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DONSKER'S INVARIANCE PRINCIPLE FOR LIE GROUPS[1]

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This paper establishes a functional central limit theorem for Lie groups under a mixing hypothesis. The main theorem generalizes results by Patrick Billingsley for Euclidean space and the author for the general linear group.

1. Description of the theorem. In 1951, Donsker [2] stated and proved a remarkable and valuable generalization to the central limit theorem. As he stated it:

> We consider a sequence $S_1, S_2, \ldots$ of partial sums of independent, identically distributed random variables $X_1, X_2, X_3, \ldots$ each having mean 0 and standard deviation $1 \ldots$. The object of this paper is to show that if $\{Y_n\}$ is a family of random variables such that Y_n is a function of $S_1, S_2, \ldots, S_n$, then under very weak restrictions the limiting distribution of Y_n is the same as the distribution of a related functional on C, i.e., the limiting distribution of Y_n is independent of the distribution of the X's.

In this instance, C is Wiener space. The statement of the theorem predates Prohorov's work [8] on probability measures on function space. Thus, the main result, as Donsker states it, is deceptively cumbersome for such an elegant theorem. We state Donsker's theorem in modern terminology as follows.

Let S be a complete and separable metric space with metric ρ and let $D([0,\infty); S)$ denote the set of paths in S which are right continuous and have left limits at each point in $[0,\infty)$. Endow this space with the Skorohod topology. Let $\{X_k;\, k \geq 1\}$ be an independent and identically distributed sequence of random variables having common mean 0 and common variance 1. Define, for each $a \geq 1$,

$$B^a(t) = \frac{1}{\sqrt{a}} \sum_{k=1}^{[at]} X_k, \tag{1.1}$$

and let $\mathbf{P}^a$ be the distribution of B^a on $D([0,\infty); \mathbf{R})$. Now, Donsker's theorem reads:

THEOREM 1.1. *$\mathbf{P}^a$ converges weakly to Wiener measure as $a \to \infty$.*

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Donsker called his theorem an invariance principle because the limiting distribution exists independently of the random variables involved. This theorem dramatically extends the method of Erdös and Kac [3], who had calculated the limiting distribution in several cases by choosing a distribution for the X_k's which most simplifies computation and taking limits. In essence, Donsker's theorem allows us to relate asymptotic properties for the distribution of random walks to the distribution of Brownian motion.

For processes in R^d, the Lévy–Khintchine formula is a guiding principle in the study of stochastic processes having stationary and independent increments. Due to the work of Hunt [5], this theorem has a full generalization to Lie groups. Let G be a Lie group of dimension d having identity element e. Let $\mathfrak{g}$ be the Lie algebra of G, which we shall define to be the set of left invariant vector fields on G. Hunt gave a complete classification of all processes on G which satisfy the following axioms:

A1. $Z(0) = e$ almost surely.

A2. $\text{limit}_{\varepsilon \to 0} P\{Z(\varepsilon) \not\subseteq U\} = 0$ where U is any neighborhood of e.

A3. For any $0 \le t_1 \le t_2 \le \cdots \le t_n$ and every $n \in N$, the random variables

$$Z(t_1), Z(t_1)^{-1}Z(t_2), \dots, Z(t_{n-1})^{-1}Z(t_n)$$

are mutually independent.

A4. The distribution of the increment $Z(t_1)^{-1}Z(t_2)$ for $t_1 \le t_2$ depends on t_1 and t_2 only through their difference $t_2 - t_1$.

In the extension of the central limit theorem for independent and identically distributed random variables, one may relax the hypothesis of identical distribution in the manner first described by Lindeberg [7]. In 1973, Stroock and Varadhan [10] proved a functional central limit theorem for Lie groups containing as a hypothesis a Lindeberg condition. Because Stroock and Varadhan were primarily concerned with continuous processes, they began their paper by classifying all processes $\mathscr{P}$ satisfying axioms A1, A3 and

A2′. $Z(t)$ is a continuous function of t with probability 1.

To explain their classification, let $\mathbf{S}_d^+$ be the set of all nonnegative definite $d \times d$ matrices and define

$$C_0^{\uparrow}([0,\infty); \mathbf{S}_d^+) = \{A \in C([0,\infty); \mathbf{S}_d^+); A(0) = 0, A(t) \ge A(s) \text{ for } t \ge s\}.$$

If we view $\mathscr{P}$ as the set of probability measures on $C([0,\infty); G)$ satisfying A1, A2′ and A3 and topologized by weak convergence, then the Stroock–Varadhan classification is given via a one-to-one onto bicontinuous mapping

$$C_0^{\uparrow}([0,\infty); S_d^+) \times C([0,\infty); G) \leftrightarrow \mathscr{P}. \tag{1.2}$$

The action of this mapping is more readily understood if we first restrict attention to absolutely continuous functions. The derivative of an absolutely continuous function chosen from $C_0^{\uparrow}([0,\infty); \mathbf{S}_d^+)$ represents the time dependent covariance matrix with respect to some choice of basis for the Lie algebra. A choice from the second factor, if it is absolutely continuous, gives a curve in G

which is the integral curve for the drift. These two functions combine to specify an element in $\mathscr{P}$ via the infinitesimal generator for a time dependent Markov process. Stroock and Varadhan then use continuity to extend the mapping to all of $\mathscr{P}$.

A second tradition for the central limit theorem begins with a stationary sequence of random variables. For these stationary sequences, ergodicity has not yet served as a sufficiently strong hypothesis to insure asymptotically normal behavior. Rosenblatt [9], in 1956, introduced a class of stationary and ergodic sequences which he called strongly mixing. Three years later Ibragimov [6] introduced a second class of mixing sequences which he called uniformly mixing, and with these hypotheses, both Rosenblatt and Ibragimov were able to prove the central limit theorem. Today, we view these two types of mixing as the extreme points in a one parameter family of measures of mixing

$$\varphi_q(\mathscr{G}\,|\mathscr{H}) = \sup\{\|P(A|\mathscr{H}) - P(A)\|_q;\ A \in \mathscr{G}\}$$

for $1 \le q \le \infty$, for $\|\cdot\|_q$ the norm on $L^q(P)$ and for two σ-field $\mathscr{G}$ and $\mathscr{H}$. (See Ethier and Kurtz [4], Chapter 3.) These measures of the degree of independence are applied, in the case of a stationary sequence $\{X_k;\ k \in Z\}$ to the σ-fields $\mathscr{F}_n = \sigma\{X_k;\ k \le n\}$ and $\mathscr{F}^{n+m} = \sigma\{X_k;\ k \ge n+m\}$. Set $\varphi_q(m) = \varphi_q(\mathscr{F}^{n+m}|\mathscr{F}_n)$, which, for a stationary sequence, is independent of n. For each q, a summability criterion on $\varphi_q(m)$ is sufficient to prove Donsker's theorem in the presence of an appropriate moment hypothesis on $\{X_k;\ k \in \mathbf{Z}\}$. The first functional central limit theorem of this type was proved by Billingsley [1] in the uniform ($q = \infty$) mixing case. The qualitative aspect of the limit remains unchanged. However, the lack of independence results in a new variance parameter.

We state the main theorem of this paper as follows.

THEOREM 1.2. *Let* $\{F_k \in C^{0,2}([0,\infty) \times [0,1]; G);\ k \in \mathbf{Z}\}$ *be a stationary sequence satisfying*:

(i) $F_0(s,0) = e$ *with probability* 1.

(ii)

$$E\big[\sup_{0\le s<\infty}\sup_{0\le p\le 1}\|D_2F_0(s,p)\|^2 + \sup_{0\le s<\infty}\sup_{0\le p\le 1}\|D_2^2F_0(s,p)\|\big] < \infty.$$

(iii) $ED_2F_0(s,0) = 0$.

(iv) $\sum_{m=1}^{\infty}\varphi_\infty(m)^{1/2} < \infty$.

Define, *for each* $a \ge 1$,

$$Z^a(t) = F_1\left(\frac{1}{a}, \frac{1}{\sqrt{a}}\right)F_2\left(\frac{2}{a}, \frac{1}{\sqrt{a}}\right)\cdots F_{[at]}\left(\frac{[at]}{a}, \frac{1}{\sqrt{a}}\right) \tag{1.3}$$

and let $\mathbf{P}^a$ *be the distribution of* Z^a *on* $D([0,\infty), G)$. *Then the two sequences*

$$\sigma^2(s) = \operatorname*{limit}_{n\to\infty}\frac{1}{n}E\left(\sum_{k=1}^{n} D_2F_k(s,0)\right)^2 \tag{1.4}$$

$$\beta(s) = \operatorname*{limit}_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{j-1}E\big[D_2F_k(s,0), D_2F_j(s,0)\big] \tag{1.5}$$

converge. $\mathbf{P}^a$ *converges weakly to* $\mathbf{P} \in \mathscr{P}$ *as* $a \to \infty$ *where* $\mathbf{P}$ *is characterized by its infinitesimal generator* $\frac{1}{2}\sigma^2(s) + \mu(s)$. *Here* $\mu(s) = \frac{1}{2}ED_2^2F(s,0) + \beta(s)$.

