_T/\log T \to \infty$, $K_T(\log T)^{-(r+1)/(r-1)}(\log\log T)^{2/(r-1)} \nrightarrow \infty$.

An application of this remark for all $r = 2, 3, \dots$ proves the following characterization.

COROLLARY 1. *Among all distributions of nonnegative random variables having a moment generating function finite in a neighborhood of* 0, *the inverse Gaussian distributions are characterized by the fact that* (2.1) *holds for all functions* $\{K_T\}_{T\geq 0}$ *such that* $0 \leq K_T \leq T$, $K_T/T \to 0$ *and* $K_T/\log T \to \infty$ *as* $T \to \infty$.

Similar characterizations may be obtained via Theorem 2. We omit the details. Note here that the inverse Gaussian plays the same role for renewal processes as does the Gaussian distribution for partial sums. Further results concerning this distribution will be discussed in Section 5.

In the remainder of this section, we present the proofs of Theorems 4 and 5. First, we introduce some notation. Consider $\Delta_T^{\mp} = \Delta^{\mp}(T, K_T)$ and $U_n^{\mp} = U^{\pm}(n, b_n)$ as defined in the Introduction, and define the events

$$E_T(\lambda) = \left\{\Delta_T^- > K_T\left(a_T^- + \frac{1}{\mu}\right) - \left(\frac{1}{2} + \lambda\right)\sigma\mu^{-3/2}\left(2K_T^{-1}\log T\right)^{-1/2}\log\log T\right\}$$

and

$$E_T'(\lambda) = \left\{\Delta_T^- > K_T\left(a_T^-\left(\frac{1}{2} + \lambda\right) + \frac{1}{\mu}\right)\right\},$$

where $a_T^{\mp}(\theta)$ is defined as in (3.14).

The motivation for $E_T(\lambda)$ is that (4.2) holds (for Δ_T^-) iff, for any fixed $\varepsilon > 0$,

$$\lim_{T\to\infty} P(E_T(\varepsilon)) = 1 \quad \text{and} \quad \lim_{T\to\infty} P(E_T(-\varepsilon)) = 0. \tag{4.13}$$

Likewise, (4.5) holds (for Δ_T^-) iff, for any $\varepsilon > 0$,

$$P(E_T(\varepsilon)\text{ i.o.}) = 1 \quad\text{and}\quad P(E_T(-\varepsilon)\text{ i.o.}) = 0, \tag{4.14}$$

where $\{E_T(\varepsilon)\text{ i.o.}\}$ denotes the event that, for any $T_1 > 0$, there exists a $T_2 > T_1$ such that $E_{T_2}(\varepsilon)$ holds.

Note that similar definitions and arguments can be used for Δ_T^+ and (4.4). In the sequel, we will limit ourselves to the proof that Δ_T^- satisfies (4.2) [resp. (4.5)] under the hypotheses of Theorem 4 (resp. Theorem 5). The proof of the other statements in our theorems is identical by straightforward formal replacements and will be therefore omitted. In the sequel, we shall consider only Δ_T^-.

Because of the involved form of $E_T(\varepsilon)$, it is convenient to replace it by $E_T'(\varepsilon)$. The justification for this replacement is given in the following lemma.

LEMMA 5. (4.2) *holds if and only if, for any* $\varepsilon > 0$,

$$\lim_{T\to\infty} P\big(E_T'(\varepsilon)\big) = 1 \quad\textit{and}\quad \lim_{T\to\infty} P\big(E_T'(-\varepsilon)\big) = 0. \tag{4.15}$$

Likewise, (4.5) *holds if and only if, for any* $\varepsilon > 0$,

$$P\big(E_T'(\varepsilon)\ i.o.\big) = 1 \quad\textit{and}\quad P\big(E_T'(-\varepsilon)\ i.o.\big) = 0. \tag{4.16}$$

PROOF. Making use of (3.16) with $\nu = 0$ and $\lambda = \frac{1}{2} \pm 2\varepsilon$, or $\lambda = \frac{1}{2} \pm \varepsilon/2$, we see that, for all T large enough and $\varepsilon > 0$ fixed, we have

$$E_T(\varepsilon/2) \subset E_T'(\varepsilon) \subset E_T(2\varepsilon) \quad\text{and}\quad E_T(-2\varepsilon) \subset E_T'(-\varepsilon) \subset E_T(-\varepsilon/2). \tag{4.17}$$

