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ASYMPTOTIC ANALYSIS OF INVARIANT DENSITY OF RANDOMLY PERTURBED DYNAMICAL SYSTEMS

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Dedicated to Professor Wendell H. Fleming on his sixtieth birthday

The invariant density of diffusion processes which are small random perturbations of dynamical systems can be expanded in W.K.B. type, as the random effect disappears, in the set in which the Freidlin–Wentzell quasipotential $V(\cdot)$ is of C^∞-class and each coefficient which appears in the expansion is of C^∞-class.

1. Introduction. Let $X^\varepsilon(t,x)$, $t \geq 0$, $x \in R^d$, $\varepsilon > 0$ be the solution of the following stochastic differential equation:

$$dX^\varepsilon(t,x) = b(X^\varepsilon(t,x))\,dt + \varepsilon^{1/2}\,dW(t),\qquad X^\varepsilon(0,x) = x, \tag{1.1}$$

where $b(\cdot)$: $R^d \mapsto R^d$ is Lipschitz continuous and $W(t)$ is a d-dimensional Wiener process. The study of such stochastic processes is originally due to Freidlin and Wentzell [cf. Freidlin and Wentzell (1984)].

Let $p^\varepsilon(x)$ be the invariant density of diffusion processes $X^\varepsilon(t,x)$, that is, the solution of

$$\varepsilon\,\Delta p^\varepsilon(x)/2 - \operatorname{div}(b(x)p^\varepsilon(x)) = 0 \quad \text{in } R^d,\qquad \int_{R^d} p^\varepsilon(x)\,dx = 1, \tag{1.2}$$

where Δ is the d-dimensional Laplacian. In our previous work, we proved the following theorem by way of Malliavin calculus [cf. Watanabe (1987)].

THEOREM 1.1 [Mikami (1988), Corollary 1.6]. *Suppose the following conditions hold*:

(A.1) $b(\cdot) = (b^i(\cdot))_{i=1}^d$ *is of* C^∞*-class in a small neighborhood of* $o \in R^d$.
(A.2) $b(o) = o$, $\lim_{t\to\infty} X^0(t,x) = o$ *for all* $x \in R^d$ *and*

$$\sup\left\{ \sum_{i,j=1}^{d} \left(\partial b^i(o)/\partial x_j\right) e_i e_j \,\middle|\, \sum_{i=1}^{d} (e_i)^2 = 1 \right\} < 0.$$

(A.3) *There exist* ε_0, $R > 0$ *and a nonnegative* C^2*-function* $w(x)$ *which tends to* ∞ *as* $|x| \to \infty$, *so that*

$$\varepsilon\,\Delta w(x)/2 + \langle b(x), \nabla w(x)\rangle \leq -1, \qquad |x| \geq R,\ \varepsilon < \varepsilon_0.$$

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Then there exist continuous functions $R_k(\cdot)$, $k \geq 0$, *and* $c > 0$ *such that*

$$\limsup_{\varepsilon \to 0} \left| R^{\varepsilon}(x) - \sum_{i=0}^{n} \varepsilon^i R_i(x) \right| \varepsilon^{-(n+1)} < +\infty, \tag{1.3}$$

for all $n \geq 0$, *uniformly for all* $|x| < c$. *Here we put*

$$R^{\varepsilon}(x) = \varepsilon^{d/2} p^{\varepsilon}(x) \exp(V(x)/\varepsilon), \tag{1.4}$$

$$V(x) = \inf\left\{ \int_0^T |\dot{\varphi}(t) - b(\varphi(t))|^2 \, dt/2; \varphi(0) = o, \varphi(T) = x, T > 0 \right\}. \tag{1.5}$$

REMARK 1.1. Although we assumed the Hölder continuity of the first derivatives of $b(\cdot)$ on R^d in our previous work just to use Theorem 3 of Day (1988), we do not have to assume it. In fact, in his paper, Day used only the fact that the first derivatives of $b(\cdot)$ are Hölder continuous on a bounded domain which contains the set $\{X^0(t, x); t \geq 0, x \in \overline{D}\}$, where D is a bounded domain under consideration. In our case, this condition is satisfied by (A.1) and (A.2), since we take a sufficiently small neighborhood of o as D.

In this paper we prove the following results.

THEOREM 1.2. *Under assumptions* (A.1)–(A.3), *the functions* $R_k(\cdot)$, $k \geq 0$, *are of* C^{∞}*-class in some neighborhood of* $o \in R^d$.

Before we state another result, we give an additional assumption.

(A.4.r) $b(\cdot)$ is of C^{∞}-class in the set $\Omega^r = \{x \in R^d | V(x) < r\}$.

