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MAXIMAL INEQUALITIES FOR MULTIDIMENSIONALLY INDEXED SUBMARTINGALE ARRAYS[1]

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Some new maximal-type probability inequalities are developed for discrete-time multidimensionally indexed submartingales. In particular, the basic idea of Chow is abstracted and extended. This leads to a result which yields extended Kolmogorov inequalities and strong laws, extended Hájek–Rényi type inequalities competitive with Smythe and an extended Doob inequality which is counter-intuitive to a counterexample of Cairoli.

1. Introduction and preliminaries. Cairoli (1970) showed (by counterexample) that certain classical maximal probability inequalities for ordinary discrete-time submartingales are not (in general) valid for discrete-time multidimensionally indexed submartingales. However, the classical results do in fact have useful extensions in modified form to the multidimensionally indexed case, and in this paper we present such maximal probability inequalities. First we abstract and extend the martingale inequality of Chow (1960) in the form of our Theorem 2.2. Using Theorem 2.2 as a "source theorem," we obtain maximal inequalities for various special cases. Among them, Corollaries 2.4 and 2.5 are extensions of Doob's inequality and Kolmogorov's inequality, respectively. The maximal inequalities of the paper are further used to produce a variety of strong laws of large numbers in general forms. In particular, Theorem 2.8 provides a Kolmogorov strong law for a collection of independent multidimensionally indexed random variables.

We denote by $\mathbf{N}^r$ the r-dimensional positive integer lattice and we use bold symbols to denote the elements of $\mathbf{N}^r$. Thus, for example, the symbol $\mathbf{n}$ denotes the vector $(n_1, n_2, \dots, n_r)$, where r is an integer greater than or equal to 1. Regular italic letters will be used for one-dimensional elements. We will assume the usual partial ordering for the elements of $\mathbf{N}^r$, i.e., for $\mathbf{n} = (n_1, \dots, n_r)$ and $\mathbf{n}' = (n'_1, \dots, n'_r)$ in $\mathbf{N}^r$, the notation $\mathbf{n} \le \mathbf{n}'$ means that $n_i \le n'_i$ for all $i = 1, \dots, r$. For $\mathbf{i} = (i_1, \dots, i_r)$ and $\mathbf{j} = (j_1, \dots, j_r)$, the notation $\mathbf{i} < \mathbf{j}$ means that $i_s \le j_s$ for $s = 1, \dots, r$ with at least one inequality strict. Finally, the notation $\mathbf{n} \to \infty$ means that $n_j \to \infty$ for $j = 1, \dots, r$, or, equivalently, that $\min_{1 \le j \le r} n_j \to \infty$.

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Let (Ω, F, P) be a probability space. Let $\{F_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^r\}$ be a nondecreasing array of sub-σ-fields of F, i.e.,

$$F_{\mathbf{n}} \subseteq F_{\mathbf{n}'} \subseteq F, \quad \text{if } \mathbf{n} \leq \mathbf{n}',$$

and let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^r\}$ be an array of random variables such that $X_{\mathbf{n}}$ is $F_{\mathbf{n}}$-measurable and integrable for every $\mathbf{n}$. Then we say that $\{X_{\mathbf{n}}, F_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^r\}$ is a *forward martingale* if

$$E\{X_{\mathbf{n}'} | F_{\mathbf{n}}\} = X_{\mathbf{n}} \quad \text{a.s. for } \mathbf{n} \leq \mathbf{n}'. \tag{1.1}$$

As pointed out in Gut (1976), an equivalent condition to (1.1) is

$$\int_A X_{\mathbf{n}'}\, dP = \int_A X_{\mathbf{n}}\, dP \quad \text{for } A \in F_{\mathbf{n}}, \mathbf{n} \leq \mathbf{n}'. \tag{1.2}$$

If the equalities in (1.1) and (1.2) are replaced by $\geq$, then we say that $\{X_{\mathbf{n}}, F_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^r\}$ is a *forward submartingale*.

2. General results on multidimensionally indexed random variables. The following lemma, an abstract version of Theorem 1 of Chow (1960), will serve as a basic tool in our development.

