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SYMMETRIES AND FUNCTIONS OF MARKOV PROCESSES

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From the symmetries contained in the collection of excessive functions of a transient Hunt process $X(t)$ on a state space E, we construct a quotient space F, a function Φ: $E \to F$ and a time change $\tau(t)$ of $X(t)$ so that $\Phi(X(\tau_t))$ is a strong Markov process.

1. Introduction. The original motivation of this article was our curiosity about which functions of a Markov process yield a Markov process. If X_t is Markov and Φ is an injective map, then $\Phi(X_t)$ is Markov. This is usually not true if Φ is not one-to-one; only special choices of Φ will again yield a Markov process. Many authors have studied this general problem: See [1], [4], [8] and [9], for example. The special case of Brownian motion has been studied by several people: See [3], [11] and [13].

In fact, if X_t has semigroup P_t, and if Φ is a function mapping E into another state space F, then $\Phi(X_t)$ is a Markov process (relative to the appropriately completed filtration generated by X_t) if and only if: For every bounded continuous function h: $F \to \mathbb{R}$, and for each $t > 0$, there is a function g: $F \to \mathbb{R}$ such that $P_t(h \circ \Phi)(x) = g \circ \Phi(x)$ for every x in E. For example, if X_t is a rotationally symmetric Lévy process in $\mathbb{R}^d$ (Brownian motion B_t being a special case), then $\Phi(X_t) = |X_t|$ is a Markov process on $[0, \infty)$; in particular, $|B_t|$ is a d-dimensional Bessel process.

Whence come these functions Φ? Our goal is to find some systematic way to generate "Markov functions" Φ from basic properties of the Markov process X_t. Clearly, Φ need not be injective if it exploits symmetries of X_t, as the above example illustrates. Both geometrical symmetries (involving where the process travels) and temporal symmetries (involving the speed at which it travels) are important. To illustrate, consider a strictly increasing continuous additive functional A_t of B_t having continuous inverse T_t. The time-changed Brownian motion $B(T_t)$ is again a Markov process, but the Chacon–Jamison theorem (see [12]) tells us that $|B(T_t)|$ will be a Markov process only for very special choices of A_t; these choices must guarantee that two paths of $B(T_t)$ having the same modulus $|B(T_t)|$ will run at the same speed. Since Φ is a function of the spatial variable only, we cannot correct this problem simply by modifying Φ. In the following, we shall seek both a function Φ and a proper time scale. Our procedure will be to identify (by considering symmetries) a candidate function Φ, and then to construct an appropriate time scale to go with it.

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The quest for such Markov functions Φ thus leads to a study of the symmetries of X_t. We shall show that these symmetries can be found in the potential theory of the process, and that Markov functions Φ can be constructed from these symmetries. As we noted above, X_t may not be running in the proper time scale for $\Phi(X_t)$ to be Markov. The potential theory will yield a time scale T_t so that $\Phi(X(T_t))$ is a Markov process. This is a curious refinement of the conventional wisdom that the potential theory determines the geometric trajectories of the process, but not the speed at which the process runs. In fact, the potential theory expresses definite preferences for speeds which keep the process "as symmetric as possible."

We shall now summarize our hypotheses and the main theorem. We present it in condensed form here; however, beginning in Section 2, we reintroduce all constructions and hypotheses in a leisurely manner when needed so the reader can see the logical development of the theory. Let X be a transient Hunt process on an LCCB state space $(E, \mathscr{B}(E))$ with lifetime ζ and with $\mathscr{S}$ denoting its collection of excessive functions. Let $\overline{G}$ be the collection of bijections φ from E to E with φ and φ^{-1} measurable. For φ in $\overline{G}$, define $\mathscr{S}_\varphi = \{f \circ \varphi\colon f \in \mathscr{S}\}$. Let $G = \{\varphi \in \overline{G}\colon \mathscr{S}_\varphi = \mathscr{S}\}$: G is a group under composition. Fix a subgroup H of G, and use H to define F and Φ as follows. Say that x and y in E are equivalent if there is a φ in H with $\varphi(x) = y$. The collection of equivalence classes, also called H-orbits, we call F. Let $\Phi\colon E \to F$ be the map assigning to $x \in E$ the equivalence class $[x] \in F$ containing it, and endow F with the quotient topology induced by Φ. (See Section 3.) We assume that F is LCCB, and we fix a metric d on F compatible with its topology. Define $Y_t = \Phi(X_t)$.

(1.1) THEOREM. *(i) F is LCCB.*

(ii) *Y_t has no holding points.*

(iii) *Y_t is transient in the following sense: If $L^r(z)$ is the last time Y_t visits the ball of radius r about z, then $L^r(z) < \zeta$ almost surely for every z in F and $r < 1$.*

Then there is a strictly increasing continuous additive functional A_t of X_t with continuous inverse γ_t so that $Y(\gamma_t)$ is a strong Markov process. In particular, $(Y(\gamma_t), P^x)$ and $(Y(\gamma_t), P^{\varphi(x)})$ are identical in law whenever $\varphi \in H$.

To prove this theorem, we show in Sections 4 and 5 that Y_t has the same geometric trajectories under P^x and $P^{\varphi(x)}$ whenever $\varphi \in H$. We rely on Hunt's balayage theorem and a theorem of Walsh. We construct A_t in Section 6. In Section 7, we discuss the case where Y_t may not be transient. While one may hope that the analog of Theorem (1.1) can be obtained in this case by the usual "piecing together" arguments, we have not been able to do this without additional hypotheses.