The proof of Lemma 5 follows from (4.17), jointly with (4.13) and (4.14). □

Note that Δ_T^- is integer, and hence that $E_T'(\lambda)$ coincides with the event

$$E_T'(\lambda) = \left\{\Delta_T^- > \left[K_T\left(a_T^-\left(\frac{1}{2} + \lambda\right) + \frac{1}{\mu}\right)\right]\right\}. \tag{4.18}$$

In general, if $0 \le T_1 \le T \le T_2$, we always have

$$E''(T_2, K_T, \lambda) \subset E_T'(\lambda) \subset E''(T_1, K_T, \lambda), \tag{4.19}$$

where

$$E''(u, K_T, \lambda) = \left\{\Delta^-(u, K_T) > \left[K_T\left(a_T^-\left(\frac{1}{2} + \lambda\right) + \frac{1}{\mu}\right)\right]\right\}. \tag{4.20}$$

Recall that $\Delta^-(u, h) = \inf_{0 \le t \le u-h}\{N(t + h) - N(t)\}$ for $0 \le h \le u$. In order to avoid restrictions on the validity of (4.19) and (4.20), we define $\Delta^-(u, h)$ for $0 \le u \le h$ by $\Delta^-(u, h) = N(h) - N(0)$.

We will now concentrate on $\kappa(T, \lambda) = K_T(a_T^-(\frac{1}{2} + \lambda) + 1/\mu)$.

LEMMA 6. *Under the assumptions of Theorem* 5, *if* $0 < R, R', R'' < \infty$ *and* $\lambda, \lambda', \lambda''$ *are fixed, then, as* $T \to \infty$,

$$\begin{aligned} &\kappa(R''T, \lambda) - \kappa(R'T, \lambda) = O(K_T/\log T), \\ &\kappa(RT, \lambda'') - \kappa(RT, \lambda') = (1 + o(1))\sigma\mu^{-3/2}(\lambda' - \lambda'') \\ &\qquad\qquad \times\left(2K_T^{-1}\log T\right)^{-1/2}\log\log T. \end{aligned} \tag{4.21}$$

PROOF. Set $k(T) = \kappa(T, \lambda)$. By (3.21), $k(\cdot)$ is increasing in the upper tail and such that $k'(T)/k(T) = O((T \log T)^{-1})$ as $T \to \infty$. It follows that $(k(R''T) - k(R'T))/k(T) = O(1/\log T)$ as $T \to \infty$. Since (3.15) implies that $k(T) \sim \mu^{-1}K_T$ as $T \to \infty$, we have evidently $k(R''T) - k(R'T) = O(K_T/\log T)$ as $T \to \infty$, which proves the first statement in (4.21). The second statement follows from (3.16), used jointly with (3.15) and the observation that $K_{RT} \sim K_T$ as $T \to \infty$. □

We are now ready for the application of our duality argument. Define $b_n = b_n(\lambda, R)$ by

$$(4.22)\quad b_n = \left[K_T\left(a_T^-\left(\frac{1}{2} + \lambda\right) + \frac{1}{\mu}\right)\right] + 1 = [\kappa(T, \lambda)] + 1 \quad \text{for } T = Rn.$$

LEMMA 7. *Let $b_n = b_n(\lambda, R)$ be as in* (4.22). *Then* (4.5) *holds if*:

(1°) *For some $R = R' < \mu$, we have, for any $\lambda = \varepsilon > 0$,*

$$(4.23)\qquad P(U^+(n, b_n) \le K_{Rn}\ i.o.) = 1.$$

(2°) *For some $R = R'' > \mu$, we have, for any $\lambda = -\varepsilon < 0$,*

$$(4.24)\qquad P(U^+(n, b_n) \le K_{Rn}\ i.o.) = 0.$$

PROOF. We will make use of the fact that, for $u \ge K_T$ and $T \ge 0$,

$$(4.25)\qquad \begin{aligned} \{U^+(N(u) + 1, m) \le K_T\} &\subset \{\Delta^-(u, K_T) \ge m\} \\ &\subset \{U^+(N(u), m) \le K_T\}. \end{aligned}$$

