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POLAR SETS AND MULTIPLE POINTS FOR SUPER-BROWNIAN MOTION[1,2]

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We study the closed support of the measure-valued diffusions of Watanabe and Dawson. When the spatial motion is Brownian, sufficient conditions involving capacity are given for a fixed set to be hit by the k-multiple points of the support process. The conditions are close to the necessary conditions found by Dawson, Iscoe and Perkins and lead to necessary and sufficient conditions for the existence of k-multiple points. When the spatial motion is a symmetric stable process of index $\alpha < 2$, the closed support is shown to be $\mathbb{R}^d$ or $\varnothing$.

1. Introduction and statement of results. Let $M_F(\mathbb{R}^d)$ be the space of finite measures on $\mathbb{R}^d$ with the topology of weak convergence and write $m(\phi)$ for $\int \phi(x)\, dm(x)$. Let Y_t denote a right-continuous $\mathbb{R}^d$-valued Lévy process on canonical path space $(\Omega^0, \mathscr{F}^0)$, starting at x under the probability P_0^x. The associated super-Lévy process (or critical multiplicative measure-valued branching Markov process!), X_t, is an $M_F(\mathbb{R}^d)$-valued diffusion. More precisely, if A denotes the generator of Y on its domain, $\mathscr{D}(A)$, in the Banach space $C(\overline{\mathbb{R}^d})$ of continuous functions on $\mathbb{R}^d$ with a finite limit at ∞, then for each $m_0 \in M_F(\mathbb{R}^d)$ there is a unique (in law) continuous $M_F(\mathbb{R}^d)$-valued strong Markov process X such that for any $\phi \in \mathscr{D}(A)$,

$$X_t(\phi) = m_0(\phi) + \int_0^t X_s(A\phi)\, ds + Z_t(\phi),$$

(1.1) $\quad Z_t(\phi)$ is a continuous $\mathscr{F}_t^X$-martingale with square function

$$\langle Z(\phi)\rangle_t = \int_0^t X_s(\phi^2)\, ds$$

[see, e.g., Ethier and Kurtz (1986), page 406]. Here $\{\mathscr{F}_t^X\}$ is the smallest filtration satisfying the usual conditions and for which $X_t \in \mathscr{F}_t^X$. Let Q^{m_0} denote the law of X on the space $\Omega = C([0,\infty), M_F(\mathbb{R}^d))$ of continuous measure-valued paths and let $\{\mathscr{F}_t\}$ denote the canonical right-continuous filtration on this space of paths completed with respect to $\{Q^{m_0}: m_0 \in M_F\}$ as in Blumenthal and Getoor (1968).

X is shown to be the weak limit of a sequence of discrete measure-valued processes in Watanabe (1968) [see also Ethier and Kurtz (1986), Chapter 9].

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Consider μ particles starting at 0 and following independent copies of Y on $[0, \mu^{-1})$. At $t = \mu^{-1}$ each particle dies or splits into two particles independently of each other. The new generation of particles then follow i.i.d. copies of Y on $[\mu^{-1}, 2\mu^{-1})$. This scheme of alternating critical (mean 0) reproduction and Lévy migration then continues. If mass μ^{-1} is assigned to the location of each particle at time t, the resulting measure-valued processes converge weakly as $\mu \to \infty$ to the above process X with $m_0 = \delta_0$. More generally, if $K(\mu)$ particles start at $x_1^\mu, \dots, x_K^\mu$ and $\mu^{-1}\sum_{i=1}^{K(\mu)} \delta_{x_i^\mu} \to_w m_0$ as $\mu \to \infty$ then one obtains the process X in (1.1). The branching mechanism gives rise to the martingale term in (1.1) and the spatial migrations give rise to the drift term, $\int_0^t X_s(A\phi)\,ds$, in (1.1).

Assume now that X is a super-Brownian motion, that is, Y is a Brownian motion. It is known [Perkins (1988)] that for $d \geq 3$ and all $t > 0$, X_t distributes its mass uniformly (up to multiplicative constants) over a random Borel support according to the deterministic Hausdorff measure function $\phi - m$, when $\phi(x) = x^2 \log\log 1/x$. This result is extended to the canonical closed support in Perkins (1989). In some sense this reduces the study of X_t to the study of its (closed) support process. Let $S(\nu)$ denote the closed support of $\nu \in M_F(\mathbb{R}^d)$. The detailed study of the sample paths of $\{S(X_t): t \geq 0\}$, initiated in Iscoe (1988) and Dawson, Iscoe and Perkins (1989) [hereafter abbreviated (D.I.P.)] is continued here. In (D.I.P.) necessary conditions were given for a fixed set to be "hit" by $S(X_t)$ for some $t > 0$ or, more generally to be "hit by the k-multiple points of $S(X_t)$." In this work we obtain corresponding sufficient conditions which are very close to the above necessary conditions. These results together with those in (D.I.P.) give necessary and sufficient conditions for the existence of "k-multiple points" for super-Brownian motion and allow us to compute the Hausdorff dimension of the "k-multiple points."

We introduce some terminology to state these results precisely. Write S_t for the closed support $S(X_t)$ of X_t.

Definition. If $0 \leq s \leq t \leq \infty$, $\overline{R}([s,t]) \equiv \overline{R}(s,t) = \mathrm{cl}(\bigcup_{s \leq u \leq t} S_u)$ is the (closed) range of X on $[s,t]$ and $\overline{R} = \bigcup_{\delta > 0} \overline{R}(\delta, \infty)$ is the range of X. The set of k-multiple points of X is

$$\overline{R}_k = \bigcup \left\{ \bigcap_{j=1}^{k} \overline{R}(I_j) \colon I_1, \dots, I_k \text{ disjoint compact intervals in } (0,\infty) \right\}.$$

Clearly, $\overline{R} = \overline{R}_1$. $A \subset \mathbb{R}^d$ is a polar set iff $A \cap \overline{R} = \varnothing\ Q^{m_0}$-a.s. for any $m_0 \in M_F(\mathbb{R}^d)$, and A is polar for $\overline{R}_k$ iff $A \cap \overline{R}_k = \varnothing\ Q^{m_0}$-a.s. for any $m_0 \in M(\mathbb{R}^d)$. We say $X(w)$ hits A iff $A \cap \overline{R}(w) \neq \varnothing$. Note that since $X_0 = m_0$ is finite, the extinction time of X, $\zeta = \inf\{t: X_t(\mathbb{R}^d) = 0\}$ is finite Q^{m_0}-a.s.

Notation. $\mathscr{H} = \{\phi \in C([0,\infty], \mathbb{R}): \phi(0) = 0,\ \phi$ nondecreasing and strictly increasing near 0$\}$. If $\phi \in \mathscr{H}$ and $A \subset \mathbb{R}^d$, $\phi - m(A)$ denotes the Hausdorff ϕ-measure of A and $\dim A$ is the Hausdorff dimension of A [see Rogers

(1970)]. Let

$$h_\beta(r) = \begin{cases} r^\beta & \text{if } \beta > 0,\, r \geq 0, \\ (1 + \log^+ 1/r)^{-1} & \text{if } \beta = 0,\, r \geq 0. \end{cases}$$

If $d \leq 3$, X hits points [Sugitani (1987) or (D.I.P.), Theorem 1.3]. If $d \geq 4$ a necessary condition for X to hit a set was given in (D.I.P.). *Until otherwise indicated Q^{m_0} denotes the law of super-Brownian motion starting at m_0.*

THEOREM A [(D.I.P.), Theorem 1.5]. *Let $d \geq 4$, $k \in \mathbb{N}$, $A \subset \mathbb{R}^d$ and $m_0 \in M_F(\mathbb{R}^d)$. If $Q^{m_0}(A \cap \overline{R}_k \neq \varnothing) > 0$, then $(h_{d-4})^k - m(A) > 0$. In particular, $h_{d-4} - m(A) = 0$ is sufficient for A to be polar.*

The proof was based on an asymptotic estimate for $Q^{m_0}(B(x,\varepsilon) \cap \overline{R}(\delta,\infty) \neq \varnothing)$ as $\varepsilon \downarrow 0$, where $B(x,\varepsilon)$ is the open ball of radius ε centered at x. This estimate [with considerable additional effort if $k = d/(d-4)$] gives [(1.8) of (D.I.P.)]

$$\text{(1.2)} \qquad \dim \overline{R}_k \leq d - k(d-4) \quad Q^{m_0}\text{-a.s.} \quad d \geq 4,\, k \in \mathbb{N},$$

where $\dim \overline{R}_k \leq 0$ indicates $\overline{R}_k = \varnothing$. Hence we have Theorem B.

THEOREM B [(D.I.P.), Theorem 1.6]. *If $d \geq 4$, $\overline{R}_k = \varnothing$ for $k \geq d/(d-4)$ Q^{m_0}-a.s. for all $m_0 \in M_F(\mathbb{R}^d)$.*

Our sufficient condition for X to hit a set involves capacity. If $1/g \in \mathscr{H}$, A is an analytic subset of $\mathbb{R}^d$ and $\mu \in M_F(\mathbb{R}^d)$ let

$$\langle \mu, \mu \rangle_g = \iint g(|y-x|)\, d\mu(x)\, d\mu(y),$$

$$I(g)(A) = \inf\{\langle \mu, \mu \rangle_g : \mu \text{ a probability supported by } A\}.$$

The g-capacity of A is then given by $C(g)(A) = I(g)(A)^{-1}$ [see, e.g., Hawkes (1979), Section 3]. If A is compact then [see Taylor (1961), Lemma A]

$$\text{(1.3)} \qquad I(g)(A) = \liminf_{n \to \infty} \left\{ n(n-1)^{-1} \sum_{i=1}^{n} \sum_{i'=1,\, i' \neq i}^{n} g(|x_i - x_{i'}|) : x_1, \ldots, x_n \in A \right\}.$$

Let

$$g_\beta(r) = h_\beta(r)^{-1} = \begin{cases} r^{-\beta} & \text{if } \beta > 0, \\ 1 + \log^+ 1/r & \text{if } \beta = 0. \end{cases}$$

The following partial converse to Theorem A is proved in Section 5.

THEOREM 1.1. *Let $d \geq 4$; $k \in \mathbb{N}$ and m_0 be a nonzero measure in $M_F(\mathbb{R}^d)$. If A is an analytic subset of $\mathbb{R}^d$ such that $C((g_{d-4})^k)(A) > 0$, then $Q^{m_0}(A \cap \overline{R}_k \neq \varnothing) > 0$. In particular, $C(g_{d-4})(A) = 0$ is necessary for A to be polar.*

In fact Theorem 5.8 gives a lower bound for $Q^{m_0}(A \cap \overline{R}_k \neq \varnothing)$ in terms of $C((g_{d-4})^k)(A)$.

The Hausdorff measure condition in Theorem A and the capacity condition of Theorem 1.1 are close but not equivalent. One has [see, e.g., Taylor (1961), Theorems 1 and 2]

$$C((g_{d-4})^k)(A) > 0 \quad \text{implies } (h_{d-4})^k - m(A) = \infty,$$

$$(h_{d-4})^k \left(\log^+ \frac{1}{x}\right)^{-1-\varepsilon} - m(A) > 0 \text{ for some } \varepsilon > 0 \quad \text{implies}$$
$$C((g_{d-4})^k)(A) > 0, \qquad d > 4,$$

and

$$(h_0)^k \left(\log^+ \log^+ \frac{1}{x}\right)^{-1-\varepsilon} - m(A) > 0 \quad \text{for some} \quad \varepsilon > 0$$
$$\text{implies } C((g_0)^k)(A) > 0.$$

In general, this is all that can be said [Taylor (1961), Section 4]. Our conjecture is

$$(1.4) \quad C\big((g_{d-4})^k\big)(A) > 0 \Leftrightarrow Q^{m_0}\big(A \cap \overline{R}_k\big) > 0, \qquad d \geq 4, m_0 \neq 0.$$

The state of affairs regarding polar sets for super-Brownian motion is therefore similar to that for various multiparameter processes [Dynkin (1981), Evans (1987) and Le Gall, Rosen and Shieh (1989)] prior to the appearance of Fitzsimmons and Salisbury (1988). In fact the methodology of Theorem 1.1 may also be used to prove similar results for a variety of multiparameter processes, as was suggested to us by S. James Taylor.

THEOREM 1.2. *Let $Z(t_1, t_2)$ be a d-dimensional, two-parameter, α-stable sheet* [*see Bass and Pyke* (1984)] *such that $d \geq 2\alpha$. Let*

$$\overline{M}_k = \bigcup\big\{\mathrm{cl}(Z(I_1)) \cap \cdots \cap \mathrm{cl}(Z(I_k)) \colon I_1, \dots, I_k \text{ disjoint compact rectangles in } (0, \infty)^2\big\}$$

be the set of k-multiple points for Z. If A is an analytic subset of $\mathbb{R}^d$, then

$$(1.5) \qquad C\big((g_{d-2\alpha})^k\big)(A) > 0 \quad \text{implies } P\big(\overline{M}_k \cap A \neq \varnothing\big) > 0.$$

Replace 2α by $N\alpha$ in the N-parameter case. Although we have been unable to find (1.5) in the literature we will not present a proof for several reasons. First, once the general approach is demonstrated in the derivation of Theorem 1.1, the proof becomes little more than a computational exercise (albeit a rather long one). Second, our suspicion (recently confirmed by Loren Pitt) was that the experts knew how to prove (1.5) but were really interested in the converse. Finally, and most importantly, Fitzsimmons and Salisbury (1988) have recently developed general techniques which, in addition to (1.5), prove the elusive converse. Unfortunately, their elegant approach does not seem applicable to super-Brownian motion and so (1.4) remains a conjecture. A handwritten proof of Theorem 1.2 is available from the author upon request.

In Section 5 we also use Theorem 1.1 to obtain the converse inequality to (1.2) and hence obtain Theorem 1.3.

THEOREM 1.3. *If* $d \geq 4$, $k \in \mathbb{N}$ *and* $m_0(\neq 0) \in M_F(\mathbb{R}^d)$, *then*

$$\dim \overline{R}_k = d - k(d-4) \qquad Q^{m_0}\text{-}a.s.,$$

where $\dim \overline{R}_k \leq 0$ *means* $\overline{R}_k = \varnothing$. *Hence* $\overline{R}_k \neq \varnothing$ *a.s. if* $k < d/(d-4)$ *and* $\overline{R}_k = \varnothing$ *a.s. if* $k \geq d/(d-4)$.

The study of k-multiple points was suggested to us by Robert Adler who also proposed the study of a self-intersection local time (SILT) for X. Motivated by Theorem 1.3, Dynkin (1988) constructed such a SILT and thus gave another proof of the existence of k-multiple points if $k > d/(d-4)$. In fact Dynkin constructed a SILT for a more general class of superprocesses.

A more natural notion of X hitting A would be $A \cap S_t \neq \varnothing$ for some $t > 0$. That is, one would replace $\overline{R}(s,t)$, $\overline{R}$ and $\overline{R}_k$ with

$$R([s,t]) = R(s,t) = \bigcup_{s \leq u \leq t} S_u, \qquad 0 \leq s \leq t \leq \infty,$$

$$R = \bigcup_{0 < u < \infty} S_u,$$

$$R_k = \left\{ x \colon x \in \bigcap_{i=1}^{k} S_{t_i} \text{ for some } 0 < t_1 < \cdots < t_k \right\}.$$

In Section 4 we obtain estimates on $\overline{R} - R$ (Proposition 4.7 and Theorem 4.9) which show Theorems 1.1 and 1.3 are also valid with R and R_k in place of $\overline{R}$ and $\overline{R}_k$ (see Theorem 5.9). To do this, we study the discontinuities of $\{S_t \colon t \geq 0\}$.

Let $\mathscr{K}(\mathbb{R}^d)$ and $\mathscr{F}(\mathbb{R}^d)$ denote the sets of compact and closed subsets of $\mathbb{R}^d$, respectively. If A, B are nonempty sets in $\mathscr{F}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, let $d(x, A)$ denote the distance from x to A and

$$\rho_1(A, B) = \sup_{x \in A} d(x, B) \wedge 1,$$

$$\rho(A, B) = \rho_1(A, B) \vee \rho_1(B, A).$$

Let $\rho(\phi, A) = 1$ if $A \neq \varnothing$. ρ is the Hausdorff metric on $\mathscr{F}(\mathbb{R}^d)$ [or $\mathscr{K}(\mathbb{R}^d)$] [see Cutler (1984), Section 4, or Dugundji (1966), page 205]. The mapping $S \colon M_F(\mathbb{R}^d) \to \mathscr{F}(\mathbb{R}^d)$ is Borel measurable [Cutler (1984), Theorem 4.4.1] and hence so is its restriction to the measurable set

$$M_F^K(\mathbb{R}^d) = \{\nu \in M_F(\mathbb{R}^d) \colon S(\nu) \text{ compact}\}$$

as a mapping into $\mathscr{K}(\mathbb{R}^d)$.

Theorem 1.2 of (D.I.P.) shows that

(1.6) $\quad \{S_t \colon t > 0\}$ is a right-continuous $\mathscr{K}(\mathbb{R}^d)$-valued process $\qquad Q^{m_0}$-a.s.

and the proof also gives

(1.7) $\{S_t: t \geq 0\}$ is a right-continuous $\mathscr{F}(\mathbb{R}^d)$-valued process $\quad Q^{m_0}$-a.s.

It is a little more convenient to work with $\mathscr{K}(\mathbb{R}^d)$ and so (1.6) is extended as follows in Section 4.

THEOREM 1.4. *If $m_0 \in M_F(\mathbb{R}^d)$ then for Q^{m_0}-a.a. w:*

(a) *$\{S_t$: $t > 0\}$ is a $\mathscr{K}(\mathbb{R}^d)$-valued process having right-continuous paths with left-hands limits.*
(b) *$S_{t-} \supset S_t$ for all $t > 0$.*
(c) *$S_{t-} - S_t$ is empty or a singleton for all $t > 0$.*

Hence the countable set of discontinuities of $\{S_t$: $t > 0\}$ occurs when an isolated "colony" becomes extinct. The space–time set of these extinction points forms a dense subset of the graph of S (see Theorem 4.8 for a precise statement). It is easy to see that $\overline{R}_k - R_k$ is contained in the countable set of spatial extinction points and hence Theorem 1.3 remains valid if $\overline{R}_k$ is replaced by R_k. The extension of Theorem 1.1 takes a bit more work (see Theorem 4.9).