The memory given by the mixing along with the noncommutativity of the Lie group results in a new source of drift β, a limit of expectations involving the Lie bracket. We shall illustrate this in Section 2 with an example using a time homogeneous diffusion on the Euclidean group.

In stating Donsker's theorem on Lie groups, we have adopted a more geometric point of view. Thus, consider $D_2F_k(s, p)$ to be a vector in the tangent space above the $F_k(s, p)$. Then $D_2F_k(s,0)$ is in the tangent space above the identity. This generates an invariant vector field, which we shall regard as an element of the Lie algebra. Because $ED_2F_k(s,0) = 0$, $ED_2^2F_k(s,0)$ is a derivation and, thus, we may view $ED_2^2F_k(s,0)$ as an invariant vector field. In addition, we have endowed the fibers in the tangent space with a norm $\|\cdot\|$, giving these vector spaces the structure of a finite dimensional Banach space. Because all norms on this space are equivalent, the specific choice of norm is unimportant. Note that the possible limiting probability measures will be the image of C^1 functions under the Stroock–Varadhan classification mapping.

2. Outline of the proof. The Stroock–Varadhan classification theorem shows us that it is sufficient to verify the tightness of $\{\tilde{\mathbf{P}}_s^a;\ a \geq 1\}$, show that each possible limit process satisfies A1, A2′ and A3 and show that each process has the characteristics $\sigma^2(s)$ and $\mu(s)$. The technical aspects of the proof tend to obscure these basic ingredients. In particular, the tightness proof also gives A2′, the continuity of the sample paths. In order to deal with the possibility of explosions, we have chosen to work directly with a martingale problem. We first write a difference as a telescoping sum of intervals of length δ in hopes of creating a Riemann sum for the integral in the martingale. In essence, the proof that the limit process has property A3, the independence of the increments, is accomplished by setting this telescoping sum and by discarding some elements in the product to create a weaker dependence between terms in the sum. The total number of elements tends to infinity with a, but the limit is not altered because the total length of time that these products operate tends to zero as $a \to \infty$. The bulk of the proof is a Taylor series expansion. In taking limits on δ and a we find both the integral in the martingale problem and consequently, the characteristics $\mu(s)$ and $\sigma(s)$.

To outline the proof of Theorem 1.2, first note that the functions F_k and, as we shall see, the sample paths of the limit process, are continuous. Thus, all of the distributions $\{\mathbf{P}^a;\ a \geq 1\}$ are supported on paths in the connected component of the identity. Thus, we can assume that G is a connected Lie group. Let G be endowed with a left invariant metric ρ compatible with the norm $\|\cdot\|$. Then $\rho(g_1, g_2) = \rho(gg_1, gg_2)$ for all $g, g_1, g_2 \in G$. Thus, the pair (G, ρ) is a complete and separable metric space.

We begin the proof by localizing the process. For $s \geq 0$, define

$$(2.1)\qquad Z^a(s,t) = F_{[as]+1}\left(\frac{[as]+1}{a}, \frac{1}{\sqrt{a}}\right) \cdots F_{[as]}\left(\frac{[as]}{a}, \frac{1}{\sqrt{a}}\right)$$

and let $\mathbf{P}_s^a$ denote the distribution of $Z^a(s, t)$ on $D([s, \infty), G)$. Let U be a convex open neighborhood of e, $\operatorname{diam} U < \infty$, chosen so that G can be represented as a subgroup of $GL(N, \mathbf{R})$. For a path $\zeta(s, \cdot) \in D([s, \infty), G)$, define the stopping time $\tau(s, U) = \inf\{t \geq s: \zeta(s, t) \notin U\}$ and let $\tilde{\mathbf{P}}_s^a$ be the law of the process $Z^a(s, \cdot)$ stopped at time $\tau(s, U) - 1/a$. The proof of a limit theorem for the measures $\{\tilde{\mathbf{P}}_s^a;\, a \geq 1\}$ has three basic parts. In the first part, we show that this collection is relatively compact. This requires an understanding of the compact subsets of $D([0, \infty), G)$. The characterization of the compact sets is similar to the statement of the Arzela–Ascoli theorem. However, the equicontinuity property used in this theorem must be replaced by a new form of the modulus of continuity compatible with the Skorohod topology. Thus, for $\zeta \in D([0, \infty), G)$, we define

$$(2.2)\quad w'(\zeta, \delta, T) = \inf_{\{t_i\} \in \Pi(T, \delta)} \max_i \sup\{\rho(\zeta(t), \zeta(t'));\, t, t' \in [t_{i-1}, t)\},$$

where $\Pi(T, \delta)$ is the set of all partitions of $[0, T]$ chosen so that no two points of the partition ware within δ of each other. Letting $\tilde{Z}^a(s, \cdot)$ denote the stopped processes as defined on the original probability space and using $\{\tilde{\mathbf{P}}_s^a;\, a \geq 1\}$ relatively compact and $\{\tilde{Z}^a(s, \cdot);\, a \geq 1\}$ relatively compact interchangeably, we quote the following theorem from Ethier and Kurtz [4], page 128.

THEOREM 2.1. *$\{\bar{Z}^a(s, \cdot);\, a \geq 1\}$ is relatively compact if and only if the following two conditions hold*:

(i) *For every $\varepsilon > 0$ and $t > s$, there exists a compact set $\Gamma_{\varepsilon, t} \subseteq G$ such that*

$$(2.3)\quad \liminf_{a \to \infty} P\{\tilde{Z}^a(s, t) \in \Gamma_{\varepsilon, t}\} \geq 1 - \varepsilon.$$

(ii) *For every $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that*

$$(2.4)\quad \limsup_{a \to \infty} P\{w'(\tilde{Z}^a(s, \cdot), \delta, T) \geq \varepsilon\} \leq \varepsilon.$$

Both of these conditions are established using the following estimate.

LEMMA 2.2. *For $F_k(s, q) \in \frac{1}{2}U$, let $\tilde{F}_k(s, q) \in GL(N, \mathbf{R})$ denote its matrix group representation. Otherwise, let $\tilde{F}_k(s, q) = I$, the identity matrix. Let $\|\cdot\|_0$ denote the operator norm for a matrix in $GL(N, \mathbf{R})$. Then for any $s \geq 0$, and $p \in [0, 1]$*,

$$(2.5)\quad E\Big[\sup_{0 \leq q \leq p} \log\|\tilde{F}_0(s, q)\|_0\Big] \leq p^2 C, \text{ where } C = \tfrac{3}{2} E\Big[\sup_{0 \leq p \leq 1} \|D_2^2 \tilde{F}_0(s, p)\|_0\Big].$$

The proof of this lemma is essentially the same as the proof of Lemma 2.5 in [12] and so we omit it. The proof uses a Taylor series expansion having a first order term with zero expectation.

THEOREM 2.3. *$\{\tilde{Z}^a(s,\cdot); a \geq 1\}$ is relatively compact.*

By Lemma 2.2, the one dimensional process $\log \|\tilde{Z}^a(s,t)\|_0$ is dominated by a process which satisfies the hypotheses of Theorem 1.3 and this can be used to verify the hypotheses of Theorem 2.1. The proof actually shows that condition (ii) of Theorem 2.1 is satisfied for the modulus of continuity

$$w(\zeta,\delta,T) = \sup\{\rho(\zeta(t),\zeta(u)); t,u \leq T, |t-u| < \delta\}. \tag{2.6}$$

Because $w(\zeta,\delta,T) \geq w'(\zeta,\delta,T)$, this is sufficient. As a bonus, this immediately implies

THEOREM 2.4. *All limit points for $\{\tilde{Z}^a(s,\cdot); a \geq 1\}$ have a version with continuous sample paths.*

The second part of the proof involves showing that $\{\tilde{\mathbf{P}}_s^a; a \geq 1\}$ has exactly one limit point $\tilde{\mathbf{P}}_s$ and giving a sufficient description to identify $\tilde{\mathbf{P}}_s$. We begin with

THEOREM 2.5. *The limits in the definitions of $\sigma^2(s)$ in* (1.4) *and $\beta(s)$ in* (1.5) *exist.*

The proof of the limit for $\sigma^2(s)$ is essentially the same as in Billingsley [1], Lemma 3, page 172. By using the stationarity of $\{F_k; k \in Z\}$, Billingsley also shows that we may write

$$\sigma^2(s) = \sum_{k=-\infty}^{\infty} ED_2F_k(s,0)D_2F_0(s,0) = \sum_{k=-\infty}^{\infty} ED_2F_0(s,0)D_2F_k(s,0). \tag{2.7}$$

Similarly, the limit defining the bracket term $\beta(s)$ exists and may be rewritten

$$\beta(s) = \sum_{k=1}^{\infty} E[D_2F_0(s,0), D_2F_k(s,0)]. \tag{2.8}$$

The Stroock–Varadhan classification theorem states that the limit law $\mathbf{P}$ is determined by its infinitesimal generator $\frac{1}{2}\sigma^2(s) + \mu(s)$. In stating this result, they refer to a fixed basis for the Lie algebra. In the development we have chosen here, the generator is more naturally stated without introducing a basis.