The proof of (4.25) is achieved in two steps. (1) Suppose that $\Delta^-(u, K_T) < m$. Then $N(v + K_T) - N(v) < m$ for some $0 \le v \le u - K_T$. Set $N(s-) := \lim_{\varepsilon \downarrow 0} N(s - \varepsilon)$. Since $N(s)$ is constant for $S_{N(v+K_T)} \le v + K_T \le s < S_{N(v+K_T)+1}$, we have $N(S_{N(v+K_T)+1} -) - N(S_{N(v+K_T)+1} - K_T -) < m$, which implies that $S_{N(v+K_T)+1} - S_{N(v+K_T)+1-m} > K_T$ and hence that $U^+(N(u) + 1, m) > K_T$. (2) Suppose now that $U^+(N(u), m) > K_T$. Then $S_{i+m} - S_i > K_T$ for some $0 \le i \le N(u) - m$. Choose t such that $S_i + K_T < t + K_T < S_{i+m}$. Then $N(t + K_T) - N(t) \le m - 1$ and hence $\Delta^-(u, K_T) < m$. This suffices for (4.25).

By (4.16), (4.5) is equivalent to the fact that $P(E_T'(\lambda)$ i.o.$) = 1$ (resp. 0) for all $\lambda = \varepsilon > 0$ (resp. < 0). Moreover, we have, by the law of large numbers,

$$(4.26)\qquad \lim_{u \to \infty} u^{-1}N(u) = 1/\mu \quad \text{a.s.}$$

In view of (4.19), for $T_1 \le T$, we have $E''(T_1, K_T, \lambda) \supset E_T'(\lambda)$. Taking in (4.20) and (4.25) $u = T_1$ and $m = [K_T(a_T^-(\frac{1}{2} + \lambda) + 1/\mu)] + 1$, it follows evidently that $\{U^+(N(T_1), m) \le K_T\} \supset E_T'(\lambda)$. This, jointly with (4.26) and (4.22) taken with $R = R'' > \mu$ shows that

$$\begin{aligned} &P(U^+(n, b_n) \le K_{Rn} \text{ i.o.}) = 0 \quad \text{for all } \lambda < 0 \\ &\Rightarrow P(E_T'(\lambda) \text{ i.o.}) = 0 \quad \text{for all } \lambda < 0. \end{aligned}$$

A similar argument can be used to show that (4.23) implies that $P(E_T'(\lambda)$ i.o.$) = 1$ for all $\lambda = \varepsilon > 0$. We omit the details. □

For the proof of Theorem 5, we will need the following theorem, due to Deheuvels and Steinebach (1987), which we reformulate, using a different notation.

THEOREM A. *Let $\{b_n, n \geq 1\}$ be a nondecreasing integer sequence such that $1 \leq b_n \leq n$, $b_n/\log n \to \infty$ and $b_n/\log^p n \to 0$ for some $p > 1$. Assume further that there exists a real-valued sequence $\{\tilde{b}_n, n \geq 1\}$ such that*

(iii) $$b_n - \tilde{b}_n = O(b_n/\log n) \quad \text{as } n \to \infty$$

and

(iv) $$\tilde{b}_{n+1} - \tilde{b}_n = O(\tilde{b}_n/(n \log n)) \quad \text{as } n \to \infty.$$

Let α_n^- be defined as the unique negative solution of the equation in δ,

$$b_n^{-1} \log(n b_n^{-1}) = \sup_s \left\{ s\left(\delta + \frac{1}{\mu}\right)^{-1} - \log \phi(s) \right\}. \tag{4.27}$$

Then, for $U_n^+ = U^+(n, b_n)$,

$$\limsup_{n\to\infty} \frac{(2\log n)^{1/2}}{\sigma b_n^{1/2} \log\log n} \left(U_n^- - b_n\left(\alpha_n^- + \frac{1}{\mu}\right)^{-1} \right) = \frac{3}{2} \quad a.s. \tag{4.28}$$

and

$$\liminf_{n\to\infty} \frac{(2\log n)^{1/2}}{\sigma b_n^{1/2} \log\log n} \left(U_n^- - b_n\left(\alpha_n^- + \frac{1}{\mu}\right)^{-1} \right) = \frac{1}{2} \quad a.s. \tag{4.29}$$

Note that the formal change of $X_1, X_2, \ldots$ into $-X_1, -X_2, \ldots$ in Theorem A enables one to obtain similar result for U_n^-. This, in turn, will provide the results needed for Δ_T^+. In the sequel, we shall make use only of (4.29), noting that similar arguments as those we use will yield the proof of (4.4) via (4.28).

