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UNUSUAL CLUSTER SETS FOR THE LIL SEQUENCE IN BANACH SPACE[1]

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Let $S_n = X_1 + \cdots + X_n$, where $X_1, X_2, \dots$ are iid Banach-space-valued random variables with weak mean 0 and weak second moments. Let K be the unit ball of the reproducing kernel Hilbert space associated to the covariance of X. The cluster set A of $\{S_n/(2n \log\log n)^{1/2}\}$ is known to be a.s. either empty or have form αK, with $0 \le \alpha \le 1$ determined by a series condition. To show that this series condition is a complete characterization of A, examples are given to show that all $\alpha \in [0, 1)$ do occur; $A = \phi$ and $\alpha = 1$ are already known possibilities. A regularity condition is given under which A must be either ϕ or K. Under stronger moment conditions, a natural necessary and sufficient condition for $A = \phi$ is given.

1. Introduction. Let $X, X_1, X_2, \dots$ be iid random variables with law P taking values in a separable Banach space $(B, \|\cdot\|)$ and defined on some $(\Omega, \mathscr{A}, \mathbb{P})$, let $S_n := X_1 + \cdots + X_n$, and $a_n := (2n \log\log n)^{1/2}$. We say $X \in WM_0^2$ if $Ef(X) = 0$ and $Ef^2(X) < \infty$ for all $f \in B^*$. Let A be the cluster set of the sequence $\{S_n/a_n\}$ in B. It is well-known [see Kuelbs (1981)] that when $X \in WM_0^2$, there is a nonrandom closed set D such that $A = D$ a.s.

The "canonical" value of the cluster set A is the unit ball K of the reproducing kernel Hilbert space $H_P \subset B$ associated to the covariance of X. That is,

$$K = \left\{u_f\colon f \in \overline{B^*}, \|f\|_2 \le 1\right\},$$

where

$$u_f := \int x f(x)\, dP(x) \quad \text{for } f \in L^2(P),$$

$\|\cdot\|_2$ is the $L^2(P)$ norm, and $\overline{B^*}$ is the closure of the dual B^* in $L^2(P)$. K exists as a subset of B whenever $X \in WM_0^2$. An equivalent definition of K is

$$K = \left\{y \in B\colon f(y) \le \|f\|_2 \text{ for all } f \in B^*\right\}. \tag{1.1}$$

Let B_1^* be the unit ball of B^*. Then

$$\sup\left\{\|y\|\colon y \in K\right\} = \sigma := \sup\left\{\|f\|_2\colon f \in B_1^*\right\} < \infty.$$

For details, and more about K, see Goodman, Kuelbs and Zinn (1981).

The reasons K is canonical are twofold. First, intuitively, (1.1) and the one-dimensional law of the iterated logarithm (LIL) make K natural. Second,

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more concretely, a result of Kuelbs (1976) says that if there is a compact set $D \subset B$ such that

$$A = D \quad \text{a.s.} \tag{1.2}$$

and

$$d(S_n/a_n, D) \to 0 \quad \text{a.s.}, \tag{1.3}$$

then this limit set D must be K. Here $d(y, D) := \inf(\|y - z\|\colon z \in D\}$ is the distance from y to D.

When (1.2) and (1.3) hold for some compact $D \subset B$ (necessarily $D = K$) we say X satisfies the compact LIL, denoted by $X \in$ CLIL. When $\{S_n/a_n\}$ stays bounded a.s., we say X satisfies the bounded LIL, denoted by $X \in$ BLIL. By the one-dimensional LIL, $X \in$ BLIL implies $X \in WM_0^2$.

In contrast to the finite-dimensional situation, in infinite-dimensional Banach space (1.2) can hold, but (1.3) fail, for a nonempty bounded $D \subset B$. Further, $X \in$ BLIL does not ensure that $A = K$ a.s.; $A = \varnothing$ a.s. is also possible. (1.2) and (1.3) may hold with $D = K$ even if K is not compact. Examples of these phenomena were given by Goodman, Kuelbs and Zinn (1981) and by Kuelbs (1981).

In all of these examples, the cluster set is either $\varnothing$ or K a.s. There has been speculation in the literature that these might be the only possible values. Indeed, the possible cluster sets are limited by the following characterization of A, from Alexander (1989). Let $\gamma > 1$ and

$$n_k := n_k(\gamma) := [\gamma^k], \qquad I_k := I_k(\gamma) := [n_k, n_{k+1}).$$

When there is no ambiguity or when a result does not depend upon γ we will suppress the γ in this notation.

THEOREM 1.1. *Suppose $X \in WM_0^2$. Define $\alpha \in [0,1]$ by*

$$\alpha^2 := \sup\left\{\beta \geq 0\colon \sum_{k=1}^{\infty} k^{-\beta}\mathbb{P}[\|S_n/a_n\| < \delta \text{ for some } n \in I_k] = \infty \text{ for all } \delta > 0\right\} \tag{1.4}$$

whenever this set is not empty. Then

$$A = \begin{cases} \alpha K & \text{a.s. if the set in (1.4) is not empty,} \\ \varnothing & \text{a.s. if the set in (1.4) is empty.} \end{cases}$$

Thus the obvious question: Do examples exist in which $A = \alpha K$ a.s. with $0 \leq \alpha < 1$? Our main result answers that question affirmatively. Let c_0 denote the space of all sequence of real numbers converging to 0, endowed with the sup norm.

THEOREM 1.2. *For each $\alpha \in [0,1)$ there exists a bounded c_0-valued random variable X for which $A = \alpha K$ a.s.*

Without these examples, Theorem 1.1 would not of course be a true characterization of the cluster set A.

All proofs will be given in Section 2–4.

