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A NOTE ON THE RATE OF POISSON APPROXIMATION OF EMPIRICAL PROCESSES

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We obtain a probability inequality for the sup-norm distance between multivariate empirical processes and suitable constructed Poisson processes.

1. Introduction. Let $\mathbf{X}_1 = (X_1^{(1)}, \dots, X_1^{(d)})$, $\mathbf{X}_2 = (X_2^{(1)}, \dots, X_2^{(d)}), \dots$ be independent identically distributed random vectors with distribution function $F_n(\mathbf{x})$, $\mathbf{x} = (x_1, \dots, x_d)$ is defined by $F(x_1, \dots, x_d) = P\{X_1^{(1)} \le x_1, \dots, X_1^{(d)} \le x_d\}$. The empirical distribution function $F_n(\mathbf{x}) = (1/n)\#\{1 \le j \le n: X_j^{(1)} \le x_1, \dots, X_j^{(d)} \le x_d\}$. It is well-known that the d-variate empirical process $n^{1/2}(F_n(\mathbf{x}) - F(\mathbf{x}))$ converges weakly to a Gaussian process. The rate of convergence in this invariance principle has been studied by several authors. The best presently available result is due to Borisov (1982) in case of an arbitrary distribution function F. Massart (1989) gets a much better approximation when F is the uniform distribution function on $[0,1]^d$. Csörgő and Horváth (1990) and Horváth (1988) studied the limit distributions of weighted uniform empirical processes. They proved that the weighted uniform empirical process can behave like a weighted Poisson process. Assuming $d = 1$, Major (1990) showed that Poisson approximation is better for the univariate empirical process than the usual Gaussian approximations.

The main aim of our paper is to get Poisson approximations for the multivariate empirical process. We also study the rate of the approximation which will be new even for the univariate empirical process. Our method is based on the well-known Kac representation of empirical processes.

Before stating our result we need some notations. Let $\mathscr{B}$ be the Borel σ-algebra on R^d, and define $\mu(B) = \int_B dF(\mathbf{x})$, $B \in \mathscr{B}$. Let $M_n(B)$, $B \in \mathscr{B}$ be a Poisson point process with mean measure $EM_n(B) = n\mu(B)$ [cf. Karlin and Taylor (1981)]. This means that for any pairwise disjoint $B_j \in \mathscr{B}$, $1 \le j \le m$, the random variables $M_n(B_j)$, $1 \le j \le m$ are independent. Furthermore, for any $B \in \mathscr{B}$ and any $k = 0, 1, \dots$ one has

$$P\{M_n(B) = k\} = \frac{(n\mu(B))^k}{k!}\exp(-n\mu(B)).$$

Let $N_n(\mathbf{x}) = M_n((-\infty, x_1] \times \cdots \times (-\infty, x_d])$, $\mathbf{x} = (x_1, \dots, x_d)$. We say that $\{N_n(\mathbf{x}), \mathbf{x} \in R^d\}$ is a Poisson process with mean function $EN_n(\mathbf{x}) = nF(\mathbf{x})$.

THEOREM. *We can define a sequence of Poisson processes $\{\tilde{N}_n(\mathbf{x}), \mathbf{x} \in R^d\}$ with mean measure $E\tilde{N}_n(\mathbf{x}) = nF(\mathbf{x})$ such that for all $\mathbf{x}(n) = (x_1(n), \dots, x_d(n))$*

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and $t > 0$

$$P\left\{\max_{1\le i\le d}\ \sup_{-\infty<x_i\le x_i(n)}\left|n(F_n(\mathbf{x})-F(\mathbf{x}))-(\tilde N_n(\mathbf{x})-nF(\mathbf{x}))\right|>2t\right\}$$
$$\le (t^{-1}+t^{-2})n^{1/2}F(\mathbf{x}(n)),$$

if $n \ge C$, *where* C *is an absolute constant.*

Choosing $d = 1$ and $F(t) = t$, $0 \le t \le 1$, we get an improvement of Theorem 1′ of Major (1990).