LEMMA 8. *Let $\lambda = \varepsilon$ and $R > 0$ be fixed and set $b_n = [K_T(a_T^-(\frac{1}{2} + \lambda) + 1/\mu)] + 1$ for $T = Rn$. Then*

$$P(U^+(n, b_n) \leq K_{Rn} \text{ i.o.}) = 1 \text{ or } 0, \tag{4.30}$$

according as $\varepsilon > 0$ or $\varepsilon < 0$.

PROOF. First, we see that b_n given as above fulfills the conditions of Theorem A. This follows from Lemma 3 and our assumptions on K_T. A direct consequence is that (4.29) holds.

Let $\alpha_n(\lambda)$ and $a_T(\lambda)$ be defined, respectively, by (3.22) and (3.14). Set throughout $T = Rn$. By Lemmas 2 and 4, we see that (4.29) can be written equivalently as

$$\liminf_{n\to\infty} \frac{(2\log n)^{1/2}}{\sigma b_n^{1/2}\log\log n}\left(U_n^+ - b_n\left(a_T^-(\lambda) + \frac{1}{\mu}\right)^{-1}\right) = \frac{1}{2} - \lambda \quad \text{a.s. (all } \lambda). \tag{4.31}$$

This result follows from (3.23) and (3.16) via the relations

$$\frac{(2\log n)^{1/2}}{\sigma b_n^{1/2}\log\log n} b_n\left(\left(\alpha_n^-(0) + \frac{1}{\mu}\right)^{-1} - \left(a_T^-(\lambda) + \frac{1}{\mu}\right)^{-1}\right)$$
$$\simeq \frac{(2\log T)^{1/2}}{\sigma\log\log T}\mu^{3/2}K_T^{1/2}(a_T^-(\lambda) - a_T^-(0)) \to -\lambda,$$

as $T = Rn \to \infty$.

Now set $\lambda = \frac{1}{2} + \varepsilon$. We see that (4.31) and (4.22) jointly imply that, for $T = Rn$,

$$P(U^+(n, b_n) \le b_n(K_T/b_n) \text{ i.o.}) = 1 \text{ or } 0,$$

according as $\varepsilon > 0$ or $\varepsilon < 0$, thus proving (4.30). □

PROOF OF THEOREM 5. We may now collect the pieces of our puzzle. By Lemma 7, for the proof of (4.5), it suffices to prove that, for some $R = R' < \mu$, $P(U^+(n, b_n) \le K_{Rn}$ i.o.$) = 1$ for all $\lambda = \varepsilon > 0$, and that, for some $R = R'' > \mu$, $P(U^+(n, b_n) \le K_{Rn}$ i.o.$) = 0$ for all $\lambda = -\varepsilon < 0$. But this is precisely the statement of Lemma 8. We have therefore proved (4.5). The proof of (4.4) is similar, and the result follows. □

PROOF OF THEOREM 4. The proof is achieved by similar arguments as above (we omit the details), based on the following theorem, due to Deheuvels and Steinebach (1987), which we reformulate, using the notation of Theorem A. □

THEOREM B. *Let $\{b_n, n \ge 1\}$ be an integer sequence such that* $1 \le b_n \le n$, $b_n/\log n \to \infty$ *and, for some* $p > 1$, $b_n/\log^p n \to 0$ *as* $n \to \infty$. *Let* α_n^- *be defined as in* (4.27). *Then*

$$\lim_{n\to\infty} \frac{(2\log n)^{1/2}}{\sigma b_n^{1/2}\log\log n}\left(U_n^+ - b_n\left(\alpha_n^- + \frac{1}{\mu}\right)^{-1}\right) = \frac{1}{2} \quad \textit{in probability}.$$

5. Erdös–Rényi and small-size increments. We start this section by considering the case where $K_T = C\log T$ for some $0 < C < \infty$. Let c_0 and c_1 be as in (3.6). The following results have been obtained by Bacro, Deheuvels and Steinebach (1987).