2. The base process *X*. For our state space, we take E to be locally compact with a countable base (LCCB), and we shall denote its Borel field by $\mathscr{B}(E)$. Adjoint a cemetery point Δ to E as the point at ∞ if E is noncompact and as an isolated point if E is compact. We denote the extended space and Borel field by E_Δ and $\mathscr{B}(E_\Delta)$.

Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a Hunt process on $(E, \mathscr{B}(E))$ which is defined in the classical manner as discussed in I-9 of [2]. So Ω is the collection of right-continuous paths ω: $[0,\infty) \to E_\Delta$ having left limits in E_Δ and Δ as cemetery; $X_t(\omega) = \omega(t)$; $\mathscr{F}$ and $\mathscr{F}_t$ are the appropriate completions of the σ-algebras $\mathscr{F}^0 = \sigma\{X_t\colon\ t \geq 0\}$ and $\mathscr{F}_t^0 = \sigma\{X_s\colon\ s \leq t\}$; and θ_t is the shift operator on Ω characterized by $X_s(\theta_t\omega) = X_{s+t}(\omega)$. Because we are assuming X is a Hunt process, X_t is quasi-left-continuous on $[0,\infty)$. That is, $\lim_{n\to\infty} X(T_n) = X(T)$ almost surely on $\{T < \infty\}$ for any sequence (T_n) of $(\mathscr{F}_t)$-stopping times increasing to T.

Let P_t and U^α denote the semigroup and resolvent of X: these are defined by

$$P_t f(x) = P^x[f(X_t)],$$

$$U^\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x)\, dt.$$

As is usual, U^0 will be denoted U. Transience of the process will play an important role throughout this article: We assume there is a strictly positive measurable function Λ on E so that $U\Lambda(x) < \infty$ for every x in E. This hypothesis guarantees that the cone $\mathscr{S}$ of excessive functions is rich enough to be of use. Recall that a positive universally measurable function f on E is in $\mathscr{S}$ provided $P_t f \leq f$ for every $t > 0$ and $\lim_{t\to 0} P_t f = f$.

3. Symmetries in $\mathscr{S}$. Let $\overline{G}$ be the collection of bijective maps φ: $E \to E$ so that both φ and φ^{-1} are $\mathscr{B}(E)$-measurable. For each φ in $\overline{G}$, define $\mathscr{S}_\varphi = \{f \circ \varphi\colon\ f \in \mathscr{S}\}$. Our interest centers on the set $G = \{\varphi \in \overline{G}\colon \mathscr{S}_\varphi = \mathscr{S}\}$. It is simple to check that G is a group under composition.

(3.1) EXAMPLE. If X_t is Brownian motion in $\mathbb{R}^d$, then G contains the collection of rigid transformations of $\mathbb{R}^d$, often called the Euclidean group of translations, rotations and reflections. G also contains the dilations about the origin $(x \to \alpha x)$.

(3.2) EXAMPLE. If $E = \mathbb{R}^d$, and if $P_t(x, \cdot\,) = e^{-t}\varepsilon_x(\cdot)$, then every positive measurable function is excessive, and G consists of all measurable bijections from E to E.

Let H be a subgroup of G, and use it to define an equivalence relation $\sim$ on E as follows.

(3.3) DEFINITION. $x \sim y$ if and only if there is an element φ in H with $\varphi(x) = y$.

This equivalence relation partitions E into equivalence classes. In the theory of topological transformation groups, an equivalence class is called an

H-orbit. Thus, if we let $[x]$ denote the equivalence class containing the point x in E, we also refer to $[x]$ as the H-orbit of x. Let F denote the collection of H-orbits, and let Φ denote the canonical injection,

$$\Phi(x) = [x]. \tag{3.4}$$

We endow F with the quotient topology by declaring a set $A \subset F$ to be open if and only if $\Phi^{-1}(A)$ is open in E. Our first major hypothesis is the following.

(3.5) HYPOTHESIS. F is LCCB.

Hypothesis (3.5) can often be verified, but in general, F need not be Hausdorff, as one of the examples below shows.

(3.6) EXAMPLE. Let $E = \mathbb{R}^d$, $d \geq 3$, and let X be Brownian motion in E. If H is the subgroup of G which consists of rotations about 0, then F is homeomorphic to the half-line $[0, \infty)$. If H is the subgroup of G which consists of the identity map together with the flip transformation $(x_1, x_2, \dots, x_d) \to (-x_1, x_2, \dots, x_d)$, then F is homeomorphic to the half-space $\{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d: x_1 \geq 0\}$.

(3.7) EXAMPLE. Although we have not discussed any topology on the group G, it often comes with an associated topology in concrete situations. The group G is then said to be a topological group if the map $\varphi \to \varphi^{-1}$ is continuous on G and if the map $(\varphi, \psi) \to \varphi \circ \psi$ is continuous on $G \times G$. In this example, let G be a topological group, each element of which is a homeomorphism of E onto E so that $(\varphi, x) \to \varphi(x)$ is jointly continuous on $G \times E$. (Such a situation often arises in the case E is a manifold and X a diffusion on E.) If H is a *compact* subgroup of G, then F is LCCB [7].

(3.8) EXAMPLE. In this case, we do not assume G is a topological group, but we do assume that E is compact. If F is a Hausdorff space, then F is compact and metrizable since it is a continuous image of E. Unfortunately, F need not be Hausdorff, as the next example shows.

