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BAHADUR–KIEFER-TYPE PROCESSES

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We establish strong and weak laws for Bahadur–Kiefer-type processes of the form $e_n + i_n$, where i_n denotes the inverse of e_n. In particular, we provide a proof for the strong version of Theorem 1A of Kiefer (1970), together with similar results for renewal and partial sum processes.

1. Introduction and statement of results. Let $U_1, U_2, \dots$ be a sequence of independent uniform $(0,1)$ random variables. For each integer $n \geq 1$, let $G_n(s) = n^{-1}\#\{U_i \leq s\colon 1 \leq i \leq n\}$ denote the right-continuous uniform empirical distribution function based on the first n of these random variables and let

$$\alpha_n(s) = n^{1/2}(G_n(s) - s) \quad \text{for } 0 \leq s \leq 1$$

be the *uniform empirical process*. Also for each integer $n \geq 1$, let

$$H_n(s) = \inf\{t\colon G_n(t) \geq s\} \quad \text{for } 0 < s \leq 1,\ H_n(0) = H_n(0+),$$

denote the inverse or quantile function of G_n and let

$$\beta_n(s) = n^{1/2}(H_n(s) - s) \quad \text{for } 0 \leq s \leq 1$$

be the *uniform quantile process*.

For any real-valued function f defined on $[0,1]$, let $\|f\| = \sup_{0 \leq s \leq 1}|f(s)|$. Bahadur (1966) introduced the process

$$R_n(s) = \alpha_n(s) + \beta_n(s) \quad \text{for } 0 \leq s \leq 1$$

and showed that, for each fixed $0 \leq s \leq 1$,

$$a_n R_n(s) = O(1) \quad \text{a.s. as } n \to \infty,$$

where

$$a_n = n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4}.$$

The process R_n is often called the *Bahadur–Kiefer* process. Kiefer (1967) established the exact order of $R_n(s)$ by proving that, for each fixed $0 \leq s \leq 1$,

$$\limsup_{n\to\infty} \pm n^{1/4}(\log\log n)^{-3/4}R_n(s) = 2^{5/4}3^{-3/4}(s(1-s))^{1/4} \quad \text{a.s.}$$

and later demonstrated that the rate obtained by Bahadur was exact for the

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supremum norm of R_n, i.e.,

$$\limsup_{n\to\infty} a_n\|R_n\| = 2^{-1/4} \quad \text{a.s.}; \tag{1.1}$$

cf. Kiefer (1970). Kiefer (1970) also proved that

$$b_n\|R_n\|/\|\alpha_n\|^{1/2} \to_P 1 \quad \text{as } n \to \infty, \tag{1.2}$$

where

$$b_n = n^{1/4}(\log n)^{-1/2}.$$

He stated that the limit in (1.2) remains true when convergence in probability is replaced by convergence with probability 1, but did not publish a proof of this strong version of (1.2).

Shorack (1982) has given a short proof for (1.1) using the Finkelstein (1971) functional law of the iterated logarithm and a partial proof for the strong version of (1.2), based in part on the Komlós, Major and Tusnády (KMT) (1975) Kiefer process strong approximation to the uniform empirical process, i.e., he has shown that

$$\limsup_{n\to\infty} b_n\|R_n\|/\|\alpha_n\|^{1/2} \le 1 \quad \text{a.s.} \tag{1.3}$$

A similar Bahadur–Kiefer-type process arises in connection with the partial sum process and its associated renewal process. In order to describe this process, we introduce the following notation.

Let $X_1, X_2, \dots$ be a sequence of independent random variables with a common distribution function $F(x) = P(X_1 \le x)$ such that

(i) $E(X_1) = \mu > 0$;
(ii) $0 < \mathrm{Var}(X_1) = \sigma^2 < \infty$;
(iii) $E(X_1^4) < \infty$.

For each integer $n \ge 1$, set $S_n = X_1 + \cdots + X_n$, with $S_0 = 0$. For $t \ge 0$, let $N(t) = \min\{n \ge 0\colon S_{n+1} > t\}$ denote the corresponding renewal process. By Theorem 2 of KMT (1976), on a rich enough probability space there exist a Wiener process W and a sequence $X_1, X_2, \dots$, i.i.d. F, such that as $n \to \infty$,

$$n^{-1/4} \sup_{0\le t\le n} |\sigma^{-1}(S_{[t]} - t\mu) - W(t)| \to 0 \quad \text{a.s.}, \tag{1.4}$$

where $[x]$ denotes the integer part of x.

Introduce the *standardized partial sum process*

$$s_n(s) = n^{1/2}\sigma^{-1}\mu(S_n(s) - s) = n^{-1/2}\sigma^{-1}(S_{[ns]} - n\mu s) \quad \text{for } 0 \le s < \infty$$

and the *standardized renewal process*

$$r_n(s) = n^{1/2}\sigma^{-1}\mu(N_n(s) - s) = n^{-1/2}\sigma^{-1}(\mu N(n\mu s) - n\mu s) \quad \text{for } 0 \le s < \infty,$$

where

$$S_n(s) = n^{-1}\mu^{-1}S_{[ns]} \quad \text{and} \quad N_n(s) = n^{-1}N(n\mu s) \quad \text{for } n = 1, 2, \dots.$$

Also, let

$$W_n(s) = n^{-1/2}W(ns) \quad \text{for } 0 \le s < \infty.$$

Recently, using the ideas of Shorack (1982), Horváth (1984) showed that on the probability space of (1.4),

$$\limsup_{n \to \infty} a_n \|r_n + W_n\| = 2^{1/4}\sigma^{1/2}\mu^{-1/2} \quad \text{a.s.} \tag{1.5}$$

For future reference, we rewrite (1.4) as

$$\lim_{n \to \infty} n^{1/4}\|s_n - W_n\| = 0 \quad \text{a.s.} \tag{1.6}$$

Notice that $N_n(s) = -n^{-1} + \inf\{t \ge 0\colon S_n(t) > s\}$ is a kind of inverse of $S_n(s)$. Led by the fact that β_n is the inverse process of α_n and that the Bahadur–Kiefer process R_n is the sum of these two processes, we define the *Bahadur–Kiefer-type process*

$$T_n(s) = s_n(s) + r_n(s) \quad \text{for } 0 \le s < \infty.$$

By combining (1.5) and (1.6), we obtain the analogue of (1.1) for T_n:

$$\limsup_{n \to \infty} a_n \|T_n\| = 2^{1/4}\sigma^{1/2}\mu^{-1/2} \quad \text{a.s.} \tag{1.7}$$

One of the purposes of this paper is to provide a proof of the almost sure version of Kiefer's (1970) Theorem 1A stated in (1.2) and to obtain by the same method the analogous result for the Bahadur–Kiefer-type process T_n. It will become clear that the methods of proof for Theorems 1A and 1B can be modified easily to obtain similar results for other Bahadur–Kiefer-type processes formed by the sum of a process e_n on $[0,1]$ (such as α_n or s_n) and its appropriate inverse i_n (such as β_n or r_n), whenever strong approximation results like (1.4) and (1.5) hold for the process and its inverse.

Some related results concerning the limiting behaviors of processes and their inverses are given in Vervaat (1972).

Our first theorem is the almost sure version of Kiefer's (1970) Theorem 1A (note that $\|\alpha_n\| = \|\beta_n\|$).

THEOREM 1A. *We have*

$$b_n\|R_n\|/\|\beta_n\|^{1/2} \to 1 \quad a.s. \text{ as } n \to \infty. \tag{1.8}$$

Our second theorem gives the analogue of Theorem 1A for the Bahadur–Kiefer-type process T_n.

THEOREM 1B. *Under assumptions* (i), (ii) *and* (iii),

$$b_n\|T_n\|/\|r_n\|^{1/2} \to \sigma^{1/2}\mu^{-1/2} \quad a.s. \text{ as } n \to \infty. \tag{1.9}$$

The proofs of Theorems 1A and 1B are given in the next section.

Corollaries 1A and 1B below are immediate consequences of Theorems 1A and 1B, when combined with the following well-known results in which B

denotes a Brownian bridge and W a standard Wiener process:

$$\|\beta_n\| \to_d \|B\| \tag{1.10}$$

[Doob (1949) and see, e.g., Billingsley (1968), page 105];

$$\liminf_{n\to\infty}(\log\log n)^{1/2}\|\beta_n\| = 2^{-3/2}\pi \quad \text{a.s.} \tag{1.11}$$

[Mogul'skii (1979) and see, e.g., Csörgő and Révész (1981), page 159];

$$\limsup_{n\to\infty}(\log\log n)^{-1/2}\|\beta_n\| = 2^{-1/2} \quad \text{a.s.} \tag{1.12}$$

[Chung (1949) and see, e.g., Csörgő and Révész (1981), page 157];

$$\|r_n\| \to_d \|W\| \tag{1.13}$$

[see, e.g., Vervaat (1972), pages 245 and 251];

$$\liminf_{n\to\infty}(\log\log n)^{1/2}\|r_n\| = 2^{-3/2}\pi \quad \text{a.s.} \tag{1.14}$$

[(1.5) and Chung (1948) and see, e.g., Csörgő and Révész (1981), page 122];

$$\limsup_{n\to\infty}(\log\log n)^{-1/2}\|r_n\| = 2^{1/2} \quad \text{a.s.} \tag{1.15}$$

[see, e.g., Vervaat (1972), page 251].

COROLLARY 1A. *We have*

$$b_n\|R_n\| \to_d \|B\|^{1/2} \quad [\textit{Kiefer } (1970)]; \tag{1.16}$$

$$\liminf_{n\to\infty} a_n(\log\log n)^{1/2}\|R_n\| = 2^{-3/4}\pi^{1/2} \quad a.s.; \tag{1.17}$$

$$\limsup_{n\to\infty} a_n\|R_n\| = 2^{-1/4} \quad a.s. \ [\textit{Kiefer } (1970)]. \tag{1.18}$$

COROLLARY 1B. *Under assumptions* (i), (ii) *and* (iii), *we have*

$$b_n\|T_n\| \to_d \sigma^{1/2}\mu^{-1/2}\|W\|^{1/2}; \tag{1.19}$$

$$\liminf_{n\to\infty} a_n(\log\log n)^{1/2}\|T_n\| = 2^{-3/4}\sigma^{1/2}\mu^{-1/2}\pi^{1/2} \quad a.s.; \tag{1.20}$$

$$\limsup_{n\to\infty} a_n\|T_n\| = 2^{1/4}\sigma^{1/2}\mu^{-1/2} \quad a.s. \ [\textit{Horváth } (1984)]. \tag{1.21}$$

REMARK 1. If we do not assume (iii) in Theorem 1B and Corollary 1B, then [see Breiman (1967) and Csörgő and Révész (1981), page 108], one has

$$E(X_1^4) = \infty \Rightarrow \limsup_{n\to\infty} n^{1/4}\|s_n - W_n\| = \infty \quad \text{a.s.}$$

for any sequence W_n of Wiener processes defined on the same probability space as s_n.