LEMMA 2.1 (Abstract Chow lemma). *Let $B_1, B_2, \dots, B_m$ be disjoint events and $\{d_k, k \in \mathbf{N}\}$ a nonincreasing sequence of nonnegative numbers. Let $\{W_k, k \in \mathbf{N}\}$ be a sequence of nonnegative random variables satisfying*

$$P(B_k) \leq d_k \int_{B_k} W_k\, dP, \qquad 1 \leq k \leq m, \tag{i}$$

and

$$\int_{B_l} W_k\, dP \leq \int_{B_l} W_{k+1}\, dP, \qquad 1 \leq l \leq k \leq m-1. \tag{ii}$$

Let $B = \bigcup_{k=1}^m B_k$. Then

$$P(B) \leq \sum_{k=1}^{m-1} (d_k - d_{k+1}) E(W_k) + d_m E(W_m) - d_m \int_{B^C} W_m\, dP.$$

PROOF. Following Chow (1960), we can write

$$\begin{aligned} P(B) &= \sum_{k=1}^m P(B_k) \leq \sum_{k=1}^m d_k \int_{B_k} W_k\, dP \\ &= \sum_{k=1}^m d_k \int_{\bigcup_{i \leq k} B_i} W_k\, dP - \sum_{k=2}^m d_k \int_{\bigcup_{i \leq k-1} B_i} W_k\, dP, \end{aligned} \tag{2.1}$$

where the inequality follows from (i) and the equalities from the disjointness of the sets B_i. Using (ii), (2.1) can be replaced by

$$P(B) \leq \sum_{k=1}^m d_k \int_{\bigcup_{i \leq k} B_i} W_k\, dP - \sum_{k=2}^m d_k \int_{\bigcup_{i \leq k-1} B_i} W_{k-1}\, dP. \tag{2.2}$$

By rearranging terms, the right-hand side of (2.2) can be written as $\sum_{k=1}^{m-1}(d_k - d_{k+1})\int_{\bigcup_{i \le k} B_i} W_k \, dP + d_m \int_B W_m \, dP$, which is bounded by

$$\sum_{k=1}^{m-1} (d_k - d_{k+1}) E(W_k) + d_m E(W_m) - d_m \int_{B^C} W_m \, dP. \qquad \square$$

REMARK. Note that (ii) describes the submartingale property if $\{W_k\}$ is adapted.

Lemma 2.1 will be used to obtain bounds for $P\{\max_{\mathbf{k} \le \mathbf{n}} C_{\mathbf{k}} Y_{\mathbf{k}} \ge \varepsilon\}$, where $\varepsilon > 0$, $\{Y_{\mathbf{k}}, F_{\mathbf{k}}, k \in \mathbf{N}^r\}$ is a martingale and $\{C_{\mathbf{k}}, \mathbf{k} \in \mathbf{N}^r\}$ is a nonincreasing array of nonnegative numbers. In order to state the relevant theorem, some preliminary notation and constructions will be needed. Let $A = \{\max_{\mathbf{k} \le \mathbf{n}} C_{\mathbf{k}} Y_{\mathbf{k}} \ge \varepsilon\}$. For each $\mathbf{k}$ define $F_{\mathbf{k}} = \sigma\{Y_{\mathbf{i}}, \mathbf{i} \le \mathbf{k}\}$ and $A_{\mathbf{k}} = \{C_{\mathbf{i}} Y_{\mathbf{i}} < \varepsilon$ for $\mathbf{i} < \mathbf{k}$, $C_{\mathbf{k}} Y_{\mathbf{k}} \ge \varepsilon\}$. Note that $\bigcup_{\mathbf{k} \le \mathbf{n}} A_{\mathrm{k}} = A$. However, the sets $A_{\mathbf{k}}$, due to the lack of linear ordering when $r \ge 2$, are not disjoint. Using the sets $A_{\mathbf{k}}$, we can construct sets $B_{\mathbf{k}}$ which satisfy $\bigcup_{\mathbf{k} \le \mathbf{n}} B_{\mathbf{k}} = A$ and are disjoint, by applying the following algorithm.

```
Put D_0 := ∅
  m := 1
  For k_1 = 1 to n_1
    For k_2 = 1 to n_2
          .
            .
              For k_r = 1 to n_r
              Begin
                B^(1)_{k_1 ... k_r} := A_{k_1 ... k_r} − ∪_{l<m} D_l
                D_m := A_{k_1 ... k_r}
                m := m + 1
              End
```

REMARKS. (a) An explicit expression of the sets $B_{\mathbf{k}}^{(1)}$ in terms of the sets $A_{\mathbf{k}}$ is possible to derive, but such a formula is notationally messy and complicated. The algorithm orders the sets $A_{k_1, \dots, k_r}$ lexicographically and subtracts from each set $A_{k_1, \dots, k_r}$ the union of all sets whose index is lexicographically smaller than that of the set $A_{k_1, \dots, k_r}$.

(b) The order of the loop in the construction of the sets $B_{\mathbf{k}}$ is not without significance. Notice the superscript (1) in the sets $B_{\mathbf{k}}$. If the order of the loop is changed, then we get a different collection of sets $B_{\mathbf{k}}$ with a different superscript. For example, if the order in the loop were

For $k_2 = 1$ to n_2
For $k_1 = 1$ to n_1
$\vdots$
For $k_r = 1$ to n_r

then we would have constructed sets $B_{\mathbf{k}}^{(2)}$ and so on. The importance of this is that we shall in fact obtain r separate bounds for $P\{\max_{\mathbf{k} \le \mathbf{n}} C_{\mathbf{k}} Y_{\mathbf{k}} \ge \varepsilon\}$.