(3.9) EXAMPLE. Again let $E = \mathbb{R}^d$, $d \geq 3$, and let X be Brownian motion in E. For H, we take the subgroup of G consisting of dilations $x \to \alpha x$ with $\alpha > 0$. Note that the origin is fixed under each of these maps, and that each open ray $\{ax: a > 0\}$ constitutes an H-orbit. Therefore, as a *set*, F may be regarded as the union of the unit sphere and the origin in $\mathbb{R}^d$. But in the topology of F, every open set about the origin contains the whole unit sphere. Therefore, F is not Hausdorff. This difficulty can be easily dealt with in this example. Let $E = \mathbb{R}^d - \{0\}$, and let B be Brownian motion on E. (If $d \geq 3$, we can do this since X will not hit the point 0.) The collection of positive superharmonic functions is invariant under H, and in this case, F is the unit sphere and $\Phi(x) = x/|x|$.

This situation occurs in a number of examples: $E/\sim$ fails to be Hausdorff, but some minor modification of E (obtained by deleting a polar set or changing the original topology on E) leads to a Hausdorff quotient space.

4. Hunt's balayage theorem. Fix f in $\mathscr{S}$, and let $L \in \mathscr{B}(E)$. Hunt's balayage theorem states that the following equality holds except perhaps on a semipolar set:

$$P_L f(x) = \inf\{g(x) : g \in \mathscr{S}, g \geq f \text{ on } L\}. \tag{4.1}$$

Recall that $P_L f(x) = P^x[f(X(T_L)); T_L < \infty]$, where $T_L = \inf\{t > 0: X_t \in L\}$. Now fix $\varphi \in H$. We can rewrite the right side of (4.1) as

$$\inf\{g \circ \varphi^{-1}(\varphi(x)) : g \in \mathscr{S}, g \circ \varphi^{-1} \geq f \circ \varphi^{-1} \text{ on } \varphi(L)\}. \tag{4.2}$$

Because $\mathscr{S}_{\varphi^{-1}} = \mathscr{S}$, we can replace the condition "$g \in \mathscr{S}$" in (4.2) with "$g \circ \varphi^{-1} \in \mathscr{S}$," and (4.2) becomes

$$\inf\{k(\varphi(x)) : k \in \mathscr{S}, k \geq f \circ \varphi^{-1} \text{ on } \varphi(L)\}. \tag{4.3}$$

Applying Hunt's balayage theorem again, we recognize this last line as $P_{\varphi(L)}(f \circ \varphi^{-1})(\varphi(x))$. This function is excessive since $\mathscr{S}_\varphi = \mathscr{S}_{\varphi^{-1}} = \mathscr{S}$. Thus we have proved that the two excessive functions $P_L f(x)$ and $P_{\varphi(L)}(f \circ \varphi^{-1})(\varphi(x))$ are equal, except perhaps on a semipolar set; they must therefore be equal everywhere on E. This equality clearly extends to functions f of the form $f_1 - f_2$, where f_1 and f_2 are excessive functions, and from there to arbitrary Borel functions on E. We shall use the equality only in the following special case.

(4.4) PROPOSITION. *Let $h: F \to \mathbb{R}$ be $\mathscr{B}(F)$-measurable and let $L = \Phi^{-1}(M)$ for some $M \in \mathscr{B}(F)$. Then $P_L(h \circ \Phi)(x) = P_L(h \circ \Phi)(\varphi(x))$ for every φ in H.*

PROOF. If we let $f = h \circ \Phi$, then $f \circ \varphi^{-1} = f$. Also, $\varphi(\Phi^{-1}(M)) = \Phi^{-1}(M)$, so the proposition follows immediately from our discussion above. □

In other words, the function $P_L(h \circ \Phi)$ is constant on H-orbits. This naturally leads us to define a collection of operators Q_M on $(F, \mathscr{B}(F))$ by setting

$$Q_M h([x]) = P_L(h \circ \Phi)(x). \tag{4.5}$$

Here, $L = \Phi^{-1}(M)$, and because of (4.4), any representative x of $[x]$ can be used in the right side of (4.5). It is simple to check that $Q_N Q_M = Q_M$ if N and M are open sets in F with $M \subset N$. This is the basic property required for (Q_M) to be a family of hitting operators of a Markov process. With enough additional hypotheses (see [10], for example), one can construct a Markov process having these hitting operators. Our purpose is to use the process X to construct and identify concretely the process having these hitting operators.

5. Geometry of trajectories. Let $\bar{F}$ denote the one point compactification of F, and adjoin a cemetery point δ to $\bar{F}$ as an isolated point. Extend the definition of Φ by letting $\Phi(\Delta) = \delta$. If we define $Y_t = \Phi(X_t)$, then Y_t is a stochastic process taking values in $F_\delta = F \cup \{\delta\}$. Since Φ is continuous on E, Y_t is right-continuous with left limits on $[0, \zeta)$, and Y_t is quasi-left-continuous on $[0, \zeta)$. Our purpose in this section is to prove that Y_t run under P^x and Y_t run under $P^{\varphi(x)}$ have exactly the same geometric trajectories on the time interval $[0, \zeta)$, although the speeds at which these trajectories are traced out may differ. Our main tool in doing this is Proposition (3.2) in Walsh [12]. There, he assumes his processes have no intervals of constancy, and we make the same assumption. Recall $\zeta(\omega) = \inf\{t > 0: X_t(\omega) = \Delta\}$.

(5.1) HYPOTHESIS. Y_t is not constant on any open subinterval of $[0, \zeta)$ almost surely. (Equivalently, if $T^c_{[x]} = \inf\{t > 0: X_t \notin [x]\}$, then $T^c_{[x]} = 0$ almost surely.)