THEOREM C. *For any* $c_1 < C < \infty$, *let* $A \in (1/\mu, A_0)$ *and* $s^{**} < 0$ *be the solutions of the equations*

$$(5.1) \qquad 1/C = \sup_{s<0} \{s - A \log \phi(s)\} = s^{**} - A \log \phi(s^{**}).$$

Then, for $K_T = C \log T$,

$$(5.2) \qquad \limsup_{T\to\infty} \left(\frac{-\log \phi(s^{**})}{\log\log T} (\Delta_T^+ - AK_T) \right) = \frac{1}{2} \quad a.s.$$

and

$$(5.3) \qquad \liminf_{T\to\infty} \left(\frac{-\log \phi(s^{**})}{\log\log T} (\Delta_T^+ - AK_T) \right) = -\frac{1}{2} \quad a.s.$$

THEOREM D. *For any* $c_0 < C < \infty$, *let* $B \in (B_0, 1/\mu)$ *and* $s^* > 0$ *be the solutions of the equations*

$$(5.4) \qquad 1/C = \sup_{s>0} \{s - B \log \phi(s)\} = s^* - B \log \phi(s^*).$$

Then, for $K_T = C \log T$,

$$(5.5) \qquad \limsup_{T\to\infty} \left(\frac{-\log \phi(s^{*})}{\log\log T} (\Delta_T^- - BK_T) \right) = \frac{1}{2} \quad a.s.$$

and

$$(5.6) \qquad \liminf_{T\to\infty} \left(\frac{-\log \phi(s^{*})}{\log\log T} (\Delta_T^- - BK_T) \right) = -\frac{1}{2} \quad a.s.$$

In the above theorems, the existence of B, s^* and s^{**} is justified by the arguments of Section 2. A_0 and B_0 are as in (3.1). A comparison of Theorems C and D with Theorem 5 motivates a reformulation of these results in a unified way. This results in the following corollary.

COROLLARY 2. *Assume that* $\{K_T\}_{T\geq 0}$ *satisfies the assumptions of either of the Theorems* 5, *C or D. Denote by* C_T^+ *and* ζ_T^+ (*resp.* C_T^- *and* ζ_T^-) *the solutions of the equations in* γ *and* ζ (*whenever such solutions exist*) *with* $\zeta_T^+ < 0 < \zeta_T^-$ *and* $C_T^- < 0 < C_T^+$,

$$(5.7) \quad K_T^{-1} \log T = \sup_s \left\{ s - \left(\gamma + \frac{1}{\mu} \right) \log \phi(s) \right\} = \zeta - \left(\gamma + \frac{1}{\mu} \right) \log \phi(\zeta).$$

Then

$$(5.8) \quad \limsup_{T\to\infty} (\log\log T)^{-1} \left(-\left\{ \Delta_T^\pm - K_T \left(C_T^\pm + \frac{1}{\mu} \right) \right\} \log \phi(\zeta_T^\pm) + \log K_T \right) = \frac{3}{2} \quad a.s.$$

and

$$\liminf_{T\to\infty}(\log\log T)^{-1}\left(-\left\{\Delta_T^{\pm}-K_T\left(C_T^{\pm}+\frac{1}{\mu}\right)\right\}\log\phi(\zeta_T^{\pm})+\log K_T\right)$$
(5.9)
$$=\frac{1}{2}\quad a.s.$$

PROOF. Notice that $\mp\log\phi(\zeta_T^{\pm})>0$. Moreover, it is clear that (5.8) and (5.9) coincide with (5.2) and (5.3) when $K_T=C\log T$. On the other hand, when $K_T/\log T\to\infty$ as $T\to\infty$, we have by (3.10) and (3.12)

$$\text{(5.10)}\quad \log\phi(\zeta_T^{\pm})=\mp(1+o(1))\sigma^{-1}\mu^{3/2}K_T^{-1/2}(2\log T)^{1/2}\quad\text{as } T\to\infty.$$

Let $a_T^{\pm}(\lambda)$ be as in (3.14). Obviously, $C_T^{\pm}=a_T^{\pm}((\log K_T)/(\log\log T))$, so that by (3.16) and using the fact that $(\log K_T)/(\log\log T)=O(1)$, we obtain

$$\text{(5.11)}\quad \frac{\log\phi(\zeta_T^{\pm})}{\log\log T}K_T(C_T^{\pm}-a_T^{\pm})=\frac{-\log K_T}{\log\log T}+o(1)\quad\text{as } T\to\infty.\qquad\square$$

In view of (5.10) and (5.11), it is now clear that the statement of Theorem 5 is equivalent to that of Corollary 2 in the range covered by this theorem.