Theorem 1.1 leads us to look for natural conditions under which the unusual clusters sets of Theorem 1.2 do not occur, and for alternate necessary and/or sufficient conditions for $A = K$ and for $A = \varnothing$. Here are some results in these directions. The second shows that for $D = K$, (1.2) and (1.3) are not completely separate aspects of the behavior of the LIL sequence. The third shows that the unusual cluster sets can only occur when the sequence $\{P[\|S_n/a_n\| < \delta]: n \geq 1\}$ is quite irregular.

THEOREM 1.3. *Suppose $EX = 0$ and $E\|X\|^2 < \infty$. Then the following are equivalent:*

(i) $A = \varnothing$ *a.s.*
(ii) $\liminf_n \|S_n/a_n\| > 0$ *a.s.*
(iii) $\liminf_n \mathbb{E}\|S_n/a_n\| > 0$.

THEOREM 1.4. *Suppose $X \in WM_0^2$ and $d(S_n/a_n, K) \to 0$ a.s. Then $A = K$ a.s.*

THEOREM 1.5. *Assume $EX = 0$ and $E\|X\|^2 < \infty$. Suppose that for each $\delta > 0$ there exist $\rho > 1$ and $\lambda, \rho, \tau > 0$ such that*

$$\mathbb{P}[\|S_n/a_n\| < \delta] \geq \tau \mathbb{P}[\|S_l/a_l\| < \lambda]^r \tag{1.5}$$

whenever n and l satisfy $n \leq l \leq n^\rho$. Then either $A = K$ a.s. or $A = \varnothing$ a.s.

The examples of Kuelbs et al. mentioned preceding Theorem 1.1 will be shown to satisfy the hypotheses of Theorem 1.5. Therefore both K and $\varnothing$ are possible cluster sets under those hypotheses.

2. Preliminary results. The following is an extension of Theorem 2.3 of Alexander (1989).

THEOREM 2.1. *Suppose $X \in WM_0^2$ and let $h \in \overline{B^*}$ with $h \neq 0$. Then the following are equivalent:*

(i) $u_h \in A$ *a.s.*;
(ii) *for each $\beta < \|h\|_2^2$ and $\delta > 0$, there exists $\gamma > 1$ such that*

$$\sum_{k=1}^{\infty} k^{-\beta} \mathbb{P}[\|S_n/a_n\| < \delta \text{ for some } n \in I_k(\gamma)] = \infty; \tag{2.1}$$

(iii) *for each $\beta < \|h\|_2^2$, $\delta > 0$ and $\gamma > 1$, (2.1) holds.*

If also $E\|X\|^2 < \infty$ then (i)–(iii) are equivalent to:

(iv) *for each $\beta < \|h\|_2^2$, $\delta > 0$, $\gamma > 1$ and $m \geq 1$,*

$$\sum_{k=1}^{\infty} k^{-\beta} \mathbb{P}[\|S_{n_k}/a_{n_k}\| < \delta]^m = \infty. \tag{2.2}$$

For $h = 0$ the same result holds with "each $\beta < \|h\|_2^2$" replaced throughout by "$\beta = 0$."

Note that Theorem 1.1 is an immediate consequence of the equivalence of (i) and (iii), since $X \in WM_0^2$ implies $A \subset K$ by a lemma of Kuelbs (1976).

The equivalence of (i) and (iv) reinforces our earlier comment that the sequence $\{\mathbb{P}[\|S_n/a_n\| < \delta]: n \geq 1\}$ must be very irregular for the cluster set to be other than K of $\varnothing$. If that sequence were at all regular, divergence of (2.2) for all $m \geq 1$ for *some* $\beta \geq 0$, which occurs if the cluster set is not empty, would imply that $\mathbb{P}[\|S_{n_k}/a_{n_k}\| < \delta]$ approaches 0 more slowly than any positive power of k, if at all. This would mean (2.2) diverged for *all* $\beta < 1$, which by the equivalence of (i)–(iii) makes $A = K$.

In our proofs we will, without saying so explicitly each time, make statements which are only valid provided the parameter k or n is sufficiently large.

To prove Theorem 2.1, we will need several lemmas. The first two are standard results, given for completeness in Alexander (1989). The first establishes the equivalence of (ii) and (iii) of Theorem 2.1.

LEMMA 2.2. *Let $\{F_n, n \geq 1\}$ be any sequence of events and $\beta \geq 0$. Then convergence or divergence of*

$$\sum_{k=1}^{\infty} k^{-\beta}\mathbb{P}[F_n \text{ occurs for some } n \in I_k(\gamma)]$$

does not depend on $\gamma > 1$.

LEMMA 2.3. *Let $\xi_1, \xi_2, \dots$ be iid with $n^{-1}\sum_{i=1}^n \xi_i \to 0$ a.s., let $\delta > 0$ and let $\{m_k\}$ be an increasing sequence of positive integers such that for some $\varepsilon > 0$, $m_k \geq \varepsilon \sum_{j=1}^{k-1} m_j$ for all $k \geq 1$. Then*

$$\sum_{k=1}^{\infty} \mathbb{P}\left[\left\|\sum_{i=1}^{m_k} \xi_i\right\| > \delta m_k\right] < \infty.$$

LEMMA 2.4. *Let $\mathscr{F}$ be a collection of mean-zero functions on $(B, \mathscr{B}, P)$ and suppose $\{f^2: f \in \mathscr{F}\}$ satisfies the SLLN, that is,*

$$\sup_{\mathscr{F}}\left|n^{-1}\sum_{i=1}^{n}\left(f^2(X_i) - Ef^2(X)\right)\right| \to 0 \quad a.s. \tag{2.3}$$

Then for each $\delta > 4\sup_{\mathscr{F}}(Ef^2(X))^{1/2}$,

$$\sum_{k=1}^{\infty} \sup_{\mathscr{F}} \mathbb{P}\left[\left|\sum_{k=1}^{n_k} f(X_i)/a_{n_k}\right| > \delta\right] < \infty.$$

PROOF. Let (ε_i) be a Rademacher sequence (1 and -1 with probability $\frac{1}{2}$ each) independent of (X_i). Let $\mathbb{P}_\varepsilon$ and $\mathbb{E}_\varepsilon$ denote probability and expectation with the X_i's held fixed and only the ε_i's random, and vice versa for $\mathbb{P}_X$ and $\mathbb{E}_X$.