For the reader's ease, we recall Walsh's result now. Let W be the collection of paths $w: [0, \infty) \to F_\delta$ having δ as cemetery and left limits in $\overline{F} \cup \{\delta\}$ for every $t < \infty$. Let $\mathscr{W}$ be the σ-algebra on W induced by the coordinate functions. We say two elements w and w' of W are equivalent if there are positive increasing functions a and b so that $w = w' \circ a$ and $w' = w \circ b$. Let $\mathscr{T}$ be the σ-algebra of all sets $N \in \mathscr{W}$ so that if w and w' are equivalent, then $1_N(w) = 1_N(w')$. The atoms of $\mathscr{T}$ are equivalence classes, and each equivalence class is called a trajectory, while $\mathscr{T}$ itself is called the σ-algebra of spatial events. A path $w \in W$ is said to be nowhere constant if it is not constant on any open subinterval of $[0, \zeta(w))$. Let $W_0 = \{w \in W: w \text{ is nowhere constant}\}$.

Recall the metric d on F introduced above (1.1). We extend it to F_δ by setting $d(x, \delta) = 1$ for every $x \in F$. For $t \geq 0$, define random times on $(W, \mathscr{W})$ by setting

$$\sigma_t^0(w) = \inf\{s > 0: d(w(s), w(0)) > t\}.$$

If $n \geq 1$ is an integer, set $\tau^0_{n0} = 0$, $\tau^0_{n1} = \sigma^0_{2^{-n}}$ and

$$\tau^0_{nj+1} = \tau^0_{nj} + \sigma^0_{2^{-n}} \circ \theta_{\tau^0_{nj}}.$$

Walsh's result is the following.

(5.2) PROPOSITION. *Let P_1 and P_2 be two probability measures on $(W, \mathscr{W})$ such that $P_1(W_0) = P_2(W_0) = 1$. A necessary and sufficient condition for $P_1|_{\mathscr{T}} = P_2|_{\mathscr{T}}$ is that for each large enough n, for each K, and for each collection $M_1, M_2, \ldots, M_K$ of open subsets of $F \cup \{\delta\}$,*

$$P_1\big[w(\tau^0_{nk}) \in M_k,\ k \leq K\big] = P_2\big[w(\tau^0_{nk}) \in M_k,\ k \leq K\big]. \tag{5.3}$$

There is one slight difficulty to be overcome in applying this result here. The proposition is formulated for a canonical path space of right-continuous trajectories with left limits. As it stands, the process Y_t may not have a left limit in F_δ at ζ, so it cannot be transferred to the canonical space. We shall concentrate on proving (5.4) below, which is analogous to (5.3). At the end of this section, a time change argument will allow us to apply (5.2) to conclude that the restrictions of the geometrical trajectories of (Y_t, P^x) and $(Y_t, P^{\varphi(x)})$ to $[0, \zeta)$ are identical.

Define $(\mathscr{F}_t)$-stopping times on Ω as follows:

$$\sigma_t(\omega) = \inf\{s > 0: d(Y_s(\omega), Y_0(\omega)) > t\},$$

and if $n \geq 1$ is an integer, set $\tau_{n0} = 0$, $\tau_{n1} = \sigma_{2^{-n}}$ and

$$\tau_{nj+1} = \tau_{nj} + \sigma_{2^{-n}} \circ \theta_{\tau_{nj}}.$$

We shall prove that

$$P^x[Y(\tau_{nk}) \in M_k, k \leq K] = P^{\varphi(x)}[Y(\tau_{nk}) \in M_k, k \leq K], \tag{5.4}$$

whenever $M_1, M_2, \ldots, M_K$ are open subsets of F (not $F \cup \{\delta\}$). We prove this by induction on K. In the case $K = 1$, (5.4) reduces to showing that the function $P^x[Y(\sigma_{2^{-n}}) \in M]$ is constant on H-orbits. Note that

$$\begin{aligned}\sigma_{2^{-n}} &= \inf\{s > 0: d(\Phi(X_s), \Phi(X_0)) > 2^{-n}\} \\ &= \inf\{s > 0: d(\Phi(X_s), \Phi(x)) > 2^{-n}\} \qquad P^x \text{ a.s.} \\ &= \inf\{s > 0: X_s \in \Phi^{-1}(M_x)\} \qquad P^x \text{ a.s.},\end{aligned}$$

where $M_x = \{\Phi(y): d(\Phi(y), \Phi(x)) > 2^{-n}\}$. If $\varphi \in H$, then $M_x = M_{\varphi(x)}$. Therefore,

$$P^x[Y(\sigma_{2^{-n}}) \in M] = P_{\Phi^{-1}(M_x)}(1_M \circ \Phi)(x) = P_{\Phi^{-1}(M_{\varphi(x)})}(1_M \circ \Phi)(x).$$

By (4.4), this is

$$P_{\Phi^{-1}(M_{\varphi(x)})}(1_M \circ \Phi)(\varphi(x)) = P^{\varphi(x)}[Y(\sigma_{2^{-n}}) \in M],$$

and this finishes the case $K = 1$. Now we assume (5.4) is true when $K = N$. Then

$$\begin{aligned}&P^x[Y(\tau_{nk}) \in M_k, k \leq N; Y(\tau_{nN+1}) \in M] \\ &\quad = P^x\big[Y(\tau_{nk}) \in M_k, k \leq N; P^{X(\tau_{nN})}[Y(\sigma_{2^{-n}}) \in M]\big].\end{aligned} \tag{5.5}$$

Since $P^z[Y(\sigma_{2^{-n}}) \in M]$ is constant on H-orbits (from step $K = 1$), it can be written as $g \circ \Phi$ for some measurable function g: $F \to \mathbb{R}$. Therefore, (5.5) becomes

$$P^x\Big[\prod_{k \leq N} 1_{M_k}(Y(\tau_{nk}))g(Y(\tau_{nN}))\Big]. \tag{5.6}$$

Recall we are assuming (5.4) when $K = N$. That together with a monotone class argument imply that (5.6) can be rewritten as

$$\begin{aligned}&P^{\varphi(x)}\Big[\prod_{k \leq N} 1_{M_k}(Y(\tau_{nk}))g(Y(\tau_{nN}))\Big] \\ &\quad = P^{\varphi(x)}[Y(\tau_{nk}) \in M_k, k \leq N; Y(\tau_{nN+1}) \in M],\end{aligned}$$

and this completes the verification of (5.4).