REMARK 5. In view of Remark 1, page 375 of Deheuvels and Steinebach (1987), it is straightforward that the lim inf in (5.9) is also a limit in probability. Moreover, by Remark 6, page 382 of the just cited article, the arguments used in our proofs jointly with that of Bacro, Deheuvels and Steinebach (1987) extend the validity of Theorems C, D and Corollary 2 to the case where $K_T/\log n\to C$, assuming that $\{K_T\}_{T\geq 0}$ satisfies (a)–(d). We omit the details.

REMARK 6. We reconsider the case of the inverse Gaussian distribution (4.10) for increments of the form $K_T=C\log T$ for $0<C<\infty$. By (4.11), we see that $C_T^{\pm}=\pm K_T^{-1/2}(2\log T)^{1/2}$ and $\log\phi(\zeta_T^{\pm})=\mp K_T^{-1/2}(2\log T)^{1/2}$. Thus, Corollary 2, together with straightforward expansions, shows that (4.2), (4.4) and (4.5) remain valid for such increments. This phenomenon is naturally explained by the fact that, starting from a standard Wiener process $\{W(t),\ t\geq 0\}$, we may define the nth partial sum S_n of an i.i.d. sequence of inverse Gaussian random variables by $S_n=\inf\{t\geq 0\colon\ at+bW(t)\geq nc\}$, for positive a, b and c. Therefore, the renewal process associated to S_n is always very close to the original process $at+bW(t)$. An interesting problem would be to characterize all possible K_T's for which (4.2), (4.4) and (4.5) hold in this case. This gives a motivation for the study of $\Delta_T^{\pm}$ for $K_T=o(\log T)$ as $T\to\infty$.

REMARK 7. It is obvious that our duality argument can be extended to cover the situation for which $K_T=o(\log T)$, as long as one knows the corresponding behavior of $U^{\pm}(n,b_n)$ for sequences $b_n=o(\log n)$ as $n\to\infty$. Unfortunately, very few results are available at present on this problem. Recently, Mason (1989) has achieved remarkable progress in this field, by obtaining conditions depending both on the moment generating function ϕ and the behavior of the sample maximum, which ensure that $U_n^{\pm}/d_n \to 1$ a.s. for a suitable sequence d_n. As an example of what may be achieved in this direction, we present, in the following proposition, a simple consequence of his main theorem.

PROPOSITION 1. *Assume, in addition to (A)–(C) that*

(D) $$0 < \omega^- = \operatorname{ess\,inf} X < \omega^+ = \operatorname{ess\,sup} X < \infty.$$

Then, for all $\{K_T\}_{T \geq 0}$ *such that* $K_T \to \infty$ *and* $K_T/\log T \to 0$ *as* $T \to \infty$, *we have*

$$\lim_{T\to\infty} \Delta_T^{\pm}/\{K_T/\omega^{\mp}\} = 1 \quad a.s. \tag{5.12}$$

PROOF. Consider the case of $\Delta_T^- = \Delta^-(T, K_T)$. By (4.25), we have

$$\{U^+(N(T)+1, m) \leq K_T\} \subset \{\Delta^-(T, K_T) \geq m\} \subset \{U^+(N(T), m) \leq K_T\}.$$

By (4.26), for T sufficiently large, $[R_1 T] \leq N(T) \leq N(T) + 1 \leq [R_2 T]$, where $0 < R_1 < 1/\mu < R_2 < \infty$ are arbitrary (but fixed) constants. Hence, it is enough to show that, for all $0 < \varepsilon < 1$,

$$P(U^+(n, b_n) \leq b_n\omega^+(1-\varepsilon) \text{ i.o.}) = 0,$$

where $n = [R_1 T]$, $b_n = [K_T\{\omega^+(1-\varepsilon)\}] + 1$, and

$$P(U^+(n, b_n) > b_n\omega^+(1+\varepsilon) \text{ i.o.}) = 0,$$

where $n = [R_2 T]$, $b_n = [K_T\{\omega^+(1+\varepsilon)\}]$.