(c) Due to the construction of the sets $B_{\mathbf{k}}$, the set $B_{k_1 k_2 \cdots k_r}^{(1)}$ does not belong to $F_{k_1 k_2 \cdots k_r}$, but rather to the σ-field $F_{k_1 n_2 \cdots n_r}$.

(d) It turns out that construction of disjoint sets $D_{\mathbf{k}}$ such that $D_{\mathbf{k}} \in F_{\mathbf{k}}$ and $\bigcup_{\mathbf{k} \le \mathbf{n}} D_{\mathbf{k}} = A$ is not possible in general, because of the counterexample provided by Cairoli. If it were, then we could have obtained (3.8) of Doob (1953), page 317.

Now we are ready to state and prove the following result.

THEOREM 2.2. *Let $\{Y_{\mathbf{k}}, F_{\mathbf{k}}, \mathbf{k} \in \mathbf{N}^r\}$ be a martingale. Suppose that the σ-fields satisfy*

$$E\{E\{*|F_{\mathbf{k}}\}|F_{\mathbf{l}}\} = E\{*|F_{\mathbf{k} \wedge \mathbf{l}}\}, \tag{2.3}$$

where $\mathbf{k} \wedge \mathbf{l}$ denotes the minimum of $\mathbf{k}$ and $\mathbf{l}$ taken componentwise. Let $\{C_{\mathbf{k}}, \mathbf{k} \in \mathbf{N}^r\}$ be a nonincreasing array of nonnegative numbers. Then for $\varepsilon > 0$,

$$\begin{aligned} \varepsilon P\Big\{\max_{\mathbf{k} \le \mathbf{n}} C_{\mathbf{k}} Y_{\mathbf{k}} \ge \varepsilon\Big\} &\le \min_{1 \le s \le r} \Bigg\{ \sum_{\mathbf{k} \le \mathbf{n}} \big(C_{\mathbf{k}} - C_{\mathbf{k}; s; k_s+1}\big) E\big(Y_{\mathbf{k}}^+\big) \\ &\qquad - \sum_{\substack{k_i \\ i \ne s}} C_{\mathbf{k}; s; n_s} \int_{(\bigcup_{k_s=1}^{n_s} B_{k_1 \ldots k_r}^{(s)})^C} Y_{\mathbf{k}; s; n_s}^+ \, dP \Bigg\} \\ &\le \min_{1 \le s \le r} \Big\{ \sum_{\mathbf{k} \le \mathbf{n}} \big(C_{\mathbf{k}} - C_{\mathbf{k}; s; k_s+1}\big) E\big(Y_{\mathbf{k}}^+\big) \Big\}, \end{aligned} \tag{2.4}$$

where $C_{\mathbf{k}; s; \alpha} = C_{k_1 \cdots k_{s-1} \alpha k_{s+1} \cdots k_r}$ and $C_{\mathbf{k}} = 0$ if $k_i > n_i$ for some $i = 1, 2, \ldots, r$.

PROOF. We will give the proof for $r = 2$ (the case $r > 2$ can easily be established by induction). Let $d_{k_1 k_2} = C_{k_1 k_2}/\varepsilon$, $B_{k_2}^{(1)} = \bigcup_{k_1=1}^{n_1} B_{k_1 k_2}^{(1)}$. Then, by construction,

$$P\big(B_{k_1 k_2}^{(1)}\big) \le d_{k_1 k_2} \int_{B_{k_1 k_2}^{(1)}} Y_{k_1 k_2}^+ \, dP, \qquad 1 \le k_1 \le n_1.$$

Also,

$$\begin{aligned}\int_{B^{(1)}_{k_1k_2}} Y^+_{k_1+1k_2}\,dP &= \int_{B^{(1)}_{k_1k_2}} E\{Y^+_{k_1+1k_2}|F_{k_1+1k_2}\}\,dP\\ &= \int_{B^{(1)}_{k_1k_2}} E\{E\{Y^+_{k_1+1k_2}|F_{k_1+1k_2}\}|F_{k_1n_2}\}\,dP\\ &= \int_{B^{(1)}_{k_1k_2}} E\{Y_{k_1+1k_2}|F_{k_1k_2}\}\,dP \quad [\text{by } (2.3)]\\ &\geq \int_{B^{(1)}_{k_1k_2}} Y^+_{k_1k_2}\,dP\end{aligned}$$