THEOREM 1.3. *Suppose that* (A.2) *and* (A.3) *hold. Then for any* $r > 0$ *for which* (A.4.r) *holds, there exist* C^{∞}*-functions* $R_k(\cdot)$, $k \geq 0$, *such that, for any compact subset* K *of the set* $\Omega_{\infty}^r = \{x \in \Omega^r | V$ *is infinitely differentiable at* $x\}$ *and any* $n \geq 0$,

$$\limsup_{\varepsilon \to 0} \left(\sup_{x \in K} |R^{\varepsilon}(x) - \sum_{i=0}^{n} \varepsilon^i R_i(x)| \varepsilon^{-(n+1)} \right) < +\infty \tag{1.6}$$

and that, for all $k \geq 0$,

$$\langle \nabla R_k(x), -b^*(x) \rangle + F(x) R_k(x) - \Delta R_{k-1}(x)/2 = 0, \qquad x \in \Omega_{\infty}^r, \tag{1.7}$$

where we put $R_{-1}(x) \equiv 0$, $F(x) = \Delta V(x)/2 + \operatorname{div}(b(x))$ *and* $b^*(x) = -\nabla V(x) - b(x)$. *Moreover*,

$$R_0(o) = \left\{ (2\pi)^{-d} |D^2 V(o)| \right\}^{1/2}, \tag{1.8}$$

where $|D^2 V(o)|$ *denotes the determinant of the positive definite matrix* $D^2 V(o) = (\partial^2 V(o)/\partial x_i \, \partial x_j)_{i,j=1}^d$ [*cf*. *Day* (1987), *page* 131, (4.2)].

REMARK 1.2. Equation (1.7) can be obtained from the formal argument

$$
\begin{aligned}
0 &= \langle \nabla R^{\varepsilon}(x), -b^*(x)\rangle + F(x)R^{\varepsilon}(x) - \varepsilon \Delta R^{\varepsilon}(x)/2 \\
&= \sum_{i=0}^{\infty} \varepsilon^i \{\langle \nabla R_i(x), -b^*(x)\rangle + F(x)R_i(x) - \Delta R_{i-1}(x)/2\},
\end{aligned}
\tag{1.9}
$$

where we put $R^{\varepsilon}(x) = \sum_{i=0}^{\infty} \varepsilon^i R_i(x)$.

REMARK 1.3. (1.8) was obtained by Sheu by a method different from ours, under stronger assumptions than ours [cf. Sheu (1986)].

As a corollary of Theorem 1.3, we get the following result about the system of first-order partial differential equations. Before we state it, we give two other assumptions.

(A.2′) $b(o) = o$ and

$$
\sup\left\{ \sum_{i,j=1}^{d} \left(\partial b^i(o)/\partial x_j\right) e_i e_j \,\middle|\, \sum_{i=1}^{d} (e_i)^2 = 1 \right\} < 0.
$$

(A.5.r) $\lim_{t\to\infty} X^0(t,x) = o$ for all $x \in \overline{\Omega}^r$.

COROLLARY 1.4. *Suppose that* (A.2′) *holds. Then for any* $r > 0$ *for which* (A.4.r) *and* (A.5.r) *hold and any sequence* $\{C_k\}_{k=-1}^{\infty}$, *there exist unique* C^{∞}-*functions* $\tilde{R}_k(\cdot)$, $k \ge -1$, $(R_{-1}(x) \equiv C_{-1})$, *which satisfy* (1.7) *and*

$$
\tilde{R}_k(o) = C_k, \qquad k \ge -1. \tag{1.10}
$$

In particular, $\{\tilde{R}_k(\cdot)\}_{k=0}^{\infty}$ *is given by*

$$
\begin{aligned}
\tilde{R}_k(x) &= C_k \exp\left(-\int_0^{\infty} F(z(s,x))\,ds\right) \\
&\quad + \int_0^{\infty} \Delta R_{k-1}(z(s,x)) \exp\left(-\int_0^{s} F(z(u,x))\,du\right) ds/2,
\end{aligned}
\tag{1.11}
$$

for $x \in \Omega_{\infty}^r$, $k \ge 0$. *Here* $z(t,x)$, $t \ge 0$, $x \in \Omega_{\infty}^r$ *is the solution of*

$$
\begin{aligned}
\dot{z}(t,x) &= b^*(z(t,x)), \\
z(0,x) &= x
\end{aligned}
\tag{1.12}
$$

(*see Proposition* 2.2). *Moreover* $\tilde{R}_k(\cdot)$ *can be represented as a linear combination of* $R_0(\cdot), \dots, R_k(\cdot)$ *for* $k \ge 0$.

REMARK 1.4. When $n = 0$, the assumptions can be weakened [cf. Day (1987), Theorem 3].

REMARK 1.5. Ω_{∞}^r in Theorem 1.3 exists [see Proposition 2.1(a)].