The latter relations, however, are immediate by Mason's (1989) theorem which ensures that for all b_n such that $b_n \geq 1$ and $b_n/\log n \to 0$ as $n \to \infty$, one has

$$\lim_{n\to\infty} U^+(n, b_n)/\{b_n\omega^+\} = 1 \quad \text{a.s.} \tag{5.13}$$ □

The case of Δ_T^+ can be treated likewise by an obvious modification.

Note that an alternative proof of Proposition 1 may be obtained through the classical theory of runs [see, e.g., Deheuvels (1985)]. This follows from the fact that the length of the largest run of successive observations among $X_1, \dots, X_n$ which fall in the interval $[\omega^+ - \varepsilon, \omega^+]$ will with probability 1 always exceed $r \log n$ for all n large enough (for any $\varepsilon > 0$ and some $r > 0$). By letting $\varepsilon > 0$ be arbitrarily small, we obtain (5.13) for all $b_n = o(\log n)$.

6. Extensions. It is very natural to extend the results of the preceding sections to the case where the random variables $X_1, X_2, \dots$ are possibly negative (note that assumptions (A)–(C) cover the case where $P(X_1 = 0) > 0$). The first difficulty consists in finding the right definition for the renewal process, since S_n need not be nondecreasing any more. Following Heyde (1967), we consider $M_n = \max_{0 \leq i \leq n} S_i$ and $m_n = \min_{i \geq n} S_i$, for $n = 0, 1, \dots,$ and define the two nondecreasing processes $\{N_1(t),\ t \ge 0\}$ and $\{N_2(t),\ t \ge 0\}$, by

$$N_1(t) = \max\{n \ge 0: M_n \le t\} = -1 + \min\{n > 0: S_n > t\} \tag{6.1}$$

and

$$N_2(t) = \max\{n \ge 0: S_n \le t\} = -1 + \min\{n > 0: m_n > t\}. \tag{6.2}$$

Obviously, $N_1(t) \le N_2(t)$ with equality when the X_i's are nonnegative. In the following result, we give an upper bound for $N_2(t) - N_1(t)$.

THEOREM 6. *Assume that* (A)–(C) *hold, jointly with*

(E) $$s_1 = \inf\{s: \phi(s) = E(\exp(sX_1)) < \infty\} < 0.$$

In addition, assume that there exists an $s_2 \in (s_1, 0)$ *such that* $\phi'(s_2) = 0$. *Then*

$$\lim_{T\to\infty} (\log T)^{-1} \sup_{0\le t\le T} (N_2(t) - N_1(t)) = -1/\log\phi(s_2) \quad a.s. \tag{6.3}$$

PROOF. Let $\tilde{V}_n = \min_{1\le i\le n-k}\max_{1\le j\le k}(k/j)(S_{i+j} - S_i)$, where $k = [c\log n]$. Let $c > 0$ and $0 < \alpha < \lim_{s\downarrow s_1}\{\mu - m(s)\}$, where $m(s) = \phi'(s)/\phi(s)$, be related via the equation

$$\exp(-1/c) = \inf_s\{\phi(s)e^{-s(\mu-\alpha)}\}. \tag{6.4}$$

Since $m(\cdot)$ is increasing on (s_1, s_0) and $m(s_2) = 0$, the specific choice of $\alpha = \mu$ is possible in (6.4). For this choice, $c = -1/\log\phi(s_2)$.

In general, Theorems 3 and 4 in Deheuvels and Devroye (1987) [see also Theorem 2 in Csörgő and Steinebach (1981)], show that, for $k = [c\log n]$,

$$\lim_{n\to\infty}(c\log n)^{-1}\tilde{V}_n = \lim_{n\to\infty}(c\log n)^{-1}\tilde{W}_n = \mu - \alpha \quad \text{a.s.}, \tag{6.5}$$

where $\tilde{W}_n = \min_{1\le i\le n-k}\min_{1\le j\le k}(S_{i+j} - S_i + (k-j)\mu)$.