The fact that $S_{\zeta-}$ is a singleton, which follows from Theorem 1.4(c), has been proved independently by Liu (1988).

The situation for general super-Lévy processes is completely different from the above. If Y is a symmetric stable process of index $\alpha \in (0,2)$, the proof of Theorem 1.1 remains valid upon replacing $d - 4$ with $d - 2\alpha$. In fact many of the preliminary estimates are proved in this setting for future reference. The problem is that Theorem A fails completely for $\alpha < 2$. In fact $S_t = \mathbb{R}^d$ or $\varnothing$ a.s.!

NOTATION. Let ν denote the Lévy measure of the underlying Lévy process Y [see Fristedt (1974), page 248]. Let $\nu^{(k)}$ denote the k-fold convolution of ν with itself and let $\mu_1 * \mu_2$ denote the convolution of the measures μ_1 and μ_2.

THEOREM 1.5. *Assume Q^{m_0} denotes the law of the super-Lévy process starting at $m_0 \in M_F(\mathbb{R}^d)$. Then*

$$\bigcup_{k=0}^{\infty} S\left(\nu^{(k)} * X_t\right) = S_t \qquad Q^{m_0}\text{-}a.s.\ \text{for all } t > 0.$$

The proof is given in Section 3. Hence the support process propagates instantaneously to any points to which the underlying Lévy process can jump. This should be contrasted with the behavior of super-Brownian motion whose support propagates with finite speed. More precisely we have [(D.I.P.), Theorem 1.1] if $c > 2$ $\exists$ $\delta(w,c) > 0$ such that

$$S_t \subset \left\{x: d(x, S_s) < c\left((t-s)\log^+ \frac{1}{t-s}\right)^{1/2}\right\} \tag{1.8}$$
$$\text{for } 0 < t - s < \delta(w,c) \quad \text{a.s.}$$

An immediate corollary to Theorem 1.5 is Corollary 1.6.

COROLLARY 1.6. *If Y has Lévy measure ν and $\bigcup_{k=1}^{\infty} S(\nu^{(k)}) = \mathbb{R}^d$, then for all $t > 0$ and $m_0 \in M_F(\mathbb{R}^d)$,*

$$S_t \neq \varnothing \text{ implies } S_t = \mathbb{R}^d \qquad Q^{m_0}\text{-a.s.}$$

In particular this is the case if Y is a d-dimensional symmetric stable process of index $\alpha \in (0,2)$.

PROOF.

$$S_t \neq \varnothing \text{ and } \bigcup_{k=1}^{\infty} S(\nu^{(k)}) = \mathbb{R}^d \quad \text{clearly imply } \bigcup_{k=1}^{\infty} S(\nu^{(k)} * X_t) = \mathbb{R}^d.$$

The first result is therefore immediate from Theorem 1.5. If Y is an $\mathbb{R}^d$-valued symmetric stable process of index $\alpha \in (0,2)$ then $\nu(dx) = c|x|^{-d-\alpha}\,dx$ and so $S(\nu) = \mathbb{R}^d$. □

If Y is a symmetric α-stable process and $d > \alpha$, Perkins (1988) shows that for Q^{m_0}-a.a. w and all $t > 0$ there is a random Borel support $\Lambda_t(w)$ for $X_t(w)$ such that

$$(1.9)\qquad \begin{aligned} &c_1(\alpha,d)\phi_\alpha - m(A \cap \Lambda_t) \\ &\quad \leq X_t(A) \leq c_2(\alpha,d)\phi_\alpha - m(A\cap\Lambda_t) \quad \text{for all } A \in \mathscr{B}(\mathbb{R}^d). \end{aligned}$$

If $\alpha < 2$, Λ_t is dense in $\mathbb{R}^d$ for Lebesgue a.a. $t < \zeta$ a.s. by Corollary 1.6, and (1.9) is clearly false if Λ_t is replaced by S_t. This answers a question (in the negative) which was raised in Perkins (1988) and solved in the affirmative for super-Brownian motion in Perkins (1989). We conjecture that for $\alpha < 2$ there does exist a $t(w)$ such that $S_t \neq \mathbb{R}^d$ and $S_t \neq \varnothing$. Dynkin's SILT for k-multiple points exists for $k < d/(d-2\alpha)$. In light of Corollary 1.6 it is clearly of interest to study the nature of this random measure more closely.

It is natural to ask for which sets A is $X_t(A) > 0$ for some $t > 0$. If this is the case we say that X penetrates A. Reimers (1989) shows that if X is super-Brownian motion then for each fixed Borel set A, $X_t(A)$ is continuous in t a.s., and hence X penetrates A with positive probability if and only if A has positive Lebesgue measure. (In fact this "strong continuity" and characterization of penetrated sets holds for a much larger class of superprocesses including the supersymmetric α-stable processes.)

The following elementary result, which is immediate from (1.1) [see also Watanabe (1968), Section 2], will be needed.

PROPOSITION 1.7. *If X^1 and X^2 are independent super-Lévy processes with laws Q^{m_1} and Q^{m_2} (the underlying Lévy process Y is the same), then $X^1 + X^2$ has law $Q^{m_1+m_2}$.*

This result also follows from Watanabe's construction of X as a weak limit of branching Lévy processes, a construction that is used throughout this work. Some technical estimates for a system of branching Lévy processes are given in Section 2. We will use nonstandard analysis in Section 4 to represent a super-Lévy process as the standard part of a system of branching Lévy processes with infinitesimal interbranching times. The main advantage in the nonstandard model is that it retains the notions of ancestry which is lost in the standard model. Good introductions to nonstandard analysis are given by Cutland (1983) (brief and oriented towards probabilists) and Hurd and Loeb (1985) (more comprehensive). The nonstandard model is only used to study the jumps of $\{S_t\}$ in Section 4.

In Section 6 sufficient conditions are given on a set A so that for a fixed t, $A \cap S_t \neq \emptyset$ with positive probability. This result again complements the necessary conditions in (D.I.P.), Section 7.

Constants introduced in section i are denoted $c_{i.1}, c_{i.2} \dots$. These constants may depend on d and the law of Y. Other dependencies will be made explicit. $c_1, c_2, \dots$ denote unimportant constants whose value may change from line to line. $B(x, r)$ is the open ball in $\mathbb{R}^d$ centered at x and with radius r.

2. Branching Lévy processes. Let $I = \bigcup_{n=0}^{\infty} \mathbb{Z}_+ \times \{0,1\}^n$, and if $\gamma, \beta = (\beta_0, \dots, \beta_i) \in I$ write $|\beta| = i$, $\beta|j = (\beta_0, \dots, \beta_j)$ for $j \leq i$, and $\gamma < \beta$ if $\gamma = \beta|j$ for some $j \leq i$. Let $\{Y^\beta : \beta \in I\}$ be a collection of i.i.d. d-dimensional (right-continuous) Lévy processes with law P_0^0 on path space, and let $\{e^\beta : \beta \in I\}$ be i.i.d. random variables satisfying $P^2(e^\beta = 0) = P^2(e^\beta = 2) = 1/2$. These two collections are independent and are defined on a common $(\Omega^2, \mathscr{A}^2, P^2)$. If $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\Delta\}$, where Δ is added as the point at ∞, let $(\Omega^1, \mathscr{A}^1) = (\overline{\mathbb{R}^d}^{\mathbb{Z}_+}, \mathscr{B}(\overline{\mathbb{R}^d}^{\mathbb{Z}_+}))$ and $(\Omega, \mathscr{A}) = (\Omega^1, \mathscr{A}^1) \times (\Omega^2, \mathscr{A}^2)$. We will assume that Y^β and e^β are defined on Ω in the obvious way.

Fix a parameter μ in $\mathbb{N}$ and suppress dependence on μ wherever possible. Let $T = T^{(\mu)} = \{j/\mu : j \in \mathbb{Z}_+\}$ and λ be the measure on T which assigns mass μ^{-1} to each point in T. Given $w = ((x_j), w^2)$, we construct a branching particle system as follows: A particle starts at each $x_j \neq \Delta$; subsequently particles die or split into two, each with probability $1/2$, at the times in $T - \{0\}$; in between these times the particles follow i.i.d. Lévy processes. Each multi-index β in I will label a branch N^β of the resulting tree of branching Lévy processes up to $t = (|\beta| + 1)/\mu$:

$$\hat{N}_t^{\beta,\mu}\big((x_i)_{i \in \mathbb{Z}_+}, w^2\big) = \begin{cases} x_{\beta_0} + \sum_{i=0}^{|\beta|} \int 1\big(i/\mu \leq s < t \wedge ((i+1)/\mu)\big)\, dY_s^{\beta|i} & \text{if } x_{\beta_0} \neq \Delta, \\ \Delta & \text{if } x_{\beta_0} = \Delta, \end{cases}$$

$$\tau^\beta\big((x_i)_{i \in \mathbb{Z}_+}, w^2\big) = \begin{cases} 0 & \text{if } x_{\beta_0} = \Delta, \\ \min\{(i+1)/\mu : e^{\beta|i} = 0\} & \text{if } x_{\beta_0} \neq \Delta \text{ and this set is nonempty}, \\ (|\beta| + 1)/\mu & \text{otherwise}, \end{cases}$$

$$N_t^\beta = N_t^{\beta,\mu} = \begin{cases} \hat{N}_t^{\beta,\mu} & \text{if } 0 \leq t < \tau^\beta, \\ \Delta & \text{if } t \geq \tau^\beta. \end{cases}$$

NOTATION. If $\beta \in I$ and $t \geq 0$, write $\beta \sim^{\mu} t$ (or $\beta \sim t$) if only and only if $|\beta|/\mu \leq t < (|\beta| + 1)/\mu$. Let $M(\mathbb{R}^d)$ denote the set of measures on $\mathbb{R}^d$ and

$$M_F^{\mu}(\mathbb{R}^d) = \left\{\mu^{-1} \sum_{i=0}^{K} \delta_{x_i}: K = -1, 0, 1, \dots, x_i \in \mathbb{R}^d\right\} \subset M_F(\mathbb{R}^d).$$

DEFINITION. $N = N^{(\mu)}: [0, \infty) \times \Omega \to M(\mathbb{R}^d)$ is defined by

$$N_t(w) = \mu^{-1} \sum_{\beta \sim t} \delta_{N_t^{\beta}(w)} 1\big(N_t^{\beta}(w) \neq \Delta\big).$$

If $\phi: \mathbb{R}^d \to [0, \infty)$, define $\phi(\Delta) = 0$ so that $N_t(\phi) = \mu^{-1}\Sigma_{\beta \sim t}\phi(N_t^{\beta})$. Let Π^i denote the projection of Ω onto Ω^i and define a filtration on $(\Omega, \mathscr{A})$ by

$$\mathscr{A}_t = \mathscr{A}_t^{\mu} = \sigma\big(\pi^1, Y^{\beta}, e^{\beta}: |\beta| < [t\mu]\big) \vee \bigg(\bigcap_{u > t} \sigma\big(Y_s^{\beta}: |\beta| = [t\mu], s \leq u\big)\bigg).$$

If $w^1 \in \Omega^1$, let $P^{w^1} = \delta_{w^1} \times P^2$ [a probability on $(\Omega, \mathscr{A})$]. If $w^1 = (x_j) \in \Omega^1$ and $m = \mu^{-1}\Sigma_{j=0}^{\infty}\delta_{x_j}1(x_j \neq \Delta) \in M_F^{\mu}(\mathbb{R}^d)$, then (m, μ) uniquely determines P^{w^1} on $\sigma(N_t, t \geq 0)$ and we may write $P^{m,\mu}$ for the restriction of P^{w^1} to $\sigma(N_t, t \geq 0)$. In fact we will usually abuse the notation and write P^m for P^{w^1} itself, suppressing the dependence on (x_j).

Let $\mathscr{E} = \sigma(e^{\beta}: \beta \in I)$ and for $S \subset I$ let $\mathscr{G}(S) = \sigma(Y^{\beta}, e^{\beta}: \beta \in S)$. $\{\mathscr{D}_t: t \geq 0\}$ denotes the canonical filtration on the space of $\mathbb{R}^d$-valued cadlag paths. $\{P_t: t \geq 0\}$ denotes the semigroup of the Lévy process Y on the Banach space $C(\overline{\mathbb{R}^d})$ and $E_0^m(A) = \int P_0^x(A)\, dm(x)$ for $m \in M_F$ and $A \in \mathscr{D}_{\infty}$.

LEMMA 2.1. *Let $\mu \in \mathbb{N}$ and $m \in M_F^{\mu}(\mathbb{R}^d)$.*

(a) *If $\beta \in I$, $\underline{r} \in T^{(\mu)}$, $0 \leq t \leq \underline{r} + t \leq |\beta|/\mu$ and $A \in \mathscr{D}_t$, then on $\{N_{t+\underline{r}}^{\beta} \neq \Delta\}$,*

(2.1) $$P^m\big(N_{\underline{r}+\cdot}^{\beta} \in A | \mathscr{E} \vee \mathscr{A}_{\underline{r}}\big)(w) = P_0^{N_{\underline{r}}^{\beta}(w)}(Y \in A) \qquad P^m\text{-}a.s.$$

(b) *Let $\underline{t} \in T^{(\mu)}$ and $\phi: \mathbb{R}^d \to [0, \infty]$ be Borel measurable.*

(2.2)(i) $$E^m(N_{\underline{t}}(\phi)) = E_0^m(\phi(Y_{\underline{t}})).$$

(2.3)(ii) $$E^m\big(N_{\underline{t}}(\phi)^2\big) = E_0^m(\phi(Y_{\underline{t}}))^2 + E_0^m\bigg(\phi(Y_{\underline{t}})\int_{[-\mu^{-1}, \underline{t})} P_{\underline{u}^+}\phi(Y_{\underline{t}-\underline{u}^+})\, d\lambda(\underline{u})\bigg).$$

(a) is an easy extension of (D.I.P.), Lemma 2.1(a), b(i) is Lemma 2.1(d) of (D.I.P.) and b(ii) follows easily from Perkins (1988a), Proposition 2.6a(ii). The more general setting here (Y is a Lévy process) requires no change in the arguments.

THEOREM 2.2. *If $m_0 \in M_F(\mathbb{R}^d)$ and $m_{\mu} \in M_F^{\mu}(\mathbb{R}^d)$ converge weakly to m_0 as $\mu \to \infty$, then*

$$P^{m_{\mu},\mu}(N^{(\mu)} \in \cdot) \to_w Q^{m_0} \quad \text{on } D\big([0, \infty], M_F(\mathbb{R}^d)\big).$$

This theorem, essentially due to Watanabe (1968), is proved in Ethier and Kurtz (1986), page 406, when the interbranching times are i.i.d. exponential r.v.'s with mean μ^{-1}. The alterations required for deterministic branching times are routine.

If $m_0 \in M_F(\mathbb{R}^d)$ and $\phi, \psi \colon \mathbb{R}^d \to [0, \infty]$ are Borel measurable, the analogs of (2.2) and (2.3) for the limit process are

$$Q^{m_0}(X_t(\phi)) = E_0^{m_0}(\phi(Y_t)), \tag{2.4}$$

$$\begin{aligned} Q^{m_0}(X_t(\phi) X_u(\psi)) &= E_0^{m_0}(\phi(Y_t)) E_0^{m_0}(\psi(Y_u)) \\ &\quad + \int_0^t E_0^{m_0}(P_{t-s}\phi(Y_s) P_{u-s}\psi(Y_s))\, ds, \qquad 0 \le t \le u. \end{aligned} \tag{2.5}$$

(2.4) is clear from (2.2) and the above theorem, and (2.5) follows similarly from a slight extension of (2.3) [alternatively (2.5) and extensions to higher moments may be found in Dynkin (1988)].

We also require a classical result on the critical Galton–Watson branching process, $\{Z_n\}$ [Harris (1963), pages 21 and 22]: If $Z_0 = 1$ and $P(Z_1 = 0) = P(Z_1 = 2) = 1/2$, then

$$\lim_{n \to \infty} nP(Z_n > 0) = 2, \tag{2.6}$$

$$\lim_{n \to \infty} \sup_{z \ge 0} \left| P(Z_n/n > z | Z_n > 0) - e^{-2z} \right| = 0. \tag{2.7}$$

The uniform convergence in (2.7) follows form the pointwise convergence by a standard argument.

We close this section with a series of important lower bounds on the probability that $N_t(B(x_0, \varepsilon))$ is large.

LEMMA 2.3. *There is a* $\mu_1 \colon (0, \infty)^2 \times (0, 1) \to \mathbb{N}$, *increasing in the first variable and decreasing in the last variable, such that if* $\mu \ge \mu_1(K, \delta, p)$ *and*

$$p \le P_0^x(Y_\delta \in B(y, \varepsilon)), \tag{2.8}$$

then

$$P^{\delta_x/\mu}\left(N_\delta^{(\mu)}(B(y, \varepsilon)) > K\delta | N_\delta^{(\mu)}(\mathbb{R}^d) > 0\right) \ge \exp\{-4K/p\} p/4. \tag{2.9}$$

(*Here* δ_x/μ *is the measure assigning mass* μ^{-1} *to the point* x.)

PROOF. Choose $\delta, \varepsilon > 0$, $p \in [0, 1]$ and $y, x \in \mathbb{R}^d$ so that (2.8) holds, let $K > 0$ and define $M = 2K/p$. Let $\mu \in \mathbb{N}$ be fixed and

$$\{N_\delta^{(\mu)}(\mathbb{R}^d) \ge M\delta\} = \bigcup_{i=1}^{\eta} A_i,$$

where $\{A_i \colon i \le \eta\}$ are disjoint sets, each specifying one of the finite number of possibilities for $\{\gamma \sim \delta \colon \gamma_0 = x, N_\delta^\gamma \neq \Delta\}$. Choose one such A_i, say

$$w \in A_i \Leftrightarrow \{\gamma \sim \delta \colon N_\delta^\gamma(w) \neq \Delta\} = \{\gamma_j \colon j \leq N\},$$

where $N \geq M\delta\mu$ (necessarily).

