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AN ASYMPTOTIC EVALUATION OF THE TAIL OF A MULTIPLE SYMMETRIC α-STABLE INTEGRAL

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We expand a multiple symmetric α-stable integral

$$\int \cdots \int f(t_1, \dots, t_n)\, dM(t_1) \dots dM(t_n)$$

into a LePage type multiple series of transformed arrival times of a Poisson process. An exact evaluation of the limit of appropriately normalized tail distribution results from this representation.

0. Introduction. Let Z be a symmetric Lévy α-stable process on $[0,1]$ with the characteristic function

$$E \exp\{itZ(u)\} = \exp\{-u|t|^{\alpha}\}, \qquad 0 < \alpha < 2,$$

and let f be a real symmetric Borel function on $[0,1]^n$ vanishing on diagonals. A random functional

$$I_n(f) = \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_n)\, dZ(x_1) \dots dZ(x_n) \tag{0.1}$$

extends the notion of the multiple Wiener integral in a natural way. Existence and characterization problems, not necessarily restricted to the stable case, have recently attracted the attention of many authors. For a unified presentation of a classical theory due to Wiener and Itô and for further historical background we refer to Engel (1982).

Basically, a general definition of a multiple stable integral of type (0.1) proceeded by a construction of a stable product random measure, is due to Krakowiak and Szulga (1988).

However, the first characterization of integrands of a double α-stable integral in case of $\alpha \in [1,2)$ was obtained by Rosinski and Woyczynski (1986), and it was generalized to an arbitrary $\alpha \in (0,2)$ by Kwapien and Woyczynski (1987). Their condition is hardly extendable for general multiple stable integrals due to an internal complicacy even though a triple stable integration criterion of a similar nature was found by McConnell (1986).

In the present article we make a step towards a characterization of a distribution of a multiple stable integral by evaluating its limit behavior under a

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suitable normalization. We show that

$$
\begin{aligned}
&2 \lim_{x\to\infty} x^{\alpha}(\ln x)^{1-n} P(I_n(f) > x)\\
(0.2)\qquad &= \lim_{x\to\infty} x^{\alpha}(\ln x)^{1-n} P(|I_n(f)| > x)\\
&= n\alpha^{n-1}(n!)^{\alpha-2} s^{-n} \int_0^1 \cdots \int_0^1 |f(t_1,\dots,t_n)|^{\alpha}\, dt_1 \dots dt_n,
\end{aligned}
$$

provided

$$(0.3)\qquad \int_{[0,1]^n} |f|^{\alpha}(1+\delta_n(f))(\ln_+ |f|^{n-1}) < \infty,$$

where $s = \int_0^{\infty} x^{-\alpha} \sin x\, dx$, $\delta_n = 1$ if $n \neq 2$ and $\delta_2(f) = \ln_+ \ln_+ |f|$.

Observe that the first equality in (0.2) follows trivially only if n is an odd integer. In general, $I_n(f)$ is not a symmetric random variable if n is an even integer even though it behaves like such because, in a sense, it is dominated by a symmetric term.

We notice that a related result was derived by Surgailis (1985) from an interpolation theorem in Lorentz spaces. Namely, he proved that for $1 < p < \alpha < 2$,

$$(0.4)\qquad \left(E|I_n(f)|^p\right)^{1/p} \leq C\|f\|_{L^{\alpha}\log^{n-1}L(d\mu)},$$

where the r.h.s. term is a norm in a Lorentz space of random variables generated by a functional analogous to the one appearing in the r.h.s. of (0.3) (with the Lebesgue measure dx replaced by certain measure $d\mu$). A discrete counterpart of (0.4) was obtained by Rosinski and Woyczynski (1987).

The article is organized as follows. Section 1 introduces the notation and provides a collection of basic facts concerning multilinear random forms and multiple stochastic integrals. In Section 2 we prove the LePage type representation of $I_n(f)$. The distribution of products of arrival times, essential for our purpose, is studied in Section 3. Section 4 contains technical results on comparison of multiple series and multiple integrals. The asymptotic evaluation of the tail of the distribution of $I_n(f)$ is obtained in Section 5.

Although we use elementary methods a combinatoric complexity of multiple sums and integrals might suggest that some techniques seem more intrinsic than, in fact, they are. A suitable notation is introduced to avoid unnecessary misunderstandings.

1. Preliminaries. In this article $\{Z(t),\ t \in [0,1]\}$ denotes a symmetric α-stable motion, that is, a process with independent stationary increments such that $E\exp\{itZ(u)\} = \exp\{-u|t|^{\alpha}\}$, $0 < \alpha < 2$. For each $n \geq 1$, $Z(t)$ generates a random measure $M^{(n)}$ on Borel sets in $[0,1]^n$ defined as a vector measure satisfying the identity

$$M^{(n)}(A_1 \times \cdots \times A_n) = Z(A_1) \cdot \cdots \cdot Z(A_n)$$

[Krakowiak and Szulga (1988)]. Observe that only $M^{(1)}$, denoted for the sake of simplicity by M, is independently scattered, that is, its values on disjoint sets are independent random variables.

The following notation is used throughout the article:

(U_n) is a sequence of i.i.d. uniformly distributed random variables on $[0,1]$;

(X_n) is a sequence of i.i.d. exponentially distributed random variables with unit intensity;

(Γ_n) is a sequence of arrival times of a Poisson process, that is, $\Gamma_n = X_1 + \cdots + X_n$;

(ε_n) is a sequence of i.i.d. Bernoulli random variables, that is, $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$. $\mathbf{1}\{\dots\}$ will denote the indicator function of a set (or a property) $\{\dots\}$.

For the convenience of the typographer and the reader we introduce an abbreviated notation for expressions involving multiple indices. Any boldface character denotes a finite or infinite sequence, for example, $\mathbf{a} = (a_j)$, $\mathbf{j} = (j_1, \dots, j_n)$. A boldface subscript is related to a restriction of a sequence to suitable coordinates, for example, $\mathbf{a_j} = (a_{j_1}, \dots, a_{j_n})$. By definition,

$$[\mathbf{a_j}] = a_{j_1} \dots a_{j_n}.$$

We shall also write subscripts with the mathematical expectation symbol "E", for example, E_ε, E_Γ, and so forth, a convenience of which will be especially appreciated whenever Fubini's theorem is in use. We shall skip the index of stability α in all quantities used in this article.

L^p denotes the space of p-integrable random variables with usual norm (quasinorm, if $p < 1$) $\|\cdot\|_p = (E|\cdot|^p)^{1/p}$. For $k \geq 1$ we introduce a set Λ_k of all random variables for which the limit

$$\lambda_k(X) = \lim_{x\to\infty} x^\alpha (\ln x)^{-k} P(X > x)$$

exists. We shall be using frequently an observation based on the following elementary fact.