REMARK 1.6. Assume that (A.2)–(A.3) hold and that $b(\cdot)$ is of C^∞-class in R^d. Let D be a bounded domain in R^d which contains o. Put $\Gamma = \{x \in \partial D | V(x) = \min\{V(y) | y \in \partial D\}\}$. If the exterior sphere condition is satisfied on Γ, V is of C^∞-class in some neighborhood of Γ [cf. Day (1986), Theorem]. Put $\tau_D^\varepsilon = \inf\{t > 0 | X^\varepsilon(t, x) \notin D\}$. Let n_y denote the normalized outer normal at $y \in \partial D$. Let $f: \partial D \mapsto R$ be continuous. Denote by $Q^\varepsilon(dx)$ the measure on ∂D defined by

$$Q^\varepsilon(dx) = \langle n_x, b(x)\rangle p^\varepsilon(x)\, dx \Big/ \int_{\partial D} \langle n_y, b(y)\rangle p^\varepsilon(y)\, dy. \tag{1.13}$$

If the inequality

$$\limsup_{\varepsilon \to 0} \varepsilon \log \left| \int_{\partial D} f(y) P_x(X^\varepsilon(\tau_D^\varepsilon) \in dy) - \int_{\partial D} f(y) Q^\varepsilon(dy) \right| < 0 \tag{1.14}$$

holds for $x \in D$, then we can obtain the asymptotic expansion of the exit measure $P_x(X^\varepsilon(\tau_D^\varepsilon) \in dy)$ on ∂D [cf. Day (1984) and (1987)]. But we have not proven (1.14), although (1.14) holds when $d = 1$ [cf. Freidlin and Wentzell (1984), page 121].

In Section 2 we give the propositions which are necessary for the proof of our results. In Section 3 we prove the main results.

2. Propositions. In this section we state propositions which are necessary for the proof of our results.

PROPOSITION 2.1 [Day and Darden (1985), Theorems 2 and 6 and Corollary 7]. *Suppose that* (A.2′) *holds. Then for any* $r > 0$ *for which* (A.4.r) *and* (A.5.r) *hold*:

(a) Ω_∞^r *in Theorem* 1.3 *exists, contains* o *and is a dense subdomain of the set* Ω^r.

(b) *For any* $x \in \Omega_\infty^r$, *there exists a unique function* $\varphi(t)$, $-\infty < t \le 0$, *such that* $\varphi(0) = x$, $\lim_{t \to -\infty} \varphi(t) = o$, $\varphi(t) \in \Omega_\infty^r$, $-\infty < t \le 0$, *and*

$$V(x) = \int_{-\infty}^0 |\dot\varphi(t) - b(\varphi(t))|^2\, dt/2, \tag{2.1}$$

$$\dot\varphi(t) = b^*(\varphi(t)). \tag{2.2}$$

PROPOSITION 2.2 [cf. Hale (1969)]. *Suppose that* (A.2′) *holds. Then for any* $r > 0$ *for which* (A.4.r) *and* (A.5.r) *hold, any* $n \ge 0$ *and any compact subset* K *of* Ω_∞^r, *there exist* λ, $a_n = a(n, K) > 0$ *and compact set* $A = A(K)$, $K \subset A \subset \Omega_\infty^r$, *such that*

$$\begin{aligned} &|\partial^n z(t,x)/\partial x_{i_1} \cdots \partial x_{i_n}| \le a_n \exp(-\lambda t), \\ &\qquad\qquad i_1, \dots, i_n = 1, \dots, d,\ t \ge 0,\ x \in K, \end{aligned} \tag{2.3}$$

$$z(t,x) \in A, \qquad t \ge 0,\ x \in K. \tag{2.4}$$

PROOF. (When $n = 0$.) Take $r_0 > 0$ so that V is of C^∞-class in the set $\{x; |x| < 2r_0\}$. Since

$$z(t,x) = \{\exp(Bt)\}x + \int_0^t \{\exp(B(t-s))\}(b^*(z(s,x)) - Bz(s,x))\,ds, \tag{2.5}$$

$$|z(t,x)| \le |x|\exp(-(2\lambda - C_0)t) \le r_0\exp(-\lambda t), \qquad |x| < r_0,\ t \ge 0, \tag{2.6}$$

for r_0 sufficiently small, where $B = (\partial b^*(o)^i/\partial x_j)_{i,j=1}^d$,

$$C_0 = r_0 d^2 \max\left\{|\partial^2 b^*(x)^i/\partial x_j\,\partial x_k|;\ i,j,k = 1,\dots,d,\ |x| < r_0\right\}/2$$

and $2\lambda > 0$ is determined by the real parts of the eigenvalues of $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ [cf. Day (1987), (4.2)].

Before we proceed further, we prove (2.4).

PROOF OF (2.4). Take $T = T(K) > 0$ such that for all $x \in K$, $|z(t,x)| < r_0$ for some $t \in [0,T]$ [cf. Freidlin and Wentzell (1984), page 110, Lemma 2.2]. Suppose that there exist $x_n \in K$ and $t_n \in [0,T]$ such that $\mathrm{dist}(z(t_n,x_n), \partial\Omega_\infty^r) \to 0$ as $n \to \infty$. Then there exist $x \in K$, $|y| \le r_0$, $t \in [0,T]$ and a convergent subsequence $\{x_{n'}\}_{n'=1}^\infty$ such that, as $n' \to +\infty$,

$$\sup_{0 \le s \le T} |z(s,x_{n'}) - z(s,x)| \to 0, \tag{2.7}$$

$$z(T,x_{n'}) \to z(T,x) = y, \tag{2.8}$$

$$z(t_{n'},x_{n'}) \to z(t,x) \in \partial\Omega_\infty^r \tag{2.9}$$

[cf. Freidlin and Wentzell (1984), Chapter 4], which is a contradiction to Proposition 2.1(b) and completes the proof of (2.4). □

From (2.4), (2.3) holds when $n = 0$.

