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UNIQUENESS OF THE INFINITE CLUSTER FOR STATIONARY GIBBS STATES[1]

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We prove, in all dimensions, that for a stationary Gibbs state with finite range or rapidly decreasing interaction, there is at most one infinite percolation cluster. This implies that the connectivity function is bounded away from 0.

1. Introduction. This article deals with global features of percolation in Gibbs models. We shall consider the d-dimensional lattice $\mathbb{Z}^d$, in each site of which there is a spin variable which can be up or down. The stochastic distribution of spins will be described by a Gibbs state, a probability measure on the set of all possible configurations of spins, the conditional probabilities of which are given by the Gibbs formula [(2.2) in Section 2]; for a general introduction see [15], [21] or [23]. The model is built up starting from a given interaction between the spins which determines the conditional probabilities. For some interactions there can be more than one state having the same conditional probabilities. This, then, represents the phenomenon of phase transition ([10] and [22]) and hence the interest of the Gibbs models in equilibrium statistical mechanics. Some of these states may not be stationary ([5] and [25]), but in the present article we will consider only the ones which are. We also assume that the interaction has finite range or decreases sufficiently rapidly [viz. it satisfies (2.1)].

Percolation is described as follows. In any configuration of spins we consider two nearest-neighbour sites as connected if both the spins are up. The theory of percolation, then, deals with the probabilistic description of the maximal connected components of the set of sites in which the spins are up. These components are called clusters and percolation arises when there is at least one cluster containing infinitely many sites (an infinite cluster) with positive proability. (See [17] for a general reference.) To study the global properties of percolation, we consider the number N of infinite clusters. We prove that in fact the infinite cluster is unique when it exists (meaning that the possible values of N are only 0 and 1) for the Gibbs states we are considering (Section 3). This implies that when percolation arises the probability of any two points being connected by a chain of spins up is bounded away from 0 (Section 4).

The study of percolation in Gibbs models is interesting not only in itself but also for the techniques it has contributed. These have been used for example to prove the absence of a nonstationary two-dimensional Ising measure ([1], [14] and [23]) and in the study of large deviations ([24]). We hope that the techniques developed in this article will also prove of use in statistical mechanics.

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Next we review the previous results in the study of the number of infinite clusters. For the Bernoulli model, where the probability of each spin being up equals p independent of the other spins, the solution to the uniqueness problem (i.e., the proof that N can assume only the values 0 or 1 with probability 1) was given in [8] and [13] for $d = 2$ and in [3] for any dimension d. This last work deals with many other quantities related to N as functions of p. But in spite of the remarkable findings of a relation between the analytical properties of some thermodynamical quantities and the uniqueness of the cluster (see also [6] and [16] and especially [26] for a similar point of view) the proof is quite involved. A simplified proof which does not make use of the variation of the parameter p is given in [11]. We follow some ideas of these works and in particular we generalize a sort of large deviation property proved in [3] and adapted in [11] (see Section 3, Lemma 3, below).

Our article generalizes also [4] where the uniqueness problem was solved for some Gibbs models in dimension 2. A generalization of [4] in a different direction (and for dimension 2) was given in [12] under the assumption that the measure is FKG ([9]), ergodic and has some geometrical properties.

A different approach was used in [19] to show that under general conditions N can assume only one of the values 0, 1 or ∞. No system satisfying the conditions in [19] has so far been shown to have $N = \infty$ with positive probability. The present article shows that this case does not occur for the stationary Gibbs states with finite range or rapidly decreasing interaction.

The assumption that the Gibbs state is stationary seems essential for uniqueness. On the other hand we believe that the result holds (in the stationary case) also when certain local configurations of spins are excluded or for long-range interactions not satisfying (2.1) (see the discussion in Section 4). Nevertheless we cannot treat these cases.

2. Preliminaries. We consider the d-dimensional lattice $\mathbf{Z} = \mathbb{Z}^d$. A *configuration* is an element $\omega = (\omega_x)_{x \in \mathbf{Z}}$ of the *configuration space* $\Omega = \{-1, 1\}^{\mathbf{Z}}$. For every subset $S \subseteq \mathbf{Z}$ define $\Omega_S = \{-1, 1\}^S$; Ω_S can be endowed with the product topology and all the measures μ on Ω_S will be considered as defined on its Borel σ-algebra; E_μ will denote the expectation with respect to μ. For a subset $M \supset S$ we have maps $\alpha_{S,M}$: $\Omega_M \to \Omega_S$ defined by $\alpha_{S,M}\omega = (\omega_x)_{x \in S}$, transforming the measure μ on Ω_M into the measure $\alpha_{S,M}\mu = \mu \circ \alpha_{S,M}^{-1}$ on Ω_S; we denote $\alpha_S = \alpha_{S,\mathbf{Z}}$. We also have the group $G = \mathbf{Z}$ of translations of $\mathbf{Z}$ which generates maps τ^a: $\Omega_S \to \Omega_{S-a}$ defined by $(\tau^a\omega)_x = \omega_{x+a}$ for all $a \in G$, $S \subseteq \mathbf{Z}$ and $x \in S - a := \{x \in \mathbf{Z}\colon\ x + a \in S\}$. Then consider *boxes* $B = B_m = \{(x_1, \dots, x_d) \in \mathbf{Z}\colon -m \le x_i \le m,\ i = 1, \dots, d\}$ of linear size $m \in \mathbb{N}$ and denote expressions like $\lim_{m \to \infty} a_{B_m}$ as $\lim_{B \uparrow \infty} a_B$. This will replace the usual limit in the sense of van Hove (see [22], Chapter 3.9).