Then by Lemma 2.7 of Giné and Zinn (1984), for each $\eta > 0$ and $f \in \mathscr{F}$,

$$\begin{aligned} \mathbb{P}&\left[\left|\sum_{i=1}^{n_k} f(X_i)/a_{n_k}\right| > \delta\right] \\ &\leq \mathbb{E}_X\mathbb{P}_\varepsilon\left[\left|\sum_{i=1}^{n_k} \varepsilon_i f(X_i)/a_{n_k}\right| > \delta/4\right] \\ &\leq 2\mathbb{E}_X \exp\left(-\delta^2(\log\log n_k)/16 n_k^{-1}\sum_{i=1}^{n_k} f^2(X_i)\right) \\ &\leq 2\mathbb{P}\left[\sup_{\mathscr{F}}\left|n_k^{-1}\sum_{i=1}^{n_k}\left(f^2(X_i) - Ef^2(X)\right)\right| > \eta\right] \\ &\quad + 2\exp\left(-\delta^2(\log\log n_k)/16\left(\eta + \sup_{\mathscr{F}} Ef^2(X)\right)\right). \end{aligned} \tag{2.4}$$

The second term on the right side of (2.4) is summable if η is small; the first is summable by (2.3) and Lemma 2.3. □

Lemma 2.5. *If B is separable, $EX = 0$ and $E\|X\|^2 < \infty$, then $\mathscr{G} := \{f^2\colon f \in B_1^*\}$ satisfies the SLLN, that is,* (2.3) *holds for $\mathscr{F} = B_1^*$.*

Proof. For each $x \in B$ consider the random element φ_x of $l^\infty(\mathscr{G})$ given by $\varphi_x(f^2) := f^2(x) - Ef^2(X)$. If $x_n \to x$ in B, then $\varphi_{x_n} \to \varphi_x$ in $l^\infty(\mathscr{G})$; it follows that the random element φ_X takes values in a separable subspace of $l^\infty(\mathscr{G})$. Since $E\|\varphi_X\|_{l^\infty(\mathscr{G})} \leq 2E(\|X\|^2) < \infty$, the SLLN applies to φ_X. □

Let $b_n := (2\log\log n)^{1/2}$.

Lemma 2.6. *Let $\delta > 0$ and let m and n be positive integers satisfying $m \leq n^{(\log n)^3 - 1}$ (equivalently, $b_{mn} \leq 2b_n$). Then*

$$\mathbb{P}[\|S_{mn}/a_{mn}\| < \delta] \leq \mathbb{P}\left[\|S_n/a_n\| < 3\delta m^{1/2}\right]^m + m(m-1)\sup_{f\in B_1^*}\mathbb{P}\left[|f(S_n/a_n)| > \delta m^{-1/2}\right].$$

Proof.

$$\begin{aligned} \mathbb{P}&[\|S_{mn}/a_{mn}\| < \delta] \\ &\leq \mathbb{P}\left[\left\|\left(S_{rn} - S_{(r-1)n}\right)/a_n\right\| < 3\delta m^{1/2} \text{ for all } r \leq m\right] \\ &\quad + \mathbb{P}\left[\left\|\left(S_{rn} - S_{(r-1)n}\right)/a_n\right\| \geq 3\delta m^{1/2} \text{ for some } r \leq m;\ \|S_{mn}/a_{mn}\| < \delta\right] \\ &\leq \mathbb{P}\left[\|S_n/a_n\| < 3\delta m^{1/2}\right]^m \\ &\quad + m\mathbb{P}\left[\|S_n/a_n\| \geq 3\delta m^{1/2};\ \|S_{mn}/a_n\| < 2\delta m^{1/2}\right]. \end{aligned} \tag{2.5}$$

Now given $X_1, \dots, X_n$ for which $\|S_n/a_n\| < 3\delta m^{1/2}$, there exists $f \in B_1^*$ for which $f(S_n/a_n) \geq 3\delta m^{1/2}$. If the last event in (2.5) occurs, then

$$|f((S_{mn} - S_n)/a_n)| > \delta m^{1/2}.$$

It follows that the last probability in (2.5) is bounded above by

$$\sup_{g \in B_1^*} \mathbb{P}\left[|g(S_{mn} - S_n)/a_n| > \delta m^{1/2}\right]$$

and the lemma follows. □

LEMMA 2.7. *Suppose $EX = 0$ and $E\|X\|^2 < \infty$. Let m be a positive integer, $\gamma > m$, and $\delta > 4\sup_{f \in B_1^*}(Ef^2(X))^{1/2}$. Then*

$$\sum_{k=1}^{\infty} \mathbb{P}\left[\|S_{mn_k}/a_{mn_k}\| \geq 6\gamma^2\delta m^{-1/2};\ \|S_n/a_n\| < \delta \right. \tag{2.6}$$

$$\left. \text{for some } n \in I_{k+1}\right] < \infty.$$

PROOF. Note that $mn_k < n_{k+1}$. Given $X_1, \dots, X_{mn_k}$ for which $\|S_{mn_k}/a_{mn_k}\| \geq 6\gamma^2\delta m^{-1/2}$, let $f \in B_1^*$ satisfy $f(S_{mn_k}/a_{mn_k}) \geq 6\gamma^2\delta m^{-1/2}$. Then $f(S_{mn_k}/a_{n_{k+2}}) > 3\delta$. By a standard argument, conditionally on these X_i's, the probability in (2.6) is bounded above by

$$\begin{aligned} P&\left[\max_{n \in I_{k+1}} \left|f\left((S_n - S_{mn_k})/a_{n_{k+2}}\right)\right| > 2\delta\right] \\ &\leq 2\mathbb{P}\left[\left|f\left(S_{n_{k+2}}/a_{n_{k+2}}\right)\right| > \delta\right] \\ &\leq \sup_{g \in B_1^*} \mathbb{P}\left[\left|g\left(X_{n_{k+2}}/a_{n_{k+2}}\right)\right| > \delta\right] \end{aligned}$$

so that this last bound also holds unconditionally. The result now follows from Lemmas 2.4 and 2.5. □