Lemma 1.1. *Let X and Y be positive random variables. Suppose that X has a regularly varying tail, that is, there is a number $\theta > 0$ such that for every number $a > 1$,*

$$\lim_{x\to\infty} \frac{P(X > ax)}{P(X > x)} = a^{-\theta}.$$

Suppose that the tail of X dominates the tail of a random variable Y in the sense that

$$\lim_{x\to\infty} \frac{P(Y > x)}{P(X > x)} = 0.$$

Then

$$\lim_{x\to\infty} \frac{P(X + Y > x)}{P(X > x)} = \lim_{x\to\infty} \frac{P(X - Y > x)}{P(X > x)} = 1.$$

PROOF. Clearly, for any σ, $0 < \sigma < 1$, we have

$$\begin{aligned}\limsup_{x\to\infty}\frac{P(X+Y>x)}{P(X>x)} &\le \limsup_{x\to\infty}\frac{P(Y>\sigma x)+P(X>(1-\sigma)x)}{P(X>x)}\\ &\le \limsup_{x\to\infty}\frac{P(Y>\sigma x)}{P(X>\sigma x)}\frac{P(X>\sigma x)}{P(X>x)}\\ &\quad+\limsup_{x\to\infty}\frac{P(X>(1-\sigma)x)}{P(X>x)}\\ &=(1-\sigma)^{-\theta}.\end{aligned}$$

Since, obviously,

$$\liminf_{x\to\infty}\frac{P(X+Y>x)}{P(X>x)}\ge 1,$$

the first part of the lemma follows. The second part can be proved in a similar way. □

COROLLARY 1.2. *Let $X \in \Lambda_k$. Then, under assumptions of Lemma* 1.1, $X + Y \in \Lambda_k$ *and* $\lambda_k(X+Y) = \lambda_k(X)$.

The remainder of the section contains a collection of basic properties of *multilinear random forms* which are defined as formal sums

$$\langle g,\mathbf{X}\rangle = \sum_{\mathbf{j}\in\mathbf{N}^n} g(\mathbf{j})[\mathbf{X}_\mathbf{j}],$$

where g is a real function on $\mathbf{N}^n$ and $\mathbf{X} = (X_j)$ is a sequence of real random variables. Let $D_n = \{\mathbf{i} = (i_1,\dots,i_n) \in \mathbf{N}^n\colon i_1 < \cdots < i_n\}$, and observe that if a function g is *symmetric*, that is, $g(\mathbf{j}) = g(\mathbf{j}\cdot\pi)$ for every $\mathbf{j}\in\mathbf{N}^n$ and for every permutation π of the sequence $(1,\dots,n)$, and g *vanishes on diagonals* of $\mathbf{N}^n$, that is, $g(\mathbf{j}) = 0$ whenever at least two entries of $\mathbf{j}$ are equal, then

$$\langle g,\mathbf{X}\rangle = n!\sum_{\mathbf{j}\in D_n} g(\mathbf{j})[\mathbf{X}_\mathbf{j}].$$

For this reason we consider tetrahedral multilinear forms only, that is, related to functions g with a domain in D_n. We say that a multilinear random form $\langle g,\mathbf{X}\rangle$ converges if

$$\sum_{\mathbf{j}\in D_n} g(\mathbf{j})[\mathbf{X}_\mathbf{j}]\mathbf{1}\{j_n\le u\}$$

converges in an appropriate sense as $u\to\infty$. In general, most of the properties of multilinear random forms for independent symmetric random variables follow from their counterparts for Bernoulli random variables by virtue of Fubini's theorem. We quote below several results useful for our purposes.

THEOREM 1.3. *Let* $\mathbf{X} = (X_j)$ *be a sequence of independent symmetric random variables and* g *be a real function on* D_n. *The following statements hold.*

(i) [Krakowiak and Szulga (1986b)] *A random multilinear form* $\langle g, \mathbf{X} \rangle$ *converges a.s. if and only if it converges in probability if and only if*

$$\sum_{\mathbf{j} \in D_n} g(\mathbf{j})^2 [\mathbf{X}_\mathbf{j}]^2 < \infty \quad a.s.$$

(ii) [Krakowiak and Szulga (1986b)] *Let* $\mathbf{X} = \varepsilon$ *be a Bernoulli sequence. Then for any* $p > 0$ *there is a finite constant* C_p *such that*

$$C_p^{-n} \left(\sum_{\mathbf{j} \in D_n} g(\mathbf{j})^2 \right)^{p/2} \le E|\langle g, \varepsilon \rangle|^p \le C_p^n \left(\sum_{\mathbf{j} \in D_n} g(\mathbf{j})^2 \right)^{p/2} \tag{1.1}$$

(*generalized Khinchine inequality*).

(iii) Contraction principle [Krakowiak and Szulga (1986b)]. *If* h *is a real function on* D_n *such that* $|h| \le 1$, *then a.s. convergence of* $\langle g, \mathbf{X} \rangle$ *implies a.s. convergence of* $\langle g \cdot h, \mathbf{X} \rangle$. *Moreover, if* $\{X_j\} \subset L^p$, *then there is a constant* $C > 0$ *depending only on* n *and* p *such that*

$$E|\langle g \cdot h, \mathbf{X} \rangle|^p \le CE|\langle g, \mathbf{X} \rangle|^p.$$

(iv) [Krakowiak and Szulga (1986a)] *Let* $\{X_j\} \subset L^p$. *Suppose that* (g_m) *is a sequence of real finite valued functions on* D_n *such that the sequence* $(\langle g_m, \mathbf{X} \rangle)$ *converges in* L^p *for some* $p \in [0, \infty]$. *Then there is a real function* g *on* D_n *such that the multilinear random form* $\langle g, \mathbf{X} \rangle$ *converges in* L^p *and it forms an* L^p*-limit of the sequence* $(\langle g_m, \mathbf{X} \rangle)$.

For a real positive function Φ on $\mathbf{R}_+$ we define, for $r > 0$,

$$\Phi_r(x) = \Phi(x^r), \qquad x \ge 0.$$

Following Kallenberg (1975), we consider a class $\mathscr{K}_r$ of real positive functions Φ on R_+ satisfying the following properties:

(K.1) $\Phi(0) = 0$;

(K.2) Φ is a concave and increasing function;

(K.3) Φ_r belongs to Kallenberg's class $F_1 \cup F_2$, that is, either it is concave or it is absolutely continuous with the concave derivative Φ_r' vanishing at the origin.