To introduce the measures which we shall be dealing with, we need an *interaction*

$$\Phi\colon \bigcup_{S \text{ finite} \subset \mathbf{Z}} \Omega_S \to \mathbb{R}$$

invariant under translations, that is, satisfying $\Phi(\tau^a\omega) = \Phi(\omega)$ for all $a \in G$. Let $B = B_R$ be a box, $R \in \mathbb{N}$ and let

$$a_R = \sum_{S \text{ finite}, \mathbf{0} \in S, S \cap (\mathbf{Z} \setminus B) \neq \varnothing} \sup_{\omega \in \Omega_S} |\Phi(\omega)|.$$

We will mainly be concerned with interactions for which

$$R^{2d-2} a_R \to 0 \quad \text{as } R \to \infty. \tag{2.1}$$

This represents our (rapidly decreasing) interaction; we say that an interaction has *finite range* if there exists an integer R such that $a_R = 0$; the smallest of these integers will be called the length of interaction.

For any two disjoint sets B, $M \subseteq \mathbf{Z}$, B finite, the *energy* U_B and the *interaction energy* $W_{B,M}$ for an interaction Φ are real functions on Ω_B and Ω defined by, respectively,

$$U_B(\omega) = \sum_{S \subseteq B} \Phi(\alpha_{S,B}\omega)$$

and

$$W_{B,M}(\omega) = \sum_{S \text{ finite} \subseteq (B \cup M):\, S \cap M \neq \varnothing,\, S \cap B \neq \varnothing} \Phi(\alpha_S \omega).$$

DEFINITION 1. Let Φ be a (stationary) interaction. A *Gibbs state* for Φ is a probability measure μ on Ω such that

$$\alpha_S \mu(\omega) = \int_{\Omega_{\mathbf{Z} \setminus S}} M_{S,\eta}(\omega) \alpha_{\mathbf{Z} \setminus S} \mu(d\eta) \tag{2.2}$$

for all finite $S \subset \mathbf{Z}$, where for all $\eta \in \Omega_{\mathbf{Z} \setminus S}$, $M_{S,\eta}$ is the probability measure on Ω_S defined by

$$M_{S,\eta}(\omega) = (Z_{S,\eta})^{-1} \exp[-U_S(\omega) - W_{S,\mathbf{Z} \setminus S}(\omega \vee \eta)],$$

where $Z_{S,\eta}$ is a normalizing factor and $\omega \vee \eta \in \Omega$ is defined by

$$\alpha_S(\omega \vee \eta) = \omega \quad \text{and} \quad \alpha_{\mathbf{Z} \setminus S}(\omega \vee \eta) = \eta.$$

We say that a Gibbs state is stationary if $\tau^a \mu = \mu$ for all $a \in G$. Note that the stationarity of a Gibbs state is not implied by that of Φ (see [5] and [25]).

If μ is stationary we say that it is a *finite-range Gibbs state* if it has finite range and we call it a *long-range Gibbs state* if it satisfies (2.1). Observe that the measures $M_{S,\eta}$ are the conditional probabilities $\mu(\cdot \mid \eta)$ on Ω_S (see [22], Section 1.7).

We will consider percolation of *nearest-neighbour* points, which are elements of $\mathbf{Z}$ at distance 1. Let S be a subset of $\mathbf{Z}$. A *chain* in S is a sequence of elements of S such that successive terms are nearest neighbours, and two points $x, y \in S$ are *connected* by a chain in S if the chain contains the two points. S is connected if every two of its points are connected by a chain in S.

Let $M \supset S$ be a subset of $\mathbf{Z}$ and let $\omega \in \Omega_m$ be any configuration. A *cluster* of ω *in* S is a maximal connected subset $C \subseteq S \cap \omega^{-1}(+1)$; if $S = \mathbf{Z}$ we say that C

is a cluster of ω. *Percolation* occurs for a measure P on Ω when there is a positive probability to have a cluster containing infinitely many points, an *infinite cluster*. The *number of distinct infinite clusters* in ω is denoted by $N(\omega)$, where $N: \Omega \to \mathbb{N} \cup \{\infty\}$, and we study this quantity for the Gibbs states.

THEOREM 1. *Let μ be a long-range Gibbs state on Ω. Then with probability* 1, *N assumes only the values* 0 *or* 1.

The theorem will be proved in two steps. First we prove the result for finite-range Gibbs states (Section 3); in this way the main ideas will be more clearly exposed and the result for the long-range states will follow by imitating the first step (Section 4).