Let $X_j = 1(N_\delta^{\gamma_j} \in B(y, \varepsilon))$ and define

$$(2.10)\qquad \begin{aligned} q &= P^{\delta_x/\mu}\left(\sum_{j=1}^{N} X_j \geq KN/M|A_i\right) \\ &\leq P^{\delta_x/\mu}\big(N_\delta^{(\mu)}(B(y,\varepsilon)) \geq K\delta|A_i\big). \end{aligned}$$

(2.1) with $r = 0$ and (2.8) imply

$$p \leq P_0^x(Y_\delta \in B(y,\varepsilon)) = E^{\delta_x/\mu}\left(\sum_{j=1}^{N} X_j/N|A_i\right) \leq q + (1-q)K/M.$$

Rearranging the above, one sees that

$$(2.11)\qquad q \geq (p - K/M)(1 - K/M)^{-1}.$$

By (2.7) there is a $\mu_0(M, \delta)$, increasing in M, such that

$$P^{\delta_x/\mu}\Big(N_\delta^{(\mu)}(\mathbb{R}^d) \geq M\delta \big| N_\delta^{(\mu)}(\mathbb{R}^d) > 0\Big) \geq e^{-2M}/2 \quad \text{if } \mu \geq \mu_0(M,\delta).$$

Let $\mu_1(K, \delta, p) = \mu_0(2K/p, \delta)$, assume $\mu \geq \mu_1$ and combine the above with (2.10) and (2.11) to conclude

$$\begin{aligned} P^{\delta_x/\mu}\Big(N_\delta^{(\mu)}(B(y,\varepsilon)) \geq K\delta \big| N_\delta^{(\mu)}(\mathbb{R}^d) > 0\Big) &\geq e^{-2M}2^{-1}(p - K/M)(1-K/M)^{-1} \\ &\geq \exp\{-4K/p\}p/4. \end{aligned}$$

□

LEMMA 2.4. *For every* $K, \delta > 0$ *and* $p \in (0, 1]$ *there is a* $\mu_2(K, p, \delta) \in \mathbb{N}$ *such that if* $0 < \varepsilon' < \varepsilon$,

$$(2.12)\qquad p \leq P_0^0(Y_\delta \in B(0, \varepsilon - \varepsilon')),$$

and $\mu \geq \mu_2$, *then for any* $m \in M_F^\mu$, $t - \delta \in T^{(\mu)}$ *and* $M > 0$,

$$\begin{aligned} &P^m\big(N_t^{(\mu)}(B(x_0,\varepsilon)) \geq K\delta | \mathscr{A}_{t-\delta}\big) \\ &\quad \geq (1 - e^{-M})(4M\delta)^{-1} p e^{-4K/p} N_{t-\delta}^{(\mu)}(B(x_0,\varepsilon')) \\ &\qquad \times 1\big(N_{t-\delta}^{(\mu)}(B(x_0,\varepsilon')) \leq 4\delta M p^{-1} e^{4K/p}\big) \qquad P^m\text{-}a.s. \end{aligned}$$

PROOF. Let K, δ, p, ε', ε, t, m, x_0 and M be as above, $\mu \in \mathbb{N}$, and assume (2.12). Let $\mathbb{N} \in \mathbb{Z}_+$ and $\{\gamma_i \colon i \leq N\} \subset \{\gamma \colon \gamma \sim t - \delta\}$. Set

$$A = \{w \colon \{\gamma_i \colon i \leq N\} = \{\gamma \sim t - \delta \colon N_{t-\delta}^\gamma \in B(x_0, \varepsilon')\}\} \in \mathscr{A}_{t-\delta}$$

and

$$(2.13)\qquad \begin{aligned} B_i &= \left\{w \colon \mu^{-1} \sum_{\gamma \sim t,\, \gamma > \gamma_i}{}' 1\big(N_t^\gamma \in B(N_{t-\delta}^{\gamma_i}, \varepsilon - \varepsilon')\big) > K\delta\right\} \\ &= \{w \colon N_{t-\delta}^{\gamma_i} \neq \Delta\} \cap C_i, \end{aligned}$$

where

$$C_i = \left\{ w: \mu^{-1} \sum_{\gamma \sim t,\, \gamma > \gamma_i} 1\Big(\hat{N}_t^{\gamma} - \hat{N}_{t-\delta}^{\gamma_i} \in B(0, \varepsilon - \varepsilon'),\ \prod_{|\gamma_i| \le j < |\gamma|} e^{\gamma|j} \ne 0 \Big) > K\delta \right\}.$$

If $S_i = \{\gamma: \gamma > \gamma_i\}$, $i \le N$, then $\{S_i\}$ are disjoint sets and are all disjoint from $\{\gamma: |\gamma| < \mu(t - \delta)\}$. It follows that $\{\mathscr{G}(S_i): i \le N\}$ are mutually independent σ-fields and are jointly independent of $\mathscr{A}_{t-\delta}$. Since $C_i \in \mathscr{G}(S_i)$ and $B_i \cap A = C_i \cap A$ [by (2.13)] we see that if $\mu \ge \mu_1(K, p, \delta)$, then w.p.1,

$$\begin{aligned}
&P^m\big(N_t^{(\mu)}(B(x_0, \varepsilon)) \ge K\delta | \mathscr{A}_{t-\delta}\big) 1_A \\
&\quad \ge P^m\left(\bigcup_{i=1}^N B_i | \mathscr{A}_{t-\delta} \right) 1_A = P^m\left(\bigcup_{i=1}^N C_i | \mathscr{A}_{t-\delta} \right) 1_A \\
&\quad = 1_A - 1_A \prod_{i=1}^N (1 - P^m(C_i | \mathscr{A}_{t-\delta})) = 1_A - 1_A \prod_{i=1}^N (1 - P^m(C_i)) \\
&\quad = 1_A - 1_A \prod_{i=1}^N \Big(1 - P^{\mu^{-1}\delta_0}\big(N_\delta^{(\mu)}(B(0, \varepsilon - \varepsilon')) > K\delta\big)\Big) \\
&\quad \ge 1_A - 1_A\Big(1 - (p/4) e^{-4K/p} P^{\delta_0/\mu}\big(N_\delta^{(\mu)}(\mathbb{R}^d) > 0\big)\Big)^N \quad \text{(Lemma 2.3)}.
\end{aligned}$$

(2.6) shows that for large enough μ, depending only on δ, $P^{\mu^{-1}\delta_0}(N_\delta^{(\mu)}(\mathbb{R}^d) > 0) \ge (\mu\delta)^{-1}$. Therefore there is a $\mu_2(K, p, \delta)$ such that if $\mu \ge \mu_2(K, p, \delta)$, then w.p.1,

$$\begin{aligned}
&P^m\big(N_t^{(\mu)}(B(x_0, \varepsilon)) \ge K\delta | \mathscr{A}_{t-\delta}\big) 1_A \\
&\quad \ge 1_A\Big(1 - \big(1 - (p/4) e^{-4K/p} (\mu\delta)^{-1}\big)^{\mu N_{t-\delta}^{(\mu)}(B(x_0, \varepsilon'))}\Big) \\
&\quad \ge 1_A\Big(1 - \exp\big\{-p(4\delta)^{-1} e^{-4K/p} N_{t-\delta}^{(\mu)}(B(x_0, \varepsilon'))\big\}\Big) \\
&\quad \ge 1_A (1 - e^{-M}) M^{-1} p (4\delta)^{-1} e^{-4K/p} N_{t-\delta}^{(\mu)}(B(x_0, \varepsilon')) \\
&\qquad \times 1\big(N_{t-\delta}^{(\mu)}(B(x_0, \varepsilon')) \le 4\delta M p^{-1} e^{4K/p}\big).
\end{aligned}$$

Since Ω is a finite disjoint union of sets of the form A (up to P^{m_0}-null sets), this completes the proof. □

LEMMA 2.5. *If* $0 < K$, $0 < \delta < t$, $0 < \varepsilon' < \varepsilon$, $p = P_0^0(|Y_\delta| < \varepsilon - \varepsilon')$ *and* $M = pe^{-4K/p}(t - \delta)\delta^{-1}$, *then*

$$\begin{aligned}
&Q^{m_0}(X_t(B(y, \varepsilon)) \ge K\delta) \\
&\quad \ge ((1 - e^{-M})/16)\big[P_0^{m_0}(Y_{t-\delta} \in B(y, \varepsilon'))(t - \delta)^{-1} \wedge 1\big],
\end{aligned}$$

for all $y \in \mathbb{R}^d$, $\varepsilon > 0$ *and* $m_0 \in M_F(\mathbb{R}^d)$.

PROOF. Fix $K, \delta, t, \varepsilon, \varepsilon'$ as above and $\mu \ge \mu_2(K, p, \delta)$, $\mu \in \mathbb{N}$. Let $\underline{t} = [\mu t]/\mu$, $\underline{\delta} = [\mu\delta]/\mu$ and assume μ is also large enough so that $0 < \underline{\delta} < \underline{t}$.

We suppress dependence on μ wherever possible. Let $m \in M_F^\mu(\mathbb{R}^d)$, $\underline{M} = pe^{-4K/p}(\underline{t} - \underline{\delta})\underline{\delta}^{-1}$ and

$$M' = pe^{-4K/p}\big((\underline{t} - \underline{\delta}) \vee E^m(N_{\underline{t}-\underline{\delta}}(B(y,\varepsilon')))\big)\underline{\delta}^{-1}.$$

Lemma 2.4 implies that

$$\begin{aligned} P^m(N_{\underline{t}}(B(y,\varepsilon)) \ge K\underline{\delta}) \ge{}& (1 - e^{-M'})(4M'\underline{\delta})^{-1} pe^{-4K/p} \\ & \times\Big[E^m(N_{\underline{t}-\underline{\delta}}(B(y,\varepsilon'))) \\ & \quad - p(4M'\delta)^{-1}e^{-4K/p}E^m\big(N_{\underline{t}-\underline{\delta}}(B(y,\varepsilon'))^2\big)\Big]. \end{aligned} \tag{2.14}$$

From (2.3) we see that

$$\begin{aligned} & E^m\big(N_{\underline{t}-\underline{\delta}}(B(y,\varepsilon'))^2\big) \\ & \quad \le E^m(N_{\underline{t}-\underline{\delta}}(B(y,\varepsilon')))^2 + (\underline{t} - \underline{\delta} + \mu^{-1})E^m(N_{\underline{t}-\delta}(B(y,\varepsilon'))) \\ & \quad \le E^m(N_{\underline{t}-\underline{\delta}}(B(y,\varepsilon')))\big[E^m(N_{\underline{t}-\underline{\delta}}(B(y,\varepsilon'))) + 2(\underline{t} - \underline{\delta})\big]. \end{aligned}$$

Substitute this into (2.14) and use (2.2) and the definition of M' to conclude

$$\begin{aligned} & P^m(N_{\underline{t}}(B(y,\varepsilon)) \ge K\underline{\delta}) \\ & \quad \ge (1 - e^{-M'})4^{-1}[(\underline{t} - \underline{\delta}) \vee P_0^m(Y_{\underline{t}-\underline{\delta}} \in B(y,\varepsilon'))]^{-1} \\ & \qquad \times E^m(N_{\underline{t}-\underline{\delta}}(B(y,\varepsilon')))[1 - \tfrac{1}{2} - \tfrac{1}{4}] \\ & \quad \ge (1 - e^{-\underline{M}})16^{-1}\Big[\big(P_0^m(Y_{\underline{t}-\underline{\delta}} \in B(y,\varepsilon'))(\underline{t} - \underline{\delta})^{-1}\big) \wedge 1\Big]. \end{aligned}$$

Let $m = m^\mu$ converge weakly to $m_0 \in M_F(\mathbb{R}^d)$ as $\mu \to \infty$. Now use the facts that $N_{\underline{t}}^{(\mu)} \to_w X_t$ as $\mu \to \infty$ (Theorem 2.2) and $X_t(\partial B(y,\varepsilon)) = 0$ a.s., together with standard weak convergence arguments, to complete the proof. □

3. Instantaneous propagation of the support process. We are ready for the proof of Theorem 1.5. Recall X_t is a super-Lévy process. We may assume without loss of generality that the underlying Lévy process Y has a nonzero Lévy measure, ν.

NOTATION. $\mathscr{B}_{\mathbb{Q}} = \{B(y,r): y \in \mathbb{Q}^d,\ r \in \mathbb{Q}^{>0}\}$.

PROOF OF THEOREM 1.5. Fix $m_0 \in M_F(\mathbb{R}^d)$, $t > 0$ and work with respect to $P = Q^{m_0}$ on the canonical space of continuous $M_F(\mathbb{R}^d)$-valued paths. Let $K_n = [(\log n)/5]$ ($[x]$ denotes the integer part of x) for $n \ge e^5$, and

$$A_n = \{X_{t-2^{-n}}(B(y,\varepsilon)) \ge K_n 2^{-n-1}\},$$

where $y \in \mathbb{R}^d$ and $\varepsilon > 0$ are fixed. Let $\varepsilon' \in (0,\varepsilon)$ and use the Markov property and Lemma 2.5 with $t - \delta = \delta = 2^{-n-2}$ to see that

$$\begin{aligned} P(A_{n+1} \mid \mathscr{F}_{t-2^{-n}}) &= Q^{X_{t-2^{-n}}}\big(X_{2^{-n-1}}(B(y,\varepsilon)) \ge K_{n+1}2^{-n-2}\big) \\ &\ge \big((1 - e^{-M_{n+1}})/16\big) \\ &\quad \times\Big[\big(P_0^{X_{t-2^{-n}}}(Y(2^{-n-2}) \in B(y,\varepsilon'))2^{n+2}\big) \wedge 1\Big], \end{aligned} \tag{3.1}$$

where $M_n = p_n e^{-4K_n/p_n}$, $p_n = P_0^0(|Y(2^{-n-1})| < \varepsilon - \varepsilon')$. Let $\varepsilon'' \in (0, \varepsilon')$ and $\phi\colon [0,\infty] \to [0,1]$ be a decreasing C^∞ function satisfying $\phi|_{[0,\varepsilon'']} = 1$ and $\phi|_{[\varepsilon',\infty]} = 0$. Recall A and $\mathscr{D}(A)$ denote the infinitesimal generator of Y on $C(\mathbb{R}^d)$ and its domain. If $\phi^y(z) = \phi(|z-y|)$, then $\phi^y \in \mathscr{D}(A)$ and

$$(3.2)\quad A\phi^y(x) = \int \phi^y(z+x)\,d\nu(z) \ge \nu(B(y-x,\varepsilon'')) \quad \text{for } |y-x| \ge \varepsilon'$$

[see Fristedt (1974), page 250]. In particular,

$$(3.3)\qquad \lim_{t\downarrow 0} E_0^x(\phi^y(Y_t))t^{-1} = A\phi^y(x),$$

where the convergence is uniform in $x \in \mathbb{R}^d$. The expectation on the left side of (3.3) is less then or equal to $P_0^x(Y_t \in B(y,\varepsilon'))t^{-1}$. Therefore (3.2) and (3.3) imply that w.p.1,

$$(3.4)\quad \begin{aligned} &\liminf_{n\to\infty} \int 1(|x-y| \ge \varepsilon') P_0^x\big(Y(2^{-n-2}) \in B(y,\varepsilon')\big)2^{n+2}\,dX_{t-2^{-n}}(x) \\ &\quad\ge \liminf_{n\to\infty} \int 1(|x-y| \ge \varepsilon') A\phi^y(x)\,dX_{t-2^{-n}}(x) \\ &\quad= \int 1(|x-y| \ge \varepsilon') A\phi^y(x)\,dX_t(x) \\ &\quad\ge \int 1(|x-y| \ge \varepsilon')\nu(B(y-x,\varepsilon''))\,dX_t(x), \end{aligned}$$

where in the next to last line we have used the continuity of X_t and $X_t(\partial B(y,\varepsilon')) = 0$ a.s. Suppose now that $X_t(B(y,\varepsilon'')) > 0$. Then, except for a P-null set, we have

$$\begin{aligned} &\liminf_{n\to\infty} \int 1(|x-y| < \varepsilon') P_0^x\big(Y(2^{-n-2}) \in B(y,\varepsilon')\big)2^{n+2}\,dX_{t-2^{-n}}(x) \\ &\quad\ge \liminf_{n\to\infty} 2^{n+2} X_{t-2^{-n}}(B(y,\varepsilon''))\,P_0^0\big(\big|Y(2^{-n-2})\big| < \varepsilon' - \varepsilon''\big) = \infty. \end{aligned}$$

Use the above together with (3.4) and let $\varepsilon'' \uparrow \varepsilon'$ to see that w.p.1,

$$(3.5)\quad \begin{aligned} &\liminf_{n\to\infty} P_0^{X_{t-2^{-n}}}\big(Y(2^{-n-2}) \in B(y,\varepsilon')\big)2^{n+2} \\ &\quad\ge \int 1(|x-y| \ge \varepsilon')\nu(B(y-x,\varepsilon'))\,dX_t(x) + \infty \cdot 1(X_t(B(y,\varepsilon')) > 0) \\ &\quad= (\tilde{\nu} * X_t)(B(y,\varepsilon')), \end{aligned}$$

where $\tilde{\nu} = \nu + \infty \cdot \delta_0$. (3.5) and (3.1) together imply that for a.a. w and large enough n (depending on w)

$$\begin{aligned} P(A_{n+1} | \mathscr{F}_{t-2^{-n}}) &\ge \big((1 - e^{-M_{n+1}})/32\big)\big(\tilde{\nu} * X_t(B(y,\varepsilon')) \wedge 1\big) \\ &\ge (n+1)^{-1}64^{-1}\big(\tilde{\nu} * X_t(B(y,\varepsilon')) \wedge 1\big). \end{aligned}$$

The extended Borel–Cantelli lemma [Breiman (1968), Corollary 5.29] shows that (let $\varepsilon' \uparrow \varepsilon$)

(3.6) w.p.1 $\tilde{\nu} * X_t(B(y,\varepsilon)) > 0$ implies A_n occurs infinitely often.

Let $0 < \varepsilon < \varepsilon''$, $\varepsilon' = (\varepsilon + \varepsilon'')/2$ and

$$\begin{aligned} T_m &= \inf\{t - 2^{-n} \geq t - 2^{-m} : X_{t-2^{-n}}(B(y,\varepsilon)) \geq K_n 2^{-n-1}\} \wedge t \\ &= t - 2^{-n_m}, \end{aligned}$$

where $n_m \in \{m, m+1, \dots, \infty\}$, $m \in \mathbb{N}$. (3.6) implies

(3.7) w.p.1 $\tilde{\nu} * X_t(B(y,\varepsilon)) > 0$ implies $n_m < \infty$ for all $m \in \mathbb{N}$.