Now we must modify Y_t so that we can apply (5.2). Fortunately at this point, we are interested only in the *geometric* behavior of the trajectories of Y_t on $[0, \zeta)$, so we can use a time change. Define a strictly increasing continuous process C_t by

$$C_t(\omega) = \int_0^t \max\left(1, (\zeta(\omega) - s)^{-1}\right) 1_{\{\zeta(\omega) > s\}}\, ds.$$

Then $C_0 = 0$, $C_{\zeta-} = \infty$ and $C_{t+s} = C_t + C_s \circ \theta_t$ a.s. However, C_t is not adapted, so it is only a raw additive functional. Let $\pi_t = \inf\{s > 0\colon C_s > t\}$, and let $\theta_t^* = \theta(\pi_t)$. Since C_t is strictly increasing and continuous, the processes $Y_t^* = Y_{\pi(t)}$ and $X_t^* = X_{\pi(t)}$ have the same geometric trajectories as the restrictions to $[0, \zeta)$ of Y_t and X_t. Since $C_{\zeta-} = \infty$ a.s., Y_t^* and X_t^* have infinite lifetimes. In addition, $\theta_t^* \theta_s^* = \theta_{t+s}^*$, and $X_t^* \circ \theta_s^* = X_{t+s}^*$. Thus Y_t^* is right-continuous with left limits, and we let $\Psi\colon \Omega \to W$ be given by defining $\Psi(\omega)$ to be the path $t \to Y_t^*(\omega)$. Fix $\varphi \in H$. For $N \in \mathscr{W}$, let $P_1(N) = P^x[\Psi^{-1}(N)]$, and let $P_2(N) = P^{\varphi(x)}[\Psi^{-1}(N)]$. Recalling that X_t^* and the restriction of X_t to $[0, \zeta)$ have the same trajectories, we see that for $M_1, M_2, \dots, M_k$ open sets in F,

$$P_1\left[w(\tau_{nk}^0) \in M_k,\, k \le K\right] = P^x\left[Y(\tau_{nk}) \in M_k,\, k \le K\right].$$

Similarly,

$$P_2\left[w(\tau_{nk}^0) \in M_k,\, k \le K\right] = P^{\varphi(x)}\left[Y(\tau_{nk}) \in M_k,\, k \le K\right].$$

Thus by (5.4), (5.3) holds whenever $M_1, \dots, M_k$ are open subsets of F. But in fact, this is enough. If we set $\xi(w) = \inf\{t\colon w(t) = \delta\}$, then $\xi = \infty$ a.s. P_1 and a.s. P_2. Therefore, (5.3) holds for $M_1, \dots, M_k$ open subsets of $F \cup \{\delta\}$.

6. The time scale (Part 1). We have determined in Section 5 that Y_t has the same geometric trajectories when run under P^x and $P^{\varphi(x)}$ for $\varphi \in H$. We show in this section that we can choose a time scale γ_t for X so that $\Phi(X(\gamma_t))$ is Markov. In particular, the law of $(\Phi(X(\gamma_t)), P^x)$ is the same as the law of $(\Phi(X(\gamma_t)), P^{\varphi(x)})$ whenever $\varphi \in H$. We rely on potential theory and transience to do this, so we add the following assumption to those made in previous sections. If $z \in F$ and $r > 0$, let $B^r(z) = \{x \in F\colon d(x, z) < r\}$, and let

$$L^r(z) = \sup\{t > 0\colon Y_t \in B^r(z)\} = \sup\{t > 0\colon X_t \in \Phi^{-1}(B^r(z))\},$$

where we take $\sup \varnothing = 0$.

(6.1) HYPOTHESIS. $L^r(z) < \zeta$ almost surely for every $z \in F$ and $r < 1$.

This is equivalent to requiring that each $\Phi^{-1}(B^r(z))$ be a transient set for the process X.

Fix a countable collection of points (q_j) which is dense in F, and let $B_j^r = B^r(q_j)$ and $L_j^r = L^r(q_j)$. Define

$$\tilde{A}_t = \sum_j 2^{-j} \int_0^1 1_{\{0 < L_j^r \le t\}}\, dr.$$

This is a raw additive functional of X with potential $P^x[\tilde{A}_\infty] \le 1$. Let A_t be the dual predictable projection of $\tilde{A}_t$ for the process X. The next result is proved as in [5].

(6.2) PROPOSITION. *A_t is continuous almost surely.*

PROOF. Assume $J = \{(t, w): A_{t-}(w) \neq A_t(w)\}$ is not evanescent, and let T be an $(\mathscr{F}_t)$-predictable time whose graph $[T]$ is contained in J. Then

$$0 < P^x \int 1_{[T]}(s)\, dA_s = P^x \int 1_{[T]}(s)\, d\tilde{A}_s.$$

Thus for some $j \geq 1$, $r > 0$, $s > 0$,

$$P^x\left[0 < L_j^r = L_j^{r+s} = T\right] > 0. \tag{6.3}$$

Since T is predictable, $X_{T-} = X_T$. By (6.3), Y_T is in the intersection of the boundaries of B_j^r and B_j^{r+s}. Since this intersection is empty, we conclude that J must be evanescent. □

The next proof is more complicated than the analogous result (1.5) in [5]. This is due to the improper time scale in which Y_t is running.