(When $n = 1$.) Put $B(x) = (\partial b^*(x)^i/\partial x_j)_{i,j=1}^d$. Denote by e_i the d-dimensional vector whose i-component $= 1$ and j-component $= 0$ for $j \ne i$, $i = 1,\dots,d$. Let $Y(t,x) = (Y(t,x)_{ij})_{i,j=1}^d$ be the solution of

$$\dot{Y}(t,x) = B(z(t,x))Y(t,x), \quad Y(0,x) = \text{identity matrix}. \tag{2.10}$$

Then in the same way as in the case in which $n = 0$ [take $B(x)$ instead of $b^*(x)$],

$$\left(\sum_{i,J=1}^d (Y(t,x)_{ij})^2\right)^{1/2} \le \exp(-2\lambda t)\exp(2c_0 t) \le \exp(-\lambda t), \qquad |x| < r_0,\ t \ge 0, \tag{2.11}$$

for r_0 sufficiently small. Since $\partial z(t,x)/\partial x_i = Y(z(t,x))e_i$, $i = 1,\dots,d$, $t \ge 0$, $|x| < r_0$,

$$|\partial z(t,x)/\partial x_i| \le \exp(-\lambda t), \qquad i = 1,\dots,d,\ t \ge 0,\ |x| < r_0, \tag{2.12}$$

which implies that (2.3) holds when $n = 1$, since

$$\partial z(t+T,x)/\partial x_i = \sum_{j=1}^{d} \partial z(t,y)/\partial y_i|_{y=z(T,x)}\, \partial z^j(T,x)/\partial x_i, \quad i = 1,\dots,d,\ t > 0,\ x \in K, \tag{2.13}$$

$$\sup\{|\partial z^i(T,x)/\partial x_j|;\ i,j = 1,\dots,d,\ x \in K\} < +\infty \tag{2.14}$$

[from (2.4) and (2.10)].

(When $n \geq 2$.) Inductively we can show that, for any $n \geq 2$, there exist functions $F_{(i_1,\dots,i_n)}(s,x)$, which is a polynomial of derivatives of $z(t,x)$ with respect to x up to $(n-1)$th order of degree greater than or equal to 2 with coefficients $\partial^k B(z(t,x))/\partial z_{j_1} \cdots \partial z_{j_k}$, $k = 1,\dots,n-1$, $j_1,\dots,j_k = 1,\dots,d$ [from (2.10)], and $C_n > 0$ such that

$$\partial^n z(t,x)/\partial x_{i_1} \cdots \partial x_{i_n} = \int_0^t B(z(s,x))\, \partial^n z(s,x)/\partial x_{i_1} \dots \partial x_{i_n}\, ds + \int_0^t F_{(i_1,\dots,i_n)}(s,x)\, ds, \tag{2.15}$$

$$\partial^n z(t,x)/\partial x_{i_1} \cdots \partial x_{i_n} = \int_0^t Y(t,x) Y(s,x)^{-1} F_{(i_1,\dots,i_n)}(s,x)\, ds, \tag{2.16}$$

$$|F_{(i_1,\dots,i_n)}(t,x)| \leq C_n \exp(-2\lambda t), \quad i_1,\dots,i_n = 1,\dots,d,\ t \geq 0,\ |x| < r_0. \tag{2.17}$$

Hence, for all $n \geq 2$, $i_1,\dots,i_n = 1,\dots,d$, $t \geq 0$, $|x| < r_0$,

$$|\partial^n z(t,x)/\partial x_{i_1} \cdots \partial x_{i_n}| \leq C_n \lambda^{-1} \exp(-\lambda t), \tag{2.18}$$

which implies (2.3) in the same way as in the case in which $n = 1$. □

3. Proof of main results. In this section we prove our results. By Theorem 3 of Day (1988), we can assume that $b(\cdot)$: $R^d \mapsto R^d$ has bounded derivatives of all orders, which implies the smoothness of $p^\varepsilon(\cdot)$.

PROOF OF THEOREM 1.2. Take r_0 in the proof of Proposition 2.2 small enough so that $3r_0 < c$, where c is a constant in Theorem 1.1. We change b^* outside the set $\{x;\ |x| < 2r_0\}$ so that b^* has bounded derivatives of all orders. Let $z^\varepsilon(t,x)$, $t \geq 0$, $|x| < r_0$ be the solution of the following stochastic differential equation:

$$dz^\varepsilon(t,x) = b^*(z^\varepsilon(t,x))\, dt + \varepsilon^{1/2}\, dW(t), \quad z^\varepsilon(0,x) = x. \tag{3.1}$$

Put $\sigma = \inf\{t > 0;\ |z^\varepsilon(t,x)| \geq 2r_0\}$.