Note that no ergodicity has been assumed for μ and thus N is not forced to assume one value with probability 1 (see [19]). Of course this is only a matter of exposition, as our results carry over from the ergodic states to all the stationary states via the ergodic decomposition (see [22], Appendix 5, or [20]).

Let x be a point in $\mathbf{Z}$. First we reduce the problem to the study of the number of distinct infinite clusters containing at least one nearest-neighbour of x, which we denote by $N_{x,\infty}(\omega)$, $\omega \in \Omega$. Note that $N_{x,\infty} \in \{0, 1, \dots, 2d\}$. For a measure μ the quantity we are interested in is $E(x, \infty) = E_{\mu}((N_{x,\infty} - 1)I_{\{N_{x,\infty} > 0\}})$, where I_Q denotes the characteristic function of the event Q. The next lemma applies obviously to finite and long-range Gibbs states.

LEMMA 1. *Let μ be a Gibbs state with interaction Φ whose interaction energy $W_{S, \mathbf{Z}\setminus S}(\omega)$ is finite for all finite $S \subset \mathbf{Z}$, $\omega \in \Omega$. Then $\mu(N > 1) = 0$ is equivalent to $E(\mathbf{0}, \infty) = 0$.*

PROOF. Since $E(\mathbf{0}, \infty) > 0$ implies immediately that $\mu(N > 1) > 0$ we simply have to prove the reversed implication, for which it is enough to show that $\mu(N_{\mathbf{0},\infty} > 1) > 0$ if $\mu(N > 1) > 0$. A proof of this is based on the σ-additivity and the stationarity of μ and the fact that changing the configurations from an event with positive probability in a finite nonrandom collection of sites leaves the probability of the event positive, which is provided by the finiteness of $W_{S, \mathbf{Z}\setminus S}$.

A detailed proof is given in Proposition 1.1 in [3] and in Proposition 9 and Theorem 1 in [19]. □

In view of Lemma 1 the proof of Theorem 1 is equivalent to $E(\mathbf{0}, \infty) = 0$. To make the exposition more elegant, we introduce a measure μ^x on Ω defined by $\mu^x = \mu + \mu \circ \beta_x$, where $\beta_x: \Omega_S \to \Omega_S$ is defined by $(\beta_x(\omega))_x = -\omega_x$ and $(\beta_x(\omega))_y = \omega_y$ for $y \neq x$, $x \in \mathbf{Z}$.

Then we have $E(\mathbf{0}, \infty) \le E^{\mathbf{0}}(\mathbf{0}, \infty) := E_{\mu^{\mathbf{0}}}((N_{\mathbf{0},\infty} - 1)I_{\{N_{\mathbf{0},\infty} > 0\}})$. For all Gibbs states whose interaction energy $W_{S, \mathbf{Z}\setminus S}$ is finite for any finite set S there exists a constant $K > 0$ such that $E^{\mathbf{0}}(\mathbf{0}, \infty) \le KE(\mathbf{0}, \infty)$ and thus Theorem 1 holds if and only if $E^{\mathbf{0}}(\mathbf{0}, \infty) = 0$.

Let $R \in \mathbb{N}$. We will also consider the sublattice $L = L_R \subset \mathbf{Z}$ of points $(x_1, \dots, x_d)$ such that $x_i = KR$, for $i = 1, 2, \dots, d$ and $K \in \mathbb{Z}$. When no confusion arises we will omit the index R.

Let now $B = B_m$ be a given box and consider a second box $A(B)$ of linear size equal to the integer part of $m - \sqrt{m}$; note that $\lim_{B \uparrow \infty}[|A(B)|/|B|] = 1$, where $|B|$ denotes the cardinality (the volume) of B.

Next we introduce the family $\underline{C}_B$ of subsets of B which are connected and contain a nearest neighbour of a point of $\mathbf{Z} \setminus B$. For $C \in \underline{C}_B$ we denote by $\overline{C}$ the set of all the points of B contained in C or nearest neighbours of a point of C, and by $F_C \subseteq \Omega_B$ the set of all the configurations ω of which C is a cluster in B; a fortiori $\omega_x = -1$ for x in $\overline{C} \setminus C$.

For x in a finite set $S \subset \mathbf{Z}$ we shall also consider the number $N_{x,\,\partial S}(\omega)$ of distinct clusters in S of a configuration $\omega \in \Omega_S$ containing at least one nearest neighbour of x and one neighbour of some point in $\mathbf{Z} \setminus S$. Then let

$$E(x, \partial S) = E_{\alpha_B \mu^x}\big((N_{x,\,\partial S} - 1) I_{\{N_{x,\,\partial S} > 0\}}\big).$$

The next lemma again holds for measures that are more general than long-range Gibbs state.