A close look at our proofs in the sequel shows that they are essentially invalid if, for some $\varepsilon > 0$, $E(X^{4-\varepsilon}) = \infty$ (see Remark 6). At present, the extension of Theorem 1B to this case is an open problem.

REMARK 2. An easy extension of the methods used to prove Theorems 1A and 1B shows that the following results hold for the processes R_n and T_n.

Let q be any positive and continuous function on [0, 1] [in particular, $q(s)$ and $q(1-s)$ have limits as $s \downarrow 0$, which are finite and strictly positive]. We have

$$b_n \|R_n/q^{1/2}\| / \|\beta_n/q\|^{1/2} \to 1 \quad \text{a.s. as } n \to \infty \tag{1.22}$$

and, under the conditions in Theorem 1B,

$$b_n \|T_n/q^{1/2}\| / \|r_n/q\|^{1/2} \to \sigma^{1/2}\mu^{-1/2} \quad \text{a.s. as } n \to \infty. \tag{1.23}$$

The second purpose of our paper is to characterize the functions q for which (1.22) and (1.23) hold in probability. The motivation for this comes from the study of the so-called $\|\cdot/q\|$-metrics [see, e.g., Shorack and Wellner (1986), Chapter 11], which are used to investigate the convergence in distribution of weighted empirical processes in relation with tests of goodness of fit. First, we introduce the following classes of functions.

Let $\mathbb{Q}$ denote the class of bounded left-continuous functions with right-hand limits on (0, 1) that are positive and bounded away from zero on $(\delta, 1-\delta)$ for all $0 < \delta < 1/2$, nondecreasing in a right neighborhood of 0 and nonincreasing in a left neighborhood of 1. Denote likewise by $\mathbb{Q}_0$ the class of bounded left-continuous functions with right-hand limits that are positive and bounded away from zero on $(\delta, 1]$ for all $0 < \delta < 1$ and nondecreasing in a right neighborhood of zero.

For any positive function q on (0, 1) and $\varepsilon > 0$, set

$$I(q,\varepsilon) = \int_0^{1/2} t^{-1} \exp\left(-\frac{\varepsilon q^2(t)}{t}\right) dt$$

and

$$J(q,\varepsilon) = \int_0^{1/2} t^{-1} \exp\left(-\frac{\varepsilon q^2(1-t)}{t}\right) dt.$$

Any $q \in \mathbb{Q}$ such that the integrals $I(q,\varepsilon)$ and $J(q,\varepsilon)$ are finite [respectively any $q \in \mathbb{Q}_0$ such that $I(q,\varepsilon)$ is finite], for all $\varepsilon > 0$, will be called a *Chibisov–O'Reilly* (COR) [respectively (COR_0)] function and any $q \in \mathbb{Q}$ such that the integrals $I(q,\varepsilon)$ and $J(q,\varepsilon)$ are finite [respectively any $q \in \mathbb{Q}_0$ such that $I(q,\varepsilon)$ is finite], for all $\varepsilon > 0$ sufficiently large, will be called an *Erdös–Feller–Kolmogorov–Petrovskii* (EFKP) [respectively (EFKP_0)] function. The above formulations of these classes of functions are due to Csörgő, Csörgő, Horváth and Mason (CsCsHM) (1986) [see also Chibisov (1964), O'Reilly (1974), Erdös (1942), Petrovskii (1935), Feller (1943) and Itô and McKean (1965), Section 1.8].

Let

$$\tilde{R}_n(t) = R_n(t),$$
$$\tilde{\beta}_n(t) = \beta_n(t) \quad \text{for } 1/(n+1) \le t \le n/(n+1)$$

and

$$\tilde{R}_n(t) = \tilde{\beta}_n(t) = 0 \quad \text{elsewhere;}$$

likewise, let

$$\tilde{T}_n(t) = T_n(t),$$
$$\tilde{r}_n(t) = r_n(t) \quad \text{for } 1/n \le t \le 1$$

and

$$\tilde{T}_n(t) = \tilde{r}_n(t) = 0 \quad \text{elsewhere.}$$

Our third and fourth theorems give weighted versions of Theorems 1A and 1B, together with a characterization of when convergence in probability holds. Let $\log^+ u = \max(1, \log u)$ for $u > 0$ and $\log^+ 0 = 1$.

THEOREM 2A. *Let $q \in \mathbb{Q}$. Then $n^{1/4}\|\tilde{R}_n/\{2q \log^+(n^{1/2}/q)\}^{1/2}\| = O_P(1)$ if and only if q is an EFKP function, in which case we have*

$$(1.24)\quad n^{1/4}\|\tilde{R}_n/\{2q \log^+(n^{1/2}/q)\}^{1/2}\| - \|\tilde{\beta}_n/q\|^{1/2} \to_P 0 \quad \text{as } n \to \infty.$$

Moreover, $q \in \mathbb{Q}$ is not EFKP if and only if

$$n^{1/4}\|\tilde{R}_n/\{2q \log^+(n^{1/2}/q)\}^{1/2}\| \to_P \infty \quad \text{as } n \to \infty.$$

THEOREM 2B. *Let $q \in \mathbb{Q}_0$. Then, under assumptions* (i), (ii) *and* (iii), *$n^{1/4}\|\tilde{T}_n/\{2q \log^+(n^{1/2}/q)\}^{1/2}\| = O_P(1)$ if and only if q is an $EFKP_0$ function in which case we have*

$$(1.25)\quad n^{1/4}\|\tilde{T}_n/\{2q \log^+(n^{1/2}/q)\}^{1/2}\| - \sigma^{1/2}\mu^{-1/2}\|\tilde{r}_n/q\|^{1/2} \to_P 0 \quad \text{as } n \to \infty.$$

Moreover, $q \in \mathbb{Q}_0$ is not $EFKP_0$ if and only if

$$n^{1/4}\|\tilde{T}_n/\{2q \log^+(n^{1/2}/q)\}^{1/2}\| \to_P \infty \quad \text{as } n \to \infty.$$

REMARK 3. An immediate consequence of Theorems 2A and 2B is that, for any $q \in \mathbb{Q}$, $n^{1/4}\|\tilde{R}_n/\{2q \log^+(n^{1/2}/q)\}^{1/2}\|$ converges in distribution to a nondegenerate random variable if and only if q is an EFKP function, in which case we have

$$(1.26)\qquad n^{1/4}\|\tilde{R}_n/\{2q \log^+(n^{1/2}/q)\}^{1/2}\| \to_d \|B/q\|^{1/2},$$

where B is a Brownian bridge. The same statement holds for $\tilde{T}_n$ with the Brownian bridge B replaced by a standard Wiener process W multiplied by the coefficient $\sigma\mu^{-1}$ and $q \in \mathbb{Q}_0$ being then EFKP_0.

A simple example of an EFKP function $q \in \mathbb{Q}$ is given by

$$q(s) = (s(1-s))^{1/2}\left(\log\log\left(\frac{1}{s(1-s)}\right)\right)^{1/2}.$$

For this choice of q, (1.26) yields the following weighted version of the usual Bahadur representation (1.1) for sample quantiles. We have, uniformly over all $s \in [1/(n+1), n/(n+1)]$ as $n \to \infty$,

$$H_n(s) = s + (s - G_n(s))$$

$$+n^{-3/4}(\log n)^{1/2}(s(1-s))^{1/4}\left(\log\log\left(\frac{1}{s(1-s)}\right)\right)^{1/4} \dot{O}_P(1). \tag{1.27}$$

An application of Theorems 2A and 2B gives

Corollary 2A. *Let $q \in \mathbb{Q}$ be a COR function. Then*

$$b_n\|\tilde{R}_n/q^{1/2}\| - \|\tilde{\beta}_n/q\|^{1/2} \to_P 0 \quad \text{as } n \to \infty. \tag{1.28}$$

Corollary 2B. *Let $q \in \mathbb{Q}_0$ be a COR_0 function. Then, under assumptions* (i), (ii) *and* (iii),

$$b_n\|\tilde{T}_n/q^{1/2}\| - \sigma^{1/2}\mu^{-1/2}\|\tilde{r}_n/q\|^{1/2} \to_P 0 \quad \text{as } n \to \infty. \tag{1.29}$$

Remark 4. Let $f^{(+)} = \max(f, 0)$ and $f^{(-)} = \max(-f, 0)$. The assertions of Theorem 2A and Corollary 2A (respectively, Theorem 2B and Corollary 2B) remain true with $\tilde{R}_n$ (respectively $\tilde{T}_n$) replaced by $\tilde{R}_n^{(\pm)}$ (respectively $\tilde{T}_n^{(\pm)}$). The proofs of these versions are nearly the same as the present ones.

The proofs of our theorems are given in the next section together with Propositions 1, 2, 3 and 4, which are likely to be of independent interest.

2. Proofs of the theorems. In order to prove Theorems 1A and 1B, we require Proposition 1.

Let (Ω, A, P) be a probability space on which sits a sequence $\{W_n, n \geq 1\}$ of standard Wiener processes defined on $[0, \infty)$. For any $\gamma > 1$, $a > 1$, $\eta > 0$ and $n_1 \geq 3$, denote by $\mathbb{F}(\gamma, a, \eta, n_1)$ the subclass of all sequences of real-valued functions defined on $[0, \infty)$ such that, for any sequence $\{f_n, n \geq 1\} \in \mathbb{F}(\gamma, a, n, n_1)$:

(F1) For all $n \geq n_1$,

$$\gamma^{-1}n^{1/2}/(\log^2 n) \leq \|f_n\|_n \leq \gamma n^{1/2}\log^2 n,$$

where $\|f_n\|_n = \sup_{0 \leq s \leq n}|f_n(s)|$.

(F2) For all $n \geq n_1$, we have

$$M_n(f_n) := \max\left\{\inf_{s \in I_n} f_n(s), \inf_{s \in I_n}(-f_n(s))\right\} \geq a^{-1}\|f_n\|_n,$$

for some closed interval $I_n \subset [0, n]$ of length $\eta n \exp(-(\log\log n)^2)$.

(F3) For all $n \geq n_1$, $0 \leq s + f_n(s)$ for $0 \leq s \leq n$.

Let also $\mathbb{F}$ denote the subclass of all sequences $\{f_n, n \ge 1\}$ such that for any $a > 1$ there exist $\gamma > 1$, $\eta > 0$ and $n_1 \ge 3$, for which $\{f_n, n \ge 1\} \in F(\gamma, a, \eta, n_1)$, i.e.,

$$\mathbb{F} = \bigcap_{a>1} \left(\bigcup_{\gamma>1} \bigcup_{\eta>0} \bigcup_{n_1 \ge 3} \mathbb{F}(\gamma, a, \eta, n_1) \right),$$

where we may assume $\gamma > 1$, $a > 1$, $\eta > 0$ to be rational numbers and $n_1 \ge 3$ to be integer.