Suppose that $r_0(\cdot),\dots,R_{n-1}(\cdot)$ are C^∞-solutions of (1.7) in $\{x;\ |x| < r_0\}$. Put

$$R_n^\varepsilon(x) = \left(R^\varepsilon(x) - \sum_{i=0}^{n-1} \varepsilon^i R_i(x)\right)\varepsilon^{-n}. \tag{3.2}$$

Then from (1.7) and (1.9), by the Itô formula,

$$(3.3)\qquad \begin{aligned} R_n^\varepsilon(x) &= E_x\left[R_n^\varepsilon(z^\varepsilon(\sigma\wedge T))\exp\left\{-\int_0^{\sigma\wedge T}F(z^\varepsilon(s))\,ds\right\}\right] \\ &\quad + E_x\left[\int_0^{\sigma\wedge T}(\Delta R_{n-1}(z^\varepsilon(s))/2)\exp\left\{-\int_0^s F(z^\varepsilon(u))\,du\right\}ds\right]. \end{aligned}$$

Let $\varepsilon \to 0$. Then for $|x| < r_0$,

$$(3.4)\qquad \begin{aligned} R_n(x) &= R_n(z(T,x))\exp\left\{-\int_0^T F(z(s,x))\,ds\right\} \\ &\quad + \int_0^T(\Delta R_{n-1}(z(s,x))/2)\exp\left\{-\int_0^s F(z(u,x))\,du\right\}ds. \end{aligned}$$

Put $x = o$ in (3.4). Then we have

$$(3.5)\qquad \Delta R_{n-1}(o) = 0$$

since, from Day [(1987), page 132],

$$(3.6)\qquad F(o) = 0.$$

Therefore, we have

$$(3.7)\qquad \int_0^\infty |F(z(s,x))|\,ds + \int_0^\infty |\Delta R_{n-1}(z(s,x))|\,ds < +\infty,$$

since

$$(3.8)\qquad \int_0^\infty |F(z(s,x))|\,ds \le \sup_{|y|<r_0}|\nabla F(y)|\int_0^\infty |z(s,x)|\,ds,$$

$$(3.9)\qquad \begin{aligned} &\int_0^\infty |\Delta R_{n-1}(z(s,x))|\,ds \\ &\quad \le \sup_{|y|<r_0}\left(\sum_{i,j=1}^d |\partial^3 R_{n-1}(y)/\partial y_i^2\,\partial y_j|^2\right)^{1/2}\int_0^\infty |z(s,x)|\,ds. \end{aligned}$$

Let $T \to \infty$ in (3.4). Then for $|x| < r_0$,

$$(3.10)\qquad \begin{aligned} R_n(x) &= R_n(o)\exp\left\{-\int_0^\infty F(z(s,x))\,ds\right\} \\ &\quad + \int_0^\infty(\Delta R_{n-1}(z(s,x))/2)\exp\left\{-\int_0^s F(z(u,x))\,du\right\}ds \end{aligned}$$

and $R_n(\cdot)$ is a C^∞-solution of (1.7) in $\{x;\ |x| < r_0\}$ from Proposition 2.2 and the mean value theorem. □

Proof of Theorem 1.3. First we prove asymptotic expansion of $R^\varepsilon(\cdot)$. Let K be a compact subset of Ω_∞^r, A be the corresponding compact set in Proposition 2.2, $T = \max\{\inf\{t > 0;\ |z(t,x)| < r_0/2\};\ x \in K\}$ and $\tau = \inf\{T \wedge t;\ \mathrm{dist}(z^\varepsilon(t,x), A) \ge \mathrm{dist}(A, \partial\Omega_\infty^r)/2\}$. Then in the same way as Day [(1987), Section 5] there exist $0 < \delta < (r_0/4) \wedge (\mathrm{dist}(A, \partial\Omega_\infty^r)/2)$ and $C' > 0$ such that for all $x \in K$,

$$\begin{aligned} R^\varepsilon(x) &= E_x\left[R^\varepsilon(z^\varepsilon(\tau))\exp\left\{-\int_0^\tau F(z^\varepsilon(s))\,ds\right\}\right] \\ &= E_x\left[R^\varepsilon(z^\varepsilon(T))\exp\left\{-\int_0^T F(z^\varepsilon(s))\,ds\right\};\right. \\ &\qquad\qquad \left.\sup_{0\le t\le T}|z^\varepsilon(t) - z(t,x)| < \delta\right] \\ &\quad + O(\exp(-C'/\varepsilon)) \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{3.11}$$

If we modify $V(\cdot)$ outside Ω_∞^r so that $V(\cdot)$ has bounded derivatives of all orders, then there exist $z^k(t,x)$, $k \ge 0$, $T \ge t \ge 0$, such that for any $p \ge 0$ and $n \ge 0$,

$$\begin{aligned} \limsup_{\varepsilon\to 0} \left(\sup\left\{E_x\left[\left|z^\varepsilon(t) - \sum_{i=0}^{2n+1}\varepsilon^{i/2}z^i(t,x)\right|^p\right]\varepsilon^{-(n+1)};\right.\right. \\ \left.\left. x \in K, 0 \le t \le T\right\}\right) < +\infty \end{aligned} \tag{3.12}$$

[cf. Freidlin and Wentzell (1984), page 56, Theorem 2.2]. This, together with Theorem 1.2, completes the proof of the asymptotic expansion of $R^\varepsilon(\cdot)$.