As in Alexander (1989), we need to decompose X into two parts. Let $\Pi = \{E_0, \dots, E_J\}$ be bounded partition of B, that is, a partition in which E_0 is the only unbounded block, and $p_j := P(E_j)$. Let $\mathscr{S}$ denote the (finite) σ-algebra generated by Π, set

$$X' := E(X|\mathscr{S}), \qquad X'' := X - E(X|\mathscr{S}),$$

and let S_n' and S_n'' be the corresponding sums. Note that $E(X|\mathscr{S})$ is well-defined when X falls in E_0^c; when $X \in WM_0^2$ but $E\|X\|$ is not finite, and X falls in E_0, we define $E(X|\mathscr{S})$ to be the weak mean

$$y_0 := -p_0^{-1}P(E_0^c)E(X|E_0^c),$$

which satisfies

$$f(y_0) = E(f(X)|X \in E_0) \quad \text{for all } f \in B^*.$$

The following is Lemma 2.12 in Alexander (1989).

LEMMA 2.8. *Suppose* $X \in WM_0^2$, $\theta > 0$, $0 \le \mu < \beta \le 1$ (*or* $\mu = \beta = 0$), Λ *is a bounded partition of* B *and for every* $\delta > 0$

$$\sum_{k=1}^{\infty} k^{-\beta} \mathbb{P}[\|S_n/a_n\| < \delta \text{ for some } n \in I_k] = \infty.$$

Then the bounded partition Π *can be chosen so that* Π *refines* Λ *and*

$$\sum_{k=1}^{\infty} k^{-\mu} \mathbb{P}[\|S_n''/a_n\| < \theta \text{ for some } n \in I_k] = \infty.$$

The following is a variant of Lemma 2.15 of Alexander (1989). The proof is essentially similar, but simplified considerably by the assumption $E\|X\| < \infty$, which eliminates the need to condition separately on the variable T_{0n} of the earlier article.

LEMMA 2.9. *Suppose* $E\|X\| < \infty$. *Then for all* $\delta > 0$ *and* $\gamma > 1$,

$$\sum_{k=1}^{\infty} \left\{ \mathbb{P}\left[\|S_{n_k}''/a_{n_k}\| < \delta\right] - \mathbb{P}\left[\|S_{n_k}''/a_{n_k}\| < 2\delta \,|\, \|S_{n_k}'/a_{n_k}\| < \delta\right] \right\}^+ < \infty.$$

The proof of the following for $m = 1$ is essentially contained in the proof of Theorem 2.3 of Alexander (1989)—see (2.29) through (2.30) of that article—though Lemma 2.9 above must be used where Lemma 2.15 of that article was used. The argument there is easily modified to handle $m > 1$.

LEMMA 2.10. *Suppose* $X \in WM_0^2$, $\delta > 0$, $\beta \ge 0$, m *is a positive integer and*

$$\sum_{k=1}^{\infty} k^{-\beta} \mathbb{P}\left[\|S_{n_k}''/a_{n_k}\| < \delta\right]^m = \infty.$$

Then

$$\sum_{k=1}^{\infty} k^{-\beta} \mathbb{P}\left[\|S_{n_k}/a_{n_k}\| < 3\delta\right]^m = \infty.$$

PROOF OF THEOREM 2.1. The equivalence of (i), (ii) and (iii) is the content of Proposition 2.6 of Alexander (1989). It is clear that (iv) $\Rightarrow$ (iii), so we must show that (iii) $\Rightarrow$ (iv).

So suppose (iii) holds and $EX = 0$, $E\|X\|^2 < \infty$. Fix $\theta > 0$, $\gamma > 1$, β with $0 \le \beta < \|h\|_2^2$ ($\beta = 0$ if $h = 0$) and a positive integer m. Fix an integer r such that $\gamma^r > m$. Let Λ be bounded partition of B such that

$$E\left\|X - E(X|\mathscr{S})\right\|^2 < \theta^2/16 \tag{2.7}$$

whenever Π is a refinement of Λ. By Lemma 2.8, Π can be chosen so that Π

refines Λ and

$$\sum_{k=1}^{\infty} k^{-\beta}\mathbb{P}\left[\|S_n''/a_n\| < \theta \text{ for some } n \in I_{k+1}(\gamma^r)\right] = \infty.$$

Let n_k denote $n_k(\gamma^r)$. By (2.7), $\sup_{f \in B_1^*}(Ef^2(X''))^{1/2} < \theta/4$, so by Lemma 2.7,

$$\infty = \sum_{k=1}^{\infty} k^{-\beta}\mathbb{P}\left[\|S_{mn_k}''/a_{n_k}\| < 6\gamma^2\theta m^{-1/2}\right].$$

Therefore by Lemma 2.6,

$$\infty = \sum_{k=1}^{\infty} k^{-\beta}\mathbb{P}\left[\|S_{n_k}''/a_{n_k}\| < 18\gamma^2\theta\right]^m$$
$$+m(m-1)\sum_{k=1}^{\infty} \sup_{f\in B_1^*} \mathbb{P}\left[\left|f(S_{n_k}/a_{n_k})\right| > 6\gamma^2\theta m^{-1}\right].$$

Since θ is arbitrary and $n_k(\gamma^r) = n_{kr}(\gamma)$, the result now follows from Lemmas 2.10, 2.4 and 2.5. □

3. Examples with unusual cluster sets. The underlying idea of these examples is the following. Let us illustrate with $\alpha = 1/\sqrt{3}$. We wish to make the series in (2.2), with $m = 1$, diverge if and only if $\beta \le \frac{1}{3}$. This will occur if the probability in (2.2) is almost 1 when k is (nearly) a perfect cube, and very small otherwise. This is accomplished using a random variable for which $E\|S_{n_k}/a_{n_k}\|$ stays bounded away from 0 except when k is near a perfect cube, where it drops rapidly to near 0 and then rises rapidly away again. Here "rapidly" means at almost rate $n_k^{1/2}$.