It is easy to see that (K.1) and (K.2) above imply that any Φ in $\mathscr{K}_r$ satisfies Δ_2 condition, that is, for any $c > 0$ there is $0 < d(c) < \infty$ such that for any $x > 0$,

$$\Phi(cx) \le d(c)\Phi(x). \tag{1.2}$$

LEMMA 1.4. *Let* Φ *be a function from* $\mathscr{K}_r$, $r > 0$, *and* ε *be a Bernoulli sequence. Then there is a constant* $C > 0$ *such that for every* n*-dimensional tetrahedral random array* $[X(\mathbf{i}), \mathbf{i} \in D_n]$ *independent of* ε *the following inequality holds*:

$$E\Phi_r(|\langle X, \varepsilon \rangle|) \le C^n \sum_{\mathbf{i}} E\Phi_r(|X(\mathbf{i})|).$$

PROOF. The statement follows from Lemma 2.1 in Kallenberg (1975) as multilinear tetrahedral Bernoulli forms are martingales. □

EXAMPLES. In the article we shall make use of the following functions:

(i) $\Phi(x) = x$, $\Phi \in \mathscr{K}_r$, $r \geq \alpha$;
(ii) $\Phi(x) = x/\ln^{\delta}(\mathbf{a} + x)$, $\Phi \in \mathscr{K}_\alpha$ for **a** large enough.

REMARK 1.5. Once a random multilinear form $\langle g, \mathbf{X} \rangle$ in symmetric random variables $\mathbf{X} = (X_j)$ converges a.s., it converges unconditionally, that is, regardless of any deterministic permutation of its entries. This follows immediately from Fubini's theorem and the generalized Khinchine inequality.

2. LePage's representation of a multiple stable integral. In 1984 Marcus and Pisier elaborated upon the results of LePage (1980) and LePage, Woodroofe and Zinn (1981) on series representation of stable processes, and it follows from Lemma 1.4 of Marcus and Pisier (1984), that for any function $f \in L^{\alpha}([0,1])$,

$$\int_0^1 f(t)\, dZ(t) =_D s^{-1/\alpha} \sum f(U_j) \Gamma_j^{-1/\alpha} \varepsilon_j, \tag{2.1}$$

where $s = \int_0^\infty x^{-\alpha} \sin x\, dx$, and $\mathbf{U}, \mathbf{\Gamma}, \boldsymbol{\varepsilon}$ are independent of each other, and the series in the r.h.s. of (2.1) converges a.s. and in L^p, $p < \alpha$.

In particular, one obtains a series representation of a stable motion

$$(Z(t), 0 \leq t \leq 1) =_D \left(s^{-1/\alpha} \sum \mathbf{1}\{U_j \leq t\} \Gamma_j^{-1/\alpha} \varepsilon_j, 0 \leq t \leq 1\right), \tag{2.2}$$

and therefore a counterpart of (2.1) for a multiple stable integral is expected to hold. A possibility of such a representation, at least for $n = 2$ and $n = 3$, was mentioned in the paper of McConnell (1986).

The aim of this section is to extend LePage's representation to the multiple stable integral.

Recall that a symmetric vanishing on diagonals Borel function on $[0,1]^n$ is said to be integrable with respect to $M^{(n)}$ if there is a sequence (f_m) of simple functions converging in Lebesgue measure to f such that multiple stochastic integrals $I_n(f_m) = \int_{[0,1]^n} f_m\, dM^{(n)}$ (defined in a usual way) converge in probability (or equivalently, in L^p, $0 < p < \alpha$). The limit is denoted by $I_n(f)$ or by either of the integrals

$$\int_{[0,1]^n} f\, dM^{(n)} = \int_0^1 \cdots \int_0^1 f(t_1, \dots, t_n) M(dt_1) \dots M(dt_n)$$

[see Krakowiak and Szulga (1988) for details].

THEOREM 2.1. *For any symmetric vanishing on diagonals Borel function f on $[0,1]^n$,*

$$\int_{[0,1]^n} f\, dM^{(n)} =_D s^{-n/\alpha} \sum_{\mathbf{j} \in N^n} f(\mathbf{U_j}) [\mathbf{\Gamma_j}]^{-1/\alpha} [\boldsymbol{\varepsilon}_\mathbf{j}], \tag{2.3}$$

where the integral exists and the series converges unconditionally a.s., or equivalently, in L^p, $0 < p < \alpha$, *at the same time. The sequences* $\mathbf{U}$, Γ *and* ε *are independent of each other.*

PROOF. By virtue of LePage's representation we may choose an α-stable random measure, and a fortiori, a product random measure $M^{(n)}$ generated by the α-stable process $Z(t) = s^{-1/\alpha}\sum \mathbf{1}\{U_j \le t\}\Gamma_j^{-1/\alpha}\varepsilon_j$. Denoting the multiple series appearing in (2.3) by $S_n(f)$ whenever it makes sense, we infer immediately that formula (2.3) holds a.s. for simple functions.

Suppose that $I_n(f)$ exists. By definition, there is a sequence of simple functions (f_m) converging in Lebesgue measure to f, and such that $I_n(f_m)$ converges in L^p, $0 \le p < \alpha$, to a random variable Y in L^p. Hence $S_n(f_m)$ converges in L^p to Y. By Theorem 1.3(iv), $Y = S_n(g)$ a.s. for some function $g \in L^p$, and the multiple series $S_n(g)$ converges a.s. by part (i) of that theorem. For $\mathbf{U}$ and $\mathbf{\Gamma}$ being fixed, $S_n(g)$ is a Bernoulli multilinear form. Therefore we infer from Fubini's theorem, Theorem 1.3(i) and the generalized Khinchine inequality (1.1) that $S_n(g)$ converges in L^1 and thus

$$\begin{aligned} g(\mathbf{U_j})[\mathbf{\Gamma_j}]^{-1/\alpha} &= E_\varepsilon S_n(g)[\boldsymbol{\varepsilon}_\mathbf{j}] \\ &= \lim_m E_\varepsilon S_n(f_m)[\boldsymbol{\varepsilon}_\mathbf{j}] \\ &= \lim_m f_m(\mathbf{U_j})[\mathbf{\Gamma_j}]^{-1/\alpha} \\ &= f(\mathbf{U_j})[\mathbf{\Gamma_j}]^{-1/\alpha} \end{aligned}$$

$(\mathbf{U}, \Gamma)$ a.s. Therefore $f = g$ almost everywhere on $[0,1]^n$ and $I_n(f) = S_n(f)$ a.s.