by the submartingale property. Thus, the requirements of Lemma 2.1 are met and we have

$$\begin{aligned}P(A) &= P\left(\bigcup_{k_2=1}^{n_2} B^{(1)}_{k_2}\right)\\ &= \sum_{k_2=1}^{n_2} P\left(B^{(1)}_{k_2}\right)\\ &\leq \sum_{k_2=1}^{n_2}\left\{\sum_{k_1=1}^{n_1-1}(d_{k_1k_2}-d_{k_1+1k_2})E\left(Y^+_{k_1k_2}\right)\right.\\ &\qquad\left.+d_{n_1k_2}E\left(Y^+_{n_1k_2}\right)-d_{n_1k_2}\int_{(B^{(1)}_{k_2})^C}Y^+_{n_1k_2}\,dP\right\}\end{aligned}$$

or

$$\begin{aligned}P(A) \leq \frac{1}{\varepsilon}\sum_{k_2=1}^{n_2}&\left\{\sum_{k_1=1}^{n_1-1}(C_{k_1k_2}-C_{k_1+1k_2})E\left(Y^+_{k_1k_2}\right)+C_{n_1k_2}E\left(Y^+_{n_1k_2}\right)\right.\\ &\left.-C_{n_1k_2}\int_{(B^{(1)}_{k_2})^C}Y^+_{n_1k_2}\,dP\right\}.\end{aligned}\tag{2.5}$$

Now, by interchanging the order of the loop in the construction of the sets $B^{(1)}_{k_1k_2}$, we can construct analogous sets $B^{(2)}_{k_1k_2} \in F_{n_1k_2}$ and thus obtain

$$\begin{aligned}P(A) \leq \frac{1}{\varepsilon}\sum_{k_1=1}^{n_1}&\left\{\sum_{k_2=1}^{n_2-1}(C_{k_1k_2}-C_{k_1k_2+1})E\left(Y^+_{k_1k_2}\right)\right.\\ &\left.+C_{k_1n_2}E\left(Y^+_{k_1n_2}\right)-C_{k_1n_2}\int_{(B^{(2)}_{k_1})^C}Y^+_{k_1n_2}\,dP\right\}.\end{aligned}\tag{2.6}$$

Theorem 2.2 follows from the two inequalities (2.5) and (2.6). □

REMARKS. (a) The above inequality reduces, for $r = 1$, to the familiar Chow inequality, as given in Theorem 1 of Chow (1960).

(b) Condition (2.3) is also known as condition F4 [see Cairoli and Walsh (1975)].

If the array of the coeffficients $C_{\mathbf{k}}$ has a special structure, then the bound in (2.4) can take a nice form, convenient for establishing asymptotic results as well as other maximal inequalities. Thus, for example, we have the following result.

COROLLARY 2.3. *Assume the hypotheses of Theorem 2.2 and that $\{Y_{k_1k_2}\}$ is nonnegative. Assume also that there exists a number $\alpha \geq 2$ such that $C_{k_1k_2} \geq \alpha C_{k_1k_2+1}$ or $C_{k_1k_2} \geq \alpha C_{k_1+1k_2}$ for all k_1, k_2. Then*

$$\varepsilon P\Big\{\max_{(k_1,k_2)\leq(n_1,n_2)} C_{k_1k_2}Y_{k_1k_2} \geq \varepsilon\Big\} \leq \frac{\alpha}{\alpha-1} \sum_{(k_1,k_2)\leq(n_1,n_2)} (\Delta_F C_{k_1k_2}) E(Y_{k_1k_2}),$$

where $\Delta_F C_{k_1k_2}$ denotes the forward symmetric difference of the $C_{k_1k_2}$'s, i.e.,

$$\Delta_F C_{k_1k_2} = C_{k_1k_2} - C_{k_1+1k_2} - C_{k_1k_2+1} + C_{k_1+1k_2+1}$$

and $C_{k_1k_2} = 0$ if $k_i > n_i$ for $i = 1, 2$.

PROOF. Let $C^*_{k_1k_2} = C_{k_1k_2} - C_{k_1k_2+1}$. Then $\{C^*_{k_1k_2}, (k_1, k_2) \in \mathbf{N}^2\}$ is a nonincreasing array of nonnegative numbers. By applying Theorem 2.2 to the nonnegative martingale $\{Y_{k_1k_2}\}$ and to the array $\{C^*_{k_1k_2}, (k_1, k_2) \in \mathbf{N}^2\}$ the result follows. □