Lemma 2. *Let μ be any stationary probability measure on Ω. Then*

$$E(\mathbf{0}, \infty) \le E^{\mathbf{0}}(\mathbf{0}, \infty) = \lim_{B \uparrow \infty} |A(B) \cap L|^{-1} \sum_{\omega \in \Omega_B} \sum_{C \in \underline{C}_B:\ \omega \in F_C} \sum_{x \in \overline{C} \cap L \cap A(B)} (-\omega_x) \alpha_B \mu^x(\omega).$$

Proof. For any $K \in \{0, 1, \dots, 2d\}$ the sequence of events $\{N_{\mathbf{0},\,\partial B} = K\}$ converges to $\{N_{\mathbf{0},\,\infty} = K\}$ as $B \uparrow \infty$ and this yields

$$E^{\mathbf{0}}(\mathbf{0}, \infty) = \lim_{B \uparrow \infty} E^{\mathbf{0}}(\mathbf{0}, \partial B).$$

Next we make use of the stationarity of μ. Let $x \in \mathbf{Z}$ and let Λ^1, Λ^2 and B be boxes such that $\Lambda^1 + x \subset B \subset \Lambda^2 + x$. Then $N_{x,\,\partial(\Lambda^1 + x)} \ge N_{x,\,\partial B} \ge N_{x,\,\partial(\Lambda^2 + x)}$ yields

$$\begin{aligned} \lim_{B \uparrow \infty} E^{\mathbf{0}}(\mathbf{0}, \infty) &= \lim_{B \uparrow \infty} \sum_{K=2}^{2d} \alpha_{(B+x)} \mu^x\big(N_{x,\,\partial(B+x)} \ge K\big) \\ &= \lim_{B \uparrow \infty} E^x(x, \partial B) \qquad (2.3) \\ &= \lim_{B \uparrow \infty} |A(B) \cap L|^{-1} \sum_{x \in A(B) \cap L} E^x(x, \partial B), \end{aligned}$$

where $E^x(x, \partial B)$ is defined to be 0 if $x \notin B$. The last equality can be obtained by observing that $\mathrm{dist}(\partial B, \partial A(B)) \to +\infty$ as $B \uparrow \infty$.

For a box B and $x \in B$ the cylinder $F_x^i \subset \Omega_B$ is the set of all the configurations ω such that $\omega_x = i$, $i \in \{-1, +1\}$. Furthermore, recall that $N_{x,\,\partial B}(\omega) \ge 2$

implies $\omega_x = -1$. Then it is easy to see that

$$\sum_{x \in A(B) \cap L} E^x(x, \partial B)$$

$$= \sum_{x \in A(B) \cap L} \Bigg[\sum_{\omega \in F_x^{-1}} I_{\{N_{x,\partial B}(\omega) \geq 2\}} N_{x,\partial B}(\omega) \alpha_B \mu^x(\omega) - \alpha_B \mu^x\big(F_x^{-1} \cap \{N_{x,\partial B} \geq 2\}\big) \Bigg]$$

$$= \sum_{x \in A(B) \cap L} \Bigg[\sum_{\omega \in F_x^{-1}} N_{x,\partial B}(\omega) \alpha_B \mu^x(\omega) - \alpha_B \mu^x\big(F_x^{-1} \cap \{N_{x,\partial B} \geq 1\}\big) \Bigg].$$

Since $\alpha_B \mu^x(F_x^{-1} \cap \{N_{x,\partial B} \geq 1\}) = \alpha_B \mu^x(F_x^{+1} \in \{N_{x,\partial B} \geq 1\})$, the last expression equals

$$\sum_{\omega \in \Omega_B} \Bigg[\sum_{x \in A(B) \cap L \cap \omega^{-1}(-1)} N_{x,\partial B}(\omega) \alpha_B \mu^x(\omega) - \sum_{x \in A(B) \cap L \cap \omega^{-1}(+1)} I_{\{N_{x,\partial B} \geq 1\}}(\omega) \alpha_B \mu^x(\omega) \Bigg]$$

$$= \sum_{\omega \in \Omega_B} \sum_{C \in \underline{C}_B:\, \omega \in F_C} \Bigg[\sum_{x \in \overline{C} \cap (A(B) \cap L)} (-\omega_x) \Bigg] \alpha_B \mu^x(\omega).$$

In the last equality note that in the sum over $C \in \underline{C}_B$ the sites $x \in A(B) \cap L$ such that $\omega_x = +1$ are counted only once if they belong to a cluster, while if $\omega_x = -1$ the site x is counted exactly $N_{x,\partial B}(\omega)$ times. This proves the lemma. □

3. Uniqueness of the infinite cluster for finite-range Gibbs states. Throughout the section we consider only finite-range Gibbs states. Let R be the length of interaction for such a state. Fix a box $\overline{B} = B_R$ of linear size R: we shall always consider boxes B such that $\overline{B} + x \subset B$ for all $x \in A(B)$. We shall also consider the sublattice $L = L_R$.