For each $j \ge 1$ let m_j be a positive integer, $c_j > 0$ and $p_j \in [0, 1/2]$. Let $s_j := \sum_{i=1}^{j} m_j$, and $L_j := [s_{j-1}, s_j)$. For each $l \in L_j$ let ξ_i^l, $i \ge 1$, be iid with

$$\xi_1^l = \begin{cases} c_j & \text{with probability } p_j, \\ -c_j & \text{with probability } p_j, \\ 0 & \text{with probability } 1 - 2p_j, \end{cases} \tag{3.1}$$

independent for distinct l. Let

$$X_i := \left(\xi_i^1, \xi_i^2, \dots\right), \qquad i \ge 1,$$
$$S_n^l := \sum_{i=1}^{n} \xi_i^l,$$
$$Z_{jn} := \max_{l\in L_j}|S_n^l/n^{1/2}|.$$

Note that Z_{jn} is the maximum of m_j independent sums of iid real random variables, and

$$\|S_n/n^{1/2}\| = \max_j Z_{jn}.$$

Let us examine heuristically the behavior of $\mathbb{E}Z_{jn}$ as n varies. We will not prove anything yet formally, and will not use these heuristics in our latter formal proofs, so we will make statements here which are actually only correct up to a constant. Suppose c_j and p_j are small and m_j is large. Then roughly,

$$(3.2)\qquad \mathbb{E}Z_{jn} \approx \begin{cases} c_j n^{1/2} & \text{for } n \le (\log m_j)/(\log p_j^{-1}), \\ c_j(\log m_j)/n^{1/2}\log((\log m_j)/np_j) & \\ \qquad \text{for } (\log m_j)/(\log p_j^{-1}) \le n \le (\log m_j)/p_j, & \\ c_j(2p_j \log m_j)^{1/2} & \text{for } n \ge (\log m_j)/p_j. \end{cases}$$

The first range on n is the "deterministic regime," where with high probability S_n^l achieves its maximum possible value $c_j n^{1/2}$ for some $l \in L_j$. The second is the "Poisson regime," where $\mathbb{P}[|S_n^l/n^{1/2}| > \mathbb{E}Z_{jn}]$ can be approximated by a Poisson probability for $l \in L_j$. This last is the "Gaussian regime," where this same probability admits a Gaussian approximation. Note that $\mathbb{E}Z_{jn}$ rises like $n^{1/2}$ until $n = (\log m_j)/(\log p_j^{-1})$, then falls like $n^{-1/2}$ (up to a log factor) as n increases to $(\log m_j)/p_j$, then levels out.

The variability of each Z_{jn} is very small, so $\|S_n/n^{1/2}\|$ behaves like $\max_j \mathbb{E}Z_{jn}$ with very high probability.

Let us use $\gamma = 2$, that is, $n_k = 2^k$. The parameters c_j, p_j and m_j will be chosen so that

$$c_j n^{1/2} \sim (\log\log n)^{1/2} \quad \text{for } n = (\log m_j)/(\log p_j^{-1}) \sim 2^j = n_j,$$

that is, so that $\mathbb{E}Z_{jn}$ reaches its peak value $(\log\log n_j)^{1/2}$ near $n = n_j$. Thus Z_{jn} is responsible for keeping $\mathbb{E}\|S_n/a_n\|$ away from 0 when $n \approx 2^j = n_j$.

If several consecutive L_j's are "omitted" by setting $c_j = 0$, then $\mathbb{E}\|S_n/a_n\|$ will drop toward 0 that nearly an $n^{-1/2}$ rate for n near the corresponding values 2^j, by (3.2). For slightly larger j, if the L_j's are no longer omitted, $\mathbb{E}\|S_n/a_n\|$ will be pushed back higher at nearly an $n^{1/2}$ rate for n near the corresponding values 2^j, until it is bigger than 1, where it levels out.

The choice of which L_j's to omit thus controls the subsequence of values k for which $E\|S_{n_k}/a_{n_k}\|$ is small, that is, $\mathbb{P}[\|S_n/a_n\| < \delta]$ is large. Appropriate choice of this subsequence leads to any desired value of the α of (1.4).

Keeping this in mind, we present the formal proof. We will need Bernstein's inequality [see Bennett (1962)]: for $\eta_1, \dots, \eta_n$ iid mean-zero random variables bounded in magnitude by $c > 0$ with $\mathrm{var}(\eta_1) \le s^2$ and $M > 0$,

$$(3.3)\qquad \mathbb{P}\left[\left|n^{-1/2}\sum_{i=1}^n \eta_i\right| > M\right] \le 2\exp(-M^2/2s^2(1 + Mc/3n^{1/2}s^2))$$

$$\le 2\exp(-M^2/4s^2) + 2\exp(-Mn^{1/2}/2c).$$

PROOF OF THEOREM 1.2. Let us first consider $\alpha > 0$. Fix $0 < \alpha < 1$ and set $\gamma = 2$ and

$$Q := \left\{ \left[m^{1/\alpha^2} \right] : m \geq 1 \right\},$$

$$R := \bigcup_{j \in Q} \left[j - [\log_2 j], j + [\log_2 j] \right],$$

$$p_j := j^{-1} \log j,$$

$$m_j := \left[1 + \exp(2^{j+2} \log j) \right],$$

$$c_j := \begin{cases} 2\left(2^{-j} \log j\right)^{1/2} & \text{if } j \notin R, \\ 0 & \text{if } j \in R. \end{cases}$$

We will use the notation of the above heuristic.