As an application of Corollary 2.3, we obtain a Hájek–Rényi-type inequality for a special case. Let $\{X_{ij}, (i,j) \in \mathbf{N}^2\}$ be an array of independent random variables with $E(X_{ij}) = 0$ and $E(X_{ij}^2) < \infty$ for each $(i,j) \in \mathbf{N}^2$. Let $\{C_{ij}, (i,j) \in \mathbf{N}^2\}$ be a nonincreasing array of nonnegative numbers satisfying $C_{ij} \geq \alpha C_{i+1j}$ or $C_{ij} \geq \alpha C_{ij+1}$ for all i, j and for some $\alpha \geq 2$. Put $S_{ij} = \sum_{(k_1,k_2)\leq(i,j)} X_{k_1k_2}$. Then, given $\varepsilon > 0$, we can apply Corollary 2.3 to obtain

$$\varepsilon^2 P\Big\{\max_{(i,j)\leq(n_1,n_2)} C_{ij}|S_{ij}| \geq \varepsilon\Big\} \leq \frac{\alpha^2}{\alpha^2-1} \sum_{(i,j)\leq(n_1,n_2)} \big(\Delta_F C_{ij}^2\big) E\big(S_{ij}^2\big).$$

Observe that

$$\sum_{(i,j)\leq(n_1,n_2)} \big(\Delta_F C_{ij}^2\big) E\big(S_{ij}^2\big) = \sum_{(i,j)\leq(n_1,n_2)} C_{ij}^2 E\big(\Delta S_{ij}^2\big),$$

where

$$\Delta S_{ij}^2 = S_{ij}^2 - S_{i-1j}^2 - S_{ij-1}^2 + S_{i-1j-1}^2,$$

with $S_{i0} = S_{0j} = 0$. By the independence of the random variables X_{ij}, $E(\Delta S_{ij}^2) = E(X_{ij}^2)$ and thus we obtain the final form of our Hájek–Rényi-type inequality,

$$\varepsilon^2 P\Big\{\max_{(i,j)\leq(n_1,n_2)} C_{ij}|S_{ij}| \geq \varepsilon\Big\} \leq \frac{\alpha^2}{\alpha^2-1} \sum_{(i,j)\leq(n_1,n_2)} C_{ij}^2 E\big(X_{ij}^2\big).$$

We should point out that the bound we obtain here for this special case is sharper than the one obtained by applying Smythe's Hájek–Rényi-type inequality [see Smythe (1974)], or Shorack and Smythe's inequality [Shorack and Smythe (1976)]. Smythe's inequality gives $64\sum_{(i,j)\le(n_1,n_2)}C_{ij}^2E(X_{ij}^2)$ as a bound, whereas the bound obtained above is no greater than $\frac{4}{3}\sum_{(i,j)\le(n_1,n_2)}C_{ij}^2E(X_{ij}^2)$.

COROLLARY 2.4. *Let* $\{Y_{\mathbf{k}}, F_{\mathbf{k}}, \mathbf{k}\in\mathbf{N}^r\}$ *be a martingale satisfying* (2.3). *Then*

$$\varepsilon P\Big\{\max_{\mathbf{k}\le\mathbf{n}} Y_{\mathbf{k}} \ge \varepsilon\Big\} \le \min_{1\le s\le r}\left\{\sum_{\substack{k_i\\ i\ne s}}\int_{\bigcup_{k_s=1}^{n_s}B_{k_1\cdots k_r}^{(s)}} Y_{\mathbf{k};s;n_s}^{+}\,dP\right\}. \tag{2.7}$$

PROOF. Immediate from Theorem 2.2 with $C_{\mathbf{k}} = 1$ for all $\mathbf{k}\le\mathbf{n}$. □

REMARK. The above corollary is an extension of one of the three Doob inequalities, namely inequality (3.8) of Doob (1953), page 317. Thus, despite Cairoli's counterexample, we see that Doob's inequality indeed has an extension to the case of multidimensional index. Although (2.7) appears cumbersome, in fact it is powerful and efficient enough to yield easily the following extended Kolmogorov-type inequality.

COROLLARY 2.5. *Let* $\{Y_{\mathbf{k}}, F_{\mathbf{k}}, \mathbf{k}\in\mathbf{N}^r\}$ *be a martingale satisfying* (2.3). *Then for* $\varepsilon > 0$,

$$\varepsilon^2 P\Big\{\max_{\mathbf{k}\le\mathbf{n}}|Y_{\mathbf{k}}| \ge \varepsilon\Big\} \le 4^{r-1}E\big(Y_{\mathbf{n}}^2\big). \tag{2.8}$$