In fact, if $\sup_{0\le t\le T}|z^\varepsilon(t) - z(t,x)| < \delta$, then $|z^\varepsilon(T)| < 3r_0/4$ and, for any $n \ge 0$,

$$\begin{aligned} &\left|E_x\left[R^\varepsilon(z^\varepsilon(T))\exp\left\{-\int_0^T F(z^\varepsilon(s))\,ds\right\};\ \sup_{0\le t\le T}|z^\varepsilon(t) - z(t,x)| < \delta\right]\right. \\ &\quad -E_x\left[\sum_{k=0}^{n}\varepsilon^k R_k(z^\varepsilon(T))\sum_{k=0}^{2n+1}\left(\int_0^T[F(z(s,x)) - F(z^\varepsilon(s))]\,ds\right)^k(k!)^{-1}\right. \\ &\qquad\qquad \left.\left.\times\exp\left\{-\int_0^T F(z(s,x))\,ds\right\};\ \sup_{0\le t\le T}|z^\varepsilon(t) - z(t,x)| < \delta\right]\right| \\ &\le \sup_{|z|<r_0}\left|R^\varepsilon(z) - \sum_{k=0}^{n}\varepsilon^k R_k(z)\right|\exp\big(T\sup\{|F(z)|; z \in R^d\}\big) \\ &\quad + \sup\left\{\sum_{k=0}^{n}|R_k(z)|; |z| < r_0\right\}\exp\big(3T\sup\{|F(z)|; z \in R^d\}\big) \\ &\quad \times T^{2(n+1)}\sup_{0\le t\le T}E_x\left[|F(z^\varepsilon(t)) - F(z(t,x))^{2(n+1)}\right](\{2(n+1)\}!)^{-1} \\ &= O(\varepsilon^{n+1}) \end{aligned} \tag{3.13}$$

and, for any $i, j = 0, \ldots, 2n+1$,

$$\Bigg| E_x\Bigg[R_i(z^\varepsilon(T))\left(\int_0^T [F(z(s,x)) - F(z^\varepsilon(s))]\,ds\right)^j;\ \sup_{0\le t\le T}|z^\varepsilon(t) - z(t,x)| < \delta\Bigg]$$

$$- E_x\Bigg[\sum_{k=0}^{2n+1}(k!)^{-1}\sum_{i_1,\ldots,i_k=1}^{d}\frac{\partial^k R_i(z(T,x))}{\partial z_{i_1}\cdots\partial z_{i_k}}(z^\varepsilon(T) - z(T,x))^{i_1}\times\cdots$$

$$\times(z^\varepsilon(T) - z(T,x))^{i_k}\left(\int_0^T [F(z(s,x)) - F(z^\varepsilon(s))]\,ds\right)^j\Bigg]\Bigg|$$

$$\le (\{2(n+1)\}!)^{-1}\sup\Big\{|\partial^{2(n+1)}R_i(z)/\partial z_{i_1}\cdots\partial z_{i_{2(n+1)}}|;$$

$$|z| < r_0,\ i_1,\ldots,i_{2(n+1)} = 1,\ldots,d\Big\} \tag{3.14}$$

$$\times d^{2(n+1)}E_x\big[|z^\varepsilon(T) - z(T,x)|^{2(n+1)}\big]\big(2T\sup\{|F(z)|; z\in R^d\}\big)^j$$

$$+ 2(n+1)d^{2(n+1)}$$

$$\times\sup\Big\{|\partial^k R_i(z)/\partial z_{i_1}\cdots\partial z_{i_k}|;\ |z| < r_0,$$

$$0\le k\le 2n+1,\ i_1,\ldots,i_k = 1,\ldots,d\Big\}$$

$$\times\big(2T\sup\{|F(z)|; z\in R^d\}\big)^j\bigvee_{m=0}^{2n+1}\big(E_x\big[|z^\varepsilon(T) - z(T,x)|^{2m}\big]\big)^{1/2}$$

$$\times\left(P_x\left(\sup_{0\le t\le T}|z^\varepsilon(T) - z(T,x)| > \delta\right)\right)^{1/2}$$

$$= O(\varepsilon^{n+1}),$$

uniformly with respect to $x \in K$, as $\varepsilon \to 0$. Then we can use (3.5) in Freidlin and Wentzell [(1984), page 61]. Since $R^\varepsilon(x) = R^{(-\varepsilon^{1/2})^2}(x) = R^{(\varepsilon^{1/2})^2}(x)$, the coefficients of odd power of $\varepsilon^{1/2}$ equal 0 [cf. Freidlin and Wentzell (1984), page 61].