LEMMA 2.6. *There exist continuous functions $\sigma_i: [0,\infty) \to \mathfrak{g}$ for $i = 1,2,\ldots,m \leq d$ which are linearly independent and satisfy*

$$\sum_{i=1}^{m} \sigma_i^2(s) = \sigma^2(s). \tag{2.9}$$

If the operator σ^2 is nondegenerate, the mapping $\sigma^2 \mapsto (\sigma_1,\sigma_2,\ldots,\sigma_d)$ is analytic. Now Lemma 2.6 follows by approximation. Next, we consider the

Stratonovich differential equation

$$(2.10)\quad dZ(u) = \sum_{i=1}^{m} \sigma_i(u)Z(u) \circ dB_i(u) + \mu(u)Z(u)\,du, \qquad Z(s) = g,$$

where $B_1, \dots, B_m$ are independent scalar Brownian motions. This equation has a unique strong solution up to a possible explosion time. This implies by a theorem of Yamada and Watanabe [13] the uniqueness of the following martingale problem $\mathscr{M}(s, g)$ until the exit time $\tau(s, gU)$:

$$(2.11)\qquad \tilde{P}_{s,g}\{\zeta : \zeta(s, s) = g\} = 1.$$

For $f \in C^\infty(G; \mathbf{R})$ with compact support

$$(2.12)\qquad f(\zeta(s,t)) - \int_s^t \left(\tfrac{1}{2}\sigma^2(u) + \mu(u)\right) f(\zeta(s,u))\,ds, \qquad s \le t,$$

is a $\tilde{\mathbf{P}}_{s,g}$-martingale.

Stroock and Varadhan continue by showing that the explosion cannot take place. This argument applies here. Thus the major task involves proving the following.

THEOREM 2.7. *Let* $\tilde{\mathbf{P}}_s^\infty$ *be a limit point for* $\{\tilde{\mathbf{P}}_s^a;\, a \ge 1\}$. *Then* $\tilde{\mathbf{P}}_s^\infty$ *is a solution to the martingale problem* $\mathscr{M}(s, e)$ *until the exit time* $\tau(s, U)$.

Until the exit time $\tau(s, gU)$ the martingale problem $\mathscr{M}(s, g)$ is also well posed. Its unique solution is

$$(2.13)\qquad \tilde{\mathbf{P}}_{s,g}(A) = \tilde{\mathbf{P}}_{s,e}\{\zeta : g^{-1}\zeta(s, \cdot) \in A\}.$$

Thus the conditions of Theorem 6.6.1 in Stroock and Varadhan [11] are satisfied and we can patch together the local solutions. Therefore, the martingale problem $\mathscr{M}(0, e)$ is well posed on $[0, \infty)$. but this is just a restatement of Theorem 1.2.

3. An extension of the theorem and an example. Let H be a closed subgroup of G. Then there exists a natural analytic structure on the left coset space $N = G/H$ converting it into an analytic manifold. The manifold N, called a homogeneous space, is included in Hunt's theorem. We can also include this extension in the central limit theorem.

THEOREM 3.1. *Let* $\{F_k \in C^{0,2}([0, \infty) \times [0,1]; N);\, k \in Z\}$ *be a stationary sequence. Then Theorem* 1.2 *holds with* (ii) *replaced by*

(ii)′ *For some* $\delta \ge 0$,

$$E\left[\sup_{0 \le s < \infty}\ \sup_{0 \le p \le 1} \|D_2F_0(s,p)\|^{2+\delta} + \sup_{0 \le s < \infty}\ \sup_{0 \ge p \le 1} \|D_2^2F_0(s,p)\|\right] < \infty,$$

and (iv) *replaced either by*

(iv)′ $\sum_{m=1}^\infty \varphi_\infty(m)^{1/(2+\delta)} < \infty$

or by

(iv)″ $\sum_{m=1}^\infty \varphi_q(m)^{\delta/(1+\delta)} < \infty$, *where* $q = (2 + \delta)/(1 + \delta)$.

We have also included cases with slower mixing rates and higher moments. The proofs are substantially the same, differing only by the use of Hölder inequality for mixing sequences (see Ethier and Kurtz [4], Chapter 7).

We conclude with an example on $E(2)$, the Euclidean group on $\mathbf{R}^2$. Recall that for the time homogeneous independent case,

$$\sigma^2 = EF_0'(0)^2 \quad \text{and} \quad \mu = \tfrac{1}{2}EF_0''(0). \tag{3.1}$$

For the two-dependent case,

$$\sigma^2 = EF_1'(0)F_0'(0) - EF_0'(0)^2 + EF_0'(0)F_1'(0) \tag{3.2}$$

and

$$\mu = \tfrac{1}{2}EF_0''(0) + E[F_0'(0), F_1'(0)]. \tag{3.3}$$

Let $R: [-1,1] \to E(2)$ and $T_x: [-1,1] \to E(2)$ be defined as follows: $R(p)$ is rotation by an angle p and $T_x(p)$ is translation by p in the x-direction. Set $\tau_x = T_x(0)$ and $\rho = R_x'(0)$; then set $[\rho, \tau_x] = \tau_y$ where τ_y is the vector field for translation in the y-direction. Let $\{G_k;\ k \in \mathbf{Z}\}$ be an i.i.d. sequence where each $G_k(p)$ takes on each of the four values $R(p)$, $R(-p)$, $T_x(p)$ and $T_x(-p)$ with probability $\frac{1}{4}$. For each $M_1, M_2 \in E(2)$, define $\varphi(M_1, M_2) = M_1M_2$ and $F_k(p) = \varphi(G_{2k}(p), G_{2k+1}(p))$. The F_k are bounded and independent. So the moment and mixing assumptions are easily satisfied. $F_0(0) = G_0(0)G_1(0) = e \cdot e = e$ and $EF_0'(0) = E[G_0(0)G_1'(0) + G_0'(0)G_1(0)] = 0$. Thus Theorem 1.2 applies and

$$Z^a(t) = F_1\left(\frac{1}{\sqrt{a}}\right)F_2\left(\frac{1}{\sqrt{a}}\right) \cdots F_{[at]}\left(\frac{1}{\sqrt{a}}\right)$$

converges weakly to the process $Z(t)$, a diffusion with

$$\sigma^2 = \rho^2 + \tau_x^2 \quad \text{and} \quad \mu = -\tau_x. \tag{3.4}$$

The product $R(p)T_x(p)R(-p), T_x(-p) \approx \exp(-p^2\tau_y)$ for p near zero. To improve the chance for appearance of this product, we define

$$\tilde{\varphi}(M_1, M_2) = \begin{cases} T_x(p)M_2, & \text{if } M_1 = R(p), \\ T_x(-p)M_2, & \text{if } M_1 = R(-p), \\ R(-p)M_2, & \text{if } M_1 = T_x(p), \\ R(p)M_2, & \text{if } M_1 = T_x(-p) \end{cases} \tag{3.5}$$

and set $\tilde{F}_k(p) = \tilde{\varphi}(G_k(p), G_{k+1}(p))$. Then $\{F_k;\ k \in \mathbf{Z}\}$ and $\{\tilde{F}_k;\ k \in \mathbf{Z}\}$ have the same one-dimensional distributions. $\{\tilde{F}_k;\ k \in \mathbf{Z}\}$ is stationary and two-dependent, and thus theorem 1.2 applies:

$$\tilde{Z}^a(t) = \tilde{F}_1\left(\frac{1}{\sqrt{a}}\right)\tilde{F}_2\left(\frac{1}{\sqrt{a}}\right) \cdots \tilde{F}_{[at]}\left(\frac{1}{\sqrt{a}}\right)$$

converges weakly to a diffusion $\tilde{Z}(t)$ on $E(2)$. Because G_3 is independent of the other three factors

$$EF_0'(0)F_1'(0) = EF_1'(0)F_0'(0) = 0. \tag{3.6}$$

Also,

$$E[F_0'(0), F_1'(0)] = [\rho, \tau_x] = -\tau_y. \tag{3.7}$$

Thus

$$\sigma^2 = \rho^2 + \tau_x^2 \quad \text{and} \quad \mu = -\tau_y - \tau_x. \tag{3.8}$$

4. The details of the proof. We begin with a technical lemma intended to make the proof of Theorem 2.3 more readable.

LEMMA 4.1.

$$\begin{aligned} \lim_{\delta \to 0} \frac{1}{\delta} \sup_{0 \le u \le \delta} \sup_{0 \le s < \infty} \operatorname{var}\Bigg(& \sum_{k=1}^{[au]} \frac{1}{\sqrt{a}} \frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta] + k}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} \\ & - \sum_{k=1}^{[au]} \frac{1}{\sqrt{a}} \frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta]}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} \Bigg) = 0. \end{aligned} \tag{4.1}$$