Suppose now that the series $S_n(f)$ converges in probability [or equivalently, by Theorem 1.3(i), almost surely]. For $k = 1, 2, \ldots,$ and $x \ge 0$ define

$$H_k(x) = -H_k(-x) = \begin{cases} 2^{-k}i & \text{if } x \in [2^{-k}i, 2^{-k}(i+1)),\ i = 0, 1, \ldots, 2^{2k-1}, \\ 0 & \text{if } x \ge 2^k. \end{cases}$$

We observe that $0 \le |x| - |H_k(x)| < 2^{-k}$ and thus applying the contraction principle [Theorem 1.3(iii)], we infer that the series $S_n(H_k(f))$ converges in probability. Further, $(I_n(H_k(f));\ n \in \mathbf{N})$ is a Cauchy sequence in L^0 because by virtue of Fubini's theorem and Lebesgue's dominated convergence theorem we have that

$$\begin{aligned} &\lim_{k, m \to \infty} E \min\left(1, |I_n(H_k(f) - H_m(f))|^2\right) \\ &\quad = \lim_{k, m \to \infty} E \min\left(1, |S_n(H_k(f) - H_m(f))|^2\right) \\ &\quad \le \lim_{k, m \to \infty} E_{\mathbf{U}, \Gamma} \min\left(1, E_\varepsilon |S_n(H_k(f) - H_m(f))|^2\right) \\ &\quad = \lim_{k, m \to \infty} E_{\mathbf{U}, \Gamma} \min\left(1, \sum_{\mathbf{j}} \left|H_k(f(\mathbf{U_j})) - H_m(f(\mathbf{U_j}))\right|^2 [\mathbf{\Gamma_j}]^{-2/\alpha}\right) = 0. \end{aligned}$$

Since $(H_k(f))$ is a sequence of simple functions converging almost everywhere to f then the latter identity implies the existence of $I_n(f)$. Moreover, it follows from the first part of the proof that $I_n(f) = S_n(f)$ a.s.

Unconditional convergence is a general feature of random multilinear forms in symmetric random variables (cf. Remark 1.5). □

3. Products of Poisson arrivals. Most of the properties of products of arrival times of a Poisson process presented in this section are part of a mathematical folklore. For the sake of convenience we collect them in one place.

LEMMA 3.1. *For $n \geq 1$ we have*

$$\lim_{t \to 0} \frac{P(\Gamma_1 \cdot \cdots \cdot \Gamma_n \leq t)}{t(-\ln t)^{n-1}} = \frac{1}{(n-1)!\, n!}.$$

PROOF. The identity is trivial for $n = 1$. Let $n \geq 2$. Using the well-known formula for converting arrival times of a Poisson process into i.i.d. uniformly distributed r.v. [cf., e.g., Karlin (1969), page 183], we check that

$$g_n(t) \overset{\mathrm{df}}{=} \int_0^\infty t^{-1} P(\Gamma_1/x \cdot \cdots \cdot \Gamma_{n-1}/x \leq t/x^n | \Gamma_n = x) e^{-x} x^{n-1}/(n-1)!\, dx$$

$$= \int_0^\infty t^{-1} P(U_1 \cdot \cdots \cdot U_{n-1} \leq t/x^n) e^{-x} x^{n-1}/(n-1)!\, dx.$$

Applying the well-known formula for the Erlang distribution [cf. Feller (1971), page 11], we obtain for any $s > 0$,

$$P(U_1 \cdot \cdots \cdot U_n \leq s) = P(X_1 + \cdots + X_n \geq -\ln s)$$

$$= \begin{cases} s \sum_{k=0}^{n-1} (-\ln s)^k / k! & \text{if } s \leq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, by elementary calculations

$$g_n(t) = (n-1)!^{-1} \sum_{k=0}^{n-2} n^k/(k+1)!\, f_{k+1}(t^{1/n}) + o(1), \tag{3.1}$$

where

$$f_k(s) = s \int_1^\infty (\ln y)^k e^{-sy}\, dy, \qquad s > 0,\ k \geq 0.$$

Elementary calculus (l'Hospital formula, change of variables of integration, induction, etc.) shows that

$$\lim_{s \to 0} f_k(s)/(-\ln s)^k = 1.$$

Together with (3.1) this completes the proof. □

COROLLARY 3.2. *Let $X \in L^\alpha$ be a nonnegative random variable independent of Γ. Then for $\mathbf{j} = (1, 2, \dots, n)$ and $a > 1$,*

$$\lambda_{n-1}\left(X\left[\Gamma_{\mathbf{j}}\right]^{-1/\alpha}\right) = \frac{\alpha^{n-1}EX^\alpha}{(n-1)!\,n!}.$$

We conclude this section with an observation that for any number $\beta \geq 1$ there is a constant $K > 0$ such that for every sequence $\mathbf{j} = (j_1, \dots, j_n) \in D_n$ such that $j_1 > n\beta$ we have

$$E\left[\Gamma_{\mathbf{j}}\right]^{-\beta} \leq K[\mathbf{j}]^{-\beta}. \tag{3.2}$$

Indeed, this follows by the Hölder inequality from the well-known estimate

$$E(\Gamma_j)^{-\beta} = \Gamma(j-\beta)/(j-1)! \leq Kj^{-\beta}.$$

4. Some estimates of multiple sums. For an $i \geq 1$ let $d_n(i)$ be the number of $\mathbf{i}$'s $\in N^n$ such that $[\mathbf{i}] = i$, and let $\mathscr{D}_n(k) = \sum_{i=1}^k d_n(i)$. It is well-known [cf., e.g., Titchmarsh (1951), page 263] that for some $C > 0$, any $k \geq 2$,

$$\mathscr{D}_n(k) \leq Ck(\ln k)^{n-1}. \tag{4.1}$$

Let Ψ: $\mathbf{R}_+ \to \mathbf{R}_+$ be a nonincreasing function, and let $1 \leq a < b \leq \infty$ be integers. Applying the formula of summation by parts, we obtain

$$\begin{aligned}
\sum_{\mathbf{i} \in N^n,\ a \leq [\mathbf{i}] < b} \Psi([\mathbf{i}]) &= \sum_{i=a}^{b-1} \Psi(i)\, d_n(i) \\
&= \sum_{i=a}^{b-1} \Psi(i)(\mathscr{D}_n(i) - \mathscr{D}_n(i-1)) \\
&= \left[\Psi(b-1)\mathscr{D}_n(b-1) - \Psi(a)\mathscr{D}_n(a-1)\right] \\
&\quad - \sum_{i=a}^{b-2} \mathscr{D}_n(i)(\Psi(i+1) - \Psi(i)) \\
&\leq \Psi(b-1)\mathscr{D}_n(b-1) + \sum_{i=a}^{b-2} (\Psi(i) - \Psi(i+1))\mathscr{D}_n(i) \\
&\leq C\left[\Psi(b-1)b(\ln b)^{n-1} + \sum_{i=a}^{b-2} (i+1)(\ln(i+1))^{n-1}\right. \\
&\qquad \left. \times (\Psi(i) - \Psi(i+1))\right].
\end{aligned} \tag{4.2}$$