The strong Markov property shows that on $\{n_m < \infty\}$,

$$\begin{aligned} &P\big(X_t(B(y,\varepsilon'')) < 2^{-n_m-1} | \mathscr{F}_{T_m}\big)(w) \\ &\quad = Q^{X_{T_m}(w)}\big(X(2^{-n_m(w)})(B(y,\varepsilon'')) < 2^{-n_m(w)-1}\big) \\ &\quad \leq Q^{X_{T_m}(w)/K_{n_m(w)}}\big(X(2^{-n_m(w)})(B(y,\varepsilon'')) < 2^{-n_m(w)-1}\big)^{K_{n_m(w)}} \end{aligned} \tag{3.8}$$

(by Proposition 1.7).

Use Lemma 2.5 with $t - \delta = \delta = 2^{-n_m(w)-1}$, $\varepsilon = \varepsilon''$ and $K = 1$ to bound the right side of (3.8) by [write $Z_m(w)$ for $X_{T_m}(w)/K_{n_m(w)}$]

$$\Big(1 - (1 - e^{-M_m(w)})(16)^{-1}\big(P_0^{Z_m(w)}(Y_{2^{-n_m(w)-1}} \in B(y,\varepsilon'))2^{n_m(w)+1} \wedge 1\big)\Big)^{K_{n_m}(w)},$$

where $M_m(w) = p_m(w)e^{-4/p_m(w)}$, $p_m(w) = P_0^0(|Y_{2_m^{-n(w)-1}}| < \varepsilon'' - \varepsilon')$. Note that

$$p_m(w) \geq \inf_{t \leq 2^{-m-1}} P_0^0(|Y_t| < \varepsilon'' - \varepsilon') \equiv p_m^1.$$

If m is large enough so that $p_m^1 \geq 1/2$, then on $\{n_m < \infty\}$ we have (recall $\varepsilon'' - \varepsilon' = \varepsilon' - \varepsilon$)

$$\begin{aligned} &P\big(X_t(B(y,\varepsilon'')) < 2^{-n_m-1} | \mathscr{F}_{T_m}\big)(w) \\ &\quad \leq \Big(1 - \big(1 - \exp\{-e^{-8}/2\}\big)16^{-1} \\ &\qquad \times \Big(\big(X_{T_m}(w)(B(y,\varepsilon))K_{n_m(w)}^{-1} p_m(w)2^{n_m(w)+1}\big) \wedge 1\Big)\Big)^{K_m} \\ &\quad \leq q^{K_m}, \end{aligned}$$

where $q = 1 - (1 - \exp\{-e^{-8}/2\})/32 \in (0,1)$. Integrate the above inequality over $\{n_m < \infty\} = \{T_m < t\}$ to see

$$P\big(X_t(B(y,\varepsilon'')) \geq 2^{-n_m-1}, T_m < t\big) \geq (1 - q^{K_m})P(T_m < t).$$

From this and (3.7) we derive

$$P(X_t(B(y,\varepsilon'')) > 0) \geq (1 - q^{K_m})P(\tilde{\nu} * X_t(B(y,\varepsilon)) > 0).$$

Let $m \to \infty$ and $\varepsilon'' \downarrow \varepsilon$ and use $X_t(\partial B(y,\varepsilon)) = 0$ a.s. to conclude

$$P(X_t(B(y,\varepsilon)) > 0) \geq P(\tilde{\nu} * X_t(B(y,\varepsilon)) > 0).$$

The event on the left is a subset of that on the right and so w.p.1 for any $B \in \mathscr{B}_{\mathbb{Q}}$, $X_t(B) > 0$ iff $\tilde{\nu} * X_t(B) > 0$, and hence

$$\text{w.p.1} \quad S(\nu * X_t) \subset S(\tilde{\nu} * X_t) = S_t. \tag{3.9}$$

If $\mu_1, \mu_2, \mu_3 \in M_f(\mathbb{R}^d)$, $S(\mu_1) \subset S(\mu_2)$ implies $S(\mu_1 * \mu_3) \subset S(\mu_2 * \mu_3)$. Use this and (3.9) to see

$$S(\nu^{(2)} * X_t) = S(\nu * (\nu * X_t)) \subset S(\nu * X_t) \subset S_t \quad \text{a.s.}$$

Iterate to get $S(\nu^{(k)} * X_t) \subset S_t$ for all $k \in \mathbb{N}$ a.s. and therefore

$$\bigcup_{k=0}^{\infty} S(\nu^{(k)} * X_t) = S_t \quad \text{a.s.} \qquad \square$$

In addition to the symmetric stable case considered in the Introduction it is interesting to consider the above result when $m_0 = \delta_0$ and Y is a Poisson process with rate λ so that $\nu = \lambda\delta_1$. The theorem implies that if $t > 0$ is fixed then w.p.1 $k \in S_t$ implies $\{k, k+1, \dots\} \subset S_t$. The process becomes extinct as $\min(S_t)$ increases to $+\infty$.

COROLLARY 3.1. *If Y is a Poisson process with rate $\lambda > 0$, then Q^{δ_0}-a.s. there are $(\mathscr{F}_t)$-stopping times $\{T_k: k \in \mathbb{Z}_+\}$ such that $T_0 = 0$, $\{T_k\}$ increases to ζ a.s.,*

$$k \in S_t \subset \{k, k+1, \dots\} \quad \textit{for all } t \textit{ in } [T_k, T_{k+1}), \qquad k \in \mathbb{Z}_+ \ \textit{a.s.} \tag{3.10}$$

and

$$S_t = \{k, k+1, \dots\} \quad \textit{for Lebesgue a.a. } t \textit{ in } [T_k, T_{k+1}), k \in \mathbb{Z}_+ \ \textit{a.s.} \tag{3.11}$$

PROOF. It is clear from Theorem 2.2 that for any $m_0 \in M_F$,

$$S(m_0) \subset \{k, k+1, \dots\} \quad \text{implies } S_t \subset \{k, k+1, \dots\} \quad \text{for all } t \geq 0 \qquad Q^{m_0}\text{-a.s.} \tag{3.12}$$

Define

$$T_k = \inf\{t: X_t(\{0, \dots, k-1\}) = 0\}.$$

Then clearly $T_k \uparrow \zeta$ a.s. and

$$Q^{\delta_0}\big(X_t(\{0, \dots, k-1\}) = 0 \text{ for all } t \geq T_k | \mathscr{F}_{T_k}\big)$$
$$= Q^{X(T_k)}\big(S_t \subset \{k, k+1, \dots\} \text{ for all } t \geq 0\big) = 1 \quad \text{a.s.}$$

by (3.12) with $m_0 = X(T_k)$. It follows that

$$S_t \subset \{k, k+1, \dots\} \quad \text{for all } t \geq T_k \qquad Q^{\delta_0}\text{-a.s.}$$

(3.10) is now immediate from the definition of T_k. Choose w so that the conclusion of Theorem 1.5 holds for all $t > 0$ outside an exceptional Lebesgue null set $\Lambda(w)$. Since $\nu = \lambda\delta_1$, (3.10) and Theorem 1.5 imply

$$S_t \supset \bigcup_{j=0}^{\infty} S(\delta_j * X_t) \supset \{k, k+1, \dots\} \quad \text{for } t \in \Lambda(w) \cap [T_k, T_{k+1}). \qquad \square$$

REMARK. It is not hard to use (1.1) to set up a stochastic differential equation for $(X_t(0), X_t(\{0\}^c))$ from which one may infer via a comparison argument that

$$P\big(X_t(\{0\}^c) = 0 \text{ for some } t < T_1\big) > 0.$$

Hence the exceptional set of t in (3.11) is necessary.

4. On the discontinuities of the support of super-Brownian motion. Throughout this section X is a d-dimensional ($d \geq 1$) super-Brownian motion starting at $m_0 \in M_F$. Our immediate objective is Theorem 1.4. A deterministic lemma will give the existence of left-hand limits for $\{S_t: t \geq 0\}$.

NOTATION. If $F \subset \mathbb{R}^d$ and $\varepsilon > 0$, let $A^\varepsilon = \{x: d(x, A) \leq \varepsilon\}$, where $d(x, A)$ is the distance from x to A.

LEMMA 4.1. *Assume $f: (0, \infty) \to \mathscr{K}(\mathbb{R}^d)$ satisfies*

(4.1) $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ *such that* $0 \leq u - t < \delta$ *implies* $f(u) \subset f(t)^\varepsilon$.

Then f possesses left- and right-hand limits at all t in $(0, \infty)$.

PROOF. If $N \in \mathbb{N}$ is fixed, (4.1) implies there is a compact ball B so that $f([N^{-1}, N]) \subset \mathscr{K}(B)$. $\mathscr{K}(B)$ is compact [Dugundji (1966), page 253, Exercise 6, Section 4]. Therefore if $\lim_{s \uparrow t} f(s)$ does not exist in $\mathscr{K}(\mathbb{R}^d)$ there are sequences $\{s_n\}, \{u_n\}$, increasing to t such that $s_n < u_n < s_{n+1}$, $\lim f(u_n) = K_1$, and $\lim f(s_n) = K_2$ in $\mathscr{K}(B)$ [or equivalently $\mathscr{K}(\mathbb{R}^d)$], where $K_1 \neq K_2$. Suppose $x \in K_2 - K_1$. If $\varepsilon \in (0, d(x, K_1)/2)$ choose δ as in (4.1). For large enough n, $0 < s_{n+1} - u_n < \delta$ and so (4.1) implies $f(s_{n+1}) \subset f(u_n)^\varepsilon$. Since $x \in K_2 = \lim f(s_n)$, we also have $d(x, f(s_{n+1})) < \varepsilon/2$ for large n and hence $d(x, f(u_n)) < 3\varepsilon/2$ for large n. Let $n \to \infty$ to see $d(x, K_1) \leq 3\varepsilon/2$ which contradicts the choice of ε. Therefore $K_2 \subset K_1$ and by symmetry $K_2 = K_1$. This contradicts the choice of K_1 and K_2 and hence proves the existence of left-hand limits of f. By making some trivial changes in this argument one obtains the existence of right-hand limits. $\square$

PROPOSITION 4.2. *$\{S_t: t > 0\}$ is a $\mathscr{K}(\mathbb{R}^d)$-valued $\mathscr{F}_t$-optional process which has right-continuous paths with left-hand limits Q^{m_0}-a.s. Moreover,*

(4.2) $$S_{t-} \supset S_t \quad \textit{for all } t > 0 \qquad Q^{m_0}\text{-a.s.}$$

PROOF. Theorem 4.4.1 of Cutler (1984) implies S is $\mathscr{F}_t$-optional. The previous lemma and the one-sided Lévy modulus for S_t (1.8) imply the a.s. existence of left- and right-hand limits. The right-continuity was proved in

(D.I.P.) [see (1.6)]. If $X_t(B(x,\varepsilon)) > 0$, then $X_u(B(x,\varepsilon)) > 0$ for u near t by the continuity of X. (4.2) is therefore obvious. □

A nonstandard construction of X is used to finish the proof of Theorem 1.4. We work in on w_1-saturated enlargement of a superstructure containing $\mathbb{R}$. Let $\{N^\beta\colon \beta \in I\}$ be the system of branching Brownian motions constructed on $(\Omega, \mathscr{A})$ in Section 2. Fix an infinite μ in $^*\mathbb{N} - \mathbb{N}$ and consider the internal collection of branching *-Brownian motions $\{N^\beta\colon \beta \in {}^*I\}$ and the internal $^*M(\mathbb{R}^d)$-valued process $N = N^{(\mu)}$, all defined on the internal measure space $(^*\Omega, {}^*\mathscr{A})$. If $m_\mu = \mu^{-1}\sum_{i=0}^{K}\delta_{x_i} \in {}^*M_F^\mu(\mathbb{R}^d)$, $^*P^{m_\mu}$ (or more precisely $^*P^{(x_i)}$) is the internal probability on $(^*\Omega, {}^*\mathscr{A})$ defined as in Section 2. We abuse the notation slightly and write

$$(^*\Omega, \mathscr{F}, P^{m_\mu}) = (^*\Omega, L(^*\mathscr{A}), L(^*P^{m_\mu})),$$

for the Loeb space constructed from $(^*\Omega, {}^*\mathscr{A}, {}^*P^{m_\mu})$ [see Loeb (1975) and Cutland (1983)].

NOTATION. st_d denotes the standard part map on the near-standard points in $^*\mathbb{R}^d$, $^\circ t$ denotes the standard part of a finite t in $^*[0,\infty)$ and st_M denotes the standard part map on $\mathrm{ns}(^*M_F(\mathbb{R}^d))$.

THEOREM 4.3. *Let $m_0 \in M_F(\mathbb{R}^d)$ and choose $m_\mu \in {}^*M_F^\mu(\mathbb{R}^d)$ such that $\mathrm{st}_M(m_\mu) = m_0$. There is a unique (up to indistinguishability) continuous M_F-valued process X on $(^*\Omega, \mathscr{F}, P^{m_\mu})$ such that*

$$X_{^\circ t}(A) = L(N_t)(\mathrm{st}_d^{-1}(A)), \tag{4.3}$$
$$\textit{for all } A \textit{ in } \mathscr{B}(\mathbb{R}^d) \textit{ and } t \textit{ in } \mathrm{ns}(^*[0,\infty))\ P^{m_\mu}\text{-a.s.}$$

Moreover, $P^{m_\mu}(X \in \cdot) = Q^{m_0}(\cdot)$

This is Theorem 2.3 of (D.I.P.). We also need the following nonstandard Lévy modulus for $N^\beta_\cdot$.

NOTATION. $h(u) = (u \log^+ 1/u)^{1/2}$.

THEOREM 4.4 [Theorem 4.7 of (D.I.P.)]. *If m_μ is as in Theorem 4.3, then for P^{m_μ}-a.a. w and every $c > 2$ there is a $\delta(w,c) > 0$ such that*

$$|N_t^\beta - N_s^\beta| \le \mathrm{ch}(t-s), \quad \textit{for all } s,t \in {}^*[0,\infty] \tag{4.4}$$
$$\textit{and } \beta \sim t \textit{ satisfying } 0 < t - s \le \delta(w,c) \textit{ and } N_t^\beta \ne \Delta.$$

Now choose $m_\mu \in {}^*M_F^\mu$ such that $\mathrm{st}_M(m_\mu) = m_0$ and $\mathrm{st}(S(m_\mu)) = S(m_0)$ and write P and $^*P$ for P^{m_μ} and $^*P^{m_\mu}$, respectively. Let $\underline{t} = [t\mu]/\mu$ for

$t \in {}^*[0,\infty)$ and let $\mathscr{F}_t^0 = \sigma(\mathscr{A}_t)$ for $t \in [0,\infty)$. For each $B \in \mathscr{B}_{\mathbb{Q}}$ and $r \in \mathbb{Q}^{\geq 0}$, define an internal ${}^*M_F$-valued process by

$$N_t^{r,B}(A) = \mu^{-1} \sum_{\gamma \sim t+\underline{r}} 1(N_{\underline{r}}^{\gamma} \in {}^*B, N_{t+\underline{r}}^{\gamma} \in A).$$

$N^{r,B}$ records the future evolution of the descendants of those particles which are in ${}^*B$ at time $\underline{r}$. If C_1 and C_2 are internal *-Borel measurable subsets of ${}^*D([0,\infty), M_F)$ and $\nu|_B$ denotes the restriction of a measure ν to a set B, then clearly

$$(4.5) \qquad {}^*P(N^{r,B} \in C_1 | \mathscr{A}_{\underline{r}})(w) = {}^*P^{N_{\underline{r}}|_{*B}(w)}(N \in C_1) \qquad {}^*P\text{-a.s.}$$

More generally if B_1 and B_2 are disjoint in $\mathscr{B}_{\mathbb{Q}}$ then by conditioning on the disjoint sets $\{\gamma \sim \underline{r}: N_{\underline{r}}^{\gamma} \in {}^*B_i\}$, $i = 1, 2$, it is easy to see that

$$(4.6) \quad {}^*P(N^{r,B_i} \in C_i, i = 1,2 | \mathscr{A}_{\underline{r}})(w) = \prod_{i=1}^{2} {}^*P^{N_{\underline{r}}|_{*B_i}(w)}(N \in C_i) \qquad {}^*P\text{-a.s.}$$

(4.5) and Theorem 4.3 (with $m_\mu = N_{\underline{r}}|{}^*B$) show that for each $r \in \mathbb{Q}^{\geq 0}$ and $B \in \mathscr{B}_{\mathbb{Q}}$ there is a unique (up to indistinguishability) continuous M_F-valued process $X_t^{r,B}$ on $({}^*\Omega, \mathscr{F}, P)$ such that

$$(4.7) \qquad X_{\circ t}^{r,B}(A) = L(N_t^{r,B})(\mathrm{st}_d^{-1}(A)) \quad \text{for all } A \in \mathscr{B}(\mathbb{R}^d) \text{ and } t \in \mathrm{ns}({}^*[0,\infty)) \ P\text{-a.s.}$$

(Technically this is just a.s. S-continuity of $N^{r,B}$.) Take standard parts in (4.5), and use Hoover and Perkins (1983), Lemma 3.3, the fact that $\mathrm{st}_M(N_{\underline{r}}|_{*B}) = X_r|_B$ a.s. [recall $X_r(\partial B) = 0$ a.s.] and Theorem 4.3 to see

$$(4.8) \qquad P(X^{r,b} \in C | \mathscr{F}_r^0)(w) = Q^{X_r|_B(w)}(C) \quad \text{a.s. for all Borel measurable } C \text{ in } C([0,\infty], M_F).$$

Similarly (4.6) shows that:

(4.9) If $B_1 \cap B_2 = \varnothing$, X^{r,B_1} and X^{r,B_2} are conditionally independent given $\mathscr{F}_r^0$.

NOTATION. $\zeta(r, B) = \inf\{t \geq 0: X_t^{r,B}(\mathbb{R}^d) = 0\} + r$.

$$S_t^{r,B} = S(X_t^{r,B}).$$

In the rest of this section we will work with respect to P, but in view of Theorem 4.3, may transfer these results onto path space under Q^{m_0} by trivial measurability arguments.

LEMMA 4.5. *For P-a.a. w if $t > 0$, $x \in S_{t-} - S_t$ and $B = B(y, 2\delta_1) \in \mathscr{B}_{\mathbb{Q}}$ satisfies $X_t(B(y, 3\delta_1)) = 0$ and $x \in B(y, \delta_1)$, then there is an $\varepsilon = \varepsilon(w, B, t) > 0$*

such that $t = \zeta(r, B)$ *and* $x \in S^{r,B}((\zeta(r, B) - r) -)$ *for all* $r \in (t - \varepsilon, t) \cap \mathbb{Q}^{\geq 0}$.