PROPOSITION 1. *With the above notation, we have with probability* 1 *for all* $\{f_n, n \ge 1\} \in \mathbb{F}$,

$$\liminf_{n\to\infty} Z_n(f_n) \ge 1, \tag{2.1}$$

where

$$Z_n(f_n) = \{\|f_n\|_n \log n\}^{-1/2} \sup_{0 \le t \le n} |W_n(t + f_n(t)) - W_n(t)|.$$

PROOF. We shall make use of the following two inequalities. For a standard Wiener process W defined on $[0, \infty)$, there exist constants $A > 0$, $B > 0$, $u_0 > 0$ and $T_0 > 0$ such that for all $u \ge u_0$ and $T \ge T_0$,

$$P\Big(\sup_{0 \le x \le T} |W(x+1) - W(x)| \le u \Big) \le \exp(-ATue^{-u^2/2}) \tag{2.2}$$

and

$$P\Big(\sup_{0 \le x \le T} \sup_{0 \le s \le 1} |W(x+s) - W(x)| > u \Big) \le BTue^{-u^2/2}. \tag{2.3}$$

Both inequalities are implied by Lemma 2.3 in Révész (1982) and by the fact that $1 - \exp(-z) \le z$ for all $z \ge 0$.

Define, for any $a > 1$, $\gamma > 1$, $\delta > 0$ and $n \ge 3$,

$$h_n(k) = \gamma^{-1} a^k n^{1/2} / \log^2 n$$

for $k = -2, -1, 0, \dots, k(n) := [\log_a(\gamma^2 \log^4 n)] + 1$ and

$$I_n(m) = \Big[m\delta n \exp\big(-(\log\log n)^2\big), (m+1)\delta n \exp\big(-(\log\log n)^2\big)\Big]$$

for $m = 0, 1, \dots, m(n) := [\delta^{-1} \exp((\log\log n)^2)] + 1$.

Set

$$\Delta_n(a, \gamma, \delta) = \min_{-2 \le k \le k(n)} \min_{0 \le m \le m(n)} \sup_{s \in I_n(m)} |W_n(s + h_n(k)) - W_n(s)| / \{h_n(k) \log n\}^{1/2}$$

and, with $\rho > 0$,

$$D_n(a, \gamma, \rho) = \max_{-2 \le k \le k(n)} \sup_{0 \le s \le 2n} \sup_{0 \le t \le \rho h_n(k)} |W_n(s + t) - W_n(s)| / \{h_n(k) \log n\}^{1/2}.$$

LEMMA 1. *With probability* 1, *for all rational numbers* $a > 1$, $\gamma > 1$ *and* $\delta > 0$,

$$\liminf_{n \to \infty} \Delta_n(a, \gamma, \delta) \geq 1. \tag{2.4}$$

PROOF. Choose any $0 < \varepsilon < 1$. We have

$$\begin{aligned} P\big(\Delta_n(a,\gamma,\delta) \leq (1-\varepsilon)^{1/2}\big) &= P\left(\bigcup_{k=-2}^{k(n)} \bigcup_{m=0}^{m(n)} \left\{ \sup_{s \in I_n(m)} |W_n(s + h_n(k)) - W_n(s)| \right.\right. \\ &\qquad\qquad \left.\left. \leq ((1-\varepsilon) h_n(k) \log n)^{1/2} \right\} \right) \qquad (2.5) \\ &\leq \sum_{k=-2}^{k(n)} \sum_{m=0}^{m(n)} P\left(\sup_{0 \leq x \leq Y_n(k)} |W(x+1) - W(x)| \leq ((1-\varepsilon)\log n)^{1/2} \right), \end{aligned}$$

where $Y_n(k) := \delta n h_n^{-1}(k) \exp(-(\log\log n)^2) \geq Y_n := n^{1/2 - \varepsilon/4}$ for $k = -2, -1, \dots, k(n)$ and all large n, and where $W(\cdot)$ denotes a standard Wiener process. Thus, by (2.2), the right-hand side of inequality (2.5) is, for all large n, less than or equal to

$$8k(n)m(n)\exp\big(-AY_n(1-\varepsilon)^{1/2}(\log n)^{1/2} n^{-(1-\varepsilon)/2}\big) \leq \exp(-n^{\varepsilon/8}).$$

Since this last expression is summable in $n \geq 1$, an application of the Borel–Cantelli lemma in combination with a, γ and δ assumed to be rational numbers completes the proof of Lemma 1. □

LEMMA 2. *With probability* 1, *for all rational numbers* $a > 1$, $\gamma > 1$ *and* $\rho > 0$,

$$\limsup_{n \to \infty} D_n(a, \gamma, \rho) \leq 2\rho^{1/2}. \tag{2.6}$$

PROOF. Notice that

$$\begin{aligned} P\big(D_n(a,\gamma,\rho) > 2\rho^{1/2}\big) \leq \sum_{k=-2}^{k(n)} P\left(\sup_{0 \leq s \leq 2n} \sup_{0 \leq t \leq \rho h_n(k)} |W_n(s+t) - W_n(s)| > 2\{\rho h_n(k) \log n\}^{1/2} \right), \end{aligned}$$

which, since $2n/(\rho h_n(k)) \leq n^{3/4}$ for all $k = -2, -1, \dots, k(n)$ and all large n, is less than or equal to

$$4k(n) P\left(\sup_{0 \leq x \leq n^{3/4}} \sup_{0 \leq s \leq 1} |W(x+s) - W(x)| > 2(\log n)^{1/2} \right),$$

which by inequality (2.3), again for large n, is less than or equal to

$$4k(n) B n^{3/4} \big(2(\log n)^{1/2}\big) \exp(-2\log n) \leq n^{-6/5},$$

which is summable in $n \geq 1$. The Borel–Cantelli lemma along with a, γ and ρ assumed to be rational numbers finishes the proof of Lemma 2. □

We are now prepared to complete the proof of Proposition 1. Choose any $\{f_n,\ n \ge 1\} \in \mathbb{F}(\gamma, a, \eta, n_1)$, where $\gamma > 1$, $a > 1$, $\eta > 0$ are rational numbers and $n_1 \ge 3$ is integer.

By (F1), we can find for all $n \ge n_1$ a $1 \le k = k_n \le k(n)$ such that

$$a^{-1} h_n(k) \le \|f_n\|_n \le h_n(k) \tag{2.7}$$

and by (F2) and (2.7), for all $n \ge n_1$,

$$a^{-2} h_n(k) \le M_n(f_n) \le h_n(k). \tag{2.8}$$

Set $\delta = \eta/6$, and recall that $I_n(m)$ has length $\delta n \exp(-(\log\log n)^2)$ and I_n has length $\eta n \exp(-(\log\log n)^2)$. It follows that we may choose an $1 \le m \le m(n)$ such that $I_n(m) \subset I_n$ and $I_n(m-1) \subset I_n$. Next, observe that $h_n(k(n)) = O(n^{1/2}\log^2 n)$, so that we may select an $n_2 \ge n_1$ so large that, for all $n \ge n_2$,

$$h_n(k) \le h_n(k(n)) \le \tfrac{1}{2}\delta n \exp\bigl(-(\log\log n)^2\bigr). \tag{2.9}$$

Let $Z_n(f_n)$ be as in Proposition 1. Since $I_n \subset [0, n]$, (2.7) implies obviously that

$$Z_n(f_n) \ge \{h_n(k)\log n\}^{-1/2} \sup_{s \in I_n} |W_n(s + f_n(s)) - W_n(s)|. \tag{2.10}$$

We shall now give a lower bound for the right-hand side of (2.10).

CASE 1. If $M_n(f_n) = \inf_{s \in I_n} f_n(s)$, then by (2.7) and (2.8), for all $n \ge n_2$ and $s \in I_n$,

$$a^{-2} h_n(k) \le M_n(f_n) \le f_n(s) \le \|f_n\|_n \le h_n(k),$$

and hence $|f_n(s) - h_n(k)| \le \rho h_n(k)$ for $s \in I_n$ and $\rho = 1 - a^{-2}$. Thus, by the triangle inequality and using the fact that $I_n(m) \subset I_n$, we have

$$\sup_{s \in I_n} |W_n(s + f_n(s)) - W_n(s)| \ge \sup_{s \in I_n(m)} |W_n(s + h_n(k)) - W_n(s)| - \sup_{0 \le s \le 2n} \sup_{0 \le t \le \rho h_n(k)} |W_n(s+t) - W_n(s)|.$$

CASE 2. If $M_n(f_n) = \inf_{s \in I_n}(-f_n(s))$, then by (2.7) and (2.8), for all $n \ge n_2$ and $s \in I_n$,

$$-h_n(k) \le -\|f_n\|_n \le f_n(s) \le -M_n(f_n) \le -a^{-2} h_n(k) = -h_n(k-2),$$

and hence $|f_n(s) + h_n(k-2)| \le \rho h_n(k)$ for $s \in I_n$ and $\rho = 1 - a^{-2}$. Moreover, by (F3) and the above inequalities, for $n \ge n_2$ and $s \in I_n$,

$$0 \le s + f_n(s) \le s - h_n(k-2),$$

so that, for $n \ge n_2$,

$$\sup_{s \in I_n} |W_n(s + f_n(s)) - W_n(s)| \ge \sup_{s \in I_n} |W_n(s - h_n(k-2)) - W_n(s)| - \sup_{0 \le s \le 2n} \sup_{0 \le t \le \rho h_n(k)} |W_n(s+t) - W_n(s)|.$$

Since we have chosen m such that $I_n(m-1) \cup I_n(m) \subset I_n$, the fact that $h_n(k-2) = a^{-2}h_n(k) < h_n(k(n)) \le \frac{1}{2}\delta n \exp(-(\log\log n)^2)$ implies that

$$I_n(m-1) \subset \{s - h_n(k-2): s \in I_n\}.$$

Thus,

$$\sup_{s\in I_n} |W_n(s + f_n(s)) - W_n(s)| \ge \sup_{s\in I_n(m-1)} |W_n(s + h_n(k-2)) - W_n(s)|$$
$$- \sup_{0<s<2n} \sup_{0<t<\rho h_n(k)} |W_n(s+t) - W_n(s)|.$$

Hence, in either case, the right-hand side of (2.10) is greater than or equal to

$$a^{-1}\Delta_n(a,\gamma,\delta) - D_n(a,\gamma,\rho).$$

Therefore, by applying Lemmas 1 and 2 (recall that $\rho = 1 - a^{-2}$), we get with probability 1, *uniformly over all sequences* $\{f_n, n \ge 1\} \in \mathbb{F}(\gamma, a, \eta, n_1)$,

$$\liminf_{n\to\infty} Z_n(f_n) \ge a^{-1} - 2(1 - a^{-2})^{1/2}.$$

The fact that γ, a and η are rational, and n_1 integer, jointly with the remark that $a^{-1} - 2(1-a^{-2})^{1/2}$ can be chosen arbitrarily close to 1 for a suitable choice of $a > 1$ completes the proof of Proposition 1. □

REMARK 5. A close look at the proof of Proposition 1 shows that the validity of (2.1) can be extended to cover a larger class of sequences $\{f_n, n \ge 1\}$. For instance, we may replace $\log^2 n$ in (F1) and $(\log\log n)^2$ in (F2) by $\log^R n$ and $(\log\log n)^R$, respectively, for any constant $R > 1$. We limit ourselves to the statement above, which will be more than sufficient for our needs.