Define the set $\Omega(x)$ of the *local environments* around a point $x \in L$ as $\Omega_{((\overline{B}+x) \setminus \{x\})}$. Let $\sigma \in \Omega(\mathbf{0})$; define for a Gibbs state μ

$$T_\sigma = \mu\big(\omega_0 = +1 \mid \alpha_{\overline{B} \setminus \{0\}} \omega = \sigma\big),$$

$$U_\sigma = \mu\big(\omega_0 = -1 \mid \alpha_{\overline{B} \setminus \{0\}} \omega = \sigma\big);$$

furthermore for $C \in \underline{C}_B$ and $\omega \in F_C$ let

$m_\sigma^C(\omega)$ = number of sites $x \in C \cap L$ such that the local environment around x is σ,

$l_\sigma^C(\omega)$ = number of sites $x \in (\overline{C} \setminus C) \cap L$ such that the local environment around x is σ,

$M^c(\omega) = \sum_{\sigma \in \Omega(\mathbf{0})} [U_\sigma^{-1} l_\sigma^C(\omega) - T_\sigma^{-1} m_\sigma^C(\omega)]$.

In the independent case we can choose $L = \mathbf{Z}$, $\overline{B} = \{\mathbf{0}\}$ so that

$$M^C(\omega) = \left[\frac{|\overline{C} - C|}{1-p} - \frac{|C|}{p}\right].$$

In this case the property of M^C stated in the next lemma was already proved in [3] and adapted in [11]. Note that we will make no use of the stationarity of the measure.

LEMMA 3. *Let μ be a Gibbs state with finite-range interaction (with length R). Then there exists $H = H(\{T_\sigma\}_{\sigma \in \Omega(\mathbf{0})}) > 0$ such that for all B, $n \in \underline{N}$ and $\varepsilon > 0$ there exists a probability measure $\overline{\mu}_\varepsilon$ on Ω_B satisfying*

$$Q_{B,n}(\varepsilon) = \sum_{C,n}^{*} \alpha_B \mu\left(\{M^C \geq \varepsilon n\} \cap F_C\right) \leq e^{-\varepsilon^2 nH} \sum_{C,n}^{*} \overline{\mu}_\varepsilon(F_C),$$

where $\Sigma_{C,n}^*$ means $\Sigma_{C \in C_B:\, |\overline{C}| = n}$.

PROOF. First note that for all $\gamma > 0$,

$$Q_{B,n}(\varepsilon) \leq e^{-\varepsilon n \gamma} \sum_{C,n}^{*} E_{\alpha_B\mu}\left[e^{\gamma M^C} I_{(F_C)}\right]. \tag{3.1}$$

Next rewrite the measure of a configuration $\omega \in \Omega_B$ using the choice of L and the Markov property induced by the finiteness of the range of Φ to obtain

$$\begin{aligned} &E_{\alpha_B\mu}\left[e^{\gamma M^C} I_{F_C}\right] \\ &\quad = \sum_{\omega \in F_C} e^{\gamma M^C(\omega)} \prod_{\sigma \in \Omega(0)} (T_\sigma)^{m_\sigma^C(\omega)} (U_\sigma)^{l_\sigma^C(\omega)} \alpha_{B \setminus (L \cap \overline{C})} \mu\left(\alpha_{B \setminus (L \cap \overline{C}),\, B}\omega\right). \end{aligned} \tag{3.2}$$

Define for $C \in \underline{C}_B$ and for all $\omega \in \Omega_B$,

$$\overline{\mu}_C(\omega) = \prod_{\sigma \in \Omega(\mathbf{0})} \zeta(\gamma)^{l_\sigma^C(\omega)} \tau(\gamma)^{m_\sigma^C(\omega)} \alpha_{B \setminus (L \cap \overline{C})} \mu\left(\alpha_{B \setminus (L \cap \overline{C}),\, B}\omega\right),$$

where

$$\zeta(\gamma) = \left[\frac{U_\sigma e^{\gamma U_\sigma^{-1}}}{Z_\sigma}\right],$$

$$\tau(\gamma) = \left[\frac{T_\sigma e^{-\gamma T_\sigma^{-1}}}{Z_\sigma}\right],$$

$$Z_\sigma = U_\sigma e^{\gamma U_\sigma^{-1}} + T_\sigma e^{-\gamma T_\sigma^{-1}}.$$

Then

$$E_{\alpha_B\mu}\left[e^{\gamma M^C} I_{F_C}\right] = \sum_{\omega \in F_C} \prod_{\sigma \in \Omega(\mathbf{0})} (Z_\sigma)^{m_\sigma^C(\omega) + l_\sigma^C(\omega)} \overline{\mu}_C(\omega). \tag{3.3}$$

It is easy to see that for any $a \in [0,1]$ there exists $H(a) > 0$ such that

$$\left(ae^{\gamma a^{-1}} + (1-a)e^{-\gamma(1-a)^{-1}}\right) \le e^{\gamma^2/H(a)} \quad \text{for all } \gamma \in \mathbb{R}.$$

For any $\sigma \in \Omega(\mathbf{0})$ observe that $T_\sigma + U_\sigma = 1$ and let $H' = \min_{\sigma \in \Omega(\mathbf{0})} H(T_\sigma) > 0$ which yields $Z_\sigma \le e^{\gamma^2/H'}$.