Applying once more the formula of summation by parts, we obtain

$$\sum_{\mathbf{i}\in N^n,\ a\le[\mathbf{i}]<b} \Psi([\mathbf{i}])$$

$$\le C\Big\{\Psi(b-1)b(\ln b)^{n-1} - \Big[\Big(\Psi(b-1)b(\ln b)^{n-1} - \Psi(a)(a+2)(\ln(a+2))^{n-1}\Big)\Big] - \sum_{i=a}^{b-3}\Psi(i+1)\big[(i+3)(\ln(i+3))^{n-1} - (i+2)(\ln(i+2))^{n-1}\big]\Big\}$$

$$= C\Big\{\Psi(a)(a+2)(\ln(a+2))^{n-1} + \sum_{i=a}^{b-3}\Psi(i+1)\big[(i+3)(\ln(i+3))^{n-1} - (i+2)(\ln(i+2))^{n-1}\big]\Big\}.$$

It is easy to check that for every $x \ge 1$,

$$(x+1)(\ln(x+1))^{n-1} - x(\ln x)^{n-1} \le C_1(\ln(x+1))^{n-1}$$

for some finite positive constant C_1. We conclude finally that

$$\sum_{\mathbf{i}\in N^n,\ a\le[\mathbf{i}]<b} \Psi([\mathbf{i}]) \le C_2\Big[\Psi(a)(a+1)(\ln(a+1))^{n-1} + \sum_{i=a}^{b-3}\Psi(i+1)(\ln(i+1))^{n-1}\Big] \tag{4.3}$$

for some $0 < C_2 < \infty$. The following estimates are now easy to obtain.

LEMMA 4.1. (i) *For every* $u \ge 2$,

$$\sum_{[\mathbf{i}]\le u} 1 \le Cu(\ln u)^{n-1}. \tag{4.4}$$

(ii) *For every* $u \ge 2$,

$$\sum_{[\mathbf{i}]\le u} [\mathbf{i}]^{-1} \le C(\ln u)^{n}. \tag{4.5}$$

(iii) *Let* $F(x) = x^{-1}(\ln(\theta + x^{-1}))^{-\delta}$, $\delta \ge 1$. *For any* $\theta > e$ *big enough to make* F *nonincreasing and any* $u_0 > e$, *for every* $u \ge u_0$,

$$\sum_{[\mathbf{i}]\le u} F([\mathbf{i}]/u) \le \begin{cases} Cu(\ln u)^{n-1}\ln\ln u & \text{if } \delta = 1,\\ Cu(\ln u)^{n-1} & \text{if } \delta > 1.\end{cases} \tag{4.6}$$

(iv) *Let* $\beta > 1$. *For any* $u > e$,

$$\sum_{[\mathbf{i}]> u} [\mathbf{i}]^{-\beta} \le Cu^{1-\beta}(\ln u)^{n-1}. \tag{4.7}$$

Throughout, C is a finite positive constant independent of u.

PROOF. (i) Follows trivially from (4.1).
(ii) An immediate consequence of (4.3) with $\Psi(x) = x^{-1}$.
(iii) Putting $\Psi(x) = F(x/u)$, we use (4.3) and monotonicity of Ψ to get

$$
\begin{aligned}
\sum_{[\mathbf{i}]\le u} F([\mathbf{i}]/u) &= \sum_{[\mathbf{i}]\le u} \Psi([\mathbf{i}]) \\
&\le C_2\Big\{u(\ln(\theta+u))^{-\delta}\cdot 2(\ln 2)^{n-1} \\
&\qquad + \sum_{i\le u}(\ln(i+1))^{n-1}u(i+1)^{-1}\big[\ln\big(\theta+u(i+1)^{-1}\big)\big]^{-\delta}\Big\} \\
&\le C_2\Big\{2(\ln 2)^{n-1}u + (\ln(u+1))^{n-1}\sum_{i\le u} u(i+1)^{-1} \\
&\qquad \times\big[\ln\big(\theta+u(i+1)^{-1}\big)\big]^{-\delta}\Big\} \\
&\le C_2\Big\{2(\ln 2)^{n-1}u + (\ln(u+1))^{n-1} \\
&\qquad \times\int_1^{u+1}\frac{u}{x+1}\left(\ln\left(\theta+\frac{u}{x+1}\right)\right)^{-\delta}dx\Big\} \\
&= C_2\Big\{2(\ln 2)^{n-1}u + u(\ln(u+1))^{n-1} \\
&\qquad \times\int_{2/u}^{(u+2)/u} y^{-1}\big(\ln(\theta+y^{-1})\big)^{-\delta}dy\Big\},
\end{aligned}
$$

and (4.6) easily follows.
(iv) Follows immediately from (4.3). □

5. Asymptotic evaluation of the tail. The main result of the article (Theorem 5.3) is stated and proved in this section, but first we study certain properties of tetrahedral multilinear forms of the type

$$S_n^{\sim} = \sum_{\mathbf{j}\in D_n}[\boldsymbol{\varepsilon}_{\mathbf{j}}][\boldsymbol{\Gamma}_{\mathbf{j}}]^{-1/\alpha}X_{\mathbf{j}},$$

where $\mathbf{X}$ is an array of identically distributed random variables which is independent of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\Gamma}$ sequences. We introduce some useful decompositions of the series

$S_n^{\sim}$. Put

$$T_{n,m} = \sum_{\mathbf{j} \in D_n,\ j_1 \geq m} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} X_{\mathbf{j}}.$$

We decompose $T_{n,m}$ into two summands as follows:

$$\begin{aligned} T_{n,m} &= T'_{n,m} + T''_{n,m} \\ &= \sum_{\mathbf{j} \in D_n,\ j_1 \geq m} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} X_{\mathbf{j}} \mathbf{1}\{|X_{\mathbf{j}}|^\alpha \leq [\mathbf{j}]\} \\ &\quad + \sum_{\mathbf{j} \in D_n,\ j_1 \geq m} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} X_{\mathbf{j}} \mathbf{1}\{|X_{\mathbf{j}}|^\alpha > [\mathbf{j}]\}. \end{aligned}$$

PROPOSITION 5.1. (a) *Let $r > \alpha$, $m \geq m_0 > nr/\alpha$, $n \geq 1$. Then there is a finite constant $C' > 0$ depending only on α, r, m_0 and n and independent of m and the law of $\{X_{\mathbf{j}}, \mathbf{j} \in D_n\}$ such that*

$$E|T'_{n,m}|^\alpha \leq C'\Big\{E\Big[|X_{\mathbf{j}}|^\alpha\Big(1 + \big(\ln_+ |X_{\mathbf{j}}|\big)^{n-1}\Big)\Big]\Big\}^{\alpha/r}.$$

(b) *Consider $\Phi(x) = x/(\ln(a+x))^{n-1}$ with a chosen large enough to have Φ in $\mathscr{K}_\alpha$. Let $n > 1$ and $m \geq m_0 > n$. Then there is a finite constant $C'' > 0$ depending only on α, m and n, and independent of m and the law of $\{X_{\mathbf{j}}, \mathbf{j} \in D_n\}$ such that*

$$E\Phi_\alpha(|T''_{n,m}|) \leq \begin{cases} C''E\Big[|X_{\mathbf{j}}|^\alpha\Big(1 + \big(\ln_+ |X_{\mathbf{j}}|\big)^{n-1}\Big)\Big] & \text{if } n > 2, \\ C''E\Big[|X_{\mathbf{j}}|^\alpha\big(1 + \ln_+ |X_{\mathbf{j}}| \ln_+ |\ln |X_{\mathbf{j}}||\big)\Big] & \text{if } n = 2. \end{cases}$$