Smoothness can be proven by induction from Theorem 1.2, (3.4) and Proposition 2.2.

(1.7) can be proven in the following way. From (3.10),

$$R_n(z(t,x)) = R_n(o)\exp\left\{-\int_t^\infty F(z(s,x))\,ds\right\} + \int_t^\infty (\Delta R_{n-1}(z(s,x))/2)\exp\left\{-\int_t^s F(z(u,x))\,du\right\}ds. \tag{3.15}$$

Differentiate both sides of (3.15) with respect to t and put $t = 0$. Then we have (1.7).

At last we prove (1.8). For $r_0 > \delta > 0$,

$$\begin{aligned} 1 &= \int_{R^d} p^\varepsilon(x)\,dx \\ &= \int_{|x|>\delta} p^\varepsilon(x)\,dx + \int_{|x|<\varepsilon^{5/12}} R^\varepsilon(x)\varepsilon^{-d/2}\exp(-V(x)/\varepsilon)\,dx \\ &\quad + \int_{\varepsilon^{5/12}\le|x|\le\delta} R^\varepsilon(x)\varepsilon^{-d/2}\exp(-V(x)/\varepsilon)\,dx \\ &:= (1) + (2) + (3). \end{aligned} \tag{3.16}$$

Then $(1) = (1)^\varepsilon \to 0$ as $\varepsilon \to 0$ [cf. Freidlin and Wentzell (1984), page 131]. $(3) = (3)^\varepsilon \to 0$ as $\varepsilon \to 0$, since

$$\begin{aligned} (3) &\le \sup_{|x|\le\delta} R^\varepsilon(x)\int_{\varepsilon^{5/15}\le|x|\le\delta} \varepsilon^{-d/2} \\ &\qquad\qquad \times\exp\{-(V(x) - V(o) - \langle\nabla V(o), x\rangle)/\varepsilon\}\,dx \\ &\le \sup_{|x|\le\delta} R^\varepsilon(x)C(\delta)\varepsilon^{-d/2} \\ &\qquad \times\exp\left(-\inf\{\langle D^2V(y)x, x\rangle/\varepsilon;\ |x| \ge \varepsilon^{5/12},\ |y| < \delta\}/2\right) \\ &\le \sup_{|x|\le\delta} R^\varepsilon(x)C(\delta)\varepsilon^{-d/2} \\ &\qquad \times\exp\left(-\inf\{\langle D^2V(y)z, z\rangle;\ |z| = 1,\ |y| < \delta\}\varepsilon^{-1/6}/2\right)\to 0 \\ &\qquad\qquad\qquad \text{as} \quad \varepsilon\to 0, \end{aligned} \tag{3.17}$$

if δ is sufficiently small, because $D^2V(o)$ is positive definite. Here $C(\delta)$ denotes the volume of the set $\{x;\ |x| < \delta\}$. With respect to $(2) = (2)^\varepsilon$, we have

$$\limsup_{\varepsilon\to 0}\,(2)^\varepsilon \le R_0(o)\left\{(2\pi)^d|D^2V(o)|^{-1}\right\}^{1/2}, \tag{3.18}$$

$$\liminf_{\varepsilon\to 0}\,(2)^\varepsilon \ge R_o(o)\left\{(2\pi)^d|D^2V(o)|^{-1}\right\}^{1/2}. \tag{3.19}$$

Since $(1) + (2) + (3) = 1$, we have (1.8).

(3.18) can be proven in the following way:

$$
\begin{aligned}
(2) \le & \sup_{|x| \varepsilon^{5/12}} R^{\varepsilon}(x) \int_{|x| \le \varepsilon^{5/12}} \varepsilon^{-d/2} \\
& \qquad \times \exp\{-(V(x) - V(o) - \langle \nabla V(o), x \rangle)/\varepsilon\}\, dx \\
\le & \sup_{|x| \le \varepsilon^{5/12}} R^{\varepsilon}(x) \varepsilon^{-d/2} \int_{|x/\varepsilon^{1/2}| \le \varepsilon^{-1/12}} dx \exp\{-\langle D^2 V(o)x, x \rangle / 2\varepsilon\} \\
& \qquad \times \exp\Big(d^3 \sup\big\{|\partial^3 V(z)/\partial z_i\, \partial z_j\, \partial z_k|; \\
& \qquad\qquad |z| < \varepsilon^{5/12}, i, j, k = 1, \dots, d\big\} |x/\varepsilon^{4/12}|^2/3!\Big) \\
\le & \sup_{|x| \le \varepsilon^{5/12}} R^{\varepsilon}(x) \big\{(2\pi)^d |D^2 V(o)|^{-1}\big\}^{1/2} \big\{(2\pi)^d |D^2 V(o)|^{-1}\big\}^{-1/2} \\
& \times \int_{|y| \le \varepsilon^{-1/12}} \exp\{-\langle D^2 V(o)y, y \rangle / 2\}\, dy \\
& \qquad \times \exp\Big(d^3 \sup\big\{|\partial^3 V(z)/\partial z_i\, \partial z_j\, \partial z_k|; \\
& \qquad\qquad |z| < \varepsilon^{5/12}, i, j, k = 1, \dots, d\big\} \varepsilon^{1/4}/3!\Big) \\
\to & R_0(o) \big\{(2\pi)^d |D^2 V(o)|^{-1}\big\}^{1/2} \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}
\tag{3.20}
$$

(3.19) can be proven in the same way. □

REMARK 3.1. Put $x = o$ in (3.4) with $n = 0$. Then we get (3.6), since $R_0(o) \neq 0$ from (1.8).