Let $C \in \underline{C}_B$ be such that $|\bar{C}| = n$ and let $\omega \in F_C$; then $\sum_{\sigma \in \Omega(\mathbf{0})}(m_\sigma^c(\omega) + l_\sigma^C(\omega)) \le 2n/R$, where R is the number which occurs in the definition of the sublattice L. Collecting (3.1), (3.3) and the last remark, we have

$$Q_{B,n}(\varepsilon) \le e^{-\varepsilon n\gamma} \sum_{C,n}^{*} \exp\left(\frac{\gamma^2 2n}{RH'}\right)\bar{\mu}_C(F_C). \tag{3.4}$$

Next put $\gamma = \gamma_\varepsilon = H'R\varepsilon/4$ and $H = H'R/8$.

Then define

$$\mu_\varepsilon(\omega) = \prod_{\sigma \in \Omega(\mathbf{0})} \zeta(\gamma_\varepsilon)^{l_\sigma(\omega)} \tau(\gamma_\varepsilon)^{m_\sigma(\omega)} \alpha_{B \setminus L}\mu(\alpha_{B\setminus L, B}\omega),$$

where

$m_\sigma(\omega)$ = number of points x of $A(B) \cap L$ such that $\omega_x = +1$ and the local environment is σ,

$l_\sigma(\omega)$ = number of points x of $A(B) \cap L$ such that $\omega_x = -1$ and the local environment is σ.

Using the Markov property of μ, it is easy to see that

$$\bar{\mu}_C(F_C) = \bar{\mu}_\varepsilon(F_C),$$

which yields

$$Q_{B,n}(\varepsilon) \le e^{-\varepsilon^2 nH} \sum_{C,n}^{*} \bar{\mu}_\varepsilon(F_C)$$

and proves the lemma. □

By collecting the previous lemmas, the proof of Theorem 1 for finite-range Gibbs states is now straightforward.

Proof of Theorem 1 (for finite-range Gibbs states). In view of Lemma 1 we have to prove $E(\mathbf{0},\infty) = 0$. First we rewrite the estimation made in Lemma 2 for the Gibbs measure μ.

Let B be a box, $\omega \in \Omega_B$, $x \in A(B)$ and $\sigma \in \Omega(\mathbf{0})$. If $\omega_x = +1$ (or $\omega_x = -1$) and the local environment around x is σ, then $\alpha_B\mu^x(\omega) = T_\sigma^{-1}\alpha_B\mu(\omega)$ [resp. $U_\sigma^{-1}\alpha_B\mu(\omega)$]. Furthermore perform the sum in $C \in \underline{C}_B$ according to the size of $\bar{C}$, noting that from the definition of $A(B)$ it follows that the size of $\bar{C}$ is at least the integer part of $m - \sqrt{m}$, where $B = B_m$, which we will call $n(B)$. This yields

$$E(\mathbf{0},\infty) \le \lim_{B\uparrow\infty} |A(B) \cap L|^{-1} \sum_{n=n(B)}^{|B|} \sum_{C,n}^{*} \sum_{\omega \in F_C} M^C(\omega)\alpha_B\mu(\omega). \tag{3.5}$$

Now we want to make use of the estimation given in Lemma 3. Let $\varepsilon > 0$ be a positive constant. For $C \in \underline{C}_B$ such that $|\overline{C}| = n$ denote by Q^+ the event $\{M^C \geq \varepsilon n\} \cap F_C$ and by Q^- the event $\{M^C < \varepsilon n\} \cap F_C$. In (3.5) divide the sum over $\omega \in F_C$ in Q^+ and Q^-. The first term can be estimated from Lemma 3 by noting that $M^C \leq 2n/T$, where $T = \min_{\sigma \in \Omega(\mathbf{0})}(T_\sigma, U_\sigma)$ and $n = |\overline{C}|$, as

$$\sum_{n=n(B)}^{|B|} \sum_{C,n}^{*} \sum_{\omega \in Q^+} M^C(\omega)\alpha_B \mu(\omega)$$
$$\leq \sum_{n=n(B)}^{|B|} \frac{2n}{T} Q_{B,n}(\varepsilon)$$
$$\leq \sum_{n=n(B)}^{|B|} \frac{2n}{T} e^{-\varepsilon^2 nH} \sum_{C,n}^{*} \bar{\mu}_\varepsilon(F_C)$$
$$\leq \frac{2}{T} e^{-\varepsilon^2 n(B) H} 2d|B|,$$

where $\bar{\mu}_\varepsilon$ and H were defined in Lemma 3; in the last inequality we have used that for any probability measure μ on Ω_B, $\sum_{n=1}^{|B|} n \sum_{C,n}^{*} \mu(F_C) \leq 2d|B|$. Then

$$\lim_{B \uparrow \infty} |A(B) \cap L|^{-1} \frac{2}{T} e^{-\varepsilon^2 n(B) H} 2d|B| = 0$$

because $n(B) \to \infty$ and $|B|\,|A(B) \cap L|^{-1} \to R^d$ when $B \uparrow \infty$, where R is the length of interaction.