PROOF. Note that $E[d/dr\|\tilde{F}_k(s, r) - I\|_0|_{r=0}] = 0$. Thus by expanding the variance of the sum and using the Hölder inequality for mixing we find that

$$\begin{aligned} & \frac{1}{a\delta} \sum_{k=1}^{[au]} \sum_{l=1}^{[au]} \operatorname{cov}\Bigg(\frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta]}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} - \frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta] + k}{a}, r\right) - I \right\|_0 \Bigg|_{r=0}, \\ & \qquad \frac{d}{dr} \left\| \tilde{F}_l\left(s + \frac{[ja\delta] + l}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} - \frac{d}{dr} \left\| \tilde{F}_l\left(s + \frac{[ja\delta]}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} \Bigg) \\ & \le \frac{2}{a\delta} \sum_{k=1}^{[au]} \sum_{l=1}^{[au]} \varphi_\infty(|k - l|)^{1/2} E\Bigg[\Bigg(\frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta] + k}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} \\ & \qquad - \frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta]}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} \Bigg)^2 \Bigg]^{1/2} \\ & \qquad \times E\Bigg[\Bigg(\frac{d}{dr} \left\| \tilde{F}_l\left(s + \frac{[ja\delta] + l}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} - \frac{d}{dr} \left\| \tilde{F}_l\left(s + \frac{[ja\delta]}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} \Bigg)^2 \Bigg]^{1/2} \end{aligned} \tag{4.2}$$

$$\begin{aligned} & \le \frac{2}{a\delta} \left(2 \sum_{k=1}^{[a\delta]} \sum_{l=1}^{k-1} \varphi_\infty(|k - l|)^{1/2} + a\delta \varphi_\infty(0)^{1/2} \right) \\ & \qquad \times \sup\left\{ E\left[\left(\frac{d}{dr} \|\tilde{F}_0(s, r) - I\|_0 \Big|_{r=0} - \frac{d}{dr} \|\tilde{F}_0(s', r) - I\|_0 \Big|_{r=0} \right)^2 \right]; |s - s'| \le \delta \right\} \\ & \le 2\left(2 \sum_{k=1}^{\infty} \varphi_\infty(k)^{1/2} + 1 \right) \\ & \qquad \times \sup\left\{ E\left[\left(\frac{d}{dr} \|\tilde{F}_0(s, r) - I\|_0 \Big|_{r=0} - \frac{d}{dr} \|\tilde{F}_0(s', r) - I\|_0 \Big|_{r=0} \right)^2 \right]; |s - s'| \le \delta \right\}, \end{aligned}$$

which tends to zero as $\delta \to 0$ by the dominated convergence theorem. □

PROOF OF THEOREM 2.3. Take $\Gamma_{\varepsilon,t} = \overline{U}$. Then for all a,

$$P\{\tilde{Z}^a(s,t) \in \overline{U}\} = 1, \tag{4.3}$$

satisfying condition (i) of Theorem 2.1. For condition (ii), if $g_1, g_2 \in U$, then for some constant K,

$$\rho(g_1, g_2) \le K\|\tilde{g}_1 - \tilde{g}_2\|_0,$$

where $\tilde{g}_i$ is the matrix representative for g_i, $i = 1, 2$:

$$\begin{aligned}
&w(\tilde{Z}^a(s,\cdot), \delta, T)\\
&\quad = \sup\{\rho(\tilde{Z}^a(s,t_1), \tilde{Z}^a(s,t_2)); s \le t_1, t_2 \le T+s, |t_1 - t_2| < \delta\}\\
&\quad \le K \sup\{\|\tilde{Z}^a(s,t_1) - \tilde{Z}^a(s,t_2)\|_0; s \le t_1, t_2 \le T+s, |t_1 - t_2| < \delta\}\\
&\quad \le K \sup\left\{\sum_{j=[at_1]+1}^{[at_2]} \left\|\tilde{F}_j\left(\frac{j}{a}, \frac{1}{\sqrt{a}}\right) - I\right\|_0 \left\|\tilde{Z}^a\left(s, \frac{j-1}{a}\right)\right\|_0;\right.\\
&\qquad\qquad \left. s \le t_1, t_2 \le T+s, |t_1 - t_2| < \delta\right\}\\
&\quad \le K(K \operatorname{diam} U + 1) \sup\left\{\left|\sum_{j=[at_1]+1}^{[at_2]} \frac{1}{\sqrt{a}} \frac{d}{dr}\left\|\tilde{F}_j\left(\frac{j}{a}, r\right) - I\right\|_0\right|_{r=0};\right.\\
&\qquad\qquad \left. s \le t_1, t_2 \le T+s, |t_1 - t_2| < \delta\right\}\\
&\qquad + \sup\left\{\sum_{j=[at_1]+1}^{[at_2]} \int_0^{1/\sqrt{a}} \left(\frac{1}{\sqrt{a}} - r\right) \frac{d^2}{dr^2}\left\|\tilde{F}_j\left(\frac{j}{a}, r\right) - I\right\|_0 dr;\right.\\
&\qquad\qquad \left. s \le t_1, t_2 \le T+s, |t_1 - t_2| < \delta\right\} \qquad (4.4)\\
&\quad \le K(K \operatorname{diam} U + 1)\\
&\qquad \times \left(2 \sup_{0 \le j \le T/\delta} \left\{\sup_{0 \le u \le \delta} \left|\frac{1}{\sqrt{a}} \sum_{k=1}^{[au]}\right.\right.\right.\\
&\qquad\qquad \left.\left. \times \frac{d}{dr}\left\|\tilde{F}_{[(s+j\delta)a]+k}\left(s + \frac{[ja\delta]+k}{a}, r\right) - I\right\|_0\right|_{r=0}\right\}\\
&\qquad + \sup\left\{\frac{1}{2a} \sum_{j=[at_1]+1}^{[at_2]} \sup_{0 \le p \le 1} \sup_{0 \le u < \infty} \|D_2^2 F_j(u,p)\|_0;\right.\\
&\qquad\qquad \left.\left. s \le t_1, t_2 \le T+s, |t_1 - t_2| < \delta\right\}\right).
\end{aligned}$$

Choose δ sufficiently small so that

$$\tag{4.5} \begin{aligned} \frac{T}{\delta} \sup_{0 \le j \le T/\delta} \Bigg| & P\Bigg\{ 2K(K \operatorname{diam} U + 1) \\ & \quad \times \sup_{0 \le u \le \delta} \left| \frac{1}{\sqrt{a}} \sum_{k=1}^{[au]} \frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta] + k}{a}, r \right) - I \right\|_0 \Bigg|_{r=0} \right| > \varepsilon \Bigg\} \\ & - P\Bigg\{ 2K(K \operatorname{diam} U + 1) \\ & \quad \times \sup_{0 \le u \le \delta} \left| \frac{1}{\sqrt{a}} \sum_{k=1}^{[au]} \frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta]}{a}, r \right) - I \right\|_0 \Bigg|_{r=0} \right| > \varepsilon \Bigg\} \Bigg| < \varepsilon \end{aligned}$$

and

$$\tag{4.6} \begin{aligned} \operatorname*{limit}_{a \to \infty} P\Bigg\{ & K(K \operatorname{diam} U + 1) \sup \Bigg\{ \frac{1}{2a} \sum_{j=[at_1]+1}^{[at_2]} \sup_{0 \le p \le 1} \sup_{0 \le u \le \delta} \| D_2^2 F_j(u, p) \|_0; \\ & s \le t_1, t_2 \le T + s, |t_1 - t_2| < \delta \Bigg\} > \varepsilon \Bigg\} = 0. \end{aligned}$$

Inequality (4.5) follows from Lemma 4.1. Inequality (4.6) follows from the ergodic theorem if

$$\delta < \varepsilon / K(K \operatorname{diam} U + 1) E\left[\sup_{0 \le p \le 1} \sup_{0 \le u < \infty} \| D_2^2 F_0(u, p) \|_0 \right].$$

Therefore, if $|||\cdot|||_0 = \sup_{0 \le p \le 1} \sup_{0 \le u < \infty} \| \cdot \|_0$,

$$\tag{4.7} \begin{aligned} & P\{ w(\tilde{Z}^a(s, \cdot), \delta, T) > \varepsilon \} \\ & \le P\Bigg\{ K(K \operatorname{diam} U + 1) \\ & \quad \times 2 \sup_{0 \le u \ne \delta} \left\{ \left| \frac{1}{\sqrt{a}} \sum_{k=1}^{[au]} \frac{d}{dr} \left\| \tilde{F}_{[(s+j\delta)a]+k}\left(s + \frac{[ja\delta] + k}{a}, r \right) - I \right\|_0 \Bigg|_{r=0} \right| > \varepsilon \right\} \Bigg\} \\ & \quad + P\Bigg\{ K(K \operatorname{diam} U + 1) \sup \Bigg\{ \frac{1}{2a} \sum_{j=[at_1]+1}^{[at_2]} |||D_2^2 F_j|||_0; \\ & \qquad s \le t_1, t_2 \le T + s, |t_1 - t_2| < \delta \Bigg\} > \varepsilon \Bigg\} \end{aligned}$$

$$\leq \frac{T}{\delta} \sup_{0 \leq j \leq T/\delta} \Bigg\{ P\Bigg\{ K(K \operatorname{diam} U + 1)$$

$$\times 2 \sup_{0 \leq u \leq \delta} \left\{ \left| \frac{1}{\sqrt{a}} \sum_{k=1}^{[au]} \frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta]}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} \right| \right\} > \varepsilon \Bigg\} \Bigg\} + \varepsilon$$

$$+ P\Bigg\{ K(K \operatorname{diam} U + 1) \sup\Bigg\{ \frac{1}{2a} \sum_{j=[at_1]+1}^{[at_2]} |||D_2^2 F_j|||_0;$$

$$s \leq t_1, t_2 \leq T + s, |t_1 - t_2| < \delta \Bigg\} > \varepsilon \Bigg\}.$$