PROOF OF THEOREM 1A. On account of (1.3), we need only show that

$$\liminf_{n\to\infty} b_n \|R_n\| / \|\beta_n\|^{1/2} \ge 1 \quad \text{a.s.} \tag{2.11}$$

Without loss of generality we can assume that we are on the probability space of Theorem 3 of KMT (1975) on which there sit a sequence of independent uniform (0, 1) random variables $U_1, U_2, \dots,$ and a sequence of Brownian bridges $B_1, B_2, \dots,$ such that

$$\|B_n - \alpha_n\| = O(n^{-1/2}\log n) \quad \text{a.s.} \tag{2.12}$$

Now, as in (1.6) of Shorack (1982) we have [see also (2.45) in the sequel]

$$\|R_n - (\alpha_n - \alpha_n(H_n))\| = n^{-1/2} \quad \text{a.s.} \tag{2.13}$$

and, recalling (1.11),

$$\liminf_{n\to\infty} (\log\log n)^{1/2} \|\beta_n\| = 2^{-3/2}\pi \quad \text{a.s.} \tag{2.14}$$

Thus by (2.12), (2.13) and (2.14), to establish (2.11) it suffices to prove that

$$\liminf_{n\to\infty} b_n \|B_n(H_n) - B_n\| / \|\beta_n\|^{1/2} \ge 1 \quad \text{a.s.} \tag{2.15}$$

We can write each B_n as

$$B_n(s) = n^{-1/2}(W_n(sn) - sW_n(n)), \qquad 0 \le s \le 1,$$

where W_n is a standard Wiener process on $[0, \infty)$. Next, rewrite H_n as

$$H_n(s) = n^{-1}(f_n(sn) + sn) \ge 0 \quad \text{for } 0 \le s \le 1, \tag{2.16}$$

where $f_n(t) = n^{1/2}\beta_n(t/n)$ for $0 \le t \le n$.

By (2.14) we have

$$\liminf_{n \to \infty} n^{-1/2}(\log\log n)^{1/2}\|f_n\|_n = 2^{-3/2}\pi \quad \text{a.s.} \tag{2.17}$$

and, by (1.12),

$$\limsup_{n \to \infty} n^{-1/2}(\log\log n)^{-1/2}\|f_n\|_n = 2^{-1/2} \quad \text{a.s.} \tag{2.18}$$

Thus, $\{f_n,\ n \ge 1\}$ satisfies (F3), by (2.16), and satisfies (F1) for any $\gamma > 1$ and all n_1 sufficiently large, by (2.17) and (2.18), with probability 1.

For (F2), we make use of Theorem 02 in Stute (1982) [see also Theorem 1 in Mason (1984)] jointly with (1.1), to show that, for any sequence $0 \le \kappa_n \le n$ such that

(a) $\kappa_n \uparrow \infty$ and $\kappa_n/n \downarrow 0$ as $n \to \infty$,
(b) $\kappa_n/\log n \to \infty$ and $(\log(n/\kappa_n))/\log\log n \to \infty$ as $n \to \infty$,

we have almost surely as $n \to \infty$

$$\sup_{0 \le t \le n - \kappa_n} \sup_{0 \le s \le \kappa_n} |f_n(t+s) - f_n(t)| = (1 + o(1))(2\kappa_n \log(n/\kappa_n))^{1/2}. \tag{2.19}$$

It is straightforward that $\kappa_n = \eta n \exp(-(\log\log n)^2)$ satisfies (a) and (b) (the Csörgő–Révész–Stute conditions). Therefore, by (2.17) and (2.19), we have almost surely for n large,

$$\sup_{u, v \in I_n} |f_n(u) - f_n(v)| \le n^{1/2} \exp\left(-\tfrac{1}{4}(\log\log n)^2\right) = o(\|f_n\|_n),$$

uniformly over all intervals $I_n \subset [0, n]$ of length κ_n. Hence (F_2) holds and $\{f_n,\ n \ge 1\} \in \mathbb{F}$ almost surely.

Now with the above notation,

$$b_n\|B_n(H_n) - B_n\|/\|\beta_n\|^{1/2} = \sup_{0 \le t \le n} |W_n(t + f_n(t)) - W_n(t) - n^{-1}f_n(t)W_n(n)|/\{\|f_n\|_n \log n\}^{1/2}.$$

Next observe that by (2.18) and the easily verified fact [use Borel–Cantelli and the inequality $P(n^{-1/2}|W_n(n)| \le u) \ge 2u^{-1}(2\pi)^{-1/2}e^{-u^2/2}$ for $u > 0$] that $n^{-1/2}|W_n(n)| = O((\log n)^{1/2})$ a.s., we have

$$n^{-1}\|f_n\|_n^{1/2}|W_n(n)|(\log n)^{-1/2} = O\left(n^{-1/4}\log n\right) \to 0 \quad \text{a.s.}$$

Hence, by Proposition 1 we conclude (2.15) and (2.11), which finishes the proof of Theorem 1A. □

PROOF OF THEOREM 1B. We shall assume that we are on a probability space such that (1.6) holds.

By the law of the iterated logarithm (1.15) for renewal processes we have

$$\limsup_{n\to\infty} (\log\log n)^{-1/2}\|r_n\| = 2^{1/2} \quad \text{a.s.} \tag{2.20}$$

and, by (1.14),

$$\liminf_{n\to\infty} (\log\log n)^{1/2}\|r_n\| = 2^{-3/2}\pi \quad \text{a.s.} \tag{2.21}$$

Next, observe that for $0 \le s < \infty$,

$$s_n(N_n(s)) + r_n(s) = n^{-1/2}\sigma^{-1}(S_{N(n\mu s)} - n\mu s) \le 0.$$

Since for all $t \ge 0$, $S_{N(t)} \le t < S_{N(t)+1}$, we have with probability 1 for n large (see Lemma 9 for details)

$$b_n\|s_n(N_n) + r_n\|/\|r_n\|^{1/2} \le \sigma^{-1}n^{-1/4}(\log n)^{-1/2} \max_{1\le i\le 2n} \left(|X_i|/\|r_n\|^{1/2}\right).$$

Since $E(X_1^4) < \infty$ implies that, as $n \to \infty$,

$$n^{-1/4} \max_{1\le i\le 2n} |X_i| \to 0 \quad \text{a.s.},$$

we see by (2.21) that

$$b_n\|s_n(N_n) + r_n\|/\|r_n\|^{1/2} = o(1) \quad \text{a.s.} \tag{2.22}$$

REMARK 6. A crucial step in the proof of Theorem 1B is to prove, in addition to (2.22), that

$$b_n\|s_n - W_n\|/\|r_n\|^{1/2} = o(1) \quad \text{a.s.}, \tag{2.23}$$

which, by (2.21), can be reduced to

$$n^{1/4}(\log n)^{-1/2}(\log\log n)^{1/4}\|s_n - W_n\| = o(1) \quad \text{a.s.}$$

An application of Theorem 2.6.6 and Lemma 2.6.1 in Csörgő and Révész (1981) shows that this condition holds whenever

(iv) $E(X_1^4(\log^+|X_1|)^{-2}(\log^+\log^+|X_1|)) < \infty$.

Moreover, (2.22) also holds under (iv) (see Lemma 9). Thus we see that we may replace (iii) [i.e., $E(X_1^4) < \infty$] by (iv) in the statement of Theorem 1B. This, however, only gives a minor technical improvement to the statement of this theorem.

By (2.22), it follows that in (1.9) we have

$$b_n\|T_n\|/\|r_n\|^{1/2} = b_n\|s_n + r_n\|/\|r_n\|^{1/2} = b_n\|s_n - s_n(N_n)\|/\|r_n\|^{1/2} + o(1) \quad \text{a.s.}$$

Now by (1.6) and (2.21) this last expression equals

$$b_n\|W_n(N_n) - W_n\|/\|r_n\|^{1/2} + o(1) \quad \text{a.s.}$$

Thus, to prove (1.9) it is enough to show that

$$\lim_{n\to\infty} b_n \|W_n(N_n) - W_n\|/\|r_n\|^{1/2} = \sigma^{1/2}\mu^{-1/2} \quad \text{a.s.} \tag{2.24}$$

The fact that

$$\liminf_{n\to\infty} b_n \|W_n(N_n) - W_n\|/\|r_n\|^{1/2} \geq \sigma^{1/2}\mu^{-1/2} \quad \text{a.s.} \tag{2.25}$$

follows from Proposition 1 by writing for $0 < s < 1$,

$$N_n(s) = s + \sigma\mu^{-1}n^{-1/2}r_n(s) := s + n^{-1}f_n(ns).$$

By (2.20) and (2.21), $\{f_n, n > 1\}$ satisfies (F1). (F3) is straightforward, whereas for (F2) we proceed as in the proof of Theorem 1A with the formal replacements of Theorem 02 of Stute (1982) by Theorem 1.2.1 of Csörgő and Révész [(1981), page 30] [see also Csörgő and Révész (1979)] and of (1.1) by (1.5). Thus $\{f_n, n \geq 1\} \in \mathbb{F}$ and (2.25) follows from (2.1).

To verify that

$$\limsup_{n\to\infty} b_n \|W_n(N_n) - W_n\|/\|r_n\|^{1/2} \leq \sigma^{1/2}\mu^{-1/2} \quad \text{a.s.,} \tag{2.26}$$

we use the same argument as given in Shorack (1982) substituting the application of (1.11) by that of (1.14), and (3.8) of Shorack (1982) by Theorem 1.2.1 in Csörgő and Révész [(1981), page 30]. For the sake of brevity, we omit the routine details.

Statements (2.25) and (2.26) complete the proof of (2.24), (1.9) and Theorem 1B. □

For the proof of Theorems 2A and 2B we need the following two analytic propositions. Let X denote any continuous real-valued function defined on $(-\infty, \infty)$ such that, for all $0 \leq a < b \leq 1$,

$$\lim_{h\downarrow 0} \sup_{\substack{a\leq u\leq b\\ |u-v|\leq h}} \frac{|X(u) - X(v)|}{(2h\log(1/h))^{1/2}} = \lim_{h\downarrow 0} \sup_{a\leq u\leq b} \frac{|X(u\pm h) - X(u)|}{(2h\log(1/h))^{1/2}} = 1. \tag{2.27}$$

For any choice of $0 \leq a < b \leq 1$ let $\mathbb{C}[a,b]$ denote the set of real-valued continuous functions f defined on $[a,b]$.