The sum over ω in Q^- can be estimated similarly using that $M^C < \varepsilon n$ and that $Q^- \subset F_C$ to obtain

$$\lim_{B \uparrow \infty} |A(B) \cap L|^{-1} \sum_{n=n(B)}^{|B|} \sum_{C,n}^{*} \sum_{\omega \in Q^-} M^C(\omega)\alpha_B \mu(\omega)$$
$$\leq \lim_{B \uparrow \infty} |A(B) \cap L|^{-1} \varepsilon 2d|B| = \varepsilon 2dR^d.$$

Thus $E(\mathbf{0}, \infty)$ is smaller than any positive value and this proves the theorem. □

4. Uniqueness for long-range Gibbs states and related results. We show in this section that also for the long-range Gibbs states the infinite cluster is unique if it exists and this ends the proof of Theorem 1. The proof is essentially an imitation of the one above for finite-range Gibbs states.

We will therefore sketch a proof which follows step by step the proof of Sections 2 and 3. The main changes will be in the choice of the constants in Lemma 3.

PROOF OF THEOREM 1. Consider a long-range Gibbs state μ, Lemma 1 applies and thus we have to prove $E(\mathbf{0}, \infty) = 0$. Consider again the lattice $L_R \cap \mathbf{Z}$. Later we will have to let $R \to \infty$ as the interaction now has infinite range. Lemma 2 obviously applies and the next step is to rewrite the estimation it provides.

Consider $\sigma_\infty \in \Omega_{\mathbf{Z}\setminus\{0\}}$ as environment (no longer local) of a point and define the conditional probabilities T_{σ_∞} and U_{σ_∞}, the functions $m^C_{\sigma_\infty}$ and $l^C_{\sigma_\infty}$ for $C \in \underline{C}_B$ as before, and define

$$M^C = \sum_{\sigma_\infty \in \Omega_{\mathbf{Z}\setminus\{0\}}} \left[U^{-1}_{\sigma_\infty} l^C_{\sigma_\infty} - T^{-1}_{\sigma_\infty} m^C_{\sigma_\infty}\right]$$

(the sum is finite as we only consider points of a box B). We start from the proof of the first part of Theorem 1. Formula (3.5) holds replacing $\sum_{\omega \in F_C} M^C(\omega)\alpha_B\mu(\omega)$ with $\int_{F_C} M^C(\omega)\mu(d\omega)$, where F_C is considered as event of Ω. Let $\varepsilon > 0$ and define Q^+ and Q^- as above. To estimate the expression concerning the integral in Q^+, we imitate Lemma 3. Equation (3.1) still holds. Let σ_R be an R-local environment (in $\Omega_R = \Omega_{B_R\setminus\{0\}}$) and let $\sigma_0 = \sigma_{\{0\}} = 1$; define

$$T_{\sigma_R} = \frac{\exp\left(-U_{\{0\}}(\sigma_0) - W_{\{0\}, B_R\setminus\{0\}}(\sigma_0 \vee \sigma_R)\right)}{\sum_{\omega_0 = 1, -1}\exp\left(-U_{\{0\}}(\omega_0) - W_{\{0\}, B_R\setminus\{0\}}(\omega_0 \vee \sigma_R)\right)}.$$

T_{σ_R} and $U_{\sigma_R} = 1 - T_{\sigma_R}$ are formal conditional probabilities given σ_R. To estimate the expectation which appears in (3.1), we approximate T_{σ_∞} and U_{σ_∞} by T_{σ_R} and U_{σ_R}. Recall the definition of a_R and observe that $T_{\sigma_R} e^{-2a_R} \le T_{\sigma_\infty} \le T_{\sigma_R} e^{2a_R}$ and that a similar inequality holds for U_{σ_R} and U_{σ_∞} when $\alpha_{B_R\setminus\{0\}, \mathbf{Z}\setminus\{0\}}\sigma_\infty = \sigma_R$. Let

$$M^C_R = \sum_{\sigma_R \in \Omega_R} \left[U^{-1}_{\sigma_R} l^C_{\sigma_R} - T^{-1}_{\sigma_R} m^C_{\sigma_R}\right]$$

and note that

$$M^C \le M^C_R + \left(\frac{e^{2a_R} - 1}{T}\right)\frac{2n}{R},$$

where $T = \inf_R \inf_{\sigma_R}(T_{\sigma_R}, U_{\sigma_R}) > 0$ and we have used that $2n/R$ is larger than $|L_R \cap \overline{C}|$. Thus in (3.2) we can replace M^C by M^C_R and $T_{\sigma_\infty}, U_{\sigma_\infty}$ by $T_{\sigma_R}, U_{\sigma_R}$ by adding an extra factor $(e^{2\gamma(e^{2a_R}-1)T^{-1}} e^{4a_R})^{n/R}$. The following steps are made by substituting σ_R to σ. Only note that H' can be taken as the infimum over σ_∞, so it no longer depends on R; again $H' > 0$ as $T > 0$. Equation (3.4) now reads

$$Q_{B,n}(\varepsilon) \le \sum_{C,n}^{*} \exp\left[n\left(-\varepsilon\gamma + \frac{\gamma^2 2}{RH'} + \frac{2\gamma(e^{2a_R} - 1)}{RT} + \frac{4a_R}{T}\right)\right]\bar{\mu}_C(F_C).$$