Note that $E[(d/dr\|\tilde{F}_k(u, r) - I\||_{r=0})^2] \leq E[|||D_2\tilde{F}_k|||_0^2] < \infty$ and that $E[d/dr\|\tilde{F}_k(u, r) - I\|_0|_{r=0}] = 0$. Thus for $0 \leq j \leq T/\delta$,

$$\frac{1}{\sqrt{a}} \sum_{k=1}^{[au]} \frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta]}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} \tag{4.8}$$

converge weakly as $a \to \infty$ to a mean 0 Wiener process on the interval $[0, \delta]$ (see [3], page 175). Because the variance parameter is bounded by $E[|||D_2\tilde{F}_k|||_0^2]$,

$$\begin{aligned} &\operatorname*{limit}_{\delta \to 0} \operatorname*{limit}_{a \to \infty} \frac{T}{\delta} \sup_{0 \leq j \leq T/\delta} P\Bigg\{ K(K \operatorname{diam} U + 1) \\ &\qquad \times 2 \sup_{0 \leq u \leq \delta} \left| \frac{1}{\sqrt{a}} \sum_{k=1}^{[au]} \frac{d}{dr} \left\| \tilde{F}_k\left(s + \frac{[ja\delta]}{a}, r\right) - I \right\|_0 \Bigg|_{r=0} \right| > \varepsilon \Bigg\} = 0. \end{aligned} \tag{4.9}$$

Therefore by (4.6), (4.7) and (4.9),

$$\limsup_{a \to \infty} P\{ w(\tilde{Z}^a(s, \cdot), \delta, T) > \varepsilon \} \leq \varepsilon, \tag{4.10}$$

satisfying condition (ii) of Theorem 2.1 and thus completing the proof of Theorem 2.3. □

Proof of Theorem 2.7. For $t > s$, set $\mathscr{N}_s^t = \sigma\{\zeta(s, u); s \leq u \leq t\}$ and define

$$\Psi_{s,t}(\zeta) = \Psi(\zeta(s, s_1), \zeta(s, s_2), \ldots, \zeta(s, s_k)),$$

where $\Psi: G^k \to [0, 1]$ is a measurable mapping and $s < s_1 < \cdots < s_k < t$. Then $\Psi_{s,t}$ is $\mathscr{N}_s^t$ measurable. Choose a subsequence of $\{\tilde{\mathbf{P}}_s^a; a \geq 1\}$ with weak limit $\tilde{\mathbf{P}}_s^\infty$. Let $\mathbf{E}_s^\infty$ denote $\tilde{\mathbf{P}}_s^\infty$-expectation and $\mathbf{E}_s^a$ denote $\tilde{\mathbf{P}}_s^a$-expectation. Writing τ for

$\tau(U, s)$, we must show, for any $s \le t < u$ and for any choice of $\Psi_{s,t}$, that

$$\mathbf{E}_s^\infty\Big[\Big(f(\zeta(s, u \wedge \tau)) - f(\zeta(s, t \wedge \tau)) - \int_{t\wedge\tau}^{u\wedge\tau} \big(\tfrac{1}{2}\sigma^2(v) + \mu(v)\big) f(\zeta(s, v))\, dv\Big)\Psi_{s,t}(\zeta)\Big] = 0. \tag{4.11}$$

Choose a sequence δ which tends to zero in such a way that $\tilde{\mathbf{P}}_s^\infty\{\tau = i\delta;\, i \in \mathbf{N}\} = 0$, let a tend to infinity along the sequence described above and set $T_a = \inf\{t > s\colon Z^a(s, t) \notin U\}$. Then, because $\tilde{\mathbf{P}}_s^\infty$ is supported on the continuous paths,

$$\begin{aligned}
&\mathbf{E}_s^\infty[(f(\zeta(s, u \wedge \tau)) - f(\zeta(s, t \wedge \tau)))\Psi_{s,t}(\zeta)] \\
&\quad = \lim_{\delta\to 0} \mathbf{E}_s^\infty\Big[\Big(f\Big(\zeta\Big(s, \delta\Big(\Big[\frac{u}{\delta}\Big] \wedge \Big[\frac{\tau}{\delta}\Big]\Big)\Big)\Big) - f\Big(\zeta\Big(s, \delta\Big(\Big[\frac{t}{\delta}\Big] \wedge \Big[\frac{\tau}{\delta}\Big]\Big)\Big)\Big)\Big)\Psi_{s,t}(\zeta)\Big] \\
&\quad = \lim_{\delta\to 0}\lim_{a\to\infty} \mathbf{E}_s^a\Big[\sum_{i=[t/\delta]}^{[u/\delta]-1} \big(f(\zeta(s, \delta(i+1))) - f(\zeta(s, \delta i)) I_{\{\tau > \delta(i+1)\}}\big)\Psi_{s,t}(\zeta)\Big] \\
&\quad = \lim_{\delta\to 0}\lim_{a\to\infty} E\Big[\sum_{i=[t/\delta]}^{[u/\delta]-1} \big(f(Z^a(s, \delta i) Z^a(\delta i, \delta(i+1))) - f(Z^a(s, \delta i)) I_{\{T_a > \delta(i+1)\}}\big)\Psi_{s,t}(Z^a)\Big] \\
&\quad = \lim_{\delta\to 0}\lim_{a\to\infty} E\Big[\sum_{i=[t/\delta]}^{[u/\delta]-1} \big(f(Z^a(s, \delta i - l(a)) Z^a(\delta i, \delta(i+1))) - f(Z^a(s, \delta i - l(a))) I_{\{T_a > \delta(i+1)\}}\big)\Psi_{s,t}(Z^a)\Big],
\end{aligned} \tag{4.12}$$

where $l(a) \to 0$ as $a \to \infty$. To explain this, note that the absolute value for the difference in corresponding terms in each of the last two sums is bounded by

$$2\|Df\|_\infty E[\rho(Z^a(s, \delta i - l(a)), Z^a(s, \delta i)) \wedge \operatorname{diam} U] \tag{4.13}$$

and the term inside the expectation tends to zero in probability as $a \to \infty$ because the oscillation of $Z^a(s, \cdot)$ tends to zero in probability. $\|Df\|_\infty$ is the maximum value of the derivative of f,

$$\begin{aligned}
&\mathbf{E}_s^\infty[(f(\zeta(s, u \wedge \tau)) - f(\zeta(s, t \wedge \tau)))\Psi_{s,t}(\zeta)] \\
&\quad = \lim_{\delta\to 0}\lim_{a\to\infty} E\Big[\sum_{i=[t/\delta]}^{[u/\delta]-1} \big(f(Z^a(s, \delta i - l(a)) Z^a(\delta i, \delta(i+1))) - f(Z^a(s, \delta i - l(a))) I_{\{T_a > \delta i - l(a)\}}\big)\Psi_{s,t}(Z^a)\Big].
\end{aligned} \tag{4.14}$$

This equality holds because the difference of this sum and the last sum has only two nonzero terms, which are bounded in absolute value by

$$E\left[\|Df\|_\infty w(Z^a(s,\cdot),\delta,u-s)\wedge 2\|f\|_\infty\right] \tag{4.15}$$

which by (4.10) is uniformly bounded in a and tends to zero as $\delta\to 0$. Choose a term from the sum in (4.14) and define

$$\begin{aligned} G_i(p_1,\dots,p_{[a\delta]}) = f\Bigg(&Z^a(s,\delta i-l(a))F_{[a\delta i]+1}\left(\frac{[a\delta i]+1}{a},p_1\right)\\ &\times\cdots\times F_{[a\delta(i+1)]}\left(\frac{[a\delta(i+1)]}{a},p_{[a\delta]}\right)\Bigg). \end{aligned} \tag{4.16}$$

Then

$$\begin{aligned} &f(Z^a(s,\delta i-l(a))Z^a(\delta i,\delta(i+1)))-f(Z^a(s,\delta i-l(a)))\\ &\quad= G_i\left(\frac{1}{\sqrt{a}},\dots,\frac{1}{\sqrt{a}}\right)\\ &\quad= G_i(0,\dots,0)+\frac{1}{\sqrt{a}}\sum_{j=1}^{[a\delta]}(\partial_j G_i)(0,\dots,0)\\ &\qquad+\sum_{j=1}^{[a\delta]}\sum_{k=1}^{[a\delta]}\int_0^{1/\sqrt{a}}\left(\frac{1}{\sqrt{a}}-p\right)(\partial_j\partial_k G_i)(p,\dots,p)\,dp. \end{aligned} \tag{4.17}$$