PROOF. Choose w outside a P-null set so that there is a $\delta(w, 3) > 0$ satisfying (4.4) (with $c = 3$), and such that (4.3) and (4.7) hold, the latter for all $r \in \mathbb{Q}^{\geq 0}$ and $B \in \mathscr{B}_{\mathbb{Q}}$. Assume $B = B(y, 2\delta_1)$ and $x \in S_{t-} - S_t$ satisfy the hypotheses of the lemma. If $B_\rho = B(y, \rho\delta_1)$, the fact that $x \in S_{t-} \cap B_1$ means there is an $\varepsilon = \varepsilon(w, B, t) > 0$ such that

$$X_u(B_1) > 0 \quad \text{for } u \in (t - \varepsilon, t). \tag{4.10}$$

We may choose ε so that, in addition,

$$\varepsilon < \delta(w, 3) \quad \text{and} \quad 3h(\varepsilon) < \delta_1. \tag{4.11}$$

Let $r \in (t - \varepsilon, t) \cap \mathbb{Q}^{\geq 0}$. If $s \in [r, t]$, and $\gamma \sim s$ then (4.11) implies

$$^{\circ}|N_s^\gamma - N_{\underline{r}}^\gamma| \leq 3h(s - r) < \delta_1 \tag{4.12}$$

and hence $N_s^\gamma \in {}^*B(y, \rho)$ implies $N_{\underline{r}}^\gamma \in {}^*B$ whenever $\rho \leq \delta_1$. Therefore

$$N_{s-\underline{r}}^{r,B}({}^*B(y, \rho)) = N_s({}^*B(y, \rho)) \quad \text{whenever } \rho \leq \delta_1.$$

Now use (4.3) and (4.7) to obtain

$$\begin{aligned} X_s(B_1) &= \lim_{\rho \uparrow \delta_1} {}^{\circ}N_s({}^*B(y, \rho)) \\ &= \lim_{\rho \uparrow \delta_1} {}^{\circ}N_{s-\underline{r}}^{r,B}({}^*B(y, \rho)) \quad \text{(by the above)} \\ &= X_{s-r}^{r,B}(B_1) \qquad \forall\, s \in [r, t]. \end{aligned} \tag{4.13}$$

This together with (4.10) shows that $\zeta(r, B) \geq t$. For the opposite inequality use (4.12) with $s = t$ to see that $N_{\underline{r}}^\gamma \in {}^*B$ implies ${}^{\circ}N_t^\gamma \in B_3$ and hence [use (4.3)] ${}^{\circ}N_{t-\underline{r}}^{r,B}({}^*\mathbb{R}^d) \leq X_t(B_3) = 0$. Therefore $X_{t-r}^{r,B}(\mathbb{R}^d) = 0$ [by (4.7)] and we have $t \geq \zeta(r, B)$. (4.13) and $X_s \geq X_{s-r}^{r,B}$ [by (4.3) and (4.7)] implies $S_s \cap B_1 = S_{s-r}^{r,B} \cap B_1$ $\forall\, s \in [r, t]$. Let $s \uparrow t$ (omitting another null set of w to ensure the limits exist) to see

$$S_{t-} \cap B_1 = S_{(t-r)-}^{r,B} \cap B_1.$$

Since x belongs to the set on the left, it belongs to $S_{(t-r)-}^{r,B}$ and the proof is complete. □

PROPOSITION 4.6. *For a.a.* w *for all* $t > 0$, $\text{card}(S_{t-} - S_t) = 0$ *or* 1.

PROOF. Since $X_t(\mathbb{R}^d)$ is the diffusion with generator $\frac{1}{2}x\,d^2/dx^2$,

$$Q^m(\zeta \leq t) = \exp\{-2m(\mathbb{R}^d)/t\} \qquad \text{[Knight (1981), page 100]}.$$

(4.8) therefore implies

$$P(\zeta(r, B) - r \leq t | \mathscr{F}_r^0) = \exp\{-2X_r(B)/t\}.$$

If B_1 and B_2 are disjoint balls in $\mathscr{B}_{\mathbb{Q}}$ and $r \leq 0$, $\zeta(r, B_1)$ and $\zeta(r, B_2)$ are conditionally independent given $\mathscr{F}_r^0$ by (4.9). Since their conditional laws are atomless we have

$$P\big(\zeta(r,B_1) = \zeta(r,B_2) > r \text{ for some } r \in \mathbb{Q}^{\geq 0} \text{ and disjoint balls } B_1, B_2 \text{ in } \mathscr{B}_{\mathbb{Q}}\big) = 0.$$

Fix w outside this P-null set and such that the conclusion of the previous lemma holds. Suppose that x, y are distinct points in $S_{t-} - S_t$ for some $t > 0$. There are disjoint balls $B_1 = B(x_1, 2\delta_1)$, $B_2 = B(y_1, 2\delta_1)$ in $\mathscr{B}_{\mathbb{Q}}$ such that $X_t(B(x_1, 3\delta_1)) = X_t(B(y_1, 3\delta_1)) = 0$ and $x \in B(x_1, \delta_1)$, $y \in B(y_1, \delta_1)$. By the choice of w there is an $r \in \mathbb{Q}^{\geq 0}$, $r < t$ such that $\zeta(r, B_1) = \zeta(r, B_2) = t$. This contradicts the choice of w and hence completes the proof. □

Theorem 1.4 is now immediate from Propositions 4.2 and 4.6.

Let

$$\mathscr{D} = \left\{(A,B) \in \mathscr{K}(\overline{\mathbb{R}^d})^2 \colon A \supset B,\ A - B \text{ is empty or a singleton}\right\},$$

and define $\psi\colon \mathscr{D} \to \overline{\mathbb{R}^d}$ by

$$\psi(A,B) = \begin{cases} x & \text{if } A - B = \{x\}, \\ \Delta & \text{if } A = B. \end{cases}$$

It is easy to check that $\mathscr{D}$ is a closed subset of $\mathscr{K}(\mathbb{R}^d)^2$ and ψ is measurable [its restriction to $\{(A,B) \in \mathscr{D}\colon A \neq B\}$ is continuous]. Let

$$Z(t,w) = \begin{cases} \psi(S_{t-}(w), S_t(w)) & \text{if } (S_{t-}, S_t) \in \mathscr{D} \text{ and } t > 0, \\ \Delta & \text{otherwise}. \end{cases}$$

Theorem 1.4 shows Z is an $(\mathscr{F}_t)$-optional process on $[0,\infty) \times \Omega$ and w.p.1 equals Δ except at the countably many jump times of S_t. For such a time $Z_t(w)$ is the location at which a "colony" becomes extinct at time t. With (4.8) in mind we also define (for each $r \in \mathbb{Q}^{\geq 0}$, $B \in \mathscr{B}_{\mathbb{Q}}$)

$$Z^{r,B}(t) = \begin{cases} \psi\big(S_{t-}^{r,B}, S_t^{r,B}\big) & \text{if } \big(S_{t-}^{r,B}, S_t^{r,B}\big) \in \mathscr{D} \text{ and } t > 0, \\ \Delta & \text{otherwise}, \end{cases}$$

$$Z(r,B) = Z^{r,B}(\zeta(r,B) - r).$$

(4.8) and Theorem 1.4 show that

$$\text{if } \zeta(r,B) > r \text{ then } Z(r,B) \in \mathbb{R}^d \quad \text{a.s.} \tag{4.14}$$

$Z(r, B)$ is the location at which descendants of those individuals in B at time r, become extinct.

PROPOSITION 4.7. *For P-a.a. w and all $0 \leq s < t \leq \infty$, $\overline{R}(s,t) - R(s,t)$ is countable and satisfies*

$$\begin{aligned} \overline{R}(s,t) - R(s,t) &\subset \{Z(u) \in \mathbb{R}^d \colon s < u \leq t\} \\ &\subset \{Z(r,B)\colon r \in \mathbb{Q} \cap (s,t),\ \zeta(r,B) > r,\ B \in \mathscr{B}_{\mathbb{Q}}\}. \end{aligned} \tag{4.15}$$

PROOF. Choose w outside a null set so that the conclusions of Theorem 1.4 and Lemma 4.5, and (1.7) hold. If $x \in \overline{R}(s,t) - R(s,t)$, choose $u_n \in [s,t]$

and $x_n \in S_{u_n}$ so that $\lim x_n = x$. By taking a subsequence we may assume $u_n \downarrow u \in [s,t]$ or $u_n \uparrow u \in [s,t]$. The former case would imply $x \in S_u$ by right-continuity of $\{S_u\}$ [use (1.7) if $u = 0$], contradicting $x \notin R(s,t)$. Hence we are in the latter case and so $x \in S_{u-} - S_u$ which implies $x = Z(u)$ for some $u \in (s,t)$. Assume, in addition, that w is in the set of probability 1 for which $\{S_t^{r,B}: t \geq 0\}$ satisfies the conclusion of Theorem 1.4 for all $r \in \mathbb{Q}^{\geq 0}$ and $B \in \mathscr{B}_{\mathbb{Q}}$ [use (4.8)]. The second inclusion in (4.15) is then immediate from Lemma 4.5. The countability of $\bar{R}(s,t) - R(s,t)$ is obvious from (4.15). □

NOTATION. If $r \in \mathbb{Q}^{\geq 0}$ and $B \in \mathscr{B}_{\mathbb{Q}}$, let

$$\underline{\zeta}(r,B) = \min\{t: N_t^{r,B}({}^*\mathbb{R}^d) = 0\} + \underline{r}$$

and

$$I^{r,B} = \left\{\gamma \sim \underline{\zeta}(r,B) - \mu^{-1}: N^\gamma\left(\underline{\zeta}(r,B) - \mu^{-1}\right) \neq \Delta,\ N_{\underline{r}}^\gamma \in {}^*B\right\}.$$

Since $N^{r,B}(\underline{\zeta}(r,B) - \underline{r})({}^*\mathbb{R}^d) = 0$, (4.7) implies $\zeta(r,B) \leq {}^\circ\underline{\zeta}(r,B)$ a.s. The opposite inequality is clear from (4.5) and Lemma 4.9 of (D.I.P.) so that

$$ {}^\circ\underline{\zeta}(r,B) = \zeta(r,B) < \infty \quad \text{a.s.} \tag{4.16}$$

The definition of $I^{r,B}$ implies it is nonempty if $\underline{\zeta}(r,B) > \underline{r}$. By lexicographically ordering I and choosing the "first" $\gamma \in I^{r,\bar{B}}$ if $\underline{\zeta}(r,B) > \underline{r}$, and any fixed $\gamma_0 \in I$ otherwise, we may define a random index $\gamma^{\bar{r},B}(w)$ such that

$$\gamma^{r,B} \text{ is measurable w.r.t. the internal algebra } \mathscr{E} \vee \mathscr{A}_{\underline{r}} \quad \left(\text{recall } \mathscr{E} \text{ is generated by } \{e^\beta: \beta \in I\}\right) \tag{4.17}$$

and

$$\underline{\zeta}(r,B) > \underline{r} \quad \text{implies } \gamma^{r,B} \in I^{r,B}.$$

Lemma 4.9 of (D.I.P.), (4.4), (4.16) and the above imply

$$\begin{gathered} \text{if } \zeta(r,B) > r \text{ then } \gamma^{r,B} \in I^{r,B} \text{ and} \\ \operatorname{st}_d\left(N^{\gamma^{r,B}}\left(\underline{\zeta}(r,B) - \mu^{-1}\right)\right) = Z(r,B) \quad \text{a.s.} \end{gathered} \tag{4.18}$$

[Here Lemma 4.9 of (D.I.P.) and (4.4) together imply

$$\operatorname{st}_d\left(N^{\gamma^{r,B}}\left(\underline{\zeta}(r,B) - \mu^{-1}\right)\right) \in S^{r,B}\left(\left({}^\circ\underline{\zeta}(r,B) - r\right) -\right) \quad \text{if } {}^\circ\underline{\zeta}(r,B) > r.]$$

DEFINITION. The closed graph of S is

$$G(w) = \{(t,x): x \in S_{t-}, t > 0\} \cup \{0\} \times S_0.$$

$G(w)$ is a.s. a closed subset of $[0,\infty) \times \mathbb{R}^d$ by Theorem 1.4 and (1.7). We are now ready to prove that the set of extinction points $(t, Z(t))$ in $[0,\infty) \times \mathbb{R}^d$ is dense in the graph of S for $d \geq 3$.

THEOREM 4.8. *If $d \geq 3$, the countable set*

$$H(w) = \{(t, Z(t)): t > 0, Z(t) \in \mathbb{R}^d\}$$

is a dense subset of $G(w)$ a.s.

PROOF. Clearly $H(w) \subset G(w)$. Fix $r \in \mathbb{Q}^{\geq 0}$ and $B \in \mathscr{B}_{\mathbb{Q}}$. We claim

$$\text{(4.19)} \qquad \begin{gathered} \zeta(r,B) > r \text{ implies} \\ (\zeta(r,B), Z(r,B)) = (\zeta(r,B), Z(\zeta(r,B))) \in H \text{ a.s.} \end{gathered}$$

Define X^{r,B^c} as for $X^{r,B}$ (but with B^c in place of B). As for (4.8) and (4.9) we have

$$\text{(4.20)} \qquad \begin{gathered} P(X^{r,B^c} \in C | \mathscr{F}_r^{\,0})(w) = Q^{X_r|_{B^c(w)}}(C) \quad \text{a.s.} \\ \text{for all measurable } C \text{ in } C([0,\infty], M_F) \end{gathered}$$

and

(4.21) $X^{r,B}$ and X^{r,B^c} are conditionally independent given $\mathscr{F}_r^{\,0}$.

Theorem 3.1 of (D.I.P.) implies

$$\text{(4.22)} \qquad Q^m(x \in S_t) = 0 \qquad \forall\, x \in \mathbb{R}^d,\ t > 0 \text{ and } m \in M_F.$$

Condition on $\mathscr{F}_r^{\,0} \vee \sigma(X^{r,B})$ and use (4.14) and (4.20)–(4.22) to conclude that outside a P-null set, N_1, $\zeta(r,B) > r$ implies $Z(r,B) \in \mathbb{R}^d - S(X^{r,B^c}_{\zeta(r,B)-r})$. Finally, (4.3) and (4.7) show that outside of a null set, N_2, $X_t = X^{r,B^c}_{t-r} + X^{r,B}_{t-r}$ $\forall\, t \geq r$, and the conclusion of Theorem 1.4 holds. Choose $w \notin N_1 \cup N_2$ so that $\zeta(r,B) > r$. Then $Z(r,B) \in S^{r,B}((\zeta(r,B) - r) -) \subset S(\zeta(r,B) -)(X^{r,B}_{t-r} \leq X_t$ $\forall\, t \geq r)$, and $Z(r,B) \notin S_{\zeta(r,B)}$ because $Z(r,B) \notin S(X^{r,B^c}_{\zeta(r,B)-r})$ and $X_{\zeta(r,B)} = X^{r,B^c}(\zeta(r,B) - r)$. This proves $Z(r,B) \in S((\zeta(r,B)) -) - S(\zeta(r,B))$ and hence (4.19).

If $r_1 < r_2$, r_1 and $r_2 \in \mathbb{Q}^{\geq 0}$, let

$$J(r_1,r_2) = \{N_{r_1}^{\gamma}\colon \gamma \sim r_1, \exists\, \beta \sim r_2 \text{ s.t. } \beta > \gamma \text{ and } N_{r_2}^{\beta} \neq \Delta\},$$

$$\Gamma(r_1,r_2) = \operatorname{card}(J(r_1,r_2)).$$

Then conditional on $\mathscr{F}_{r_1}^{0}$, $\Gamma(r_1,r_2)$ is a Poisson r.v. with mean $2X_{r_1}(\mathbb{R}^d)(r_2 - r_1)^{-1}$. Therefore for a.a. w,

$$\text{(4.23)} \qquad \Gamma(r_1,r_2) < \infty \quad \text{for all } r_1, r_2 \in \mathbb{Q}^{\geq 0} \text{ with } r_1 < r_2.$$

Choose w outside a null set so that, in addition to (4.23) and (4.18), the following conditions hold:

$$\text{(4.24)} \qquad \begin{gathered} X_t(\{x\}) = 0 \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d \\ \text{[Theorems A and B of Perkins (1988)]}, \end{gathered}$$

$$\text{(4.25)} \qquad \text{(4.4) holds with } c = 3,$$

$$\text{(4.26)} \qquad \begin{gathered} \operatorname{st}_d(S(N_t)) \supset S(X_{{}^\circ t})\ \forall\, t \in \operatorname{ns}({}^*[0,\infty)) \text{ and} \\ \operatorname{st}_d(S(N_r)) = S(X_r)\ \forall\, r \in \mathbb{Q}^{\geq 0} \\ \text{[Lemma 4.8 and (4.26) of (D.I.P.) and the choice of } m_\mu], \end{gathered}$$

$$\text{(4.27)} \qquad \begin{gathered} \zeta(r,B) > r \quad \text{implies } (\zeta(r,B), Z(r,B)) \in H(w) \\ \text{for all } r \in \mathbb{Q}^{\geq 0} \text{ and } B \in \mathscr{B}_{\mathbb{Q}}. \end{gathered}$$

In (4.27) we have used (4.19). Let $(t, x) \in G$ and $\varepsilon > 0$. Assume $x \in S_{t-}$ and $t > 0$ (a similar argument will work if $t = 0$ and $x \in S_0$) and let $r_1, r_2 \in \mathbb{Q}^{\geq 0}$ satisfy $t \in (r_1, r_2)$ and $r_2 - r_1 < \delta(w, 3) \wedge \varepsilon$ [δ as in (4.4)]. Choose $u \in (r_1, t)$ and $y \in S_u$ such that $|y - x| < \varepsilon$. Then $y = \mathrm{st}_d(N_u^\gamma)$ for some $\gamma \sim u$ and $z \equiv \mathrm{st}_d(N_{r_1}^\gamma) \in S_{r_1}$ by (4.26). Since $z \in S_{r_1}$, (4.24) implies $X_{r_1}(B(z, \delta) - \{z\}) > 0$ for all $\delta > 0$. By (4.23) we may therefore choose $B \in \mathscr{B}_{\mathbb{Q}}$ such that $B \subset B(\mathrm{st}_d(N_{r_1}^\gamma), \varepsilon)$, $B \cap S_{r_1} \neq \varnothing$ and $B \cap J(r_1, r_2) = \varnothing$. Use the definition of $J(r_1, r_2)$ and (4.26) to see that the latter two conditions imply

$$r_1 < \zeta(r_1, B) \leq r_2. \tag{4.28}$$

(4.18), (4.21) and the inclusion $B \subset B(\mathrm{st}_d(N_{r_1}^\gamma), \varepsilon)$ show that if $\gamma' = \gamma^{r_1, B}$ and $v = \underline{\zeta}(r_1, B) - \mu^{-1}$, then

$$\begin{aligned} |Z(r_1, B) - x| &\leq {}^\circ|N_v^{\gamma'} - N_{\underline{r}_1}^{\gamma'}| + {}^\circ|N_{\underline{r}_1}^{\gamma'} - N_{r_1}^{\gamma}| + {}^\circ|N_{r_1}^{\gamma} - N_u^{\gamma}| + |y - x| \\ &\leq 3h(r_2 - r_1) + \varepsilon + 3h(r_2 - r_1) + \varepsilon = 6h(\varepsilon) + 2\varepsilon. \end{aligned} \tag{4.29}$$

(4.27), (4.28) and (4.29) show that we have found a point $(\zeta(r_1, B), Z(r_1, B)) \in H$ within $6h(\varepsilon) + 3\varepsilon$ of (t, x). The result follows. □

REMARK. The condition $d \geq 3$ was only needed to derive (4.22). In fact (4.22), and hence the above result, is also true in two dimensions. One approach is to show first that $Q^{m_1}(X_{\delta+\cdot} \in A)$ and $Q^{m_2}(X_{\delta+\cdot} \in A)$ are equivalent laws. This implies that if $Q^m(x \in S_t) > 0$ for some x, then this is the case for all x, and hence S_t has positive Lebesgue measure with positive probability. The latter is shown to be false in Perkins (1989). We feel, however, that this argument is longer than the result warrants and we leave a simple derivation of (4.22) for $d = 2$ as an open problem.