(6.4) PROPOSITION. *A_t is strictly increasing almost surely.*

PROOF. Define $R = \inf\{t > 0\colon A_t > 0\}$, $p(x) = P^x[e^{-R}]$, and $C = \{x\colon p(x) = 1\}$. We need to show $C = E$. A standard proof (which we omit) shows that $R = T_C$ a.s.

Since the event $\{T_C = 0\}$ is a *spatial event* (i.e., it depends on the trajectories and not on the speeds at which they are traversed), $p(x) = P^x[T_C = 0]$ is constant on H-orbits by (5.2) and (5.4). Therefore, the sets C and C^c are unions of H-orbits.

Now assume there is at least one point $x \in E - C$. Then the whole H-orbit $[x] \in E - C$. We shall show this leads to a contradiction. Fix any point $y \in [x]$. Since $R > 0$ a.s. P^y, we have

$$0 = P^y \int 1_{(0, R]}(s)\, dA_s = P^y \int 1_{(0, R]}(s)\, d\tilde{A}_s$$

$$= \sum_j 2^{-j} \int_0^1 P^y\left[0 < L_j^r \leq R\right] dr.$$

Thus, for each $r < 1$ and each j so that $y \in \Phi^{-1}(B_j^r)$, $P^y[T_j^r \circ \theta_R < \infty] = 1$, where $T_j^r = T_{\Phi^{-1}(B_j^r)}$. Choose a subsequence $S_k = T_{j(k)}^{r(k)}$ so that $B_k = B_{j(k)}^{r(k)}$ decreases to the H-orbit $[x] \in F$, and consider the sequence $D_k = R + S_k \circ \theta_R$ of $(\mathscr{F}_t)$-optional times. This sequence increases to an optional time D and $P^y[Y_{D-} = [x]$ or $Y_D = [x]] = 1$. Note that $D < \zeta$ a.s. P^y, since otherwise $P^y[L^r([x]) = \zeta] > 0$, contradicting (6.1). Therefore, $Y_D = [x]$ a.s. P^y by quasi-left-continuity of Y on $[0, \zeta)$.

(6.5) LEMMA. *$D = R + T^* \circ \theta_R$, where $T^* = T_{\Phi^{-1}([x])}$.*

PROOF. Since $D_k \leq R + T^* \circ \theta_R$ for every k, $D \leq R + T^* \circ \theta_R$. On the other hand, $R + T^* \circ \theta_R$ is the first time after R that X_t visits $[x]$ (since $X_R \notin [x]$ a.s.). Since $D \geq R$ and $Y_D = [x]$, $D \geq R + T^* \circ \theta_R$. □

Let $L = \sup\{t > 0: X_t \in [x]\}$. So far, we have shown that

$$P^y[0 < D \leq L < \zeta, X_D \in [x]] = 1 \quad \text{for every } y \in [x]. \tag{6.6}$$

Let $\mathscr{I}$ denote the set of countable ordinals, and let $D_0 = D$. For $\alpha \in \mathscr{I}$, define

$$D_\alpha = D_{\alpha-1} + D \circ \theta_{D_{\alpha-1}} \quad \text{if } \alpha \geq 1 \text{ is not a limit ordinal,}$$

$$D_\alpha = \sup_{\beta < \alpha} D_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Applying a transfinite induction argument yields the next result.

(6.7) LEMMA. *For each $y \in [x]$,*

$$P^y\big[0 < D_\beta \leq L < \zeta, X(D_\beta) \in [x]\big] = 1 \tag{6.8}$$

for every $\beta \in \mathscr{I}$.

PROOF. Assume (6.8) holds for every $\beta < \alpha$. If $\alpha \geq 1$ is not a limit ordinal, then

$$\begin{aligned} &P^y[0 < D_\alpha \leq L < \zeta, X(D_\alpha) \in [x]] \\ &\quad = P^y\big[0 < D \circ \theta_{D_{\alpha-1}} \leq L \circ \theta_{D_{\alpha-1}} < \zeta \circ \theta_{D_{\alpha-1}}, X_D \circ \theta_{D_{\alpha-1}} \in [x]\big] \\ &\quad = P^y\big(P^{X(D_{\alpha-1})}[0 < D \leq L < \zeta, X_D \in [x]]\big) = 1, \end{aligned}$$

since $X(D_{\alpha-1}) \in [x]$ a.s. P^y for every $y \in [x]$. If α is a limit ordinal, then $D_\alpha = \sup_{\beta<\alpha} D_\beta \leq L$ a.s. P^y. Since $L < \zeta$ by (6.1), quasi-left-continuity and the fact that $[x]$ is closed in E yield $X(D_\alpha) \in [x]$ a.s. P^y for every $y \in [x]$. □

Now we can quickly obtain a contradiction. Since $\alpha \to P^x[\exp(-D_\alpha)]$ is a positive decreasing function on $\mathscr{I}$, it must eventually be constant. That is, for some $\alpha \in \mathscr{I}$, $D_\alpha = D_{\alpha+1}$ a.s. P^y. But (6.6) and (6.8) yield $P^y(D \circ \theta_{D_\alpha} > 0) = 1$, so $D_{\alpha+1} > D_\alpha$ a.s. P^y, and this is a contradiction. Therefore, $E = C$. This concludes the proof of Proposition (6.4). □