For these three terms we have

$$G_i(0,\dots,0)=0, \tag{4.18}$$

$$(\partial_j G_i)(0,\dots,0)=\left(D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},0\right)f\right)(Z^a(s,\delta i-l(a))) \tag{4.19}$$

and

$$\begin{aligned} (\partial_j\partial_k G_i)(p,\dots,p) &= (\partial_k\partial_j G_i)(p,\dots,p)\\ &= \left(D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)D_2F_{[a\delta i]+k}\left(\frac{[a\delta i]+k}{a},p\right)f\right)\\ &\quad\left(Z^a(s,\delta i-l(a))F_{[a\delta i]+1}\left(\frac{[a\delta i]+1}{a},p\right)\right.\\ &\quad\left.\times\cdots\times F_{[a\delta(i+1)]}\left(\frac{[a\delta(i+1)]}{a},p\right)\right) \end{aligned} \tag{4.20}$$

if $j \neq k$. The diagonal terms in the sum

$$(\partial_j\partial_j G_i)(p,\dots,p) = \left(D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)^2 + D_2^2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)f\right)$$

$$\left(Z^a\left(s,\delta i - l(a)F_{[a\delta i]+1}\left(\frac{[a\delta j]+1}{a},p\right)\times\cdots\times F_{[\delta(i+1)]}\left(\frac{[a\delta(i+1)]}{a},p\right)\right)\right). \tag{4.21}$$

Therefore,

$$\begin{aligned}
&E_s^\infty[(f(\zeta(s,u\wedge\tau)) - f(\zeta(s,t\wedge\tau)))\Psi_{s,t}(\zeta)]\\
&= \underset{\delta\to 0}{\text{limit}}\ \underset{a\to\infty}{\text{limit}}\, E\Bigg[\sum_{i=[t/\delta]}^{[u/\delta]-1}\frac{1}{\sqrt{a}}\sum_{j=1}^{[a\delta]}\left(D_2F_{[a\delta i]+1}\left(\frac{[a\delta i]+j}{a},0\right)f\right)(Z^a(s,\delta i - l(a)))\\
&\quad + \int_0^{1/\sqrt{a}}\left(\frac{1}{\sqrt{a}}-p\right)\Bigg(\sum_{j=1}^{[a\delta]}\left(D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)^2 + D_2^2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)\right)\\
&\quad + 2\sum_{j=1}^{[a\delta]}\sum_{k=1}^{j-1}\left(D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)D_2F_{[a\delta i]+k}\left(\frac{[a\delta i]+k}{a},p\right)\right)\\
&\quad \times f\Bigg(Z^a(s,\delta i - l(a))F_{[a\delta i]+1}\left(\frac{[a\delta i]+1}{a},p\right)\\
&\quad \times\cdots\times F_{[a\delta(i+1)]}\left(\frac{[a\delta(i+1)]}{a},p\right)\Bigg)\,dp\Bigg)I_{\{T_a>\delta i - l(a)\}}\Psi_{s,t}(Z^a)\Bigg].
\end{aligned}$$

If $l(a)$ is chosen in such a way that $al(a)$ tends to infinity with a, then the first line in this double limit vanishes in the limit on a. To see this, note that the Hölder inequality for uniformly mixing sequences implies that

$$\begin{aligned}
&\left|E\left[\frac{1}{\sqrt{a}}\sum_{j=1}^{[a\delta]}\left(D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},0\right)f\right)\times(Z^a(s,\delta i - l(a)))I_{\{T_a>\delta i - l(a)\}}\Psi_{s,t}(Z^a)\right]\right|\\
&\le 2\,d\varphi_\infty(al(a))^{1/2}E\left[\left\|\frac{1}{\sqrt{a}}\sum_{j=1}^{[a\delta]}D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},0\right)\right\|_0^2\right]\|f\|_\infty\\
&\le 2\,d\varphi_\infty(al(a))^{1/2}\left(E\left[\left\|\frac{1}{\sqrt{a}}\sum_{j=1}^{[a\delta]}D_2F_j(\delta i,0)\right\|_0^2\right]+o(\delta)\right)\|f\|_\infty
\end{aligned} \tag{4.23}$$

by a computation similar to the proof of Lemma 4.1. The expectation is bounded in a by the central limit theorem for the uniformly mixing sequence in R^d. Now let $a \to \infty$ to substantiate the claim. For the second line in this double limit note that

$$\mathbf{E}_s^\infty\left[\int_{t\wedge\tau}^{u\wedge\tau}\frac{1}{2}E\left[D_2F_0(v,0)^2+D_2^2F_0(v,0)\right]f(\zeta(s,v))\,dv\,\Psi_{s,t}(\zeta)\right]$$

$$= \lim_{\delta\to 0}\lim_{a\to\infty}\mathbf{E}_s^a\left[\sum_{i=[t/\delta]}^{[u/\delta]-1}\frac{1}{2}E\left[D_2F_0(\delta i,0)^2+D_2^2F_0(\delta i,0)\right]\times f(\zeta(s,\delta i-l(a)))\,\delta I_{\{\tau>\delta i-l(a)\}}\Psi_{s,t}(\zeta)\right]$$

$$= \lim_{\delta\to 0}\lim_{a\to\infty}\mathbf{E}_s^a\left[\sum_{i=[t/\delta]}^{[u/\delta]-1}\frac{\delta}{2}E\left[D_2F_0(\delta i,0)^2+D_2^2F_0(\delta i,0)\right]\times f(\zeta(s,\delta i-l(a)))\,\delta I_{\{\tau>\delta i-l(a)\}}\Psi_{s,t}(\zeta)\right]$$

$$= \lim_{\delta\to 0}\lim_{a\to\infty}\left[\sum_{i=[t/\delta]}^{[u/\delta]-1}\frac{\delta}{2}E\left[D_2F_0(\delta i,0)^2+D_2^2F_0(\delta i,0)\right]\times\mathbf{E}_s^a\left[f(\zeta(s,\delta i-l(a)))I_{\{\tau>\delta i-l(a)\}}\Psi_{s,t}(\zeta)\right]\right]$$

$$(4.24)\quad = \lim_{\delta\to 0}\lim_{a\to\infty}\sum_{i=[t/\delta]}^{[u/\delta]-1}E\left[\int_0^{1/\sqrt{a}}\left(\frac{1}{\sqrt{a}}-p\right)\sum_{j=1}^{[a\delta]}\left(D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)^2+D_2^2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)\right)dp\right]\times E\left[f(Z^a(s,\delta i-l(a)))I_{\{T^a>\delta i-l(a)\}}\Psi_{s,t}(Z^a)\right]$$

$$= \lim_{\delta\to 0}\lim_{a\to\infty}\sum_{i=[t/\delta]}^{[u/\delta]-1}E\left[\int_0^{1/\sqrt{a}}\left(\frac{1}{\sqrt{a}}-p\right)\sum_{j=1}^{[a\delta]}\left(D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)^2+D_2^2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)\right)dp\times f(Z^a(s,\delta i-l(a)))I_{\{T^a>\delta i-l(a)\}}\Psi_{s,t}(Z^a)\right]$$

$$= \lim_{\delta\to 0}\lim_{a\to\infty}E\left[\sum_{i=[t/\delta]}^{[u/\delta]-1}\int_0^{1/\sqrt{a}}\left(\frac{1}{\sqrt{a}}-p\right)\sum_{j=1}^{[a\delta]}\left(D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)^2+D_2^2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a},p\right)\right)dp\times f\left(Z^a(s,\delta i-l(a))F_{[a\delta i]+1}\left(\frac{[a\delta i]}{a},p\right)\times\cdots\times F_{[a\delta(i+1)]}\left(\frac{[a\delta(i+1)]}{a},p\right)\right)dp\right)I_{\{T_a>\delta i-l(a)\}}\psi_{s,t}(Z^a)\right],$$

which is exactly this second and third lines. However, the last three inequalities require a justification. For the first of these three, subtract the expression from the one above and take the absolute value. We then find the bound

$$\sum_{i=[t/\delta]-1}^{[u/\delta]} E\left[\left\| \int_0^{1/\sqrt{a}} \left(\frac{1}{\sqrt{a}} - p\right)\left(\sum_{j=1}^{[a\delta]}\left(D_2 F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a}, p\right)^2 + D_2^2 F_{[\delta i]+j}\left(\frac{[a\delta i]+j}{a}, p\right)\right)\right.\right.\right.$$

$$\left.\left.\left. - \left(D_2 F_{[a\delta i]+j}(\delta i, 0)^2 + D_2^2 f_{[a\delta i]+j}(\delta i, 0)\right)\right)\right\|_0\right] \|f\|_\infty$$