(c) *Let $n \geq 1$, $m \geq m_0 > n$. Then there is a finite constant $C''' > 0$ depending only on α, m_0 and n, and independent of m and the law of $\{X_{\mathbf{j}}, \mathbf{j} \in D_n\}$ such that*

$$E|T''_{n,m}|^\alpha \leq C'''E\Big[|X_{\mathbf{j}}|^\alpha\Big(1 + \big(\ln_+ |X_{\mathbf{j}}|\big)^n\Big)\Big].$$

PROOF. (a) By Hölder's inequality, Fubini's theorem and Lemma 1.4 we conclude that

$$E|T'_{n,m}|^\alpha \leq \big(E|T'_{n,m}|^r\big)^{\alpha/r} \leq C'_1\Bigg[\sum_{\mathbf{j} \in D_n,\ j_1 \geq m} E[\Gamma_{\mathbf{j}}]^{-r/\alpha} E\Big[|X_{\mathbf{j}}|^r \mathbf{1}\{|X_{\mathbf{j}}|^\alpha \leq [\mathbf{j}]\}\Big]\Bigg]^{\alpha/r}.$$

Using the estimate (3.2) for moments of Γ_j and Lemma 4.1(iv) (with $\beta = nr/\alpha$), we bound the latter expression from above by the quantities

$$C_2'\left[\sum_{\mathbf{j}\in D_n,\ j_1\ge m}[\mathbf{j}]^{-r/\alpha}\sum_{k=1}^{[\mathbf{j}]}E\big(|X_{\mathbf{j}}|^r\mathbf{1}\{k-1<|X_{\mathbf{j}}|^\alpha\le k\}\big)\right]^{\alpha/r}$$

$$\le C_2'\left[\sum_{k=1}^{\infty}E\big(|X_{\mathbf{j}}|^r\mathbf{1}\{k-1<|X_{\mathbf{j}}|^\alpha\le k\}\big)\sum_{[\mathbf{j}]\ge k}[\mathbf{j}]^{-r/\alpha}\right]^{\alpha/r}$$

$$\le C_3'\Bigg[E\big(|X_{\mathbf{j}}|^r\mathbf{1}\{0<|X_{\mathbf{j}}|^\alpha\le 2\}\big)$$

$$+\sum_{k=3}^{\infty}k(\ln k)^{n-1}\mathbf{P}\big(k-1<|X_{\mathbf{j}}|^\alpha\le k\big)\Bigg]^{\alpha/r}$$

$$\le C'\Big\{E\Big[|X_{\mathbf{j}}|^\alpha\big(1+(\ln_+|X_{\mathbf{j}}|)^{n-1}\big)\Big]\Big\}^{\alpha/r}.$$

(b) Applying Lemma 1.4, we get

$$E\Phi_\alpha(|T''_{n,m}|)\le C_1''\sum_{\mathbf{j}\in D_n,\ j_1\ge m}E\Big[\Phi_\alpha\big(|X_{\mathbf{j}}|[\Gamma_{\mathbf{j}}]^{-1/\alpha}\big)\mathbf{1}\{|X_{\mathbf{j}}|^\alpha>[\mathbf{j}]\}\Big]$$

$$=C_1''\sum_{\mathbf{j}\in D_n,\ j_1\ge m}E\Big[\Phi\big(|X_{\mathbf{j}}|^\alpha[\Gamma_{\mathbf{j}}]^{-1}\big)\mathbf{1}\{|X_{\mathbf{j}}|^\alpha>[\mathbf{j}]\}\Big].$$

By the concavity of Φ, independence of Γ and $\mathbf{X}$, moment inequality (3.2) and Δ_2 property (1.2) we conclude that

$$E\Big(\Phi\big(|X_{\mathbf{j}}|^\alpha[\Gamma_{\mathbf{j}}]^{-1}\big)|X_{\mathbf{j}}\Big)\le\Phi\Big(|X_{\mathbf{j}}|^\alpha E\big([\Gamma_{\mathbf{j}}]^{-1}\big)\Big)$$

$$\le C_2''\Phi\big([\mathbf{j}]^{-1}|X_{\mathbf{j}}|^\alpha\big),$$

so that

$$E\Phi_\alpha(|T''_{n,m}|)\le C_3''\sum_{\mathbf{j}\in D_n,\ j_1\ge m}E\left[\Phi\big([\mathbf{j}]^{-1}|X_{\mathbf{j}}|^\alpha\big)\sum_{k=[\mathbf{j}]}^{\infty}1\{k<|X_{\mathbf{j}}|^\alpha\le k+1\}\right].$$

Changing the order of summation and making use of Lemma 4.1(iii), we obtain in the case $n>2$,

$$E\Phi_\alpha(|T''_{n,m}|)\le C_4''\sum_{k=1}^{\infty}\sum_{j\in D_n,\ [\mathbf{j}]\le k}\Phi\big((k+1)[\mathbf{j}]^{-1}\big)\mathbf{P}\big(k<|X_{\mathbf{j}}|^\alpha\le k+1\big)$$

$$\le C_5''\sum_{k=1}^{\infty}(k+1)(\ln(k+1))^{n-1}\mathbf{P}\big(k<|X_{\mathbf{j}}|^\alpha\le k+1\big)$$

$$\le C''E\Big[|X_{\mathbf{j}}|^\alpha\big(1+(\ln_+|X_{\mathbf{j}}|)\big)^{n-1}\Big].$$

The case $n=2$ is similar.

(c) The proof of this part is completely similar to the proof of (b). □

Proposition 5.2. *Let $\{X_j\}$ be a sequence of identically disturbed random variables, independent of Γ and ε sequences, such that* $E[|X_1|^\alpha(1 + \ln_+|X_1|)] < \infty$. *Define*

$$Z^{\sim}_{n,1,i} = \Gamma_1^{-1/\alpha} \cdot \cdots \cdot \Gamma_{n-1}^{-1/\alpha} \sum_{j=i}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} X_j, \qquad n \geq 2,\ i \geq n+1.$$

Then

$$\lambda_{n-1}(|Z^{\sim}_{n,1,i}|) = 0.$$

Proof. The proof is by induction in n. For $n = 2$ we apply the contraction principle for probabilities

$$\mathbf{P}(|Z^{\sim}_{2,1,i}| > x) \leq 2\mathbf{P}\left(\Gamma_1^{-1/\alpha}\left|\sum_{j=i}^{\infty} \varepsilon_j(\Gamma_j - \Gamma_1)^{-1/\alpha} X_j\right| > x\right),$$

and Proposition 5.2 follows from the independence of Γ_1 and $\{\Gamma_j - \Gamma_1\}$ and Proposition 5.1(a) and (c) with $n = 1$, $m_0 = 2$.