PROPOSITION 2. *For any* $0 \leq a < b \leq 1$ *and* $f \in \mathbb{C}[a,b]$,

$$\begin{aligned}\lim_{T\to\infty} T^{1/4} &\sup_{a\leq s\leq b} |X(s + T^{-1/2}f(s)) - X(s)|/(\log T)^{1/2} \\ &= \sup_{a\leq s\leq b} |f(s)|^{1/2}.\end{aligned} \tag{2.28}$$

PROPOSITION 3. *For any* $f \in \mathbb{C}[0,1]$ *and* $q \in \mathbb{Q}$,

$$\sup_{T\to\infty} T^{1/4} \sup_{0<s<1} \frac{|X(s + T^{-1/2}f(s)) - X(s)|}{\{2q(s)\log^+(T^{1/2}/q(s))\}^{1/2}} = \sup_{0<s<1}\left|\frac{f(s)}{q(s)}\right|^{1/2}. \tag{2.29}$$

[Note that the limit in (2.29) is possibly infinite.]

PROOF OF PROPOSITION 2. Choose any $f \in \mathbb{C}[a, b]$. We can assume that f is not identically equal to zero, otherwise the limit in (2.28) holds trivially. Write $M = \sup_{a \le s \le b} |f(s)|$ and notice that

$$\sup_{a \le s \le b} |X(s + T^{-1/2} f(s)) - X(s)| \le \sup_{a \le u \le b} \sup_{|u-v| \le MT^{-1/2}} |X(u) - X(v)|.$$

Thus since

$$T^{1/4} (\log T)^{-1/2} \{2T^{-1/2} M \log(T^{1/2}/M)\}^{1/2} \to M^{1/2} \quad \text{as } T \to \infty,$$

we have by (2.27) (set $h = MT^{-1/2}$) that

$$\limsup_{T \to \infty} T^{1/4} \sup_{a \le s \le b} |X(s + T^{-1/2} f(s)) - X(s)| / (\log T)^{1/2} \le M^{1/2}. \tag{2.30}$$

Next, by continuity of f, for any $0 < \varepsilon < M$, we can find an $\eta > 0$ and a subinterval $[c, d] \subset [a + \eta, b - \eta]$ such that for all $c \le s \le d$, $c < d$, $m - \varepsilon \le f(s) \le m + \varepsilon$, where $m = \pm M$ is the value of f at a point where $|f|$ assumes its maximum in $[a, b]$. It is easy to see that for any such $\varepsilon > 0$ and subinterval $[c, d]$, whenever $MT^{-1/2} \le \eta$,

$$\begin{aligned} \sup_{a \le s \le b} |X(s + T^{-1/2} f(s)) - X(s)| &\ge \sup_{c \le u \le d} |X(u + T^{-1/2} f(u)) - X(u)| \\ &\ge \sup_{c \le u \le d} |X(u + T^{-1/2} m) - X(u)| \\ &\quad - \sup_{\substack{a \le u \le b \\ |u-v| \le \varepsilon T^{-1/2}}} |X(u) - X(v)|. \end{aligned}$$

By (2.27) as $T \to \infty$,

$$T^{1/4} (\log T)^{-1/2} \sup_{c \le u \le d} |X(u + T^{-1/2} m) - X(u)| \to M^{1/2}$$

and

$$T^{1/4} (\log T)^{-1/2} \sup_{\substack{a \le u \le b \\ |u-v| \le \varepsilon T^{-1/2}}} |X(u) - X(v)| \to \varepsilon^{1/2}.$$

Hence for all $\varepsilon > 0$

$$\begin{aligned} &\liminf_{T \to \infty} T^{1/4} \sup_{a \le s \le b} |X(s + T^{-1/2} f(s)) - X(s)| / (\log T)^{1/2} \\ &\qquad \ge M^{1/2} - \varepsilon^{1/2}. \end{aligned} \tag{2.31}$$

Statements (2.30) and (2.31) (note that $\varepsilon > 0$ can be chosen arbitrarily small) complete the proof of Proposition 2. □

PROOF OF PROPOSITION 3. Choose any $f \in \mathbb{C}[0,1]$ and $q \in \mathbb{Q}$. We claim that

$$\liminf_{T\to\infty} T^{1/4} \sup_{0<s<1} \frac{|X(s+T^{-1/2}f(s)) - X(s)|}{\{2q(s)\log^+(T^{1/2}/q(s))\}^{1/2}} \geq \sup_{0<s<1}\left|\frac{f(s)}{q(s)}\right|^{1/2}. \tag{2.32}$$

Assertion (2.32) is an immediate consequence of the following lemma.

LEMMA 3. *For any* $f \in \mathbb{C}[0,1]$, $q \in \mathbb{Q}$ *and* $0 < \eta < \frac{1}{2}$,

$$\lim_{T\to\infty} T^{1/4} \sup_{\eta<s\leq 1-\eta} \frac{|X(s+T^{-1/2}f(s)) - X(s)|}{\{2q(s)\log^+(T^{1/2}/q(s))\}^{1/2}} = \sup_{\eta<s\leq 1-\eta}\left|\frac{f(s)}{q(s)}\right|^{1/2} \tag{2.33}$$

PROOF. Note that our assumptions imply that $\sup_{\eta<s\leq 1-\eta}|f(s)/q(s)| < \infty$. Moreover, $q \in \mathbb{Q}$ is bounded away from zero and infinity on $[\eta, 1-\eta]$. Therefore, as $T \to \infty$, $2\log^+(T^{1/2}/q(s)) = (1+o(1))\log T$, where the "$o(1)$" is uniform over $s \in [\eta, 1-\eta]$. Hence it suffices to prove (2.33) with $2\log^+(T^{1/2}/q(s))$ replaced by $\log T$.

For any $\varepsilon > 0$ and integer $n \geq 1$, choose $\eta = \delta_0 < \delta_1 < \cdots < \delta_n < \delta_{n+1} = 1-\eta$, $\delta_i < \gamma_i \leq \delta_{i+1}$ and $\delta_i < \gamma_i^\star \leq \delta_{i+1}$ for $i = 0,\dots,n$, such that

$$\begin{aligned}(1+\varepsilon)q(\gamma_i) > \sup_{\delta_i<s\leq\delta_{i+1}} q(s) &\geq \inf_{\delta_i<s\leq\delta_{i+1}} q(s)\\ &> q(\gamma_i^\star)/(1+\varepsilon) \quad \text{for } i = 0,\dots,n.\end{aligned}$$

Note that this choice is possible by the left-continuous version of Lemma 1 in Billingsley [(1968), page 110].

By the continuity of f and X, we have evidently

$$\begin{aligned}(1+\varepsilon)^{-1/2} &\sup_{\delta_i\leq s\leq\delta_{i+1}} \frac{T^{1/4}|X(s+T^{-1/2}f(s)) - X(s)|}{\{q(\gamma_i)\log T\}^{1/2}}\\ &\leq \sup_{\delta_i<s\leq\delta_{i+1}} \frac{T^{1/4}|X(s+T^{-1/2}f(s)) - X(s)|}{\{q(s)\log T\}^{1/2}}\\ &\leq (1+\varepsilon)^{1/2} \sup_{\delta_i\leq s\leq\delta_{i+1}} \frac{T^{1/4}|X(s+T^{-1/2}f(s)) - X(s)|}{\{q(\gamma_i^\star)\log T\}^{1/2}}.\end{aligned}$$

An application of Proposition 2 shows that the left-hand side of the above inequality tends to

$$(1+\varepsilon)^{-1/2} \sup_{\delta_i\leq s\leq\delta_{i+1}}\left|\frac{f(s)}{q(\gamma_i)}\right|^{1/2} = (1+\varepsilon)^{-1/2} \sup_{\delta_i<s\leq\delta_{i+1}}\left|\frac{f(s)}{q(\gamma_i)}\right|^{1/2}$$

while the right-hand side tends to

$$(1+\varepsilon)^{1/2} \sup_{\delta_i\leq s\leq\delta_{i+1}}\left|\frac{f(s)}{q(\gamma_i^\star)}\right|^{1/2} = (1+\varepsilon)^{1/2} \sup_{\delta_i<s\leq\delta_{i+1}}\left|\frac{f(s)}{q(\gamma_i^\star)}\right|^{1/2}.$$

Now using left continuity of q and continuity of f, it is easy to conclude (2.33), completing the proof of Lemma 3. □

Let $c_1 = \limsup_{s\downarrow 0}|f(s)/q(s)|$ and $c_2 = \limsup_{s\downarrow 0}|f(1-s)/q(1-s)|$. Notice that if c_1 or c_2 is equal to infinity, (2.32) and (2.33) obviously imply (2.29). Therefore, we need only consider the case for which c_1 and c_2 are finite.

LEMMA 4. *Whenever $f \in \mathbb{C}[0,1]$ and $q \in \mathbb{Q}$ are such that both c_1 and c_2 are finite, for all $\varepsilon > 0$ there exists a $0 < \eta < \frac{1}{2}$ such that*

$$\limsup_{T\to\infty} T^{1/4} \sup_{0<s\le\eta} \frac{|X(s + T^{-1/2}f(s)) - X(s)|}{\{2q(s)\log^+(T^{1/2}/q(s))\}^{1/2}} \le (1+\varepsilon)(c_1+\varepsilon)^{1/2} \tag{2.34}$$

and

$$\limsup_{T\to\infty} T^{1/4} \sup_{1-\eta<s\le 1} \frac{|X(s + T^{-1/2}f(s)) - X(s)|}{\{2q(s)\log^+(T^{1/2}/q(s))\}^{1/2}} \le (1+\varepsilon)(c_2+\varepsilon)^{1/2}. \tag{2.35}$$

PROOF. Choose any $\varepsilon > 0$ and select a $0 < \eta < \frac{1}{2}$ such that for all $0 < s \le \eta$,

$$|f(s)| < (c_1+\varepsilon)q(s) \quad \text{and} \quad |f(1-s)| < (c_2+\varepsilon)q(1-s). \tag{2.36}$$

Now for any $0 < s \le \eta$,

$$|X(s + T^{-1/2}f(s)) - X(s)| \le \sup_{0\le u\le 1} |X(u + T^{-1/2}f(s)) - X(u)| \tag{2.37}$$

and

$$|X(1 - s + T^{-1/2}f(1-s)) - X(1-s)| \le \sup_{0\le u\le 1} |X(u + T^{-1/2}f(1-s)) - X(u)|. \tag{2.38}$$

Next, by (2.27) we have, for all T sufficiently large and *uniformly in* $0 < s \le \eta$, that the right-hand sides of inequalities (2.37) and (2.38) are less than or equal to

$$(1+\varepsilon)^{1/2}|f(s)|^{1/2}T^{-1/4}\left\{2\log\left(\frac{T^{1/2}}{|f(s)|}\right)\right\}^{1/2} \tag{2.39}$$

and

$$(1+\varepsilon)^{1/2}|f(1-s)|^{1/2}T^{-1/4}\left\{2\log\left(\frac{T^{1/2}}{|f(1-s)|}\right)\right\}^{1/2}, \tag{2.40}$$

respectively. Since $x\log^{+}(1/x) \geq x\log(1/x)$ is strictly increasing on $[0,\infty)$, expressions (2.39) and (2.40) are by (2.36), uniformly in $0 < s \leq \eta$, less than

$$(1+\varepsilon)^{1/2}(c_1+\varepsilon)^{1/2}(q(s))^{1/2}T^{-1/4}\left\{2\log^{+}\left(\frac{T^{1/2}}{(c_1+\varepsilon)q(s)}\right)\right\}^{1/2}$$

and

$$(1+\varepsilon)^{1/2}(c_2+\varepsilon)^{1/2}(q(1-s))^{1/2}T^{-1/4}\left\{2\log^{+}\left(\frac{T^{1/2}}{(c_2+\varepsilon)q(1-s)}\right)\right\}^{1/2},$$

respectively.