We now make four claims:

$$\sum_{k \in Q} k^{-\beta} = \infty \quad \text{if } \beta < \alpha^2, \tag{3.4}$$

$$\sum_{k \in R} k^{-\beta} = \infty \quad \text{if } \beta > \alpha^2, \tag{3.5}$$

$$\liminf_{k \in Q} \mathbb{P}\left[\|S_{n_k}/a_{n_k}\| < 2\varepsilon \right] = 1 \quad \text{for every } \varepsilon > 0, \tag{3.6}$$

$$\sum_{k \notin R} \mathbb{P}\left[\|S_{n_k}/a_{n_k}\| < 1 \right] < \infty. \tag{3.7}$$

Once these claims are established, it follows from (3.4) and (3.6) that (2.2) holds for $\gamma = 2$, $m = 1$ and all $\delta > 0$ and $\beta < \alpha^2$. Hence so does (2.1) (which does not depend upon γ), so $u_h \in A$ a.s. for all $h \in \overline{B^*}$ with $\|h\|_2 < \alpha$. It follows from (3.5) and (3.7) that (2.2) fails for $\gamma = 2$, $m = 1$, $\gamma = 1$ and all $\beta > \alpha^2$, so $u_h \notin A$ a.s. for all $h \in \overline{B^*}$ with $\|h\|_2 > \alpha$. It follows that $A = \alpha K$ a.s.

PROOF OF (3.4). For $\beta < \alpha^2$,

$$\sum_{k \in Q} k^{-\beta} \geq \sum_{m=1}^{\infty} m^{-\beta/\alpha^2} = \infty.$$

PROOF OF (3.5). For $\beta > \alpha^2$,

$$\begin{aligned} \sum_{k \in R} k^{-\beta} &\leq \sum_{k \in Q} 2^{\beta} k^{-\beta} (1 + 2 \log_2 k) \\ &\leq 2^{2\beta} \sum_{j=1}^{\infty} j^{-\beta/\alpha^2} \left(1 + 2\alpha^{-2} \log_2 j\right) \\ &< \infty. \end{aligned}$$

PROOF OF (3.6). Fix $\varepsilon > 0$. Let

$$Y_{jk} := b_{n_k}^{-1} Z_{jn_k} = \max_{l \in L_j} |S_{n_k}^l / a_{n_k}|$$

and for each $j \geq 1$ let U_{ji}, $i \geq 1$, be iid with the distribution (3.1) of ξ_1^l when $l \in L_j$. Let

$$v_k := k - [\log_2 k], \qquad w_k := k + [\log_2 k]$$

and fix $k \in Q$. Then

$$\begin{aligned} \|S_{n_k}/a_{n_k}\| &\leq \max_{j < v_k} Y_{jk} + \max_{j > w_k} Y_{jk} \\ &:= (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

Now (II) can be bounded deterministically: We have for $j > w_k$,

$$Y_{jk} \leq n_k c_j / a_{n_k} \leq (2n_k/\log\log n_k)^{1/2}((\log j)/2^j)^{1/2} \leq \lambda_1/k^{1/2},$$

so that (II) $\leq \lambda_1/k^{1/2} < \varepsilon$, where λ_1, and other λ_i to follow, are constants not depending on k or j.

For (I), we need an upper bound for

$$q_k := \mathbb{P}\left[\max_{j < v_k} Y_{jk} \geq \varepsilon\right].$$

By Bernstein's inequality (3.3),

$$\begin{aligned} &\mathbb{P}\left[\left|\sum_{i=1}^{n_k} U_{ji}\right| \geq \varepsilon a_{n_k}\right] \\ &\quad \leq 2\exp\left(-\varepsilon^2(\log\log n_k)/8c_j^2 p_j\right) + 2\exp\left(-\varepsilon a_{n_k}/2c_j\right) \\ &\quad \leq 2\exp\left(-\lambda_2 \varepsilon^2 (\log k) j 2^j/(\log j)^2\right) \\ &\qquad + 2\exp\left(-\lambda_3 \varepsilon (2^j/\log j)^{1/2} (2^k \log k)^{1/2}\right) \\ &\quad := 2\mu_{jk} + 2\nu_{jk}, \end{aligned}$$

so

$$q_k \leq 2 \sum_{j < v_k} m_j(\mu_{jk} + \nu_{jk}). \tag{3.8}$$

Further (if k is large, as always), for all $j < v_k$,

$$\log m_j \leq 2^{j+3} \log j \leq \left(\log \mu_{jk}^{-1}\right)/2$$

and

$$\begin{aligned} \left(\log \nu_{jk}^{-1}\right)/\log m_j &\geq \lambda_4 \varepsilon (2^k \log k)^{1/2} / 2^{j/2} (\log j)^{3/2} \\ &\geq \lambda_5 \varepsilon k^{1/2}/\log k \\ &\geq 2. \end{aligned}$$

Therefore (3.8) leads to

$$q_k \le 2 \sum_{j<v_k} \left(\mu_{jk}^{1/2} + \nu_{jk}^{1/2}\right). \tag{3.9}$$

But

$$\sum_{j<v_k} \nu_{jk}^{1/2} \le k \exp\left(-\lambda_3 \varepsilon (2^k \log k)^{1/2}\right) \to 0 \quad \text{as } k \to \infty$$

and, by considering separately the sum over $j < \log_2(2/\lambda_2 \varepsilon^2)$,

$$\sum_{j<v_k} \mu_{jk}^{1/2} \le \left(\log_2(2/\lambda_2\varepsilon^2)\right)\exp\left(-\lambda_2 \varepsilon^2 \log k\right) + k\exp(-2\log k)$$

$$\to 0 \quad \text{as } k \to \infty.$$

Thus $q_k \to 0$ as $k \to \infty$, and (3.6) follows.