Assuming that Proposition 5.2 is valid for $n - 1$, we employ once more the contraction principle for probabilities to get for $x > 0$,

$$\begin{aligned}
&\mathbf{P}(|Z^{\sim}_{n,1,i}| > x)\\
&\quad\leq 4\mathbf{P}\left[\Gamma_1^{-1/\alpha}(\Gamma_2 - \Gamma_1)^{-1/\alpha} \cdots (\Gamma_{n-1} - \Gamma_1)^{-1/\alpha}\left|\sum_{j=i}^{\infty} \varepsilon_j(\Gamma_j - \Gamma_1)^{-1/\alpha} X_j\right| > x\right]\\
&\quad= 4\int_0^\infty e^{-y}\mathbf{P}\left(|Z^{\sim}_{n-1,1,i-1}| > xy^{1/\alpha}\right) dy\\
&\quad= 4\int_0^{x^{-\alpha}(\ln x)^{(n-1)/2}} \cdots + 4\int_{x^{-\alpha}(\ln x)^{(n-1)/2}}^{\infty} \cdots.
\end{aligned}$$

Applying the assumption of the induction to the second integral in the expression above completes the proof. □

We introduce the following modulars defined on the class of Borel functions on $[0,1]^n$:

$$L^\alpha(f) \overset{\text{df}}{=} \int \cdots \int_{[0,1]^n} |f(\mathbf{x})|^\alpha \, d\mathbf{x},$$

$$L^\alpha \log^\delta L(f) \overset{\text{df}}{=} \int \cdots \int_{[0,1]^n} |f(\mathbf{x})|^\alpha\left[1 + \left(\ln_+|f(\mathbf{x})|\right)^\delta\right] d\mathbf{x}, \qquad \delta \geq 0,$$

$$L^\alpha \log L \log\log L(f) \overset{\text{df}}{=} \int \cdots \int_{[0,1]^n} |f(\mathbf{x})|^\alpha\left[1 + \ln_+|f(\mathbf{x})|\ln_+|\ln|f(\mathbf{x})||\right] d\mathbf{x}.$$

Now we formulate the main result of the article.

THEOREM 5.3. *Let* $0 < \alpha < 2$, $n \geq 2$ *and* f *be a symmetric vanishing on diagonals Borel function on* $[0,1]^n$ *such that*

$$L^\alpha \log^{n-1} L(f) < \infty \quad \text{if } n > 2,$$
$$L^\alpha \log L \log\log L(f) < \infty \quad \text{if } n = 2.$$

Let $M^{(n)}$ *be the random measure generated on Borel sets in* $[0,1]^n$ *by a symmetric* α*-stable process with independent stationary increments on* $[0,1]$. *Then* f *is* $M^{(n)}$*-integrable and its integral* $I_n(f)$ *has the property*

$$\lambda_{n-1}(|I_n(f)|) = 2\lambda_{n-1}(I_n(f)) = n\alpha^{n-1}(n!)^{\alpha-2}s^{-n}L^\alpha(f), \tag{5.1}$$

where $s = \int_0^\infty x^{-\alpha} \sin x \, dx$.

PROOF. Fubini's theorem and Theorem 1.3(i) imply that a necessary and sufficient condition for the convergence of $S_n(f)$ [equivalently, the existence of $I_n(f)$] is

$$\sum_{\mathbf{j} \in D_n} [\Gamma_\mathbf{j}]^{-2/\alpha} |f(\mathbf{U_j})|^2 < \infty \quad \text{a.s.} \tag{5.2}$$

We introduce the following partition of the set D_n:

$$D_n = \bigcup_{k=0}^{n} D_{n,k}, \tag{5.3}$$

where $D_{n,0} = \{(1,2,\dots,n)\}$, and for $k = 1,2,\dots,n$,

$$D_{n,k} = \{(1,2,\dots,n-k, j_1, j_2,\dots,j_k) \colon (j_1,\dots,j_k) \in D_k,\ j_1 \geq n-k+2\}.$$

Let us denote for $k = 0,1,\dots,n$,

$$Z_{n,k} = \sum_{\mathbf{j} \in D_{n,k}} [\varepsilon_\mathbf{j}][\Gamma_\mathbf{j}]^{-1/\alpha} f(\mathbf{U_j}),$$

$$A_{n,k} = \sum_{\mathbf{j} \in D_{n,k}} [\Gamma_\mathbf{j}]^{-2/\alpha} |f(\mathbf{U_j})|^2.$$

We will prove that $A_{n,k} < \infty$ a.s. for any $k = 0,1,\dots,n$. This would imply (5.2) and, simultaneously, the convergence of $Z_{n,k}$'s, $k = 0,1,\dots,n$.

Note that $A_{n,0} < \infty$ trivially since $D_{n,0}$ consists of only one element. Recall also that by Corollary 3.2

$$\lambda_{n-1}(|Z_{n,0}|) = 2\lambda_{n-1}(Z_{n,0}) = n\alpha^{n-1}(n!)^{-2}L^\alpha(f) \tag{5.4}$$

since $Z_{n,0}$ is a symmetric random variable. Note that in general $I_n(f)$ [or $S_n(f)$] *is not a symmetric random variable*, except the case of the odd integer n, even though it behaves like such due to its dominance by the symmetric random variable.

To complete the proof of the theorem, we have, therefore, to show that $A_{n,k} < \infty$ a.s. for $k = 1,\dots,n$, and that

$$\lambda_{n-1}(|Z_{n,k}|) = 0 \quad \text{for } k = 1,2,\dots,n. \tag{5.5}$$

Corollary 1.2, (5.4) and (5.5) would imply then (5.1).