$$\begin{aligned}(4.25)\quad &\le \sum_{i=[t/\delta]-1}^{[u/\delta]} \frac{1}{2a}\sum_{j=1}^{[a\delta]}\left(E\left[\sup_{0\le p\le 1/\sqrt{a}}\left\|D_2F_j\left(\frac{[a\delta i]+j}{a}, p\right)^2 - D_2F_j(\delta i, 0)^2\right\|_0\right]\right.\\
&\qquad \left. + E\left[\sup_{0\le p\le 1/\sqrt{a}}\left\|D_2^2F_j\left(\frac{[a\delta i]+j}{a}, p\right) - D_2^2F_j(\delta i, 0)\right\|_0\right]\right)\|f\|_\infty\end{aligned}$$

$$\le \frac{u-t}{2}\sup\left\{E\left[\sup_{0\le p\le 1/\sqrt{a}} \|D_2F_0(v_2, p)^2 - D_2F_0(v_1, 0)^2\|_0\right]\right.$$

$$\left. + E\left[\sup_{0\le p\le 1/\sqrt{a}} \|D_2^2F_0(v_2, p) - D_2^2F_0(v_1, 0)\|_0\right]; u \le v_1 \le v_2 \le t, v_2 - v_1 < \delta\right\}\|f\|_\infty,$$

which tends to zero as $a \to \infty$ and $\delta \to \infty$. Again, $\|\cdot\|_0$ denotes the operator norm and $\|\cdot\|_\infty$ the supremum norm. For the next equality we again turn to the Hölder inequality for uniform mixing. The absolute value of the difference here is bounded for each term in the sum on i by

$$\begin{aligned}(4.26)\quad &\|f\|_\infty \varphi_\infty(al(a)) E\left[\left\|\int_0^{1/\sqrt{a}}\left(\frac{1}{\sqrt{a}} - p\right)\sum_{j=1}^{[a\delta]}\left(D_2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a}, p\right)^2\right.\right.\right.\\
&\qquad \left.\left.\left. - D_2^2F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a}, p\right)\right) dp\right\|_0\right]\\
&\le \frac{\delta}{2}\|f\|_\infty\varphi_\infty(al(a))E\left[\sup_{0\le p\le 1}\sup_{0\le v<\infty}\left(\|D_2F_0(v, p)\|_0^2 + \|D_2^2F_0(v, p)\|_0\right)\right].\end{aligned}$$

Now let $a \to \infty$. For the last equality, again subtract and take the absolute value. we then find the bound

$$\begin{aligned}(4.27)\quad &\sum_{i=[t/\delta]-1}^{[u/\delta]} \frac{1}{2a}\sum_{j=1}^{[a\delta]} E\left[\sup_{0\le p\le 1/\sqrt{a}}\left\|D_2F_j\left(\frac{[a\delta i]+j}{a}, p\right)^2\right.\right.\\
&\qquad \left. - D_2^2F_j\left(\frac{[a\delta i]+j}{a}, p\right)\right\|_0 \|Df\|_\infty\\
&\qquad \left. \times \rho\left(F_1\left(\frac{[a\delta i]}{a}, p\right)\cdots F_{[a\delta]}\left(\frac{[a\delta(i+1)]}{a}, p\right), e\right) \wedge \operatorname{diam} U\right]\end{aligned}$$

$$\leq \frac{u-t}{2} \sup_{t \leq i\delta \leq u} E\left[\sup_{0 \leq p \leq 1/\sqrt{a}} \left\| D_2^2 F_j\left(\frac{[a\delta i]+j}{a}, p \right)^2 + D_2^2 F_j\left(\frac{[a\delta i]+j}{a}, p \right) \right\|_0 \|Df\|_\infty \times \sup_{0 \leq p \leq 1/\sqrt{a}} \rho\left(F_1\left(\frac{[a\delta i]+j}{a}, p \right) \cdots \times F_{[a\delta]}\left(\frac{[a\delta i]+j}{a}, p \right), e \right) \wedge \operatorname{diam} U \right].$$

The first line in the expectation is bounded uniformly in L^1 and the second converges to zero in probability uniformly in a and i as $\delta \to 0$ [see (4.8)].

Taking stock of the situation at this juncture, we find that

$$\begin{aligned}
&\mathbf{E}_s^\infty\left[\left(f(\zeta(s, u \wedge \tau)) - f(\zeta(s, t \wedge \tau)) - \int_{t\wedge\tau}^{u\wedge\tau} \frac{1}{2} E\left[D_2 F_0(v,0)^2 + D_2^2 F_0(v,0) \right] f(\zeta(s,v))\, dv \right) \Psi_{s,t}(\zeta) \right] \\
&= \operatorname*{limit}_{\delta\to 0} \operatorname*{limit}_{a\to\infty} 2E\left[\sum_{i=[t/\delta]}^{[u/\delta]-1} \left(\sum_{j=1}^{[a\delta]} \sum_{k=1}^{j-1} \int_0^{1/\sqrt{a}} \left(\frac{1}{\sqrt{a}} - p \right) \times D_2 F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a}, p \right) \times D_2 F_{[a\delta i]+k}\left(\frac{[a\delta i]+k}{a}, p \right) \right) \right. \\
&\qquad \times f(Z^a(s, \delta i - l(a))) \times F_{[a\delta i]+1}\left(\left(\frac{[a\delta i]+1}{a}, p \right) \times \cdots \right. \\
&\qquad \left.\left. \times D_2 F_{[a\delta(i+1)]}\left(\frac{[a\delta(i+1)]}{a}, p \right) dp \right) I_{\{T_a > \delta i - l(a)\}} \Psi_{s,t}(Z^a) \right].
\end{aligned} \tag{4.28}$$

In a line of argument similar to (4.24), we have

$$\begin{aligned}
&\mathbf{E}_s^\infty\left[\int_{t\wedge\tau}^{u\wedge\tau} \operatorname*{limit}_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{j-1} E\left[D_2 F_j(v,0) D_2 F_k(v,0) \right] f(\zeta(s,v))\, dv\, \Psi_{s,t}(\zeta) \right] \\
&= \operatorname*{limit}_{\delta\to 0} \operatorname*{limit}_{a\to\infty} \left[\sum_{i=[t/\delta]}^{[u/\delta]-1} E\left[\frac{1}{a} \sum_{j=1}^{[a\delta]} \sum_{k=1}^{j-1} D_2 F_j(\delta i, 0) D_2 F_k(\delta i, 0) \right] \times \mathbf{E}_s^a\left[f(\zeta(s, \delta i - l(a))) I_{\{\tau > \delta i - l(a)\}} \Psi_{s,t}(\zeta) \right] \right]
\end{aligned} \tag{4.29}$$

$$= \lim_{\delta\to 0} \lim_{a\to\infty} \Bigg[\sum_{i=[t/\delta]}^{[u/\delta]-1} E\Bigg[\frac{1}{a} \sum_{j=1}^{[a\delta]} D_2 F_{[a\delta i]+j}(\delta i, 0) D_2 F_{[a\delta i]+k}(\delta i, 0) \Bigg] \times f(Z^a(s, \delta i - l(a))) I_{\{T^a > \delta i - l(a)\}} \Psi_{s,t}(Z^a) \Bigg]$$

$$= \lim_{\delta\to 0} \lim_{a\to\infty} \mathbf{E}_s^\infty \Bigg[\sum_{i=[t/\delta]}^{[u/\delta]-1} 2E\Bigg[\int_0^{1/\sqrt{a}} \left(\frac{1}{\sqrt{a}} - p \right) \times \sum_{j=1}^{[a\delta]} \sum_{k=1}^{j-1} D_2 F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a}, p \right) \times D_2 F_{[a\delta i]+k}\left(\frac{[a\delta i]+k}{a}, p \right) dp \Bigg] \times f(Z^a(s, \delta i - l(a))) I_{\{T^a > \delta i - l(a)\}} \Psi_{s,t}(Z^a) \Bigg]$$

$$= \lim_{\delta\to 0} \lim_{a\to\infty} \mathbf{E}_s^\infty \Bigg[\sum_{i=[t/\delta]}^{[u/\delta]-1} 2E \int_0^{1/\sqrt{a}} \left(\frac{1}{\sqrt{a}} - p \right) \times \sum_{j=1}^{[a\delta]} \sum_{k=1}^{j-1} D_2 F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a}, p \right) \times D_2 F_{[a\delta i]+k}\left(\frac{[a\delta i]+k}{a}, p \right) dp \Bigg] \times f(Z^a(s, \delta i - l(a))) I_{\{T^a > \delta i - l(a)\}} \Psi_{s,t}(Z^a) \Big]$$

$$= \lim_{\delta\to 0} \lim_{a\to\infty} \Bigg[\sum_{i=[t/\delta]}^{[u/\delta]-1} 2E \sum_{j=1}^{[a\delta]} \sum_{k=1}^{j-1} \Bigg[\int_0^{1/\sqrt{a}} \left(\frac{1}{\sqrt{a}} - p \right) \times D_2 F_{[a\delta i]+j}\left(\frac{[a\delta i]+j}{a}, p \right) \times D_2 F_{[a\delta i]+k}\left(\frac{[a\delta i]+k}{a}, p \right) \Bigg] \times f(Z^a(s, \delta i - l(a))) F_{[a\delta i]+1}\left(\frac{[a\delta i]+1}{a}, p \right) \times \cdots \times F_{[a\delta(i+1)]}\left(\frac{[a\delta(i+1)]}{a}, p \right) dp\, I_{\{T_a > \delta i - l(a)\}} \Psi_{s,t}(Z^a) \Bigg].$$