The next result will be useful in the study of polar sets in the next section.

THEOREM 4.9. *If A is a Lebesgue null subset of $\mathbb{R}^d$, $\{Z(u): u > 0\} \cap A = \varnothing$ a.s.*

PROOF. Fix $r \in \mathbb{Q}^{\geq 0}$ and $B \in \mathscr{B}_{\mathbb{Q}}$, and let $C \in {}^*\mathscr{B}(\mathbb{R}^d)$. Then (4.17) and the fact that $\underline{\zeta}(r, B)$ is $\mathscr{E} \vee \mathscr{A}_{\underline{r}}$-measurable imply that on $\{\underline{\zeta}(r, B) > \underline{r}\}$,

$$\begin{aligned} &{}^*P\left(N^{\gamma^{r,B}}\left(\underline{\zeta}(r, B) - \mu^{-1}\right) \in C \,|\, \mathscr{A}_{\underline{r}} \vee \mathscr{E}\right)(w) \\ &\quad = {}^*P\left(N^{\gamma^{r,B}(w)}\left(\underline{\zeta}(r, B)(w) - \mu^{-1}\right) \in C \,|\, \mathscr{A}_{\underline{r}} \vee \mathscr{E}\right)(w) \\ &\quad = {}^*P_0^{N_{\underline{r}}^{\gamma^{r,B}(w)}}\left({}^*B\left(\underline{\zeta}(r, B)(w) - \mu^{-1} - \underline{r}\right) \in C\right) \quad \text{a.s. [Lemma 2.1(a)].} \end{aligned}$$

Take standard parts in the above, using Hoover and Perkins (1983), Lemma 3.3, and extend the resulting equality to C in $\sigma({}^*\mathscr{B}(\mathbb{R}^d))$ to get

$$\begin{aligned} &P\left(N^{\gamma^{r,B}}\left(\underline{\zeta}(r, B) - \mu^{-1}\right) \in C \,|\, \sigma(\mathscr{A}_{\underline{r}} \vee \mathscr{E})\right)(w) 1\left(\underline{\zeta}(r, B) > \underline{r}\right) \\ &\quad = L\left({}^*P_0^{N_{\underline{r}}^{\gamma^{r,B}(w)}}\right)\left({}^*B\left(\underline{\zeta}(r, B)(w) - \mu^{-1} - \underline{r}\right) \in C\right) 1\left(\underline{\zeta}(r, B) > \underline{r}\right) \quad \text{a.s.} \end{aligned}$$

Take $C = \mathrm{st}_d^{-1}(A)$, where $A \in \mathscr{B}(\mathbb{R}^d)$, use (4.18), (4.16) and the fact that $\mathrm{st}(^*B)$ is a Brownian motion starting at $^\circ x$ under $L(^*P_0^x)$ to conclude

$$P(Z(r,B) \in A, \zeta(r,B) > r) = E\Big(P_0^{\mathrm{st}_d(N_r^{y^r,B}(w))}(B(\zeta(r,B)(w) - r) \in A)$$

$$\times 1(\zeta(r,B)(w) > r)\Big).$$

In particular the left-hand side of the above is 0 if A is Lebesgue null. (4.15) now completes the proof. □

The above result allows us to transfer hitting estimates for $\overline{R}$ or $\overline{R}_k$ to estimates for R or R_k. For example we have [(D.I.P.), Theorem 1.3]

$$(4.30)\quad Q^{m_0}(x \in \overline{R}) = 1 - \exp\left\{-2(4-d)\int |y-x|^{-2}\, dm_0(y)\right\}, \qquad d \le 3.$$

(4.15) and the previous theorem show $Q^{m_0}(x \in \overline{R} - R) = 0$ and therefore

$$(4.31)\quad Q^{m_0}(x \in R) = 1 - \exp\left\{-2(4-d)\int |y-x|^{-2}\, dm_0(y)\right\}, \qquad d \le 3.$$

5. Polar sets and multiple points for super-Brownian motion. Throughout this section the underlying Lévy process Y_t will be a symmetric stable process of index α scaled so that

$$(5.1)\qquad E(e^{i\langle\theta, Y_t\rangle}) = \exp\{-t|\theta|^\alpha\}, \qquad \theta \in \mathbb{R}^d \text{ for } \alpha < 2,$$

and so that Y_t is a standard Brownian motion if $\alpha = 2$. In fact the main theorems only hold for super-Brownian motion but, with an eye to future applications, we prove several preliminary estimates in the more general stable setting. Assume $d > \alpha$, unless otherwise indicated. Recall that Y_t has a radially symmetric continuous transition density $p_t(y) = q_t(|y|)$ satisfying

$$(5.2)\qquad q_t(r) = t^{-d/\alpha} q_1(rt^{-1/\alpha}),$$

$$(5.3)\qquad q_1(\cdot) \text{ is decreasing},$$

$$(5.4)\quad 0 < c_{5.1}(R) \le q_1(r)\ \forall\, r \le R, \qquad q_1(r) \le c_{5.2}(1+r)^{-d-\alpha}\ \forall\, r \ge 0.$$

Use the fact that Y_t is equal in law to $B(\tau_t)$ where B is a d-dimensional Brownian motion and τ_t is an independent stable subordinator to see (5.3). See Blumenthal and Getoor (1960) for (5.4).

The next estimate for the system of branching stable processes introduced in Section 2 follows easily from Proposition 2.6(a)(i) and Lemma 2.7 of Perkins (1988).

LEMMA 5.1. *If $2\mu^{-1} \le \varepsilon^\alpha \le 2^{-\alpha}$ and $t \in T^{(\mu)}$ then*

$$E^m\big(N_t^{(\mu)}(B(x,\varepsilon))^2\big) \le E^m\big(N_t^{(\mu)}(B(x,\varepsilon))\big)^2 + c_{5.3}\varepsilon^\alpha E^m\big(N_t^{(\mu)}(B(x,\varepsilon))\big)$$
$$\forall\, x \in \mathbb{R}^d \text{ and } m \in M_F^\mu(\mathbb{R}^d).$$

[The additional hypothesis $\varepsilon^\alpha \le t^{-1}$ in Lemma 2.7 of Perkins (1988) is only required if $d = \alpha$.]

A simple first moment estimate shows

$$\begin{aligned} P^{m_0}(X_t(B(x_0,\varepsilon)) \ge \varepsilon^\alpha) &\le \varepsilon^{-\alpha}P_0^{m_0}(Y_t \in B(x_0,\varepsilon)) \\ &\le c_{5.2}2^d t^{-d/\alpha} m_0(\mathbb{R}^d)\varepsilon^{d-\alpha} \quad [\text{by } (5.4)]. \end{aligned} \tag{5.5}$$

In fact if $\alpha = 2$ such an upper bound is obtained in [(D.I.P., Theorem 3.1] for $P^{m_0}(X_t(B(x_0,\varepsilon)) > 0)$. The next result establishes a converse inequality to (5.5).

PROPOSITION 5.2. *Suppose $\varepsilon \in (0, 1/2]$, $m_0 \in M_F(\mathbb{R}^d)$ and*

$$t \ge \max\big(2\varepsilon^\alpha, \varepsilon^{\alpha(1-(\alpha/d))} m_0(\mathbb{R}^d)^{\alpha/d}\big). \tag{5.6}$$

Then for every x_0 in $\mathbb{R}^d$,

$$P^{m_0}(X_t(B(x_0,\varepsilon)) \ge \varepsilon^\alpha) \ge c_{5.4}\int q_{t/2}(|x - x_0|)\, dm_0(x)\varepsilon^{d-\alpha}. \tag{5.7}$$

PROOF. Let $\varepsilon \in (0, 1/2]$,

$$p_0 = p_0(\alpha, d) = P_0^0(|Y_1| < 1/2),$$

and $\mu \ge \mu_2(1, p_0, \varepsilon^\alpha) \vee 2\varepsilon^{-\alpha}$, $\mu \in \mathbb{N}$, where μ_2 is as in Lemma 2.4. Fix x_0 in $\mathbb{R}^d$, $m \in M_F^\mu(\mathbb{R}^d)$ and t such that (5.6) holds with $m_0 = m$ and

$$t - \varepsilon^\alpha \in T^{(\mu)}. \tag{5.8}$$

Lemma 2.4 (with $K = 1$, $\delta = \varepsilon^\alpha$, $\varepsilon' = \varepsilon/2$) implies that for every $M > 0$,

$$\begin{aligned} &P^m\big(N_t^{(\mu)}(B(x_0,\varepsilon)) \ge \varepsilon^\alpha \,|\, \mathscr{A}_{t-\varepsilon^\alpha}^{(\mu)}\big) \\ &\quad\ge (1 - \varepsilon^{-M})M^{-1}c_1\varepsilon^{-\alpha}N_{t-\varepsilon^\alpha}^{(\mu)}(B(x_0,\varepsilon/2)) \\ &\qquad \times 1\big(N_{t-\varepsilon^\alpha}^{(\mu)}(B(x_0,\varepsilon/2)) \le c_1^{-1}\varepsilon^\alpha M\big), \end{aligned} \tag{5.9}$$

where $c_1 = 4^{-1}p_0 e^{-4/p_0}$. Use (2.2) to see

$$\begin{aligned} E^m\big(N_{t-\varepsilon^\alpha}^{(\mu)}(B(x_0,\varepsilon/2))\big) &= \iint 1_{B(x_0,\varepsilon/2)}(y)p_{t-\varepsilon^\alpha}(y - x)\, dy\, dm(x) \\ &\ge 2^{-d/\alpha}\iint 1_{B(x_0-x,\varepsilon/2)}(z)p_{t/2}(z)\, dz\, dm(x) \\ &\qquad [\text{by } (5.6), (5.2) \text{ and } (5.3)] \\ &\ge c_2\varepsilon^d\int q_{t/2}(|x - x_0| \vee \varepsilon)\, dm(x) \\ &\qquad [\text{by an elementary argument and } (5.3)] \\ &\ge c_3\varepsilon^d\int q_{t/2}(|x - x_0|)\, dm(x) \\ &\qquad [\text{by } (5.6), (5.2) \text{ and } (5.4)]. \end{aligned} \tag{5.10}$$

Conversely, it is clear from the first line of the above, (5.6) and (5.4) that

$$E^m\big(N^{(\mu)}_{t-\varepsilon^\alpha}(B(x_0,\varepsilon/2))\big) \le c_4 t^{-d/\alpha}\varepsilon^d m(\mathbb{R}^d). \tag{5.11}$$

Lemma 5.1 together with the above shows that

$$\begin{aligned} &E^m\Big(N^{(\mu)}_{t-\varepsilon^\alpha}(B(x_0,\varepsilon/2))^2\Big)\\ &\quad\le E^m\big(N^{(\mu)}_{t-\varepsilon^\alpha}(B(x_0,\varepsilon/2))\big)\big(c_4 t^{-d/\alpha}\varepsilon^d m(\mathbb{R}^d) + c_5\varepsilon^\alpha\big). \end{aligned} \tag{5.12}$$

Take expected values in (5.9) to conclude (for any $M > 0$)

$$\begin{aligned} &P^m\big(N^{(\mu)}_t(B(x_0,\varepsilon)) \ge \varepsilon^\alpha\big)\\ &\quad\ge (1-e^{-M})M^{-1}c_1\varepsilon^{-\alpha}\Big[E^m\big(N^{(\mu)}_{t-\varepsilon^\alpha}(B(x_0,\varepsilon/2))\big)\\ &\qquad\qquad -c_1\varepsilon^{-\alpha}M^{-1}E^m\Big(N^{(\mu)}_{t-\varepsilon^\alpha}(B(x_0,\varepsilon/2))^2\Big)\Big]\\ &\quad\ge (1-e^{-M})M^{-1}c_1c_3\varepsilon^{d-\alpha}\int q_{t/2}(|x-x_0|)\,dm(x)\\ &\qquad\times\big[1-c_1c_4M^{-1}\varepsilon^{d-\alpha}t^{-d/\alpha}m(\mathbb{R}^d) - c_1c_5M^{-1}\big]\\ &\qquad\qquad [\text{by } (5.10) \text{ and } (5.12)]. \end{aligned}$$

Now set $M = M(d,\alpha) = 2c_1(c_4 + c_5)$ and use (5.6) (with $m_0 = m$) to conclude that the final term (in square brackets) is at least $1/2$. We have shown

$$P^m\big(N^{(\mu)}_t(B(x_0,\varepsilon)) \ge \varepsilon^\alpha\big) \ge c_{5.4}\int q_{t/2}(|x-x_0|)\,dm(x)\varepsilon^{d-\alpha},$$

whenever $\varepsilon \in [0,1/2]$, $\mu \ge \mu_2(1,p_0,\varepsilon^\alpha) \vee 2\varepsilon^{-\alpha}$, $x_0 \in \mathbb{R}^d$, $m \in M_F^\mu(\mathbb{R}^d)$ and t satisfies (5.6) (with $m_0 = m$) and (5.8). Theorem 2.2 and an easy weak convergence argument complete the proof. □

We want to use (2.5) to get an upper bound on $E^m(X_t(B(x,\varepsilon))X_{t'}(B(x',\varepsilon)))$ (Lemma 5.6 below). To do this, a sequence of probability estimates for the stable process Y is required.

Lemma 5.3. *For all $x, y \in \mathbb{R}^d$ and $\varepsilon, u > 0$,*

$$\begin{aligned} P_0^y(Y(u) \in B(x,\varepsilon)) \le c_{5.5}\big(&|y-x|^{-(d+\alpha)}u\varepsilon^d 1(|y-x| \ge 2(u^{1/\alpha}\vee\varepsilon))\\ &+((u^{-d/\alpha}\varepsilon^d)\wedge 1)1(|y-x| < 2(u^{1/\alpha}\vee\varepsilon))\big). \end{aligned}$$

Proof. This follows easily from (5.2)–(5.4) by considering four cases according to the relative sizes of $|y-x|$ and $2(u^{1/\alpha}\vee\varepsilon)$, and $u^{1/\alpha}$ and ε. □

LEMMA 5.4. *If $s > 0$, $0 < r \le R$ and $y, x, x' \in \mathbb{R}^d$, then*

(a) $$E_0^y\big(1(|Y_s - x| \ge r, |Y_s - x'| \ge R)|Y_s - x|^{-(d+\alpha)}|Y_s - x'|^{-(d+\alpha)}\big) \le c_{5.6}s^{-d/\alpha}r^{-\alpha}(R \vee |x - x'|)^{-(d+\alpha)},$$

(b) $$E_0^y\big(1(|Y_s - x| \ge r, |Y_s - x'| \le R)|Y_s - x|^{-(d+\alpha)}\big) \le c_{5.7}s^{-d/\alpha}\big(1(|x - x'| > 2R)R^d|x - x'|^{-(d+\alpha)} + 1(|x - x'| \le 2R)r^{-\alpha}\big),$$

(c) $$E_0^y\big(1(|Y_s - x| \le r, |Y_s - x'| \ge R)|Y_s - x'|^{-(d+\alpha)}\big) \le c_{5.8}s^{-d/\alpha}r^d(R \vee |x - x'|)^{-(d+\alpha)},$$

(d) $P_0^y(|Y_s - x| \le r, |Y_s - x'| \le R) \le c_{5.9}s^{-d/\alpha}1(|x - x'| \le 2R)r^d$.

PROOF. A simple scaling argument allows us to assume $s = 1$, and translation invariance allows us to set $y = 0$. The elementary proof now only uses the fact that Y_1 has a bounded density. We only give the details for (a) as the other cases are even simpler.