PROOF. We prove the result for $r = 2$. The case $r > 2$ follows by induction. By Corollary 2.4 we have

$$\begin{aligned}
&\varepsilon^2 P\Big\{\max_{(k_1,k_2)\le(n_1,n_2)}|Y_{k_1k_2}| \ge \varepsilon\Big\}\\
&\quad\le \min\left\{\sum_{k_2=1}^{n_2}\int_{\bigcup_{k_1=1}^{n_1}B_{k_1k_2}^{(1)}} Y_{n_1k_2}^2\,dP,\ \sum_{k_1=1}^{n_1}\int_{\bigcup_{k_2=1}^{n_2}B_{k_1k_2}^{(2)}} Y_{k_1n_2}^2\,dP\right\}\\
&\quad\le \min\left\{\sum_{k_2=1}^{n_2}\int_{\bigcup_{k_1=1}^{n_1}B_{k_1k_2}^{(1)}} \max_{k_2} Y_{n_1k_2}^2\,dP,\ \sum_{k_1=1}^{n_1}\int_{\bigcup_{k_2=1}^{n_2}B_{k_1k_2}^{(2)}} \max_{k_1} Y_{k_1n_2}^2\,dP\right\}\\
&\quad= \min\left\{\int_A \max_{k_2} Y_{n_1k_2}^2\,dP,\ \int_A \max_{k_1} Y_{k_1n_2}^2\,dP\right\}\\
&\quad\le \min\left\{E\Big(\max_{k_2} Y_{n_1k_2}^2\Big),\ E\Big(\max_{k_1} Y_{k_1n_2}^2\Big)\right\}\\
&\quad= \tfrac{1}{2}\left\{E\Big(\max_{k_2} Y_{n_1k_2}^2\Big) + E\Big(\max_{k_1} Y_{k_1n_2}^2\Big) - \left|E\Big(\max_{k_2} Y_{n_1k_2}^2 - \max_{k_1} Y_{k_1n_2}^2\Big)\right|\right\}.
\end{aligned} \tag{2.9}$$

Observe that $\{Y^2_{n_1k_2}, F_{n_1k_2}, k_2 \geq 1\}$ and $\{Y^2_{k_1n_2}, F_{k_1n_2} \geq 1\}$ are both nonnegative submartingales, so that, by Theorem 3.4 of Doob (1953), page 317, we have

$$
\begin{aligned}
&\varepsilon^2 P\Big\{\max_{(k_1,k_2)\leq(n_1,n_2)} |Y_{k_1k_2}| \geq \varepsilon\Big\} \\
&\quad\leq 4E\big(Y^2_{n_1n_2}\big) - \frac{1}{2}\left|E\Big(\max_{k_2} Y^2_{n_1k_2} - \max_{k_1} Y^2_{k_1n_2}\Big)\right| \qquad (2.10)\\
&\quad\leq 4E\big(Y^2_{n_1n_2}\big).
\end{aligned}
$$

Thus the proof is complete. □

REMARKS. (a) Corollary 2.5 has also been obtained by Smythe (1974) (see his Lemma 1.1). Other Kolmogorov-type inequalities for multidimensionally indexed martingales were proved by various other authors including Wichura (1969) and Zimmerman (1972).

(b) As (2.10) suggests, the bound obtained is not the best possible. One expects, for example, that if the martingale $Y_{k_1k_2}$ is not symmetric with respect to the two indices, the second term in the right-hand side of (2.10) will be positive.

The next corollary is a general form of the strong law of large numbers, extending the Corollary of Chow (1960).

COROLLARY 2.6. *Assume that* $\{Y_{n_1n_2}\}$ *and* $\{C_{n_1n_2}\}$ *satisfy the conditions of Corollary* 2.3. *Assume also that there exists a number* $p \geq 1$ *such that* $E(Y^p_{n_1n_2}) < \infty$, $E(\Delta Y^p_{n_1n_2}) \geq 0$,

$$\sum_{n_1,n_2} C^p_{n_1n_2} E\big(\Delta Y^p_{n_1n_2}\big) < \infty \qquad (2.11)$$

and

$$\sum_{n_1} C^p_{n_1N} E\big(Y^p_{n_1N} - Y^p_{n_1-1N}\big) < \infty, \qquad \sum_{n_2} C^p_{Nn_2} E\big(Y^p_{Nn_2} - Y^p_{Nn_2-1}\big) < \infty, \qquad (2.12)$$

for each $N \in \mathbf{N}$.