Since we assume that q is bounded on $[0,1]$, it follows that these last expressions are, uniformly in $0 < s \leq \eta$ for all large enough T, less than

$$(1+\varepsilon)(c_1+\varepsilon)^{1/2}T^{-1/4}\left\{2\log^{+}\left(\frac{T^{1/2}}{q(s)}\right)\right\}^{1/2}$$

and

$$(1+\varepsilon)(c_2+\varepsilon)^{1/2}(q(1-s))^{1/2}T^{-1/4}\left\{2\log^{+}\left(\frac{T^{1/2}}{q(1-s)}\right)\right\}^{1/2},$$

respectively. This completes the proof of Lemma 4. □

Assume that c_1 and c_2 are finite. By Lemmas 3 and 4, for every $\varepsilon > 0$ we have for all $\eta \in (0, 1/2)$ sufficiently small,

$$\begin{aligned}
&\limsup_{T\to\infty}\sup_{0<s<1}\frac{T^{1/4}|X(s+T^{-1/2}f(s))-X(s)|}{\{2q(s)\log^{+}(T^{1/2}/q(s))\}^{1/2}}\\
&\quad\leq \max\left\{(1+\varepsilon)(c_1+\varepsilon)^{1/2},(1+\varepsilon)(c_2+\varepsilon)^{1/2},\ \sup_{\eta<s\leq 1-\eta}\left|\frac{f(s)}{q(s)}\right|^{12}\right\}\\
&\quad\leq \max\left\{(1+\varepsilon)(c_1+\varepsilon)^{1/2},(1+\varepsilon)(c_2+\varepsilon)^{1/2},\ \sup_{0<s<1}\left|\frac{f(s)}{q(s)}\right|^{1/2}\right\}.
\end{aligned}$$

Since the right-hand side of this last inequality tends to $\sup_{0<s<1}|f(s)/q(s)|^{1/2}$ as $\varepsilon \downarrow 0$, on account of (2.32), we have established (2.29). This finishes the proof of Proposition 3. □

A key result for the proof of Theorems 2A and 2B is captured in the following statement, which is a direct consequence of Proposition 3.

PROPOSITION 4. *Let W be a standard Wiener process extended on $(-\infty, \infty)$. Then with probability* 1 *for all $f \in \mathbb{C}[0,1]$ and all $q \in \mathbb{Q}$,*

$$\lim_{T\to\infty} T^{1/4} \sup_{0<s<1} \frac{|W(s+T^{-1/2}f(s)) - W(s)|}{\{2q(s)\log^+(T^{1/2}/q(s))\}^{1/2}} = \sup_{0<s<1} \left|\frac{f(s)}{q(s)}\right|^{1/2}. \tag{2.41}$$

PROOF. The proof follows from Proposition 3, jointly with the fact [see, e.g., Csörgő and Révész (1981), pages 26–28] that $X = W$ is continuous and satisfies (2.27) with probability 1. □

Note for further use that in (2.41) we may choose in particular $f(s) = -W(s)$ or $f(s) = -B(s) := -W(s) + sW(1)$. Also recall the following important properties of EFKP and COR functions [cf. CsCsHM (1986), Theorems 3.3, 3.4, 4.2.1, 4.2.2 and 4.2.3].

$q \in \mathbb{Q}$ is an EFKP function *if and only if* for any Brownian bridge B there exist $\beta_0 < \infty$ and $\beta_1 < \infty$ such that, with probability 1,

$$\beta_0 = \limsup_{s\downarrow 0} |B(s)|/q(s) \quad \text{and} \quad \beta_1 = \limsup_{s\downarrow 0} |B(1-s)|/q(1-s).$$

Moreover, $q \in \mathbb{Q}$ is a COR function *if and only if* $\beta_0 = \beta_1 = 0$.

In addition, $q \in \mathbb{Q}$ is EFKP if and only if the random variable $\sup_{0<s<1}|\alpha_n(s)|/q(s)$ converges in distribution to a nondegenerate random variable which is then equal in distribution to $\sup_{0<s<1}|B(s)|/q(s)$. Moreover, on the probability space of (2.12), $\sup_{0<s<1}|\alpha_n(s) - B_n(s)|/q(s) = O_P(1)$ [respectively, $o_P(1)$] as $n \to \infty$ *if and only if* $q \in \mathbb{Q}$ is EFKP (respectively COR). Also [see CsCsHM (1986), page 72] these results remain valid with $\tilde{\beta}_n$ replacing α_n.

We shall repeatedly make use of the fact that if $q \in \mathbb{Q}$ is an EFKP function, then

$$q(s)/s^{1/2} \to \infty \quad \text{and} \quad q(1-s)/s^{1/2} \to \infty \quad \text{as } s \downarrow 0. \tag{2.42}$$

PROOF OF THEOREM 2A. Throughout the proof of Theorem 2A we shall assume that we are on the probability space of (2.12). Moreover, by enlarging the probability space if necessary, we can and do assume that each Brownian bridge B_n is of the form

$$B_n(s) = W_n(s) - sW_n(1), \qquad 0 \le s \le 1,$$

where now W_n is a standard Wiener process extended to $(-\infty, \infty)$.

First, we record the following property that this probability space possesses.

LEMMA 5. *On the probability space of* (2.12) *for all* $0 < \nu_1 < \frac{1}{2}$,

$$\sup_{0<s<1} n^{\nu_1}|\alpha_n(s) - B_n(s)|/(s(1-s))^{1/2-\nu_1} = O_P(1) \tag{2.43}$$

and, for all $0 < \nu_2 < \frac{1}{4}$,

$$\sup_{1/(n+1) \le s \le n/(n+1)} n^{\nu_2} |\beta_n(s) + B_n(s)| / (s(1-s))^{1/2-\nu_2} = O_P(1). \tag{2.44}$$

PROOF. Assertion (2.43), with $B_n(s)$ replaced by

$$\overline{B}_n(s) = B_n(s) \mathbf{1}_{\{1/n \le s \le (n-1)/n\}},$$

and assertion (2.44) are proven in Mason and van Zwet (1987). An application of Inequality 4 of Shorack and Wellner [(1986), page 873] gives for all $\lambda > 0$,

$$P\Big(\sup_{0 \le s \le 1/n} n^{\nu_1} |B_n(s)| s^{1/2-\nu_1} \ge \lambda \Big) \le n^{2\nu_1} \lambda^{-2} c_{\nu_1} \int_0^{1/n} u^{-1' + 2\nu_1} \, du,$$

for some $0 < C_{\nu_1} < \infty$. Since the same inequality holds with $B(1-s)$ replacing $B(s)$, this completes the proof of the lemma. □

The proof of Theorem 2A will be effected in the following steps, in which we realize successive approximations of the process $R_n(s)$. First, we assume that $q \in \mathbb{Q}$ is EFKP. Notice that, for $i = 1, \dots, n$, with probability 1,

$$R_n(s) = \alpha_n(s) - \alpha_n\big(s + n^{-1/2}\beta_n(s)\big) + n^{1/2}((i/n) - s) \quad \text{for } (i-1)/n < s \le i/n. \tag{2.45}$$

It follows from (2.42) and (2.45) after routine manipulations that

$$\lim_{n \to \infty} n^{1/4} \sup_{1/(n+1) \le s \le n/(n+1)} \frac{|R_n(s) - \big(\alpha_n(s) - \alpha_n(s + n^{-1/2}\beta_n(s))\big)|}{\{q(s) \log^+(n^{1/2}/q(s))\}^{1/2}} = 0 \quad \text{a.s.} \tag{2.46}$$

By (2.46) we see that in the statement of (1.24) we may replace $R_n(s)$ by $\alpha_n(s) - \alpha_n(s + n^{-1/2}\beta_n(s)) = \alpha_n(s) - \alpha_n(H_n(s))$. As a first step, we will show that we may also replace α_n successively by B_n and W_n, then β_n by $-B_n$, in order to reduce (1.24) for an EFKP function $q \in \mathbb{Q}$ to the statement

$$\left| n^{1/4} \sup_{1/(n+1) \le s \le n/(n+1)} \frac{|W_n(s) - W_n(s - n^{-1/2}B_n(s))|}{\{2q(s) \log^+(n^{1/2}/q(s))\}^{1/2}} - \sup_{1/(n+1) \le s \le n/(n+1)} \left| \frac{B_n(s)}{q(s)} \right|^{1/2} \right| = o_P(1). \tag{2.47}$$

In a second step, we investigate the validity of (2.47) using Proposition 4 and the remark that almost sure convergence implies convergence in probability.

The following sequence of lemmas is directed to achieving the first step of our proof.

LEMMA 6. *Whenever $q \in \mathbb{Q}$ is EFKP, on the probability space of* (2.12)

$$n^{1/4} \sup_{1/(n+1) \le s \le n/(n+1)} \frac{|(\alpha_n(s) - \alpha_n(H_n(s)) - (W_n(s) - W_n(s)))|}{\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = o_P(1). \tag{2.48}$$

PROOF. Choose in (2.43) any $\frac{1}{4} \le \nu_1 < \frac{1}{2}$. For an arbitrary but fixed $1 < \rho < \infty$, we have, uniformly in $1/\rho \le \lambda \le \rho$ and $0 \le s \le \frac{1}{2}$ with $\lambda s \le 1$,

$$n^{1/4} \frac{|\alpha_n(\lambda s) - B_n(\lambda s)|}{\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = \left(\frac{(ns)^{1/4-\nu_1}}{\{q(s)s^{-1/2}\log^+(n^{1/2}/q(s))\}^{1/2}} \right) O_P(1).$$

Routine computations jointly with an application of (2.42) show that the expression above is $o_P(1)$ uniformly over $1/(n+1) \le s \le 1/2$.