REMARK 3.2. $z^i(\cdot, x)$, $i \geq 0$, in (3.12) depends only on the derivatives of b^* in the curve $\{z(t, x)\}_{t \geq 0}$.

PROOF OF COROLLARY 1.4. First we prove uniqueness.

Uniqueness. Let $\{\tilde{R}_k(\cdot)\}_{k=-1}^{\infty}$ be C^{∞}-solutions of (1.7) and (1.10). Then for $x \neq o$, (3.4) holds since

$$
\begin{aligned}
\tilde{R}_k(z(t, x)) - \tilde{R}_k(x) &= \int_0^t \langle \nabla \tilde{R}_k(z(s, x)), \dot{z}(s, x) \rangle\, ds \\
&= -\int_0^t \langle \nabla \tilde{R}_k(z(s, x)), \nabla V(z(s, x)) + b(z(s, x)) \rangle\, ds \\
&= \int_0^t F(z(s, x)) \tilde{R}_k(z(s, x))\, ds \\
&\quad - \int_0^t \Delta \tilde{R}_{k-1}(z(s, x))\, ds/2.
\end{aligned}
$$

Let $t \to \infty$ in (3.4). Then we get (1.11) since $\lim_{t \to \infty} z(t, x) = o$, which implies the uniqueness.

Existence. We define $\{\tilde{R}_k(\cdot)\}_{k=-1}^{\infty}$ by (1.10) and (1.11). As we will see later, $\{\tilde{R}_k(\cdot)\}_{k=-1}^{\infty}$ is well defined. Change $b(\cdot)$ outside Ω^r so that (A.2) and (A.3) are satisfied.

By induction, we can show that for any $n \geq 0$, there exist $a_1^n, \dots, a_n^n$ such that

$$\tilde{R}_n(x) = \sum_{k=1}^{n} a_k^n R_k(x), \qquad x \in \Omega_\infty^r, \tag{3.21}$$

which completes the proof. In fact,

$$\tilde{R}_0(x) = C_0 R_0(o)^{-1} R_0(x) \quad \text{from (3.10)}. \tag{3.22}$$

Suppose that

$$\tilde{R}_n(x) = \sum_{k=0}^{n} a_k^n R_k(x) \quad \text{for some } a_k^n, \qquad k = 0, \dots, n. \tag{3.23}$$

Then from (1.11) and (3.10), we have

$$\begin{aligned}
\tilde{R}_{n+1}(x) &= C_{n+1} R_0(o)^{-1} R_0(x) \\
&\quad + \sum_{k=0}^{n} a_k^n \int_0^\infty \Delta R_k(z(s,x)) \exp\left(- \int_0^s F(z(u,x))\, du \right) ds/2 \\
&= \sum_{k=0}^{n} a_k^n \left\{ R_{k+1}(o) \exp\left(- \int_0^\infty F(z(s,x))\, ds \right) \right. \\
&\quad \left. + \int_0^\infty \Delta R_k(z(s,x)) \exp\left(- \int_0^s F(z(u,x))\, du \right) ds/2 \right\} \\
&\quad + \left\{ C_{n+1} R_0(o)^{-1} - \sum_{k=0}^{n} a_k^n R_{k+1}(o) R_0(o)^{-1} \right\} R_0(x) \\
&= \left\{ C_{n+1} R_0(o)^{-1} - \sum_{k=0}^{n} a_k^n R_{k+1}(o) R_0(o)^{-1} \right\} R_0(x) \\
&\quad + \sum_{k=1}^{n+1} a_{k-1}^n R_k(x).
\end{aligned} \tag{3.24}$$

□

REMARK 3.3. From the above proof, for a_k^n, $k = 0, \dots, n$, we have

$$a_0^{n+1} = \left(C_{n+1} - \sum_{k=0}^{n} a_k^n R_k(o) \right) R_0(o)^{-1}, \tag{3.25}$$

$$a_k^{n+1} = a_{k-1}^n = a_0^{n+1-k}, \qquad n + 1 \geq k \geq 1. \tag{3.26}$$

REMARK 3.4. If we can prove, by the analytic method, that $\Delta R_k(o) = 0$ provided that $R_0(\cdot), \dots, R_k(\cdot)$ are C^∞-solutions of (1.7), $k \geq 0$, then Corollary 1.4 can be proven by the purely analytic method. The fact that $\Delta R_k(o) = 0$,

$k \geq 0$, was obtained from the asymptotic expansion of $R^{\varepsilon}(\cdot)$ and its representation by way of the Itô formula.

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