Then

$$C_{n_1n_2} Y_{n_1n_2} \to 0 \quad a.s.\ as\ (n_1, n_2) \to \infty,$$

where $\Delta Y^p_{n_1n_2}$ *denotes the backward symmetric difference of the* $\mathbf{Y}$*'s.*

PROOF. Let $\varepsilon > 0$. By Corollary 2.3 we have

$$\frac{\alpha^p - 1}{\alpha^p}\varepsilon^p P\Big\{\sup_{(n_1,n_2)\ge(N,N)} C_{n_1n_2}Y_{n_1n_2} \ge \varepsilon\Big\}$$
$$\le \sum_{(n_1,n_2)\ge(N,N)} (\Delta_F C^p_{n_1n_2})E(Y^p_{n_1n_2})$$
$$= C^p_{NN}E(Y^p_{NN}) + \sum_{(n_1,n_2)\ge(N+1,N+1)} C^p_{n_1n_2}E(\Delta Y^p_{n_1n_2})$$
$$+ \sum_{n_2=N+1}^{\infty} C^p_{Nn_2}E(Y^p_{Nn_2} - Y^p_{Nn_2-1})$$
$$+ \sum_{n_1=N+1}^{\infty} C^p_{n_1N}E(Y^p_{n_1N} - Y^p_{n_1-1N}).$$

By condition (2.11) and the Kronecker lemma for random fields [Martikainen (1986)] we have

$$C^p_{NN} \sum_{(n_1,n_2)\le(N,N)} E(\Delta Y^p_{n_1n_2}) \to 0, \qquad N \to \infty,$$

i.e.,

$$C^p_{NN}E(Y^p_{NN}) \to 0 \quad \text{as } N \to \infty.$$

From this and conditions (2.11) and (2.12) we have that

$$P\Big\{\sup_{(n_1,n_2)\ge(N,N)} C_{n_1n_2}Y_{n_1n_2} \ge \varepsilon\Big\} \to 0 \quad \text{as } N \to \infty.$$

A standard argument completes the proof. □

The following variation of the above result avoids the rather restrictive condition on the coefficients $C_{\mathbf{k}}$, but imposes another condition.

COROLLARY 2.7. *Assume that $Y_{\mathbf{k}}$ is nonnegative and that $\{Y_{\mathbf{k}}\}$ and $\{C_{\mathbf{k}}\}$ satisfy the conditions of Theorem 2.2. Assume that there exists a number $p \ge 1$ such that $E(Y^p_{\mathbf{k}}) < \infty$. Assume also that for some $1 \le s \le r$*

$$\sum_{\mathbf{k}} C^p_{\mathbf{k}}E(Y^p_{\mathbf{k}} - Y^p_{\mathbf{k};s;k_s-1}) < \infty \quad \textit{and} \quad \sum_{\substack{k_i\\ i\ne s}} C^p_{\mathbf{k};s;N}E(Y^p_{\mathbf{k};s;N}) < \infty \tag{2.13}$$

for each $N \in \mathbf{N}$.

Then

$$C_{\mathbf{k}}Y_{\mathbf{k}} \to 0 \quad a.s.\ as\ \mathbf{k} \to \infty.$$

PROOF. As usual, we prove the result for $r = 2$. Let $\varepsilon > 0$. Assume without loss of generality that (2.13) is valid for $s = 2$, i.e.,

$$\sum_{(n_1,n_2)} C^p_{n_1n_2}E(Y^p_{n_1n_2} - Y^p_{n_1n_2-1}) < \infty \quad \text{and} \quad \sum_{n_1} C^p_{n_1N}E(Y^p_{n_1N}) < \infty$$

for each $N \in \mathbf{N}$. By Theorem 2.2,

$$\varepsilon^p P\left\{ \sup_{(n_1, n_2) \geq (N, N)} C_{n_1 n_2} Y_{n_1 n_2} \geq \varepsilon \right\}$$
$$\leq \sum_{(n_1, n_2) \geq (N, N)} \left(C^p_{n_1 n_2} - C^p_{n_1 n_2 + 1} \right) E\left(Y^p_{n_1 n_2} \right)$$
$$= \sum_{n_1 = N}^{\infty} C^p_{n_1 N} E\left(Y^p_{n_1 N} \right) + \sum_{(n_1, n_2) \geq (N, N+1)} C^p_{n_1 n_2} E\left(Y^p_{n_1 n_2} - Y^p_{n_1 n_2 - 1} \right).$$

It is clear by condition (2.13) that

$$P\left\{ \sup_{(n_1, n_2) \geq (N, N)} C_{n_1 n_2} Y_{n_1 n_2} \geq \varepsilon \right\} \to 0 \quad \text{as } N \to \infty,$$

and the result follows. □

Using standard arguments (by first showing the result for symmetric random variables and then using desymmetrization), one can show that Corollary 2.7 contains the following generalization of Kolmogorov's strong law of large numbers.

THEOREM 2.8. *Let* $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$ *where the* $X_{\mathbf{k}}$*'s are independent with* $E(X_{\mathbf{k}}) = 0$ *and* $E(X^2_{\mathbf{k}}) < \infty$ *for each* $\mathbf{k}$. *Assume that* $\sum_{\mathbf{k}} [E(X^2_{\mathbf{k}}) / |\mathbf{k}|^2] < \infty$. *Then*

$$|\mathbf{n}|^{-1} S_{\mathbf{n}} \to 0 \quad a.s. \ \mathbf{n} \to \infty.$$

REMARKS. (a) The above result has also been obtained by Klesov (1981) as an application of his strong law of large numbers for multidimensionaly indexed martingales.