PROOF OF (3.7). Since $a_{n_k}/c_k \sim 2^{k-1/2} = 2^{-1/2} n_k$,

$$\begin{aligned}\sum_{k\notin R} \mathbb{P}\left[\|S_{n_k}/a_{n_k}\| < 1\right] &\le \sum_{k \notin R} \mathbb{P}[Y_{kk} < 1] \\ &= \sum_{k\notin R}\left(1 - \mathbb{P}\left[\left|\sum_{i=1}^{n_k} U_{ki}\right| \ge a_{n_k}\right]\right)^{m_k} \\ &\le \sum_{k\notin R} \exp\left(-m_k \mathbb{P}\left[\left|\sum_{i=1}^{n_k} U_{ki}\right| \ge a_{n_k}\right]\right) \\ &\le \sum_{k\notin R} \exp(-m_k p_k^{n_k}).\end{aligned} \tag{3.10}$$

Since $n_k \log p_k^{-1} \sim 2^k \log k \sim (\log m_k)/4$, the right-hand side of (3.10) is bounded above for some $k_0 > 0$ by

$$k_0 + \sum_{k>k_0} \exp\left(-m_k^{1/2}\right) < \infty.$$

This establishes (3.7) and completes the proof of Theorem 1.2 for $\alpha > 0$.

For $\alpha = 0$ the proof is similar except that for Q we use a sequence $\{j_m\}$ growing faster than any power of m. □

By the method of proof of (3.6), and using Bennett's (1962) improvement of Bernstein's inequality to handle $v_k \le j \le w_k$, one can readily check that these examples satisfy the BLIL.

4. Proofs of Theorem 1.3–1.5. The proof of the following is contained in the proof of Proposition 2.6 of Alexander (1989); see the argument surrounding (2.8) of the article.

LEMMA 4.1. *Let $X \in WM_0^2$ and $\theta > 0$, and let Γ be a bounded partition of B. Suppose $0 \in A$ a.s. Then the partition Π can be chosen so that Π refines Γ*

and

$$\liminf_n \|S_n''/a_n\| < \theta \quad a.s.$$

PROOF OF THEOREM 1.3. If $\liminf_n \mathbb{E}\|S_n/a_n\| = 0$, then $\{S_n/a_n\}$ has a subsequence converging to 0 in probability, so $0 \in A$ a.s. Thus (i) implies (iii).

Equivalence of (i) and (ii) is immediate from Theorem 1.1, so it remains to show (iii) implies (ii). Suppose $\varepsilon := \liminf \mathbb{E}\|S_n/a_n\| > 0$. Let Γ be a bounded partition of B such that whenever Π is a refinement of Γ,

$$E\|X''\|^2 < (\varepsilon/192)^2. \tag{4.1}$$

Now fix such a refinement Π. Let $\tau > 0$ and

$$Y_i := X_i''1\left[\|X_i''\| < \tau(i/\log\log i)^{1/2}\right], \qquad Z_i := X_i'' - Y_i,$$

$$T_n := \sum_{i=1}^n Y_i, \qquad W_n := \sum_{i=1}^n Z_i.$$

By Lemma 2.3 of de Acosta (1983), which, as de Acosta points out, works for Banach-space-valued random variables,

$$W_n/a_n \to 0 \quad \text{a.s.} \tag{4.2}$$

Inequality (3.5) of de Acosta (1983), redone as a lower bound with virtually no change to the proof, says

$$\begin{aligned} &\mathbb{P}[\|T_n/a_n\| - E\|T_n/a_n\| < -\varepsilon/4] \\ &\le \exp\left(-(\varepsilon^2/64E\|X''\|^2)(2 - \exp(\varepsilon\tau/4E\|X''\|^2))\log\log n\right) \\ &\le \exp(-2\log\log n), \end{aligned} \tag{4.3}$$

provided τ is sufficiently small (depending on ε and $E\|X''\|^2$) by (4.1). Since

$$\begin{aligned} \mathbb{E}\|W_n/a_n\| &\le a_n^{-1}\sum_{i=1}^n E\|Z_i\| \\ &\le a_n^{-1}\sum_{i=1}^n \lambda_i(E\|X''\|^2)^{1/2}\tau^{-1}(i^{-1}\log\log i)^{1/2} \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{4.4}$$

for some sequence $\lambda_i \to 0$, we have $\liminf_n \mathbb{E}\|T_n/a_n\| = \varepsilon$, so that by (4.3),

$$\sum_{k=1}^{\infty} \mathbb{P}\left[\|T_{n_k}/a_{n_k}\| < \varepsilon/2\right] < \infty. \tag{4.5}$$

Since $|\mathbb{E}g(T_n/a_n)| = |\mathbb{E}g(W_n/a_n)|$ for $g \in B_1^*$, (4.4) implies

$$|\mathbb{E}g(T_n/a_n)| \to 0 \quad \text{uniformly in } g.$$

Using Bernstein's inequality (3.3), it follows that

$$\begin{aligned}&\sup_{g\in B_1^*}\mathbb{P}\big[|g(T_n/a_n)|>\varepsilon/48\big]\\ &\quad\le \sup_{g\in B_1^*}\mathbb{P}\big[|g(T_n/a_n)|-\mathbb{E}g(T_n/a_n)|>\varepsilon/96\big]\\ &\quad\le \exp(-2\log\log n).\end{aligned}\tag{4.6}$$

The proof of Lemma 2.7, using $m=1$, $\gamma=2$ and $\delta=\varepsilon/48$, and using (4.6) in place of Lemmas 2.4 and 2.5, shows that

$$\sum_{k=1}^{\infty}\mathbb{P}\big[\|T_{n_k}/a_{n_k}\|\ge\varepsilon/2;\ \|T_n/a_n\|<\varepsilon/48 \text{ for some } n\in I_{k+1}\big]<\infty.$$

With (4.5), (4.2) and the Borel–Cantelli lemma this shows

$$\liminf_n\|S_n''/a_n\|\ge\varepsilon/48 \quad \text{a.s.}$$

Since the refinement Π of Γ is arbitrary, Lemma 4.1 shows that $0\notin A$ a.s., that is, (ii) holds. □

PROOF OF THEOREM 1.4. The underlying idea is that if u_f is on the edge of K, then the only way for $f(S_n/a_n)$ to approach 1 (as it must i.o.) is for S_n/a_n to approach u_f. But a cluster point on the edge of K makes $A=K$ a.s.