From Section 5, we know that if $\varphi \in H$, then

$$P^x\big[h \circ \Phi\big(X(L_j^r -)\big); L_j^r > 0\big] = P^{\varphi(x)}\big[h \circ \Phi\big(X(L_j^r -)\big); L_j^r > 0\big],$$

whenever h is $\mathscr{B}(F)$-measurable. Therefore,

$$P^x \int h \circ \Phi(X_{s-})\, d\tilde{A}_s = P^{\varphi(x)} \int h \circ \Phi(X_{s-})\, d\tilde{A}_s \tag{6.9}$$

or

$$P^x \int h \circ \Phi(X_{s-})\, dA_s = P^{\varphi(x)} \int h \circ \Phi(X_{s-})\, dA_s. \tag{6.10}$$

By continuity of A_t, we may replace X_{s-} with X_s in (6.10). Doing this, and letting $\gamma_t = \inf\{s: A_s > t\}$, we rewrite (6.10) as

$$P^x \int h \circ \Phi(X(\gamma_s))\, ds = P^{\varphi(x)} \int h \circ \Phi(X(\gamma_s))\, ds. \tag{6.11}$$

If we let (U_A^α) denote the resolvent for the process $X(\gamma_t)$, then (6.11) becomes

$$U_A(h \circ \Phi)(x) = U_A(h \circ \Phi)(\varphi(x)).$$

That is, $U_A(h \circ \Phi)$ is constant on H-orbits, also. Since $U_A 1 = P^x[\tilde{A}_\infty] \le 1$, we have for every $\alpha < 1$ and for every function g bounded by 1,

$$U_A^\alpha(g) = \sum_{n=0}^{\infty} (-\alpha)^n (U_A)^{n+1} g(x).$$

Therefore, for every $\alpha < 1$,

$$U_A^\alpha(h \circ \Phi)(x) = \sum_{n=0}^{\infty} (-\alpha)^n (U_A)^{n+1} (h \circ \Phi)(x) \tag{6.12}$$

is also constant on H-orbits. The resolvent equation implies that $U_A^\alpha(h \circ \Phi)$ is constant on H-orbits for every $\alpha > 0$. Let P_t^A be the semigroup of $X(\gamma_t)$. By (6.12),

$$\int e^{-\alpha t} P_t^A(h \circ \Phi)(x)\, dt = \int e^{-\alpha t} P_t^A(h \circ \Phi)(\varphi(x))\, dt$$

for every φ in H, h continuous on F and $\alpha > 0$. Therefore, $P_t^A(h \circ \Phi)$ is constant on H-orbits. Define a semigroup and resolvent on F by setting

$$Q_t h([x]) = P_t^A(h \circ \Phi)(x),$$
$$V^\alpha h([x]) = U_A^\alpha(h \circ \Phi)(x).$$

Since $Q_t h([x]) = P^x[h(Y(\gamma_t))]$, $Y(\gamma_t)$ is a simple Markov process. To see it is strong Markov, we need only check that $V^\alpha h(Y(\gamma_t))$ is right-continuous whenever h is bounded and measurable. But $V^\alpha h(Y(\gamma_t)) = U_A^\alpha(h \circ \Phi)(X(\gamma_t))$, and this process is right-continuous a.s. since $U_A^\alpha(h \circ \Phi)$ is finely continuous for $X(\gamma_t)$.

7. The time scale (Part 2). In this section, we discuss the case where Y_t may be recurrent (although X_t is still transient), so we do not assume Hypothesis (6.1). We do not feel that we have been successful in extending the nice result in Section 6 to this more general setting; difficulties crop up in connection with possible explosions of additive functionals. We have tried two different approaches to the problem, and we indicate these below. The first is unsatisfactory since we are led to make a hypothesis (7.2), which we suspect is probably unverifiable most of the time. The second approach led us to Problem (7.4), which we believe is an unsolved problem in Markov process theory.

Therefore, the question of whether or not hypothesis (iii) in Theorem (1.1) may be omitted remains open, and we hope this section will provide some insight into the difficulties involved.

Before we begin discussing the two approaches, however, we illustrate a method of creating a transient process from Y_t which occasionally allows us to finesse the problem at small cost. In Section 5, we determined that Y_t has the same geometric trajectories when run under P^x and $P^{\varphi(x)}$ for $\varphi \in H$. Let K and L be two open sets in F with disjoint closures, and define

$$B_t = \sum_{0<s\leq t} 1_{\{(Y(s-), Y(s)) \in K \times L\}}.$$

Then (Y_t, B_t) has the same geometric trajectories in $F \times \mathbb{R}^+$ when run under P^x and $P^{\varphi(x)}$. Moreover, if Y_t is recurrent and has jumps, we often have the situation where $B_\infty = \infty$ a.s. So let us shift our attention from X_t and Y_t to the Markov additive processes (X_t, B_t) and (Y_t, B_t): Both processes are transient if $B_\infty = \infty$ a.s. The analysis in Section 6 shows there is a continuous additive functional A_t of (X_t, B_t) so that $(Y(\gamma_t), B(\gamma_t))$ is a Markov process. Unfortunately, A_t may not be an additive functional of X_t alone.

Approach 1. Recall (q_j) is a sequence of points dense in F, and B_j^r is the closed ball of radius r about q_j in F. Define

$$\begin{aligned} T(r,j) &= \inf\{t > 0: X_t \in \Phi^{-1}(B_j^r)\}, \\ F(r,j) &= \{x \in E: P^x[T(r,j) = 0] = 1\}, \\ T(r,j,t) &= T(r,j) \circ \theta_t, \\ G(r,j) &= \{t > 0: T(r,j,t-) = 0, T(r,j,t) > 0, X_t \in F(r,j)\}. \end{aligned}$$

To simplify our discussion of Approach 1, we assume

(7.1) HYPOTHESIS. $P^x[G(r,j) = \varnothing] = 0$ for each r, j and x.