Next choose $\gamma = \gamma_R = 2\sqrt{a_R H}$, independent of ε, and $H = H'/2$. Define $\bar{\mu}_R$ as $\bar{\mu}_\varepsilon$, where γ_R replaces γ_ε and σ_R replaces σ. Then $\bar{\mu}_C(F_C) = \bar{\mu}_R(F_C)$ still holds. The main difference is that now we fix ε depending on R, viz.

$$\varepsilon = \frac{1}{R}\left(\frac{9\sqrt{a_R}}{2\sqrt{H}} + 2\frac{e^{2a_R} - 1}{T}\right).$$

Then $Q_{B,n}(\varepsilon) \le e^{-a_R n}\sum^*_{C,n}\bar{\mu}_\varepsilon(F_C)$ so that the term concerning Q^+ converges to 0 for every fixed R.

For the other term the choice of ε implies that the limit is

$$\varepsilon 2dR^d = 2dR^{d-1}\left(\frac{9\sqrt{a_R}}{2\sqrt{H}} + 2\frac{e^{2a_R}-1}{T}\right).$$

This term converges to 0 as $R \to \infty$ as a result of the assumption on the interaction Φ, viz. (2.1). □

For measures having longer interaction Φ (a natural assumption is that $a_R \to 0$; see [22], page 13) we do not have much information. In the stationary case we expect the infinite cluster to be always unique, while for the nonstationary Gibbs states we do not have any evidence of whether uniqueness may hold or not. The only information in the latter case is provided by the following remark. The stationarity of the measure has been used only to derive (2.2), thus the second term in this formula is 0 even for the nonstationary Gibbs states [provided they satisfy (2.1)]. As in Lemma 2 it is easy to see that therefore

$$\begin{aligned} 0 &= \lim_{B\uparrow\infty} |A(B)\cap L|^{-1} \sum_{x\in A(B)\cap L} E^x(x, \partial B) \\ &\geq \lim_{B\uparrow\infty} |A(B)\cap L|^{-1} \sum_{x\in A(B)\cap L} \mu(N_{x,\infty} > 0) \geq 0. \end{aligned}$$

In words, the density of sites from the neighbours of which there is more than one distinct infinite cluster is 0 (i.e., with respect to any long-range Gibbs state).

The last part of this section is devoted to the study of the *connectivity function* $\tau(x, y)$ defined as the probability that there is a chain connecting the two points $x, y \in \mathbf{Z}$. The uniqueness of the infinite cluster implies that if percolation arises, then $\tau(x, y)$ is bounded away from 0 uniformly in x and y. In the Bernoulli model this represents a sharp transition from the nonpercolating phase, where $\tau(x, y)$ decreases exponentially with the distance $x - y$ (see [18] and [2]; the behaviour for $p = p_c = \inf\{p$: percolation arises for the Bernoulli model with parameter $p\}$ is excluded in this scheme).

The uniqueness of the cluster holds also for nonattractive potentials, for which the FKG inequality ([9]) in general does not hold. For the Gibbs states satisfying the FKG inequality it is easy to see that $\tau(\mathbf{0}, x) \geq [\mu(N_{\mathbf{0},\infty} = 1)]^2 > 0$; in general we have a less explicit bound as stated in the next proposition.

PROPOSITION 1. *Let μ be a long-range Gibbs state on Ω and let $x, y \in \mathbf{Z}$. If percolation occurs there exists a constant $c > 0$ such that $\tau(x, y) \geq c > 0$.*

PROOF. Given a box B let B^∞ be the event {there is an infinite cluster intersecting B}. If percolation occurs then there exists a box B such that

$$\mu\big((B+x)^\infty \mid N > 0\big) > \tfrac{1}{2},$$

$$\mu\big((B+Y)^\infty \mid N > 0\big) > \tfrac{1}{2}$$

and therefore

$$\mu\big((B+x)^{\infty}\cap(B+Y)^{\infty}\big)>0.$$

We simply have to connect the points x and y to the (unique) infinite cluster intersecting the two boxes $B+x$ and $B+y$. Let ψ: $\Omega\to\Omega$ be defined by $\psi(\omega)_z=+1$ for all $z\in(B+x)\cup(B+y)$ and $\psi(\omega)_z=\omega_z$ elsewhere. As already remarked in the proof of Lemma 1, the finiteness of the energies U_B and $W_{B,\mathbf{Z}\setminus B}$ provides $\mu(\psi(Q))>0$ if $\mu(Q)>0$ for all events $Q\subset\Omega$. (A detailed proof is in Proposition 9 of [19]). Hence there exists c such that
is in Proposition 9 of [19]). Hence there exists c such that

$$\tau(x,y)\geq\mu\Big(\psi\big((B+x)^{\infty}\cap(B+y)^{\infty}\big)\Big)\geq c>0. \qquad \square$$

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Note added in proof. The results of this paper have been recently extended by Burton and Keane to finite energy nearest neighbor models and further by Gandolfi, Keane and Newman to positive finite energy long range models.

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