(a) If

$$S_1 = \{y: r \le |y - x| \le |y - x'|\}, \qquad S_2 = \{y: R \le |y - x'| \le |y - x|\},$$

then the expectation in (a) (with $s = 1$, $y = 0$) is bounded by

$$(5.13)\qquad \int_{S_1} p_1(y)|y - x|^{-(d+\alpha)}|y - x'|^{-(d+\alpha)}\,dy + \int_{S_2} p_1(y)|y - x|^{-(d+\alpha)}|y - x'|^{-(d+\alpha)}\,dy.$$

On S_1, $|y - x'| \ge |x - x'|/2$ and so the first integral is bounded by

$$\|p_1\|_\infty 2^{d+\alpha}|x - x'|^{-(d+\alpha)}\int_r^\infty \rho^{-(d+\alpha)}\rho^{d-1}\,d\rho = \|p_1\|_\infty 2^{d+\alpha}\alpha^{-1}r^{-\alpha}|x - x'|^{-(d+\alpha)}.$$

A similar argument gives the same bound for the second integral in (5.13). Therefore we have shown

$$(5.14)\qquad E_0^0\big(1(|Y_1 - x| \ge r, |Y_1 - x'| \ge R)|Y_1 - x|^{-(d+\alpha)}|Y_1 - x'|^{-(d+\alpha)}\big) \le c_1 r^{-\alpha}|x - x'|^{-(d+\alpha)}.$$

We also have

$$\begin{aligned} &E_0^0\big(1(|Y_1 - x| \ge r, |Y_1 - x'| \ge R)|Y_1 - x|^{-(d+\alpha)}|Y_1 - x'|^{-(d+\alpha)}\big) \\ &\quad \le R^{-(d+\alpha)}\int 1(|y - x| \ge r)|y - x|^{-(d+\alpha)}p_1(y)\,dy \\ &\quad \le \|p_1\|_\infty R^{-(d+\alpha)}\int_r^\infty \rho^{-d-\alpha+d-1}\,d\rho \\ &\quad = \|p_1\|_\infty \alpha^{-1}R^{-(d+\alpha)}r^{-\alpha}. \end{aligned}$$

This together with (5.14) gives (a). □

LEMMA 5.5.

$$E_0^y\big(P_0^{Y_s}(Y_t \in B(x,\varepsilon))P_0^{Y_s}(Y_u \in B(x',\varepsilon))\big)$$
$$\le c_{5.10}\varepsilon^{2d}s^{-d/\alpha}\big[u\big(|x-x'|^{-(d+\alpha)} \wedge u^{-(d/\alpha+1)} \wedge \varepsilon^{-(d+\alpha)}\big)$$
$$+\varepsilon^{-d}1\big(|x-x'| \vee u^{1/\alpha} \le \varepsilon\big)\big]$$

for all $0 < \varepsilon, s, 0 \le t \le u$ *and* $x, x', y \in \mathbb{R}^d$.

PROOF. Use Lemmas 5.3 and 5.4 to bound the above expected value by

$$c_{5.5}^2\varepsilon^{2d}E_0^y\Big(\big(1\big(|Y_s - x| \ge 2(t^{1/\alpha} \vee \varepsilon)\big)|Y_s - x|^{-(d+\alpha)}t$$
$$+1\big(|Y_s - x| < 2(t^{1/\alpha} \vee \varepsilon)\big)t^{-d/\alpha} \wedge \varepsilon^{-d}\big)$$
$$\times\big(1\big(|Y_s - x'| \ge 2(u^{1/\alpha} \vee \varepsilon)\big)|Y_s - x'|^{-(d+\alpha)}u$$
$$+ 1\big(|Y_s - x'| < 2(u^{1/\alpha} \vee \varepsilon)\big)u^{-d/\alpha} \wedge \varepsilon^{-d}\big)\Big)$$
$$\le c_{5.5}^2\varepsilon^{2d}s^{-d/\alpha}\Big[c_{5.6}tu2^{-\alpha}(t^{-1} \wedge \varepsilon^{-\alpha})\big((2(u^{1/\alpha} \vee \varepsilon)) \vee |x-x'|\big)^{-d-\alpha}$$
$$+c_{5.8}(t^{-d/\alpha} \wedge \varepsilon^{-d})u2^d(t^{d/\alpha} \vee \varepsilon^d)\big((2(u^{1/\alpha} \vee \varepsilon)) \vee |x-x'|\big)^{-d-\alpha}$$
$$+c_{5.7}t(u^{-d/\alpha} \wedge \varepsilon^{-d})\Big(1\big(|x-x'| > 4(u^{1/\alpha} \vee \varepsilon)\big)$$
$$\times\big(2(u^{1/\alpha} \vee \varepsilon)\big)^d|x-x'|^{-d-\alpha}$$
$$+1\big(|x-x'| \le 4(u^{1/\alpha} \vee \varepsilon)\big)2^{-\alpha}(t^{-1} \wedge \varepsilon^{-\alpha})\Big)$$
$$+c_{5.9}(t^{-d/\alpha} \wedge \varepsilon^{-d})(u^{-d/\alpha} \wedge \varepsilon^{-d})$$
$$\times 1\big(|x-x'| \le 4(u^{1/\alpha} \vee \varepsilon)\big)2^d(t^{d/\alpha} \vee \varepsilon^d)\Big]$$
$$\le c_1\varepsilon^{2d}s^{-d/\alpha}\Big[1\big(|x-x'| > 4(u^{1/\alpha} \vee \varepsilon)\big)u|x-x'|^{-d-\alpha}$$
$$+1\big(|x-x'| \le 4(u^{1/\alpha} \vee \varepsilon)\big)\Big[u(u^{1/\alpha} \vee \varepsilon)^{-d-\alpha} + u^{-d/\alpha} \wedge \varepsilon^{-d}\Big]\Big]$$
$$\le c_2\varepsilon^{2d}s^{-d/\alpha}\big[u\big(|x-x'|^{-(d+\alpha)} \wedge u^{-(d/\alpha+1)} \wedge \varepsilon^{-(d+\alpha)}\big)$$
$$+1\big(|x-x'| \le 4(u^{1/\alpha} \vee \varepsilon)\big)u^{-d/\alpha} \wedge \varepsilon^{-d}\big]$$
$$\le c_{5.10}\varepsilon^{2d}s^{-d/\alpha}\big[u\big(|x-x'|^{-(d+\alpha)} \wedge u^{-(d/\alpha+1)} \wedge \varepsilon^{-(d+\alpha)}\big)$$
$$+1\big(|x-x'| \vee u^{1/\alpha} \le \varepsilon\big)\varepsilon^{-d}\big]. \qquad \square$$

LEMMA 5.6. *If* $\varepsilon > 0$, $t' \ge t \ge l > 0$, $x, x' \in \mathbb{R}^d$ *and* $m_0 \in M_F(\mathbb{R}^d)$, *then*

$$E^{m_0}\big(X_t(B(x,\varepsilon))X_{t'}(B(x',\varepsilon))\big)$$
$$\le c_{5.11}\varepsilon^{2d}\Big(l^{-2d/\alpha}m_0(\mathbb{R}^d)\big(m_0(\mathbb{R}^d) + l\big) \tag{5.15}$$
$$+l^{-d/\alpha}m_0(\mathbb{R}^d)\big(\varepsilon \vee |x-x'| \vee |t'-t|^{1/\alpha}\big)^{\alpha-d}\Big).$$

PROOF. Let $B = B(x, \varepsilon)$, $B' = B(x', \varepsilon')$ and $t' \geq t \geq l > 0$. By (2.4) and (2.5) the left side of (5.15) equals

$$\begin{aligned}
&P_0^{m_0}(Y_t \in B) P_0^{m_0}(Y_{t'} \in B') + \int_0^t E_0^{m_0}\big(P_0^{Y_s}(Y_{t-s} \in B) P_0^{Y_s}(Y_{t'-s} \in B')\big)\, ds \\
&\quad \leq m_0(\mathbb{R}^d)^2 \|q_1\|_\infty^2 (2\varepsilon)^{2d} t^{-2d/\alpha} + \int_0^{l/2} m_0(\mathbb{R}^d)(2\varepsilon)^{2d}(l/2)^{-2d/\alpha}\|q_1\|_\infty^2\, ds \\
&\qquad + m_0(\mathbb{R}^d) c_{5.10} \varepsilon^{2d} (l/2)^{-d/\alpha} \\
&\qquad \times \bigg[\int_{l/2}^t (t'-s)\big(|x-x'|^{-d-\alpha} \wedge (t'-s)^{-d/\alpha-1} \wedge \varepsilon^{-d-\alpha}\big)\, ds \\
&\qquad\qquad + 1\big(|x-x'| \vee (t'-t)^{1/\alpha} < \varepsilon\big) \int_{l/2}^t \varepsilon^{-d} 1(t'-s \leq \varepsilon^\alpha)\, ds\bigg] \qquad (5.16) \\
&\quad \leq c_1 \varepsilon^{2d} \bigg(m_0(\mathbb{R}^d)^2 l^{-2d/\alpha} + m_0(\mathbb{R}^d) l^{1-2d/\alpha} + m_0(\mathbb{R}^d) l^{-d/\alpha} \\
&\qquad \times \bigg(\int_{l/2}^t (t'-s)\big(|x-x'|^{-d-\alpha} \wedge (t'-s)^{-d/\alpha-1} \wedge \varepsilon^{-d-\alpha}\big)\, ds \\
&\qquad\qquad + 1\big(|x-x'| \vee (t'-t)^{1/\alpha} < \varepsilon\big) \varepsilon^{\alpha-d} \bigg)\bigg).
\end{aligned}$$

We have used (5.2) and Lemma 5.5 in the first inequality in the above. Let I denote the integral in the last inequality. Then

$$I \leq \int_{l/2}^t (t'-s)^{-d/\alpha}\, ds \leq (t'-t)^{1-d/\alpha}(d/\alpha - 1)^{-1}, \qquad d > \alpha, \tag{5.17}$$

and if $\varepsilon_0 = \varepsilon \vee |x - x'|$,

$$\begin{aligned}
I &\leq \int_{l/2}^{t'-\varepsilon_0^\alpha} (t'-s)^{-d/\alpha}\, ds + \int_{t'-\varepsilon_0^\alpha}^{t'} (t'-s)\, ds\, \varepsilon_0^{-d-\alpha} \\
&\leq \varepsilon_0^{\alpha-d}(d/\alpha - 1)^{-1} + \varepsilon_0^{-d+\alpha}/2.
\end{aligned} \tag{5.18}$$

Use (5.17) and (5.18) in (5.16) to obtain the desired conclusion. □

We are ready to give a lower bound for the probability that k-multiple points exist in a given compact set F, and, in particular ($k = 1$), that X hits F. Our approach is motivated by the solution of the corresponding classical problem for multiple points of a symmetric stable process Y [see Taylor (1966) for the necessary methodology if not the result itself]. For $k = 1$, recall that if points $x_1, \dots, x_N$ are spaced out appropriately in F then

$$\begin{aligned}
P_0^0(Y \text{ hits } F^\varepsilon) \geq{}& \sum_{i=1}^N P_0^0(Y \text{ hits } B(x_i, \varepsilon)) \\
&- \sum_{1 \leq i \neq i' \leq N}\sum P_0^0(Y \text{ hits } B(x_i, \varepsilon) \text{ and } B(x_{i'}, \varepsilon)).
\end{aligned} \tag{5.19}$$

Estimates for the hitting probabilities of balls for Y are well known and when the last term is estimated via the strong Markov property, one obtains the sufficient condition $C(g_{d-\alpha})(F) > 0$ for $P_0^0(Y \text{ hits } F)$ to be positive. Although hitting estimates for balls for super-Brownian motion are known when $d \geq 3$ [(D.I.P.), Theorem 3.2], the last term in the analog of (5.19) for X_t is not easily estimated by stopping X when it hits $B(x_i, \varepsilon)$ or $B(x_{i'}, \varepsilon)$ at time T. This is because one is then faced with analyzing the law of X_T. To circumvent this problem we usean inclusion–exclusion argument over space and time and also require $X_t(B(x_i, \varepsilon)) \geq \varepsilon^\alpha$ instead of $X_t(B(x_i, \varepsilon)) > 0$. This will mean that the only required probability estimates are Proposition 5.2 and Lemma 5.6. These estimates depend only on the two-dimensional distributions of the process. This explains why the method works in other settings where the potential theory is not well developed such as multiparameter stable processes (see Theorem 1.2).

In the classical setting, the a.s. existence of k-multiple points follows from their existence with positive probability either by scaling and the Blumenthal 0–1 law or by the fact that the process has an infinite lifetime. X does not scale as nicely and has a finite lifetime. This forces us to estimate the probability of finding a k-multiple point in $[0, 2kl]$ in terms of l so that we can let $l \downarrow 0$ and use Blumenthal's 0–1 law. The key estimate (Proposition 5.7) appears slightly more complicated because of this.

If $l > 0$ and $r \in \mathbb{N}$, let $I_r^l = [(2r-1)l, 2rl]$, and for $k \in \mathbb{N}$ and $F \subset \mathbb{R}^d$ define

$$A_n = A_n^{l,k,F} = \big\{w: X_{t_r}(B(x, 2^{-n})) \geq 2^{-n\alpha} \text{ for some } x \in F \text{ and } t_r \in I_r^l \text{ for } 1 \leq r \leq k\big\}.$$

If $c \in \mathbb{R}$, let $cF = \{cx: x \in F\}$.

PROPOSITION 5.7. *Assume $d \geq 2\alpha$. Let $k \in \mathbb{N}$, $R > 0$. There is a constant $c_{5.12}(k, R)$ such that if $l \in (0,1]$, F is a compact subset of $B(0, Rl^{1/\alpha})$ and $m_0 \in M_F(\mathbb{R}^d)$ satisfies*

$$m_0(B(0, Rl^{1/\alpha})) \geq l, \tag{5.20}$$

then

$$Q^{m_0}\big(A_n^{l,k,F} \text{ occurs for infinitely many } n\big) \geq c_{5.12}(k, R) C\big((g_{d-2\alpha})^k\big)(l^{-1/\alpha}F).$$

PROOF. Let k, R, l, F and m_0 satisfy the above conditions. By restricting m_0 to $B(0, Rl^{1/\alpha})$ and multiplying by $l/m_0(B(0, Rl^{1/\alpha}))$ (≤ 1) we may assume (use Proposition 1.7)

$$m_0(\mathbb{R}^d) = m_0(B(0, Rl^{1/\alpha})) \doteq l. \tag{5.21}$$

Choose $\{x_i^N: i \leq N\} \subset F$ so that if

$$I_N = N^{-1}(N-1)^{-1} \sum_{i \neq i'} \sum g_{d-2\alpha}\big(|x_i^N - x_{i'}^N| l^{-1/\alpha}\big)^k,$$

then [see (1.3)]

$$\lim_{N\to\infty} I_N = I(g^k_{d-2\alpha})(l^{-1/\alpha}F). \tag{5.22}$$

Let $\rho > 1$,

$$\Delta_n = \begin{cases} \rho n 2^{-n\alpha} & \text{if } d = 2\alpha, \\ \rho 2^{-n\alpha} & \text{if } d > 2\alpha, \end{cases}$$

and define

$$T_n = \left\{ (i_1\Delta_n, \dots, i_k\Delta_n) \in \prod_{r=1}^k I^l_r \colon i_j \in \mathbb{N} \right\}.$$

In what follows we will assume n is large enough so that

$$2\Delta_n \le l. \tag{5.23}$$

Let $\lambda > 0$ and choose $N_n \uparrow \infty$ so that if

$$\lambda_n = N_n \Delta_n^{-k} 2^{-nk(d-\alpha)} l^{k(2-d/\alpha)} = \begin{cases} N_n n^{-k} \rho^{-k} & \text{if } d = 2\alpha, \\ N_n 2^{-nk(d-2\alpha)} \rho^{-k} l^{k(2-d/\alpha)} & \text{if } d > 2\alpha, \end{cases}$$

then $\lim_{n\to\infty} \lambda_n = \lambda$. If $m' = m_0/k \in M_F$ and $X^1, \dots, X^k$ are i.i.d. measure-valued processes on some $(\Omega, \mathscr{F}, P)$ with common law $Q^{m'}$, then (Proposition 1.7) $X^1 + \cdots + X^k$ has law Q^{m_0}. Writing $B_{i,n}$ for $B(x_i^{N_n}, 2^{-n})$, we therefore have

$$\begin{aligned} Q^{m_0}(A_n) &\ge Q^{m_0}\left(\bigcup_{i=1}^{N_n} \bigcup_{t\in T_n} \{X_{t_r}(B_{i,n}) \ge 2^{-n\alpha} \text{ for } 1 \le r \le k\} \right) \\ &\ge P\left(\bigcup_{i=1}^{N_n} \bigcup_{t\in T_n} \{X^r_{t_r}(B_{i,n}) \ge 2^{-n\alpha} \text{ for } 1 \le r \le k\} \right) \\ &\ge \sum_{i=1}^{N_n} \sum_{t\in T_n} P\big(X^r_{t_r}(B_{i,n}) \ge 2^{-n\alpha} \text{ for } 1 \le r \le k\big) \\ &\quad - \sum_{i,i'=1}^{N_n} \sum_{t,t'\in T_n} 1((i,t) \ne (i',t')) P\big(X^r_{t_r}(B_{i,n}) \ge 2^{-n\alpha}, \\ &\qquad\qquad X^r_{t'_r}(B_{i',n}) \ge 2^{-n\alpha} \text{ for } 1 \le r \le k\big) \qquad (5.24) \\ &\ge \sum_{i=1}^{N_n} \sum_{t\in T_n} \prod_{r=1}^k Q^{m'}\big(X_{t_r}(B_{i,n}) \ge 2^{-n\alpha}\big) \\ &\quad - \sum_{i,i'=1}^{N_n} \sum_{t,t'\in T_n} 1((i,t) \ne (i',t')) 2^{2kn\alpha} \prod_{r=1}^k E^{m'}\big(X_{t_r}(B_{i,n}) X_{t'_r}(B_{i',n})\big). \end{aligned}$$

(5.21) and (5.23) allow us to apply Proposition 5.2 and conclude that for $t_r \in I_r^l$, $r \le k$,

$$Q^{m'}\left(X_{t_r}(B_{i,n}) \ge 2^{-n\alpha}\right) \ge c_{5.4} m_0(\mathbb{R}^d) k^{-1} \inf\left\{q_{t_r/2}(|y|) : |y| \le 2Rl^{1/\alpha}\right\} 2^{-n(d-\alpha)}$$
$$\ge c_1(k, R) l^{1-d/\alpha} 2^{-n(d-\alpha)}$$
$$[\text{by } (5.21), (5.2) \text{ and } (5.4)].$$

This result and Lemma 5.6 show that (5.24) implies

$$(5.25)\qquad Q^{m_0}(A_n) \ge c_2(k, R)\lambda_n - c_3(k)\lambda_n^2 - c_4(k) l^{k(1-d/\alpha)} 2^{-2nk(d-\alpha)} \sum_n,$$

where

$$\sum_n = \sum_{i, i'=1}^{N_n} \sum_{t, t' \in T_n} 1((i,t) \ne (i',t')) \times \prod_{r=1}^{k} \left(2^{-n} \vee |x_i^{N_n} - x_{i'}^{N_n}| \vee |t_r - t_r'|^{1/\alpha}\right)^{\alpha-d}.$$