Our result can be used to study the rate of convergence of Poisson approximation. Let $\mathbf{a} \in R^d$ and $\mathbf{b} \in R^d$ and define $\mathbf{ab}$ by $(a_1b_1, a_2b_2, \dots, a_db_d)$. The Prohorov–Lévy distance of measures is denoted by $\mathscr{L}$. Let ρ_n and κ_n stand for the measures generated by the processes $\{nF_n(\mathbf{tx}(n)) - nF(\mathbf{tx}(n)),\ \mathbf{t} \in (-\infty, 1]^d\}$ and $\{\tilde N_n(\mathbf{tx}(n)) - nF(\mathbf{tx}(n)),\ \mathbf{t} \in (-\infty, 1]^d\}$, respectively. Using the inequality in the theorem with $t = (n^{1/2}F(\mathbf{x}(n)))^{1/3}$, we get the following corollary.

Corollary. *If* $n^{1/2}F(\mathbf{x}(n)) \to 0$ $(n \to \infty)$, *then*

$$\mathscr{L}(\rho_n, \kappa_n) = O\left(\left(n^{1/2}F(\mathbf{x}(n))\right)^{1/3}\right)$$

as $n \to \infty$.

Remark. The random processes $\{nF_n(\mathbf{x}),\ \mathbf{x} \in R^d\}$ and $\{\tilde N_n(\mathbf{x}),\ \mathbf{x} \in R^d\}$ take on nonnegative integers and therefore the theorem implies that

$$P\left\{\max_{1\le i\le d}\ \sup_{-\infty<x_i\le x_i(n)}\left|n(F_n(\mathbf{x})-F(\mathbf{x}))-(\tilde N_n(\mathbf{x})-nF(\mathbf{x}))\right|=0\right\}$$
$$\ge 1-3n^{1/2}F(\mathbf{x}(n)),$$

if $n \ge C$.

2. Proof. Let $\{\nu(n),\ n \ge 1\}$ be a sequence of Poisson random variables with $E\nu(n) = n$, independent of $\{\mathbf{X}_i,\ i \ge 1\}$. Then

$$(2.1)\quad nF_n(\mathbf{x}) - nF(\mathbf{x}) = \sum_{i=1}^{\nu(n)} I\{X_i^{(1)} \le x_1, \dots, X_i^{(d)} \le x_d\} - nF(\mathbf{x}) + R_n(\mathbf{x}),$$

where $I\{A\}$ is the indicator of the set A and

$$R_n(\mathbf{x}) = \sum_{i=1}^{n} I\{X_i^{(1)} \le x_1, \dots, X_i^{(d)} \le x_d\} - \sum_{i=1}^{\nu(n)} I\{X_i^{(1)} \le x_1, \dots, X_i^{(d)} \le x_d\}.$$

Let

$$\tilde N_n(\mathbf{x}) = \sum_{i=1}^{\nu(n)} I\{X_i^{(1)} \le x_1, \dots, X_i^{(d)} \le x_d\}.$$

It is well-known that $\tilde N_n(\mathbf{x})$ is a Poisson process with $E\tilde N_n(\mathbf{x}) = nF(\mathbf{x})$ [cf.

Gaenssler (1983)]. Let

$$\xi_i(\mathbf{x}(n)) = I\{X_i^{(1)} \le x_1(n), \dots, X_i^{(d)} \le x_d(n)\} - F(\mathbf{x}(n)).$$

One can easily verify that

$$\max_{1 \le i \le d} \sup_{-\infty < x_i \le x_i(n)} |R_n(\mathbf{x}))| \le \left| \sum_{i=\nu(n)+1}^{n} \xi_i(\mathbf{x}(n)) \right| + |n - \nu(n)| F(\mathbf{x}(n)). \tag{2.2}$$

An elementary argument gives

$$E\left(\sum_{i=\nu(n)+1}^{n} \xi_i(\mathbf{x}(n)) \right)^2 \le E|\nu(n) - n| F(\mathbf{x}(n)) \tag{2.3}$$

and

$$E|\nu(n) - n| \le n^{1/2}, \tag{2.4}$$

if n is large enough [cf. Chung (1974), page 172]. Using (2.3), (2.4) and Chebyshev's inequality, we obtain that

$$P\left\{ \left| \sum_{i=\nu(n)+1}^{n} \xi_i(\mathbf{x}(n)) \right| > t \right\} \le t^{-2} n^{1/2} F(\mathbf{x}(n)) \tag{2.5}$$

for all $t > 0$. Now (2.4) and the Markov inequality imply that

$$P\{|\nu(n) - n| F(\mathbf{x}(n)) > t\} \le \frac{1}{t} n^{1/2} F(\mathbf{x}(n)) \tag{2.6}$$

for all $t > 0$. □

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