Thus fix $f\in B^*$ with $\|f\|_2=1$, and $\varepsilon\in(0,1)$. By the one-dimensional LIL, infinitely often both

$$f(S_n/a_n)\ge 1-\varepsilon^2 \quad\text{and}\quad d(S_n/a_n,K)<\varepsilon^2/(\|f\|_{B^*}\vee 1).$$

When this occurs, there exists $g\in\overline{B^*}$ such that $\|g\|_2\le 1$ and

$$\|S_n/a_n-u_g\|<\varepsilon^2/(\|f\|_{B^*}\vee 1)$$

so $|f(S_n/a_n)-f(u_g)|<\varepsilon^2$ and $f(u_g)\ge 1-2\varepsilon^2$. Therefore

$$\|f-g\|_2^2=\|f\|_2^2+\|g\|_2^2-2f(u_g)\le 4\varepsilon^2$$

so that $\|u_f-u_g\|\le 2\varepsilon\sigma$. Hence

$$\|S_n/a_n-u_f\|\le(2\sigma+1)\varepsilon.$$

Since ε is arbitrary, we have $u_f\in A$ a.s. But $u_f\notin\alpha K$ for any $\alpha<1$, so the result follows from Theorem 1.1. □

PROOF OF THEOREM 1.5. Suppose $A\ne K$ a.s. Then there exist $\beta<1$, $\gamma>1$ and $\delta>0$ such that the series in (2.2) converges for $m=1$. Let ρ, λ, r and τ be such that (1.5) holds. Corresponding to λ, there exist further constants ρ_0, λ_0, r_0 and τ_0 such that (1.5) holds with δ, τ, λ, r and ρ replaced by λ, τ_0, λ_0, r_0 and ρ_0, respectively. We may assume $\rho_0\le\rho$. For convenience of notation we will assume ρ_0 is an integer, but this is in no way essential to our argument. Let $n(s)$ denote $n_{\rho_0^s}$ for integers $s\ge 1$. Then for some constant t, which may vary from line to line,

$$\infty > \sum_{k=1}^{\infty} k^{-\beta}\mathbb{P}\left[\|S_{n_k}/a_{n_k}\| < \delta\right]$$

$$= \sum_{s=1}^{\infty} \sum_{\rho_0^s \le k < \rho_0^{s+1}} k^{-\beta}\mathbb{P}\left[\|S_{n_k}/a_{n_k}\| < \delta\right]$$

$$\ge t \sum_{s=1}^{\infty} \rho_0^{s(1-\beta)}\mathbb{P}\left[\|S_{n(s)}/a_{n(s)}\| < \lambda\right]^r.$$

Hence for $u := \rho_0^{-(1-\beta)/r}$,

$$\mathbb{P}\left[\|S_{n(s)}/a_{n(s)}\| < \lambda\right] = o(u^s) \quad \text{as } s \to \infty.$$

It follows from (1.5) that for $v := u^{1/r_0}$ and all $\rho_0^s \le k < \rho_0^{s+1}$,

$$\mathbb{P}\left[\|S_{n_k}/a_{n_k}\| < \lambda_0\right] \le tv^s.$$

But for $m > rr_0/(1-\beta)$,

$$\sum_{k=\rho_0}^{\infty} \mathbb{P}\left[\|S_{n_k}/a_{n_k}\| < \lambda_0\right]^m$$

$$= \sum_{s=1}^{\infty} \sum_{\rho_0^s \le k < \rho_0^{s+1}} \mathbb{P}\left[\|S_{n_k}/a_{n_k}\| < \lambda_0\right]^m$$

$$\le t \sum_{s=1}^{\infty} \left(\rho_0 v^m\right)^s$$

$$< \infty$$

since $\rho_0 v^m < 1$. Thus by Theorem 2.1, $A = \varnothing$ a.s. □

The examples mentioned before Theorem 1.1 all have the following form: The Banach space B is c_0, and $X = (\varepsilon_1\lambda_1, \varepsilon_2\lambda_2, \dots)$ for some nonrandom sequence $\lambda_i \downarrow 0$ and (ε_i) a Rademacher sequence. For such X, and $c > 0$,

$$\mathbb{P}\left[\|S_n/n^{1/2}\| \le c\right] = \prod_{j=1}^{\infty}\left(1 - \mathbb{P}\left[\left|n^{-1/2}\sum_{I=1}^{n}\varepsilon_i\right| > c/\lambda_j\right]\right)$$

$$= \exp\left(-\sum_{j=1}^{j_n}\beta_{nj}\mathbb{P}\left[\left|n^{-1/2}\sum_{i=1}^{n}\varepsilon_i\right| > c/\lambda_j\right]\right),$$

where $j_n := \max\{j\colon c/\lambda_j \le n^{1/2}\}$, for some constants β_{nj} which approach 1 uniformly in n and j as $c \to \infty$. Since $a_l/l^{1/2}$ is approximately constant over the range of l needed (for each fixed n) in Theorem 1.5, the following result shows that the hypotheses of Theorem 1.5 are satisfied for these X.

LEMMA 4.2. *There exists a universal constant R_0 such that for all $m \geq n \geq 1$ and $y \geq 1$,*

$$\mathbb{P}\left[\left|n^{-1/2} \sum_{i=1}^{n} \varepsilon_i\right| > y\right] \leq R_0 \mathbb{P}\left[\left|m^{-1/2} \sum_{i=1}^{m} \varepsilon_i\right| > y\right].$$

The proof of this lemma is an elementary exercise in approximating binomial coefficients and binomial probabilities using Stirling's formula, so we omit it.

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