If $\varepsilon > 0$,

$$S_r = \sum_{j_r < j_r'}\sum 1(j_r\Delta_n, j_r'\Delta_n \in I_r^l)\left(\varepsilon \vee (j_r'\Delta_n - j_r\Delta_n)^{1/\alpha}\right)^{\alpha-d}$$

and

$$S_r' = \sum_{j_r} 1(j_r\Delta_n \in I_r^l)\varepsilon^{\alpha-d},$$

then

$$(5.26)\qquad \begin{aligned} S_r &\le 2l^2\Delta_n^{-2} \int_0^l \left(\varepsilon \vee t^{1/\alpha}\right)^{\alpha-d} dt\, l^{-1} \\ &= 2l^{3-d/\alpha}\Delta_n^{-2} \int_0^1 \left(\varepsilon l^{-1/\alpha} \vee u^{1/\alpha}\right)^{\alpha-d} du \\ &\le c_5 l^{3-d/\alpha}\Delta_n^{-2} g_{d-2\alpha}(\varepsilon l^{-1/\alpha}) \end{aligned}$$

and

$$(5.27)\qquad S_r' \le 2l\Delta_n^{-1}\varepsilon^{\alpha-d} = 2l^{2-d/\alpha}\Delta_n^{-1}\left(\varepsilon l^{-1/\alpha}\right)^{\alpha-d}.$$

Use (5.26) and (5.27) with $\varepsilon = 2^{-n} \vee |x_i^{N_n} - x_{i'}^{N_n}|$ to see that for $n \geq n_0(l)$,

$$\begin{aligned}\sum_n &\leq c_6(k)\Bigg[\sum_{\substack{i,i'=1\\ i\neq i'}}^{N_n}\Big(l^{(3-d/\alpha)k}\Delta_n^{-2k}g_{d-2\alpha}\big((2^{-n}\vee|x_i^{N_n}-x_{i'}^{N_n}|)l^{-1/\alpha}\big)^k \\ &\qquad + l^{(2-d/\alpha)k}\Delta_n^{-k}\big((2^{-n}\vee|x_{i'}^{N_n}-x_i^{N_n}|)l^{-1/\alpha}\big)^{k(\alpha-d)}\Big) \\ &\qquad + N_n\Big(l^{(3-d/\alpha)k}\Delta_n^{-2k}g_{d-2\alpha}(2^{-n}l^{-1/\alpha})^k + l^{(3-d/\alpha)}\Delta_n^{-2}g_{d-2\alpha}(2^{-n}l^{-1/\alpha}) \\ &\qquad \times l^{(2-d/\alpha)(k-1)}\Delta_n^{-(k-1)}(2^{-n}l^{-1/\alpha})^{(k-1)(\alpha-\delta)}\Big)\Bigg] \end{aligned} \tag{5.28}$$

$$\begin{aligned} &\leq c_6(k)\Big[N_n^2 l^{(3-d/\alpha)k}\Delta_n^{-2k}\Big(I_{N_n} + I_{N_n}2^{-nk(\alpha-d)} \\ &\qquad \times\big(\Delta_n^{-1}l^{(2-d/\alpha)}g_{d-2\alpha}(2^{-n}l^{-1/\alpha})\big)^{-k}\Big) \\ &\qquad + N_n\Delta_n^{-k}l^k\Big(\big(l^{(2-d/\alpha)}\Delta_n^{-1}g_{d-2\alpha}(2^{-n}l^{-1/\alpha})\big)^k \\ &\qquad + l^{2-d/\alpha}\Delta_n^{-1}g_{d-2\alpha}(2^{-n}l^{-1/\alpha})2^{n(k-1)(d-\alpha)}\Big)\Big].\end{aligned}$$

In the last line we have used the fact that $r^{\alpha-d}g_{d-2\alpha}(r)^{-1}$ is decreasing near 0 to replace $(2^{-n} \vee |x_i^{N_n} - x_{i'}^{N_n}|)l^{-1/\alpha}$ with $2^{-n}l^{-1/\alpha}$ in the second of the four terms (at least for n large enough, depending only on l). By considering the cases $d = 2\alpha$ and $d > 2\alpha$ separately one finds that for $n \geq n_1(l)$ [$\geq n_0(l)$],

$$(\log 2)2^{-1}\rho^{-1}2^{n(d-\alpha)} \leq l^{2-d/\alpha}\Delta_n^{-1}g_{d-2\alpha}(2^{-n}l^{-1/\alpha}) \leq \rho^{-1}2^{n(d-\alpha)}.$$

Use the above in (5.28) to see that for $n \geq n_1(l)$,

$$\sum_n \leq c_7(k)\big[N_n^2 l^{(3-d/\alpha)k}\Delta_n^{-2k}I_{N_n}(1+\rho^k) + N_n\Delta_n^{-k}l^k 2^{nk(d-\alpha)}(\rho^{-k}+\rho^{-1})\big].$$

Now substitute this inequality into (5.25) to conclude [for $n \geq n_1(l)$]

$$Q^{m_0}(A_n) \geq c_2(k,R)\lambda_n - c_3(k)\lambda_n^2 - c_8(k)\lambda_n^2\rho^k I_{N_n} - c_8(k)\lambda_n\rho^{-1}.$$

Let $\rho = \max(2c_8(k)c_2(k,R)^{-1}, 1)$ and use $I_{N_n} \geq g_{2-d\alpha}(2R)^k$ to see that

$$Q^{m_0}(A_n) \geq c_9(k,R)\lambda_n - c_{10}(k,R)I_{N_n}\lambda_n^2, \qquad n \geq n_1(l),$$

and hence, if $I = I((g_{d-2\alpha})^k)(Fl^{-1/\alpha})$,

$$\liminf_{n\to\infty} Q^{m_0}(A_n) \geq c_9(k,R)\lambda - c_{10}(k,R)I\lambda^2.$$

Finally, choose $\lambda = c_9(2c_{10}I)^{-1}$ and hence obtain

$$Q^{m_0}(A_n \text{ i.o.}) \geq \limsup_{n\to\infty} Q^{m_0}(A_n) \geq c_{5.12}(k,R)I^{-1}. \qquad \square$$

As we saw in Section 3 there are k-multiple points in any nonempty set if $\alpha < 2$, and so we assume X_t is a super-Brownian motion ($\alpha = 2$) in the rest of this section. Note that the above result gives information on the amount of X_t-mass near points in F.

THEOREM 5.8. *Let $\alpha = 2$ and $d \geq 4$. There is a function $c_{5.13}$: $\mathbb{N} \times (0,\infty)^2 \to (0,\infty)$ such that if A is an analytic subset of $B(0,M)$ and $m_0(B(0,M)) > 0$ ($m_0 \in M_F(\mathbb{R}^d)$), then*

$$Q^{m_0}(A \cap \overline{R}_k \neq \varnothing) \geq c_{5.13}(k, M, m_0(B(0,M)))C((g_{d-4})^k)(A).$$

PROOF. By the inner regularity of $C((g_{d-4})^k)$ it suffices to prove the result when A is compact. Proposition 5.7 with $l = m_0(B(0,M)) \wedge 1$ and $R = Ml^{-1/2}$, implies

$$\begin{aligned} Q^{m_0}(A_n^{l,k,A} \text{ infinitely often}) &\geq c_{5.12}(k,R)C((g_{d-4})^k)(Al^{-1/2}) \\ &\geq c_{5.13}(k, M, m_0(B(0,M)))C((g_{d-4})^k)(A). \end{aligned}$$

The event on the left side implies $A^{2^{-n}} \cap (\bigcap_{r=1}^{k}\overline{R}(I_r^l)) \neq \varnothing$ infinitely often in n and hence $A \cap \overline{R}_k \neq \varnothing$ by the compactness of A. □

Theorem 1.1 is immediate from the above.

We can apply the method of Taylor (1966) and Fristedt (1967) to obtain a lower bound on the Hausdorff dimension of $\overline{R}_k$, which together with the upper bound from (D.I.P.) will prove Theorem 1.3, i.e., $\dim \overline{R}_k = d - k(d-4)$.

PROOF OF THEOREM 1.3. The upper bound on $\dim \overline{R}_k$ [including the fact that $\overline{R}_k = \varnothing$ if $d - k(d-4) \leq 0$] is given by Theorem 1.6 of (D.I.P.).

For the lower bound when $d > k(d-4)$, let us first assume the initial measure $X_0 = m_0$ satisfies

$$(5.29) \qquad m_0(B(0, 2l_n^{1/2})) \geq l_n \quad \text{for some } l_n \downarrow 0.$$

Choose $d' \leq d$ ($d' \in \mathbb{N}$) and $\alpha' \in (0,1)$ such that $d - d' + \alpha' > k(d-4)$, and let Y_t be a d'-dimensional symmetric stable process of index α', independent of the super-Brownian motion X and starting at 0. It will be convenient to work on the canonical product space of paths, $(\Omega \times \Omega', Q^{m_0} \times P_0^0)$. If $\tau(l) = \inf\{t\colon |Y_t| > l^{1/2}\}$, introduce $F_l' = \mathrm{cl}(\{Y_t\colon t < \tau(l)\})$, and if $B_r(x,a)$ denotes the r-dimensional open ball let

$$F_l = F_l' \times \mathrm{cl}(B_{d-d'}(0, l^{1/2})) \subset B_d(0, 2l^{1/2}).$$

The scaling property of Y shows that $I(g_{d-4}^k)(l^{-1/2}F_l)$ is equal in law to $I(g_{d-4}^k)(F_1)$ (the necessary measurability is easy to check). Since $\dim F_1' = \alpha'$ [Takeuchi (1964), Theorem 6], it is easy to see that $\dim F_1 = \alpha' + d - d' > k(d-4)$ and therefore $I(g_{d-4}^k)(F_1) < \infty$ a.s. [e.g., see Taylor (1961)]. An elementary argument therefore shows

$$\liminf_{n\to\infty} I(g_{d-4}^k)(F_{l_n} l_n^{-1/2}) < \infty \quad \text{for } P_0^0\text{-a.a. } w'.$$

Fix such an w' and a subsequence $l_{n_j}(w')$ along which $I(g^k_{d-4})(F_{l_{n_j}} l_{n_j}^{-1/2})$ remains bounded. Proposition 5.7 with $R = 2$ and $F = F_{l_{n_j}}(w')$ [(5.29) implies (5.20)] shows that

$$Q^{m_0}\left(\left\{w\colon \bigcap_{r=1}^{k} \overline{R}(I_r^{l_{n_j}})(w) \cap F_{l_{n_j}}(w') \neq \varnothing\right\}\right)$$

is bounded away from 0 uniformly in j. The Blumenthal 0–1 law now gives

$$Q^{m_0}\left(\left\{w\colon \bigcap_{r=1}^{k} \overline{R}(I_r^{l_{n_j}})(w) \cap F_{l_{n_j}}(w') \neq \varnothing \text{ for infinitely many } j\right\}\right) = 1.$$

Therefore $\Pi_{d'}(\overline{R}_k(w)) \cap F_1'(w') \neq \varnothing$, $Q^{m_0} \times P^0$-a.s., where $\Pi_{d'}$ denotes the projection of $\mathbb{R}^d$ onto $\mathbb{R}^{d'}$. Fubini's theorem shows that $C(g_{d'-\alpha'})(\overline{R}_k(w)) > 0$ Q^{m_0}-a.s. and hence $\dim \Pi_{d'}(\overline{R}_k(w)) \geq d' - \alpha'$ Q^{m_0}-a.s. It follows trivially that $\dim(\overline{R}_k(w)) \geq d' - \alpha'$ and, taking $d' - \alpha'$ arbitrarily close to $d - k(d-4)$, the result follows under (5.29).

In general let

$$\overline{R}_k(\delta,\infty) = \bigcup{}' \bigcap_{j=1}^{k} \overline{R}(I_j) \subset \overline{R}_k,$$

where $\bigcup'$ indicates the union is over disjoint compact intervals $I_1, \dots, I_k$ in (δ, ∞). The Markov property shows

$$(5.30)\qquad \begin{aligned} &Q^{m_0}\big(\dim(\overline{R}_k(\delta,\infty)) \geq d - k(d-4) \mid \mathscr{F}_\delta\big)(w) \\ &\quad = Q^{X_\delta(w)}\big(\dim \overline{R}_k \geq d - k(d-4)\big). \end{aligned}$$

The necessary measurability arguments are again easy to provide [see, e.g., Lemma 6.3 of (D.I.P.) or Cutler (1984)]. Theorem 6.5 of Perkins (1988) implies

$$(5.31)\qquad \limsup_{l \downarrow 0} X_\delta\big(B(x, l^{1/2})\big) l^{-1} = \infty \quad \text{for } X_\delta\text{-a.a. } x \text{ and all } \delta > 0 \ Q^{m_0} \text{ a.s.}$$

Fix w such that (5.31) holds. For small enough δ, $X_\delta \neq 0$ (recall $m \neq 0$) and we may choose x so that (5.31) holds. The previous case with $X_\delta(w)$ in place of m_0 and x in place of 0 (use translation invariance) shows that the right-hand side of (5.30) is one. Hence w.p.1 the left-hand side of (5.30) is one for small enough δ and hence

$$\dim \overline{R}_k \geq d - k(d-4) \qquad Q^{m_0}\text{-a.s.} \qquad \square$$

Finally, we show the above results hold if $\overline{R}_k$ is replaced by R_k.

THEOREM 5.9. *Assume $d \geq 4$, $k \in \mathbb{N}$, m_0 is a nonzero finite measure on $\mathbb{R}^d$ and Q^{m_0} is the law of super-Brownian motion starting at m_0.*

(a) $\dim R_k = d - k(d-4)$ *Q^{m_0}-a.s., where* $\dim R_k \leq 0$ *means* $R_k = \varnothing$.

(b) *If A is an analytic subset of $\mathbb{R}^d$ satisfying $C((g_{d-4})^k)(A) > 0$, then* $Q^{m_0}(A \cap R_k \neq \varnothing) > 0$.

PROOF. (a) It follows easily from (4.15) that

$$\overline{R}_k - R_k \subset \{Z(u) \in \mathbb{R}^d : 0 < u < \infty\} \quad \text{a.s.} \tag{5.32}$$

and therefore is countable a.s. [by (4.15) again]. (a) now follows from Theorem 1.3.

(b) Let $k(d-4) < d$ and assume first A is a Lebesgue null analytic set satisfying $C((g_{d-4})^k)(A) > 0$. Then (5.32) and Theorem 4.9 imply $Q^{m_0}((\overline{R}_k - R_k) \cap A \neq \varnothing) = 0$. Theorem 1.1 shows $Q^{m_0}(\overline{R}_k \cap A \neq \varnothing) > 0$, and thus completes the proof in this case. If A has positive Lebesgue measure, one can choose a Lebesgue null subset B such that $C((g_{d-4})^k)(B) > 0$ [e.g., use Rogers (1970), Theorem 57, and the relationship between capacity and Hausdorff measure]. The previous case shows $Q^{m_0}(R_k \cap B \neq \varnothing) > 0$ and the result follows. □

6. Polar sets for $S(X_t)$. Throughout this section Q^{m_0} denotes the law of d-dimensional super-Brownian motion starting at m_0 and $d \geq 3$. The following results are taken from Section 7 of (D.I.P.): Assume $0 < t_1 < \cdots < t_k$.

$$\bigcap_{i=1}^{k} S_{t_i} = \varnothing \qquad Q^{m_0}\text{-a.s. if } d > 3 \text{ and } k \geq 2, \text{ or } d = 3 \text{ and } k > 2. \tag{6.1}$$

$$\text{If } d = 3 \text{ then } \dim(S_{t_1} \cap S_{t_2}) \leq 1 \qquad Q^{m_0}\text{-a.s.} \tag{6.2}$$

$$\text{If } A \subset \mathbb{R}^d \text{ and } k \in \mathbb{N} \text{ satisfy } x^{k(d-2)} - m(\mathrm{A}) = 0, \text{ then } A \cap S_{t_1} \cap \cdots \cap S_{t_k} = \varnothing \qquad Q^{m_0}\text{-a.s.} \tag{6.3}$$

It is easy to use the method of Section 5 to prove converses to these results. The proofs are much simpler and are therefore omitted. For example, the inclusion–exclusion argument is now carried out only over space and not space and time. A partial converse to (6.3) is given by (b) of the following result.

THEOREM 6.1. (a) *There is a function* $c_{6.1}: \mathbb{N} \times (0,\infty)^3 \to (0,\infty)$ *such that if* A *is an analytic subset of* $B(0,M)$, $m_0(B(0,M)) \geq l^{d/2}$, *and* $0 < l \leq t_1 < \cdots < t_k < L$, *then*

$$Q^{m_0}(A \cap S_{t_1} \cap \cdots \cap S_{t_k} \neq \varnothing) \geq c_{6.1}(k,l,L,M) C\big((g_{d-2})^k\big)(A).$$

(b) *If* A *is an analytic subset of* $\mathbb{R}^d$, $0 < t_1 < \cdots < t_k$ *and* $m_0 \neq 0$, *then*

$$C\big((g_{d-2})^k\big)(A) > 0 \textit{ implies } Q^{m_0}\big(A \cap S_{t_1} \cap \cdots \cap S_{t_k} \neq \varnothing\big) > 0.$$

(6.1) implies that (6.3) and Theorem 6.1 are only of interest if $k = 1$, or $d = 3$ and $k = 2$.

The proof of Theorem 1.3 now gives the opposite inequality to (6.2) with positive probability.

THEOREM 6.2. *If* $d = 3$, $0 < t_1 < t_2$ *and* $m_0 \neq 0$ ($m_0 \in M_F$), *then*

$$Q^{m_0}\big(\dim(S_{t_1} \cap S_{t_2}) = 1\big) > 0.$$

REMARK 6.3. (a) Theorem 6.1(b) implies that a fixed set A will intersect S_1 with positive probability if A intersects a d-dimensional Brownian path $\Gamma_1 = \{B_s: 0 \le s \le 1\}$ with positive probability. Theorem 7.1 of (D.I.P.) shows that if $S_1 \neq \emptyset$, then S_1 and Γ_1 both have positive and finite Hausdorff $\phi - m$, where $\phi(x) = x^2 \log\log 1/x$. We have no explanation for this correspondence between two such apparently different sets.

(b) We conjecture that Theorem 6.1 remains valid if $d = 2$, and that the converse to Theorem 6.1(b) holds for all $d \ge 2$ (thus tightening the above correspondence).

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