Next we use the fact [cf. Wellner (1978)] that

$$\lim_{\rho \uparrow \infty} \liminf_{n \to \infty} P(1/\rho \le \lambda = H_n(s)/s \le \rho \colon 1/(n+1) \le s \le 1) = 1. \tag{2.49}$$

This, jointly with a similar argument for $1/2 \le s \le n/(n+1)$, implies that

$$n^{1/4} \sup_{1/(n+1) \le s \le n/(n+1)} \frac{|\alpha_n(s) - \alpha_n(H_n(s)) - (B_n(s) - B_n(H_n(s)))|}{\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = o_P(1). \tag{2.50}$$

In view of (2.50), the conclusion in (2.48) follows by writing $B_n(s) = W_n(s) - sW_n(1)$ and by noting, by (2.42) and (1.10), that

$$\begin{aligned}
&n^{1/4} \sup_{1/(n+1) \le s \le n/(n+1)} \frac{|H_n(s) - s|\,|W_n(1)|}{\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} \\
&\quad = |W_n(1)| \sup_{1/(n+1) \le s \le n/(n+1)} \frac{|\beta_n(s)|(ns(1-s))^{-1/4}}{\{q(s)(s(1-s))^{-1/2}\log^+(n^{1/2}/q(s))\}^{1/2}} \\
&\quad \le 2^{1/2}|W_n(1)|\,\|\beta_n\| \sup_{1/(n+1) \le s \le n/(n+1)} \{q(s)(s(1-s))^{-1/2} \\
&\qquad\qquad \times \log^+(n^{1/2}/q(s))\}^{-1/2} = o_P(1)
\end{aligned}$$

as desired. □

LEMMA 7. *Let $q \in \mathbb{Q}$ be EFKP. Choose any $0 < \nu < \frac{1}{2}$ and $K > 0$, and set $\psi_n(s) = n^{-\nu-1/2}(s(1-s))^{1/2-\nu}$. We have*

$$n^{1/4} \sup_{1/(n+1) \le s \le n/(n+1)} \sup_{\substack{|x-y| \le K\psi_n(s) \\ -1 \le x, y \le 1}} \frac{|W_n(s+x) - W_n(s+y)|}{\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = o_P(1). \tag{2.51}$$

PROOF. Recall [cf. Taylor (1974), page 199] that if W is a standard Wiener process extended to $(-\infty, \infty)$, the Lévy modulus of continuity theorem implies that, for any $-\infty < a < b < \infty$ and for any fixed $\varepsilon > 0$, there exists with probability 1 an $h_\varepsilon = h_{\varepsilon, a, b}$ such that $|u - v| \le h_\varepsilon$ and $a \le u, v \le b$ imply

$$\frac{|W(u) - W(v)|}{\{2|u - v|\log^+(1/|u - v|)\}^{1/2}} \le 1 + \varepsilon. \tag{2.52}$$

Hence, for all n large enough,

$$n^{1/4} \sup_{1/(n+1) \le s \le n/(n+1)} \sup_{\substack{|x-y| \le K\psi_n(s) \\ -1 \le x, y \le 1}} \frac{|W(s+x) - W(s+y)|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}}$$

$$\le K^{1/2}(1+\varepsilon) \sup_{1/(n+1) \le s \le n/(n+1)} (ns(1-s))^{-\nu/2} \left(\frac{q(s)}{(s(1-s))^{1/2}} \right)^{-1/2}$$

$$\times \left(\frac{\log^+\left(n^{1/2+\nu}/\left(K(s(1-s))^{1/2-\nu}\right)\right)}{\log^+(n^{1/2}/q(s))} \right)^{1/2}.$$

By splitting the interval $[1/(n+1), n/(n+1)]$ into $[1/(n+1), \delta]$, $[\delta, 1-\delta]$ and $[1-\delta, n/(n+1)]$, where $\delta > 0$ can be chosen arbitrarily small and using (2.42), routine arguments show that the expression above, maximized on each of these intervals, can be made arbitrarily small as n becomes large. This in turn suffices for (2.51) since $W_n =_d W$. □

In the sequel, we will repeatedly make use of this argument, which will be referred to as the *splitting technique*.

LEMMA 8. *Whenever $q \in \mathbb{Q}$ is EFKP, on the probability space of* (2.12)

$$n^{1/4} \sup_{1/(n+1) \le s \le n/(n+1)} \frac{|W_n(s + n^{-1/2}\beta_n(s)) - W_n(s - n^{-1/2}B_n(s))|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = o_P(1). \tag{2.53}$$

PROOF. Let $x_n(s) = n^{-1/2}\beta_n(s)$ and $y_n(s) = -n^{-1/2}B_n(s)$, and let $\psi_n(s)$ be as in Lemma 7. Taking $\nu_2 = \nu \in (0, \frac{1}{4})$ in (2.44), we see that

$$\lim_{K\uparrow\infty} \liminf_{n\to\infty} P(|x_n(s) - y_n(s)| \le K\psi_n(s) \text{ and } -1 \le x_n(s), y_n(s) \le 1$$
$$\text{for all } 1/(n+1) \le s \le n/(n+1)) = 1.$$

This, together with (2.51), suffices for (2.53). □

Lemmas 6 and 8 jointly imply that, whenever $q \in \mathbb{Q}$ is EFKP,

$$n^{1/4} \sup_{1/(n+1) \le s \le n/(n+1)} \frac{|(\alpha_n(s) - \alpha_n(H_n(s))) - (W_n(s) - W_n(s - n^{-1/2}B_n(s)))|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} \tag{2.54}$$
$$= o_P(1).$$

Note that $W_n =_d W$ and $B_n =_d B$, where $B(s) = W(s) - sW(1)$ and W is a standard Wiener process extended to $(-\infty, \infty)$. By (2.41) and the comment following Proposition 4, we obtain, for any EFKP function $q \in \mathbb{Q}$, that almost surely,

$$\lim_{n\to\infty} n^{1/4} \sup_{0<s<1} \frac{|W(s) - W(s - n^{-1/2}B(s))|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = \sup_{0<s<1} \left|\frac{B(s)}{q(s)}\right|^{1/2} < \infty.$$

Using the fact that almost sure convergence implies convergence in probability, we see that this, in turn, implies that for any EFKP function $q \in \mathbb{Q}$,

$$\left| n^{1/4} \sup_{0<s<1} \frac{|W_n(s) - W_n(s - n^{-1/2}B_n(s))|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} - \sup_{0<s<1} \left|\frac{B_n(s)}{q(s)}\right|^{1/2} \right| \tag{2.55}$$
$$= o_P(1).$$

A similar argument based on Lemma 3 shows that, under the same assumptions, we have, for any fixed $0 < \eta < \frac{1}{2}$,

$$\left| n^{1/4} \sup_{\eta<s\le 1-\eta} \frac{|W_n(s) - W_n(s - n^{-1/2}B_n(s))|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} - \sup_{\eta<s\le 1-\eta} \left|\frac{B_n(s)}{q(s)}\right|^{1/2} \right| \tag{2.56}$$
$$= o_P(1).$$

Next, obviously we have that whenever $q \in \mathbb{Q}$ is EFKP with probability 1,

$$\left| \sup_{0<s<1} \left|\frac{B(s)}{q(s)}\right|^{1/2} - \sup_{\eta<s\le 1-\eta} \left|\frac{B(s)}{q(s)}\right|^{1/2} \right| = o(1) \quad \text{as } \eta \downarrow 0. \tag{2.57}$$

By taking $0 < \eta < \frac{1}{2}$ arbitrarily small in (2.56) and (2.57), on account of (2.54) and (2.55), it follows easily that for any EFKP function $q \in \mathbb{Q}$,

$$\left| n^{1/4} \sup_{1/(n+1)\le s\le n/(n+1)} \frac{|\alpha_n(s) - \alpha_n(H_n(s))|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} \right. \tag{2.58}$$
$$\left. - \sup_{1/(n+1)\le s\le n/(n+1)} \left|\frac{B_n(s)}{q(s)}\right|^{1/2} \right| = o_P(1).$$

By all this, we see that in order to complete the proof of the first half of Theorem 2A it suffices to show that

$$(2.59)\qquad \left| \sup_{1/(n+1)\le s\le n/(n+1)} \left|\frac{B_n(s)}{q(s)}\right| - \sup_{1/(n+1)\le s\le n/(n+1)} \left|\frac{\beta_n(s)}{q(s)}\right| \right| = o_P(1).$$

Take $\nu_2 = \nu \in (0, \frac{1}{4})$ in (2.44) to obtain, uniformly in $1/(n+1) \le sn/(n+1)$,

$$\frac{|\beta_n(s) + B_n(s)|}{q(s)} \le n^{-\nu}\left(\sup_{1/(n+1)\le t\le n/(n+1)} n^{\nu}\frac{|\beta_n(t) + B_n(t)|}{(t(1-t))^{1/2-\nu}} \right) \times (s(1-s))^{-\nu}\left(\frac{(s(1-s))^{1/2}}{q(s)} \right)$$
$$= O_P(1)(ns(1-s))^{-\nu}\left(\frac{(s(1-s))^{1/2}}{q(s)} \right).$$

A simple analysis of this last term by the splitting technique used in the proof of Lemma 7, in combination with (2.44), shows that (2.59) holds.

This completes the proof of (1.24), assuming that $q \in \mathbb{Q}$ is EFKP.

Recall that $q \in \mathbb{Q}$ is EFKP if and only if, independently of $n \ge 1$, $\sup_{0<s<1}|B_n(s)/q(s)| < \infty$ a.s., which, in view of (1.24) and (2.57)–(2.59), implies that if q is EFKP, then

$$(2.60)\qquad n^{1/4}\left\| \frac{\tilde{R}_n}{\{2q\log^+(n^{1/2}/q)\}^{1/2}} \right\| = O_P(1) \quad \text{as } n \to \infty.$$

In order to finish the proof of Theorem 2A, we need only show that (2.60) holds if and only if $q \in \mathbb{Q}$ is EFKP, namely by showing that if $q \in \mathbb{Q}$ is not EFKP, then

$$(2.61)\qquad n^{1/4}\left\| \frac{\tilde{R}_n}{\{2q\log^+(n^{1/2}/q)\}^{1/2}} \right\| \to_P \infty \quad \text{as } n \to \infty.$$

Assume, from now on, that $q \in \mathbb{Q}$ is not EFKP. For any $0 < \eta < \frac{1}{2}$, let q_η be defined by $q_\eta(s) = q(s)$ for $\eta \le s \le 1-\eta$, $q_\eta(s) = q(\eta)$ for $0 \le s \le \eta$ and $q_\eta(s) = q(1-\eta)$ for $1-\eta \le s \le 1$. It is straightforward that $q_\eta \in \mathbb{Q}$ and is EFKP. Therefore, by (2.54) and (2.56), we have as $n \to \infty$,

$$n^{1/4}\left\| \frac{\tilde{R}_n}{\{2q\log^+(n^{1/2}/q)\}^{1/2}} \right\| \ge n^{1/4} \sup_{\eta<s\le 1-\eta} \frac{|\tilde{R}_n(s)|}{\{2q_\eta(s)\log^+(n^{1/2}/q_\eta(s))\}^{1/2}}$$
$$= \sup_{\eta<s\le 1-\eta}\left|\frac{B_n(s)}{q(s)}\right|^{1/2} + o_P(1).$$

The conclusion follows from the fact that, $q \in \mathbb{Q}$ not being EFKP,

$$\sup_{\eta < s \le 1-\eta} \left| \frac{B_n(s)}{q(s)} \right| =_d \sup_{\eta < s \le 1-\eta} \left| \frac{B(s)}{q(s)} \right| \to \sup_{0 < s < 1} \left| \frac{B(s)}{q(s)} \right| = \infty \quad \text{a.s. as } \eta \downarrow 0.$$

Thus, we have (2.61).