(b) Notice that for $r = 1$ the above result states the ordinary Kolmogorov strong law of large numbers.

Analogues of the above results for *reverse* martingales are not difficult to derive. Let $\{F_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^r\}$ be a nonincreasing array of sub-σ-fields of F, i.e.,

$$F \supseteq F_{\mathbf{n}} \supseteq F_{\mathbf{n}'} \quad \text{for } \mathbf{n} \leq \mathbf{n}'.$$

Then (assuming $X_{\mathbf{n}}$ is $F_{\mathbf{n}}$-measurable and integrable for every $\mathbf{n}$) we say that $\{X_{\mathbf{n}}, F_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^r\}$ is a *reverse martingale* if

(2.14) $$E\{X_{\mathbf{n}} | F_{\mathbf{n}'}\} = X_{\mathbf{n}'} \quad \text{a.s. for } \mathbf{n} \leq \mathbf{n}'.$$

As in the case of forward martingales, we have an equivalent condition to (2.14), namely

(2.15) $$\int_A X_{\mathbf{n}} \, dP = \int_A X_{\mathbf{n}'} \, dP \quad \text{for } A \in F_{\mathbf{n}'}, \mathbf{n} \leq \mathbf{n}'.$$

$\{X_{\mathbf{n}}, F_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^r\}$ is said to be a *reverse submartingale* if the equalities in (2.14) and (2.15) are replaced by $\geq$.

An analogue of Theorem 2.2 is given by

COROLLARY 2.9. *Let* $\{Y_{\mathbf{k}}, F_{\mathbf{k}}, \ k \in \mathbf{N}^r\}$ *be a reverse submartingale. Suppose that the* σ*-fields satisfy*

$$E\{E\{*|F_{\mathbf{k}}\}|F_{\mathbf{l}}\} = E\{*|F_{\mathbf{k}\vee\mathbf{l}}\}, \tag{2.16}$$

where $\mathbf{k}\vee\mathbf{l}$ *denotes the maximum of* $\mathbf{k}$ *and* $\mathbf{l}$ *taken componentwise. Let* $\{C_{\mathbf{k}}, \ \mathbf{k}\in\mathbf{N}^r\}$ *be a nondecreasing array of nonnegative numbers. Then for* $\varepsilon > 0$

$$\begin{aligned}\varepsilon P\Big\{\max_{\mathbf{n}\le\mathbf{k}\le\mathbf{N}} C_{\mathbf{k}}Y_{\mathbf{k}} \ge \varepsilon\Big\} &\le \min_{1\le s\le r}\Bigg\{\sum_{\mathbf{n}\le\mathbf{k}\le\mathbf{N}}\big(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s-1}\big)E\big(Y_{\mathbf{k}}^+\big) \\ &\qquad - \sum_{\substack{k_i\\ i\ne s}} C_{\mathbf{k};s;\mathbf{n}_s}\int_{(\bigcup_{k_s=n_s}^{N_s} B_{k_1\cdots k_r}^{(s)})^C} Y_{\mathbf{k};s;n_s}^+\,dP\Bigg\} \\ &\le \min_{1\le s\le r}\Big\{\sum_{\mathbf{n}\le\mathbf{k}\le\mathbf{N}}\big(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s-1}\big)E\big(Y_{\mathbf{k}}^+\big)\Big\},\end{aligned} \tag{2.17}$$

where $C_{\mathbf{k}} = 0$ *if* $k_i < n_i$ *for some* $i = 1, 2, \ldots, r$.

The analogue of Corollary 2.5, i.e., a Kolmogorov-type inequality for reverse martingales, can be stated as

COROLLARY 2.10. *Assume that* $\{Y_{\mathbf{k}}, F_{\mathbf{k}}, \ \mathbf{k}\in\mathbf{N}^r\}$ *is a reverse martingale satisfying the conditions of Corollary* 2.9. *Then for* $\varepsilon > 0$,

$$\varepsilon^2 P\Big\{\max_{\mathbf{n}\le\mathbf{k}\le\mathbf{N}} |Y_{\mathbf{k}}| \ge \varepsilon\Big\} \le 4^{r-1}E\big(Y_{\mathbf{n}}^2\big). \tag{2.18}$$

REMARKS. Corollaries 2.9 and 2.10, besides being of general interest, are used in Christofides and Serfling (1990) to obtain bounds for the rate of convergence in the strong law of large numbers for U-statistics, as well as to establish an invariance principle for generalized U-statistics.

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