Some words of explanation are necessary to justify this sequence of equalities. The limit on n exists for nearly the same reason that the limits defining $\sigma^2(v)$ and $\beta(v)$ exist. This limit may be combined with the limit on a by a standard theorem on weak convergence. We justify the last three equalities in each instance, by first subtracting and then taking absolute values. For the first of these three we start with the bound

$$\sum_{1=[t/\delta]}^{[u/\delta]-1} \frac{1}{a} \sum_{j=1}^{[a\delta]} \sum_{k=1}^{j-1} \sup_{0\le p\le 1/\sqrt{a}} \left\| E\left[D_2 F_j\left(\frac{[a\delta i]+j}{a}, p\right) D_2 F_k\left(\frac{[a\delta i]+k}{a}, p\right) - D_2F_j(\delta i, 0) D_2 F_k(\delta i, 0)\right]\right\|_0 \|f\|_\infty$$

$$\le \frac{u-t}{a\delta}\left(\sup\left\{ \sum_{j=1}^{[a\delta]}\sum_{k=1}^{j-1} \sup_{0\le p\le 1/\sqrt{a}} \| E[D_2F_j(v,p) D_2 F_k(v,p) - D_2F_j(v,0)D_2F_k(v,0)] \|_0;\ u\le v\le t\right\} \|f\|_\infty + o(\delta)\right)$$

$$\le \frac{u-t}{a\delta}\left(\sup_{u\le v\le t}\left\{ \sum_{j=1}^{[a\delta]}\sum_{k=1}^{j-1} \sup_{0\le p\le 1/\sqrt{a}} \big(\|E[D_2F_j(v,p)(D_2F_k(v,p) - D_2F_k(v,0))]\|_0 + \|E[(D_2F_j(v,p) - D_2F_j(v,0))D_2F_k(v,0)]\|_0\big);\ u\le v\le t\right\}\|f\|_\infty + o(\delta)\right) \tag{4.30}$$

$$\le \frac{u-t}{a\delta}\left(\sup_{u\le v\le t}\left\{ \sum_{j=1}^{[a\delta]}\sum_{k=1}^{j-1} \sup_{0\le p\le 1/\sqrt{a}} \frac{1}{a}\|E[D_2^2F_0(v,p)]\|_0 + 2\, d\varphi_\infty(j-k)^{1/2} \sup_{0\le p\le 1/\sqrt{a}} E[\|D_2F_0(v,p)\|_0^2]^{1/2} \times \sup_{0\le p\le 1/\sqrt{a}} E[\|D_2F_0(v,p) - D_2F_0(v,0)\|_0^2]^{1/2}\right\}\|f\|_\infty + o(\delta)\right)$$

$$\le \frac{\delta(u-t)}{2} \sup_{u\le v\le t}\left\{ \sup_{0\le p\le 1/\sqrt{a}} E[\|D_2^2F_0(v,p)\|_0] + \sum_{j=1}^{\infty}\varphi_\infty(j)^{1/2} \sup_{0\le p\le 1/\sqrt{a}} E[\|D_2F_0(v,p)\|_0^2]^{1/2} \times \sup_{0\le p\le 1/\sqrt{a}} E[\|D_2F_0(v,p) - D_2F_0(v,0)\|_0^2]\right\}\|f\|_\infty + \frac{o(\delta)}{\delta}.$$

Now let $a\to\infty$ and $\delta\to 0$. The next equality uses the Hölder inequality in much the same manner as in (4.27). For the last inequality use the bound

$$\frac{u-t}{2a\delta} \sup_{t\le v\le u}\sup_{0\le v\le 1/\sqrt{a}} \left| E\left[\sum_{j=1}^{[a\delta]}\sum_{k=1}^{j-1} D_2F_j\left(v+\frac{j}{a}, p\right) D_2 f_k\left(v+\frac{k}{a}, p\right) \times\left(f\left(Z^a(s, v-l(a))F_{[av]+1}\left(v+\frac{1}{a}, p\right) \times\cdots\times F_{[a(v+\delta)]}(v+\delta, p)\right) - f(Z^a(s,v))\right) \times I_{\{T_a>\delta i - l(a)\}}\Psi_{s,t}(Z^a)\right]\right|. \tag{4.31}$$

Now,

$$\limsup_{\delta\to 0}\limsup_{a\to\infty}\frac{u-t}{2a\delta}$$

$$\times \sup_{t\le v\le u}\sup_{0\le v\le 1/\sqrt{a}}\left\|E\left[\sum_{j=1}^{[a\delta]}\sum_{k=1}^{j-1} D_2F_j\left(v+\frac{j}{a},p\right)D_2F_k\left(v+\frac{k}{a},p\right)\right]\right\|<\infty$$

by the arguments in (4.30). Also

$$\left|\left(f\left(Z^a(s,v\dot{-}l(a))F_{[av+1]}\left(v+\frac{1}{a},p\right)\cdots F_{[a(v+\delta)]}(v+\delta,p)\right)\right.\right.$$

$$\left.\left.-f(Z^a(s,v))\right)I_{\{T^a>\delta i-l(a)\}}\Psi_{s,t}(Z^a)\right| \tag{4.32}$$

$$\le \|Df\|_\infty\left(\rho\left(F_{[av]+1}\left(v+\frac{1}{a},p\right)\cdots F_{[a(v+\delta)]}(v+\delta,p),e\right)\wedge \operatorname{diam}U\right)$$

and for any $\varepsilon>0$,

$$\limsup_{\delta\to 0}\limsup_{a\to\infty}\sup_{s\le v\le u}P\left\{\left(\rho\left(F_{[av]+1}\left(v+\frac{1}{a},p\right)\right.\right.\right.$$

$$\left.\left.\left.\times\cdots\times F_{[a(v+\delta)]}(v+\delta,p)\right),e\right)>\varepsilon\right\}=0. \tag{4.33}$$

So, the last equality holds by (4.31), (4.32) and (4.33). Summarizing, we find that

$$\mathbf{E}_s^\infty\left[\left(f(\zeta(s,u\wedge\tau))-f(\zeta(s,t\wedge\tau))\right.\right.$$

$$\left.\left.-\int_{t\wedge\tau}^{u\wedge\tau}\tfrac{1}{2}\left[ED_2F_0(v,0)^2+D_2^2F_0(v,0)\right]f(\zeta(s,v))\,dv\right)\Psi_{s,t}(\zeta)\right] \tag{4.34}$$

$$=\mathbf{E}_s^\infty\left[\int_{t\wedge\tau}^{u\wedge\tau}\operatorname*{limit}_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{j-1}ED_2F_j(v,0)D_2F_k(v,0)f(\zeta(s,v))\,dv\,\Psi_{s,t}(\zeta)\right].$$

Therefore

$$
\begin{aligned}
0 = \mathbf{E}_s^{\infty}\Bigg[\Bigg(& f(\zeta(s, u\wedge\tau)) - f(\zeta(s, t\wedge\tau)) \\
& - \int_{t\wedge\tau}^{u\wedge\tau} \operatorname*{limit}_{n\to\infty}\Bigg(\frac{1}{2n}\sum_{j=1}^{n} ED_2F_j(v,0)^2 + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{j-1} ED_2F_j(v,0)D_2F_k(v,0) \\
& \qquad + \frac{1}{2}D_2^2F_0(v,0)\Bigg) f(\zeta(s,v))\,dv\Bigg)\Psi_{s,t}(\zeta)\Bigg] \qquad (4.35) \\
= \mathbf{E}_s^{\infty}\Bigg[\Bigg(& f(\zeta(s, u\wedge\tau)) - f(\zeta(s, t\wedge\tau)) \\
& - \int_{t\wedge\tau}^{u\wedge\tau}\Bigg[\frac{1}{2}\operatorname*{limit}_{n\to\infty}\frac{1}{n}E\sum_{j=1}^{n} D_2F_j(v,0)^2 + \operatorname*{limit}_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{j-1} E\big[D_2F_j(v,0), D_2F_k(v,0)\big] \\
& \qquad + \frac{1}{2}D_2F_0(v,0)\Bigg] f(\zeta(s,v))\,dv\Bigg)\Psi_{s,t}(\zeta)\Bigg] \\
= \mathbf{E}_s^{\infty}\Bigg[\Bigg(& f(\zeta(s, u\wedge\tau)) - f(\zeta(s, t\wedge\tau)) - \int_{t\wedge\tau}^{u\wedge\tau}\Bigg(\frac{1}{2}\sigma^2(v) + \mu(v)\Bigg) f(\zeta(s,v))\,dv\Bigg)\Psi_{s,t}(\zeta)\Bigg],
\end{aligned}
$$

establishing (4.11), and hence Donsker's invariance principle for Lie groups. □

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