The proof will use an inductive argument and, as frequently happens with inductive arguments, it is more convenient to prove somewhat more general claim. For any $k = 1, 2, \dots, n$ and $i \geq n - k + 2$ define

$$D_{n,k,i} = \{(1,2,\dots,n-k, j_1,\dots,j_k): (j_1,\dots,j_k) \in D_k,\ j_1 \geq i\},$$

$$D^*_{n,k,i} = \{(1,2,\dots,n-k-1, i-1, j_1,\dots,j_k): (j_1,\dots,j_k) \in D_k,\ j_1 \geq i\}.$$

Let $\{X_{\mathbf{j}}, \mathbf{j} \in D_n\}$ be an array of identically distributed random variables, which is independent of Γ and ε sequences such that

$$E\left[|X_{\mathbf{j}}|^\alpha\left(1 + \left(\ln_+|X_{\mathbf{j}}|\right)^{n-1}\right)\right] < \infty \quad \text{if } n > 2$$

or

$$E\left[|X_{\mathbf{j}}|^\alpha\left(1 + \ln_+\left(|X_j|\right)\ln_+\left(\ln_+|X_j|\right)\right)\right] < \infty \quad \text{if } n = 2.$$

Finally, let

$$Y_{n,k,i} = \sum_{\mathbf{j} \in D_{n,k,i}} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} X_{\mathbf{j}},$$

$$B_{n,k,i} = \sum_{\mathbf{j} \in D_{n,k,i}} [\Gamma_{\mathbf{j}}]^{-2/\alpha} X_{\mathbf{j}}^2,$$

and $Y^*_{n,k,i}$ and $B^*_{n,k,i}$ are defined correspondingly. We will prove that $B_{n,k,i} < \infty$ a.s. for any $k = 1,\dots,n$ and $i \geq n - k + 2$ (this would imply that $A_{n,k} < \infty$ a.s. for any $k = 1,\dots,n$) and that

$$\lambda_{n-1}(|Y_{n,k,i}|) = 0 \quad \text{for any } k = 1,\dots,n \text{ and } i \geq n-k+2. \tag{5.6}$$

This would imply (5.5), since $D_{n,k} = D_{n,k,n-k+2}$. The proof is by induction in k. It is clear by Proposition 5.1(a) with $n = 1$ and $\alpha < r \leq 2$ and by Proposition 5.1(c) with $n = 1$ that $B_{n,1,i} < \infty$ a.s. for any $i \geq n + 1$. It also follows from Proposition 5.2 that (5.6) holds for $k = 1$ and any $i \geq n + 1$. This constitutes the basis of the induction. Assume now that for some $1 \leq k < n - 1$ and any $i \geq n - k + 2$, $B_{n,k,i} < \infty$ a.s. and (5.6) holds. Clearly, for any $i \geq n - k + 1$,

$$D_{n,k+1,i} = \left(\bigcup_{m=i}^{2n} D^*_{n,k,m+1}\right) \cup D_{n,k+1,2n+1}.$$

Therefore,

$$B_{n,k+1,i} = \sum_{m=i}^{2n} B^*_{n,k,m+1} + B_{n,k+1,2n+1}. \tag{5.7}$$

The assumption of the induction implies that for any $m \geq n - k + 1$,

$$B^*_{n,k,m+1} \leq B_{n,k,m+1} < \infty \quad \text{a.s.}$$

Moreover, Proposition 5.1(a) with $n = k + 1$, $\alpha < r \leq 2$ and Proposition 5.1(c) with $n = k + 1$ imply that $B_{n,k+1,2n+1} < \infty$ a.s. By (5.7) we conclude that $B_{n,k+1,i} < \infty$ a.s.

We have

$$Y_{n,k+1,i} = \sum_{m=i}^{2n} Y^*_{n,k,m+1} + Y_{n,k+1,2n+1}. \tag{5.8}$$

Clearly, for any $m \geq n - k + 1$, $|Y^*_{n,k,m+1}| \leq |Y_{n,k,m+1}|$. Therefore, the assumption of the induction implies that $\lambda_{n-1}(|Y^*_{n,k,m+1}|) = 0$ for any $m \leq n - k + 1$. Corollary 1.2 shows then that the claim $\lambda_{n-1}(|Y_{n,k+1,i}|) = 0$ would follow if we prove that $\lambda_{n-1}(|Y_{n,k+1,2n+1}|) = 0$. We have, for $x > 0$,

$$\begin{aligned}
&P(|Y_{n,k+1,2n+1}| > x)\\
&= P\left(\left|\sum_{\mathbf{j}\in D_{n,k+1,2n+1}} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} X_{\mathbf{j}}\right| > x\right)\\
&= \int_0^\infty P\left(\left|\sum_{\substack{j\in D_{k+1}\\ j_1\geq 2n+1}} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} X_{1,\dots,n-k-1,j_1,\dots,j_{k+1}}\right| > xy^{1/\alpha} \,\Big|\, \Gamma_1\cdot\cdots\cdot\Gamma_{n-k-1} = y\right) dg_{n-k-1}(y) \qquad (5.9)\\
&= \int_0^{x^{-\alpha}} P(\cdots)\, dg_{n-k-1}(y) + \int_{x^{-\alpha}}^\infty P(\cdots)\, dg_{n-k-1}(y),
\end{aligned}$$

where g_n is the distribution function of $\Gamma_1\cdot\cdots\cdot\Gamma_n$. We apply now the contraction principle [Theorem 1.3(iii)] and Fubini's theorem to conclude that for any $y > 0$,

$$\begin{aligned}
&E\left[\left|\sum_{\substack{\mathbf{j}\in D_{k+1}\\ j_1\geq 2n+1}} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} X_{1,\dots,n-k-1,j_1,\dots,j_{k+1}}\right|^\alpha \,\Big|\, \Gamma_1\cdot\cdots\cdot\Gamma_{n-k-1} = y\right]\\
&\leq CE\left[\left|\sum_{\substack{\mathbf{j}\in D_{k+1}\\ j_1\geq 2n+1}} [\varepsilon_{\mathbf{j}}]\prod_{i=1}^{k+1}\left(\Gamma_{j_i} - \Gamma_{n-k-1}\right)^{-1/\alpha} X_{1,\dots,n-k-1,j_1,\dots,j_{k+1}}\right|^\alpha\right]
\end{aligned}$$

for some $0 < C < \infty$ independent of $y > 0$. We apply now Proposition 5.1(c) with $n = k + 1$ to conclude that the αth moment above is finite. The claim $\lambda_{n-1}(|Y_{n,k+1,2n+1}|) = 0$ now follows from (5.9) and Markov inequality. This completes the inductive argument, and we know by now, therefore, that $B_{n,k,i} < \infty$ a.s. for any $k = 1,\dots, n-1$ and any $i \geq n - k + 2$, and that (5.6) holds for $k = 1,\dots, n - 1$ and any $i \geq n - k + 2$. It remains to consider the case $k = n$. We apply (5.7) with $k = n - 1$. Then $B^*_{n,n-1,m+1} \leq B_{n,n-1,m+1} < \infty$ a.s. as have been proven above. Moreover, $B_{n,n,2n+1} < \infty$ a.s. by (the proof of) Proposition 5.1(b). This shows that $B_{n,n,i} < \infty$ a.s. for any $i \geq 2$. Further, we apply (5.8) with $k = n - 1$ and, as above, the claim $\lambda_{n-1}(|Y_{n,n,i}|) = 0$ would follow once we show that $\lambda_{n-1}(|Y_{n,n,2n+1}|) = 0$. But the latter statement follows immediately from Proposition 5.1(a). This completes the proof of the theorem. □

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