Define the homogeneous random measure

$$\kappa_j^r(dt) = \sum_{s \in G(r,j)} \varepsilon_s.$$

Maisonneuve [6] showed that there is a continuous additive functional $A(r,j,t)$ with a one-potential bounded by one and a kernel K_j^r from $(E, \mathscr{B}(E)^*)$ to $(\Omega, \mathscr{F}^*)$ so that

$$P^x \int Z_s W \circ \theta_s \kappa_j^r(ds) = P^x \int Z_s K_j^r(X_s, W)\, dA(r,j,s),$$

for every positive optional process Z and every positive $\mathscr{F}^*$-measurable random variable W. Let $\mathscr{T}$ be the σ-algebra of all sets $N \in \mathscr{F}$ so that if the two paths $Y_t(w)$ and $Y_t(w')$ are time changes of each other, then $1_N(w) = 1_N(w')$. (See Section 5.) We assume the following extra condition.

(7.2) HYPOTHESIS. For each j and $r < 1$, there is a strictly positive $\mathscr{T}$-measurable random variable $W(r, j)$ so that the homogeneous random measure $W(r, j) \circ \theta_s \kappa_j^r(ds)$ has a one-potential bounded by one. For each t and j, the map $r \to \int_0^t W(r, j) \circ \theta_s \kappa_j^r(ds)$ is Lebesgue measurable on $(0, 1)$.

Maisonneuve produces a random variable $W(r, j) = (1 - e^{-T(r,j)})$ which is not $\mathscr{T}$-measurable. In order to explore the meaning of this hypothesis, let us assume in this paragraph only that X_t has a dual process $\hat{X}_t$ with respect to a duality measure ξ. Let μ be the Revuz measure of κ_j^r. The Revuz measure of the random measure $W(r, j) \circ \theta_s \kappa_j^r(ds)$ is $K_j^r(x, W(r, j))\mu(dx)$. In this case, (7.2) is equivalent to the existence of a $\mathscr{T}$-measurable $W(r, j)$ so that

$$\int u^1(x, y) K_j^r(y, W(r, j)) \mu(dy) \le 1 \quad \text{a.e. } (\xi).$$

To continue Approach 1, we assume (7.2) and define a raw additive functional $\tilde{A}_t$ by

$$\tilde{A}_t = \sum_j 2^{-j} \int_0^1 \int_0^t W(r, j) \circ \theta_s \kappa_j^r(ds)\, dr.$$

Then $\tilde{A}_t$ has a one-potential bounded by one. A proof similar to that given in (6.4) shows that its dual predictable projection A_t is strictly increasing. Let T be any $\mathscr{T}$-measurable optional time. Since

$$P^x \int_{(0,T]} h \circ \Phi(X_{s-})\, d\tilde{A}_s = P^{\varphi(x)} \int_{(0,T]} h \circ \Phi(X_{s-})\, d\tilde{A}_s,$$

we have (as in Section 6)

$$P^x \int_{(0\,T]} h \circ \Phi(X_s)\, dA_s = P^{\varphi(x)} \int_{(0,T]} h \circ \Phi(X_s)\, dA_s$$

or

$$P^x \int_{(0,A_T]} h \circ \Phi(X_{\gamma(s)})\, ds = P^{\varphi(x)} \int_{(0,A_T]} h \circ \Phi(X_{\gamma(s)})\, ds,$$

where γ_t is the continuous inverse of A_t. Now assume there is a sequence (D_n) of sets contained in F so that:

(7.3) HYPOTHESIS. (i) Each D_n is open in F; (ii) $\overline{D_n} \subset D_{n+1}$; (iii) $\zeta_n = \inf\{t: Y_t \in D_n^c\} < \infty$ a.s.; (iv) $\lim_{n\to\infty} \zeta_n \ge \zeta$ a.s.

From the discussion at the end of Section 6, we see that $Y(\gamma_t)$ is Markov on $[0, \zeta_n]$ for every n. From (iv), we obtain $Y(\gamma_t)$ is a Markov process.

Approach 2. We *do not* assume (7.1) or (7.2) any longer. However, we continue to assume (7.3). Let X_t^n and Y_t^n be X_t and Y_t killed at ζ_n. Now Y_t^n is transient, and Y_t^n has the same geometric trajectories under P^x and $P^{\varphi(x)}$ whenever $\varphi \in H$. Minor changes in the proofs in Section 6 need to be made to compensate for the fact that X_t^n is no longer a Hunt process. But once this is done, the reasoning there leads to the existence of a strictly increasing continuous additive functional A_t^n of X_t^n with inverse $\gamma(n,t)$ so that $Y^n(\gamma(n,t))$ is a strong Markov process. Can these processes be patched together? This seems to be a general problem in Markov process theory for which we have no answer. We restate it as a general problem independent of our particular setting.

(7.4) PROBLEM. Let (D_n) be a sequence of open sets in an LCCB state space F so that $\overline{D_n} \subset D_{n+1}$. For each n, let Z_t^n be a standard process on D_n, and let $T_{n+1} = \inf\{t\colon Z_t^{n+1} \notin D_n\}$. Assume there is a strictly increasing continuous additive functional A_t^{n+1} of the killed process $W_t^{n+1} = (Z_t^{n+1}, T_{n+1})$ so that if τ_t^{n+1} is the inverse of A_t^{n+1}, then $W^{n+1}(\tau_t^{n+1})$ and Z_t^n are identical in law. Is there a standard process Z_t with the following property: If $\sigma^n = \inf\{t\colon Z_t \notin D_n\}$, then (Z_t, σ^n) and (Z_t^n) have the same hitting distributions on D_n?

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Note added in proof. C. T. Shih has recently made progress in solving (7.4).

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