In the first place, choose $\alpha = \mu + \varepsilon$ for some small $\varepsilon > 0$ and observe that, as $\varepsilon\downarrow 0$, the corresponding c increases to $-1/\log\phi(s_2)$. Moreover, (6.5) implies that, for any c less than $-1/\log\phi(s_2)$, $\tilde{V}_n$ is ultimately negative with probability 1. This, in turn, jointly with (4.26) and the fact that $\log(T/\mu) \sim \log T$ as $T\to\infty$, implies that

$$\liminf_{T\to\infty}(\log T)^{-1}\sup_{0\le t\le T}(N_2(t) - N_1(t)) \ge -1/\log\phi(s_1) \quad \text{a.s.} \tag{6.6}$$

Next, we see that, if for some $\varepsilon > 0$, the event

$$\left\{\max_{0\le t\le T}(N_2(t) - N_1(t)) > -(1+\varepsilon)\frac{\log T}{\log\phi(s_2)}\right\}$$

occurs for an unbounded set of T's, then we must have $\{\tilde{W}_n < 0 \text{ i.o.}\}$ for $k = [c' \log n]$, and for some $c' > -\log \phi(s)$. But this is impossible with probability 1 by (6.5). This, jointly with (6.6), suffices for (6.3). □

REMARK 8. A consequence of Theorem 6 is that, up to the rate of $O(\log T)$, we can replace $N_1(t)$ by $N_2(t)$ in the interval $[0, T]$ without loss of generality and conversely. Note that the most often encountered definitions [see, for instance, Csörgő, Horváth and Steinebach (1987), (1.1), and Steinebach (1986), Section 1] are $N(t) = N_1(t)$ and $N(t) = N_1(t) + 1$.

Since the invariance principles in Csörgő, Horvath and Steinebach (1987) and Mason and van Zwet (1987) do not make any restriction concerning the nonnegativity of the X_i's, it follows that the validity of Theorems 1, 2 and 3 can be extended to this case for $N(t) = N_1(t)$ or $N(t) = N_2(t)$, under the assumptions (A)–(C) and (E).

In view of the results of Steinebach (1986), which give the same centering constants as in Theorems 4, 5, C and D, it appears as a likely conjecture that these theorems remain valid under (A)–(C) and (E) without any sign condition. However, it may be seen that the duality argument used in Section 4 [based mainly on (4.25)] does not generalize itself easily for possibly negative r.v.'s. In fact, one would need results concerning the limiting behavior of

$$\tilde{U}_n^{\pm} = \tilde{U}^{\pm}(n, b_n) = \pm \max_{0 \le k \le n - b_n} \left(\pm \left(M_{k+b_n} - M_k \right) \right), \tag{6.7}$$

similar to those obtained for $U^{\pm}(n, b_n)$ in Theorems A and B. Such results would enable one to cover the case of $N_1(\cdot)$ [and likewise with m replacing M, the case of $N_2(\cdot)$]. It can be seen that (4.25) holds for $N = N_1$ and $\tilde{U}^+$ replacing U^+. Moreover, we have almost surely for large n the inequalities

$$U^-(n, b_n) \le \tilde{U}^-(n, b_n) \le \tilde{U}^+(n, b_n) \le U^+(n, b_n). \tag{6.8}$$

Hence, the arguments we have used provide one-sided bounds based on (6.7), (6.8) and the modified version of (4.25). In order to complete the extension of our theorems to arbitraryX_i's, we need complete the description of $\tilde{U}^{\pm}$ by investigating the corresponding inner bounds. If Theorems A and B, jointly with the corresponding statements for $b_n = O(\log n)$, hold with $U_n^{\pm}$ replacing $U_n^{\pm}$, then our proofs can be repeated verbatim to cover the general situation discussed above. Since no result of this kind is yet available, we leave this point as an open problem.

Even though further investigations are needed to treat generalized renewal processes, our methods have the advantage of providing a simple analysis of the behavior of $\Delta_T^{\pm}$ as $T \to \infty$ in the classical case. Such techniques have been currently used in the past to obtain limit theorems for $N(t)$ knowing similar results for S_n.

Acknowledgment. We thank the referee for a careful reading of our article and for extremely helpful comments and suggestions.

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