The proof of Theorem 2A is now complete. □

PROOF OF COROLLARY 2A. Note that if $q \in \mathbb{Q}$ is COR, then for any Brownian bridge B,

$$\lim_{s \downarrow 0} B(s)/q(s) = \lim_{s \downarrow 0} B(1-s)/q(1-s) = 0 \quad \text{a.s.}$$

It follows that there exists almost surely a $0 < \Lambda < \frac{1}{2}$ such that

$$\max\left(\sup_{0 < s \le \Lambda} \left| \frac{B(s)}{q(s)} \right|, \sup_{0 < s \le \Lambda} \left| \frac{B(1-s)}{q(1-s)} \right| \right) \le \frac{1}{16} \sup_{0 < s < 1} \left| \frac{B(s)}{q(s)} \right|$$

and for any $\varepsilon > 0$, there exists a $0 < \eta = \eta_\varepsilon < \frac{1}{2}$ such that $P(\Lambda < \eta_\varepsilon) < \varepsilon/2$. Next, by (2.54), (2.56) and the arguments used in the proof of Theorem 2A, we have as $n \to \infty$,

$$n^{1/4} \sup_{\eta < s \le 1-\eta} \frac{|\tilde{R}_n(s)|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = \sup_{\eta < s \le 1-\eta} \left| \frac{B_n(s)}{q(s)} \right|^{1/2} + o_P(1),$$

$$n^{1/4} \sup_{0 < s \le \eta} \frac{|\tilde{R}_n(s)|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = \sup_{0 < s \le \eta} \left| \frac{B_n(s)}{q(s)} \right|^{1/2} + o_P(1)$$

and

$$n^{1/4} \sup_{1-\eta < s < 1} \frac{|\tilde{R}_n(s)|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = \sup_{1-\eta < s < 1} \left| \frac{B_n(s)}{q(s)} \right|^{1/2} + o_P(1).$$

Recall that if $q \in \mathbb{Q}$, then q is bounded above by a finite constant. Also, (2.42) implies for large n that $\inf_{1/(n+1) \le s \le n/(n+1)} q(s) \ge n^{-1/2}$, so that

$$1 \le \liminf_{n \to \infty} \inf_{1/(n+1) \le s \le n/(n+1)} \left(2\log^+(n^{1/2}/q(s))/\log n\right)$$

$$\le \limsup_{n \to \infty} \sup_{1/(n+1) \le s \le n/(n+1)} \left(2\log^+(n^{1/2}/q(s))/\log n\right) \le 2.$$

In addition, we have uniformly over all $\eta \le s \le 1-\eta$,

$$\lim_{n \to \infty} \left(2\log^+(n^{1/2}/q(s))\right)/\log n = 1.$$

It follows that, as $n \to \infty$,

$$n^{1/4} \sup_{\eta < s \le 1-\eta} \frac{|\tilde{R}_n(s)|}{(q(s)\log n)^{1/2}} = \sup_{\eta < s \le 1-\eta} \left| \frac{B_n(s)}{q(s)} \right|^{1/2} + o_P(1),$$

$$n^{1/4} \sup_{0 < s \le \eta} \frac{|\tilde{R}_n(s)|}{(q(s)\log n)^{1/2}} \le 2^{1/2} \sup_{0 < s \le \eta} \left| \frac{B_n(s)}{q(s)} \right|^{1/2} + o_P(1)$$

and

$$n^{1/4} \sup_{1-\eta < s < 1} \frac{|\tilde{R}_n(s)|}{(q(s)\log n)^{1/2}} \le 2^{1/2} \sup_{1-\eta < s < 1} \left| \frac{B_n(s)}{q(s)} \right|^{1/2} + o_P(1).$$

This combined with our choice of $\eta = \eta_\varepsilon$ ensures that, for all large n with probability greater than $1 - \varepsilon$,

$$n^{1/4} \sup_{0 < s \le \eta} \frac{|\tilde{R}_n(s)|}{(q(s)\log n)^{1/2}} \le \frac{1}{2} \sup_{0 < s < 1} \left| \frac{B_n(s)}{q(s)} \right|^{1/2} + \varepsilon,$$

$$n^{1/4} \sup_{1-\eta < s < 1} \frac{|\tilde{R}_n(s)|}{(q(s)\log n)^{1/2}} \le \frac{1}{2} \sup_{0 < s < 1} \left| \frac{B_n(s)}{q(s)} \right|^{1/2} + \varepsilon$$

and

$$\left| n^{1/4} \sup_{\eta < s \le 1-\eta} \frac{|\tilde{R}_n(s)|}{(q(s)\log n)^{1/2}} - \sup_{0 < s < 1} \left| \frac{B_n(s)}{q(s)} \right|^{1/2} \right| \le \varepsilon.$$

It follows evidently by our arbitrary choice of $\varepsilon > 0$ that

$$n^{1/4} \sup_{1/(n+1) \le s \le n/(n+1)} \frac{|\tilde{R}_n(s)|}{(q(s)\log n)^{1/2}} = \sup_{1/(n+1) \le s \le n/(n+1)} \left| \frac{B_n(s)}{q(s)} \right| + o_P(1).$$

This, jointly with (2.59) completes the proof of (1.28). □

PROOF OF THEOREM 2B. The proof of Theorem 2B follows along the same lines as that of Theorem 2A, with Lemmas 9 and 10 replacing, respectively, (2.46) and Lemma 5. We omit the routine details and limit ourselves to the proof of these two lemmas. A similar argument holds for Corollary 2B.

LEMMA 9. *Under the assumptions of Theorem* 2B, *whenever* $q \in \mathbb{Q}_0$ *is* $EFKP_0$,

$$(2.62) \qquad n^{1/4} \sup_{1/n \le s \le 1} \frac{\left| T_n(s) - \left(s_n(s) - s_n\left(s + \sigma\mu^{-1} n^{-1/2} r_n(s)\right)\right)\right|}{\left\{q(s)\log^+\left(n^{1/2}/q(s)\right)\right\}^{1/2}} \to 0$$

a.s. as $n \to \infty$.

PROOF. Observe that, for any $t \geq 0$, $S_{N(t)} \leq t < S_{N(t)+1}$. Moreover, we have

$$r_n(t) + s_n\big(t + \sigma\mu^{-1}n^{-1/2}r_n(t)\big) = \sigma^{-1}n^{-1/2}\big(S_{N(n\mu t)} - n\mu t\big) \leq 0.$$

Thus since $S_{N(n\mu t)+1} > n\mu t$, we have for all $s > 0$ and $n \geq 1$,

$$\begin{aligned}(2.63)\qquad &\sup_{0 \leq t \leq s} |r_n(t) + s_n\big(t + \sigma\mu^{-1}n^{-1/2}r_n(t)\big)| \\ &\leq n^{-1/2}\sigma^{-1} \max_{1 \leq i \leq N(n\mu s)+1} |X_i|.\end{aligned}$$

The law of large numbers implies that almost surely $z^{-1}N(\mu z) \to 1$ as $z \to \infty$, from which it follows that

$$(2.64)\qquad \sup_{n \geq 1}\left(\sup_{s \geq 1/n} (ns)^{-1}(N(n\mu s) + 1)\right) =: K < \infty \quad \text{a.s.}$$

Observe that $E(X_1^4) < \infty$ implies that $\sum_n P(n^{-1/4}|X_n| > \varepsilon) < \infty$ for every $\varepsilon > 0$ and so $n^{-1/4}X_n \to 0$ a.s. as $n \to \infty$. Therefore we have

$$(2.65)\qquad \sup_{m \geq 1}\left(m^{-1/4} \max_{1 \leq i \leq m} |X_i|\right) =: L < \infty \quad \text{a.s.}$$

A similar argument shows that condition (iv) in Remark 6 implies (2.22).

By (2.63), (2.64) and (2.65), we have for all $n \geq 1$ and $s \geq 1/n$,

$$n^{1/4}|r_n(s) + s_n\big(s + \sigma\mu^{-1}n^{-1/2}r_n(s)\big)| \leq LK^{1/4}\sigma^{-1}s^{1/4}.$$

It follows that, for any EFKP_0 function $q \in \mathbb{Q}_0$, we have almost surely as $n \to \infty$,

$$\begin{aligned}(2.66)\qquad &n^{1/4} \sup_{1/n \leq s \leq 1} \frac{|T_n(s) - \big(s_n(s) - s_n(n^{-1}N(n\mu s)\big)|}{\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} \\ &= O\left\{\sup_{1/n \leq s \leq 1}\left(\frac{q(s)\log^+(n^{1/2}/q(s))}{s^{1/2}}\right)^{-1/2}\right\},\end{aligned}$$

which by (2.42) is $o(1)$. This completes the proof of Lemma 9. □

LEMMA 10. *Under assumptions* (i), (ii) *and* (iii) *on the probability space of* (1.4), *for all* $0 < \nu_1 \leq 1/4$,

$$(2.67)\qquad \sup_{0 < s < 1} n^{\nu_1}|s_n(s) - W_n(s)|/s^{1/2-\nu_1} = O(1) \quad a.s.\ as\ n \to \infty,$$

and for all $0 < \nu_2 < \frac{1}{4}$,

$$(2.68)\qquad \sup_{1/n \leq s \leq 1} n^{\nu_2}|r_n(s) + W_n(s)|/s^{1/2-\nu_2} = O(1) \quad a.s.\ as\ n \to \infty,$$

where $W_n(s) = n^{-1/2}W(ns)$.

PROOF. Notice that on the probability space of (1.4) as $T \to \infty$,

$$\sigma^{-1}(S_{[T]} - T\mu) = W(T) + o(T^{1/4}) \quad \text{a.s.}$$

and, by (1.5) as $T \to \infty$,

$$\sigma^{-1}(\mu N(\mu T) - T\mu) = -W(T) + O\big(T^{1/4}(\log T)^{1/2}(\log\log T)^{1/4}\big) \quad \text{a.s.}$$

Thus, in view of Theorem 1.3.3 in Csörgő and Révész [(1981), page 40], for all $0 < \nu_1 < \frac{1}{4}$,

$$\sup_{T \ge 0} |\sigma^{-1}(S_{[T]} - T\mu) - W(T)|^{-1/2+\nu_1} < \infty \quad \text{a.s.} \tag{2.69}$$

and, for all $0 < \nu_2 < \frac{1}{4}$ and $c > 0$,

$$\sup_{T \ge c} |\sigma^{-1}(\mu N(\mu T) - T\mu) + W(T)| T^{-1/2+\nu_2} < \infty \quad \text{a.s.} \tag{2.70}$$

It is noteworthy that (2.70) is invalid for $\nu_2 = \frac{1}{4}$ [by (1.5)] and for $c = 0$ [since $N(0)$ may differ from zero]. Assertions (2.67) and (2.68) now follow from (2.69) and (2.70) by a change of variables. □

REMARK 7. It is clear that the proof of Theorem 2B may be adapted to show that versions of Theorems 2B and Corollary 2B hold almost surely. However, this requires much more than minor modifications of our present proofs. Therefore we limit ourselves at present to in probability statements. To obtain almost sure versions of Theorem 2A and Corollary 2A is a much harder problem.

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