The Annals of Probability
1990, Vol. 18, No. 3, 1222–1231

RANDOM SPACE CHANGE FOR MULTIPARAMETER POINT PROCESSES[1]

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A way of transforming a multiparameter point process into a Poisson process is given. As in the one-parameter case, the compensator characterization of the Poisson process plays an important role in random space change for the multiparameter point process. We use the characterization of planar Poisson processes in terms of the 1-compensator by Brown to derive random space-change theorems. The results obtained hold under weaker conditions than those in Merzbach and Nualart.

1. Introduction. In this paper we study the problem of transforming a class of multiparameter point processes into Poisson processes by means of random space change using compensators of the processes.

In the case of a one-parameter point process, the well-known characterization of a Poisson process by Watanabe (1964), which states that a point process N is Poisson if and only if $N_t - t$ is a martingale, leads to a random time-change theorem. In this case any point process with continuous compensator can be transformed into a Poisson process. Due to the lack of a total order in $\mathbb{R}_+^n$ for $n > 1$, several problems arise when we try to generalize these results to point processes in higher dimensions.

Cairoli and Walsh (1977) have exhibited a class of continuous two-parameter martingales which cannot be transformed into a Brownian sheet using stopping points. This explains the limitations of stopping points in $\mathbb{R}_+^2$. Nualart and Sanz (1981) have used stopping sets to transform a class of continuous two-parameter martingales, which are represented as an integral with respect to a Brownian sheet, to a Brownian sheet. Transformation of a class of two-parameter point processes, which are adapted to the natural filtration of a Poisson process, to a Poisson process was done in a similar way by Merzbach and Nualart (1986). In these cases the corresponding compensators are or are assumed to be absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}_+^n$, which means the existence of intensity for a point process. A reason for considering the natural filtration is condition (F4) of Cairoli and Walsh (1975).

In this paper, we transform a point process N using the i-compensators of N. A characterization of Poisson processes by Brown, Ivanoff and Weber (1986) is used to derive random space-change theorems. In our case we do not have to assume the (F4) condition and the filtration considered is arbitrary and

Received January 1989; revised May 1989.

[1]Research completed at the University of Western Australia. Financial support was provided by the University of Melbourne.

AMS 1980 *subject classifications*. Primary 60G55; secondary 60G44, 60G40.

Key words and phrases. Point process, Poisson process, compensator, stopping set, random space change.

the existence of intensity is not assumed. The conditions under which random space-change theorems are obtained using i-compensators are weaker than the conditions in Nualart and Sanz (1981) and Merzbach and Nualart (1986).

After introducing the necessary notation and definitions, the characterization results of Brown, Ivanoff and Weber (1986) are given in the next section. Random space-change theorems are stated and proved in Section 3. In Section 4 we outline how the results can be extended to higher dimensions, with particular explanations on three-dimension.

2. Notation and preliminaries. In $\mathbb{R}_+^2$, the positive quadrant of the plane, $\leq$ denotes the usual partial order defined by, $z \leq z'$ if $s \leq s'$ and $t \leq t'$, where $z = (s,t)$, $z' = (s',t')$. We write $z < z'$ if $s < s'$ and $t < t'$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $\mathbb{F} = \{\mathbb{F}_z,\ z \in \mathbb{R}_+^2\}$ be an increasing (i.e., $z \leq z'$ implies $\mathbb{F}_z \subseteq \mathbb{F}_{z'}$) family of sub σ-fields of $\mathscr{F}$ which is right continuous (i.e., $\mathbb{F}_z = \bigcap_{z' > z} \mathbb{F}_{z'}$) and complete (i.e., $\mathbb{F}_{(0,0)}$ contains null sets of $\mathscr{F}$). We define two other filtrations $\mathbb{F}^1$ and $\mathbb{F}^2$ by $\mathbb{F}_z^1 = \bigvee_{u \geq 0} \mathbb{F}_{(s,u)}$ and $\mathbb{F}_z^2 = \bigvee_{u \geq 0} \mathbb{F}_{(u,t)}$, respectively, for $z = (s,t)$.

Following Cairoli and Walsh (1975), the filtration $\mathbb{F}$ is said to satisfy the (F4) condition if for each $z \in \mathbb{R}_+^2$, $\mathbb{F}_z^1$ and $\mathbb{F}_z^2$ are conditionally independent given $\mathbb{F}_z$. A $\mathbb{F}$-adapted, integrable process $X = \{X_z,\ z \in \mathbb{R}_+^2\}$ is a *martingale* if $E(X_{z'}|F_z) = X_z$ for every $z < z'$. If a process X is adapted to the filtration $\mathbb{F}^i$, then we say X is i-adapted ($i = 1,2$). An i-adapted, integrable process X is an *i-martingale* ($i = 1,2$) if for each $z < z'$, $E(X(z,z']|\mathbb{F}_z^i) = 0$, where

$$X((s,t),(s',t')] = X_{(s',t')} - X_{(s,t')} - X_{(s',t)} + X_{(s,t)}.$$

Note that if X vanishes on the axes, then X is a 1-martingale if and only if $E(X_{(s',t)} - X_{(s,t)}|\mathbb{F}_{(s,t)}^1) = 0$ for each $s' > s$ and $t \geq 0$. Let $\mathscr{P}^i$ denote ($i = 1,2$) the predictable σ-algebra associated with the filtration $\mathbb{F}^i$, that is, the σ-algebra generated by the sets of the form $F \times (z,z']$, for $z < z'$, and $F \in \mathbb{F}_z^i$. If a process is measurable with respect to $\mathscr{P}^i$, then it is called an i-predictable process. A process $A = \{A_z,\ z \in \mathbb{R}_+^2\}$ is called *increasing* if $A(z,z'] \geq 0$ for every $z < z'$, and $A(s,0) = A(0,s) = 0$ for every $s \geq 0$.

An increasing process $N = \{N_z,\ z \in \mathbb{R}_+^2\}$ is called a *point process* if N is adapted, right continuous and takes values in $\mathbb{N} \cup \{\infty\}$. A simple point process is a point process whose jump sizes are 1, that is,

$$\Delta N_z = \lim_{n \to \infty} N(z - (1/n, 1/n), z] = 0 \text{ or } 1 \quad \text{for every } z \in \mathbb{R}_+^2.$$

In this paper we will consider only simple point processes and we assume for simplicity that $E(N_z) < \infty$, for every z.

The *i-compensator* ($i = 1,2$) of an $\mathbb{F}^i$-adapted point process N is defined to be an increasing, i-predictable process A^i such that $N - A^i$ is an i-martingale. When N is integrable, the existence and uniqueness of the i-compensator follows from Lemma 2.2 of Jacod (1975) [see Brown, Ivanoff and Weber (1986) for details]. The i-compensator also satisfies

$$E\left(\int_{\mathbb{R}_+^2} Y_z\, dN_z\right) = E\left(\int_{\mathbb{R}_+^2} Y_z\, dA_z^i\right) \tag{1}$$

for every bounded, positive, i-predictable process Y.

A simple point process N is called a *Poisson process* if for every Borel subset B of $\mathbb{R}_+^2$, $N(B)$ is a Poisson random variable with parameter $E(N(B))$ and for disjoint Borel subsets, $B_1, B_2, \dots, B_n$, $N(B_1), N(B_2), \dots, N(B_n)$ are independent, where $N(B)$ is defined as $\int I_B(z)\, dN_z$ (I_B is the indicator function of B), that is, $N(B)$ is the number of points in B. If $E(N(B)) = m(B)$, then N is called a *unit rate Poisson process*, where $m(B)$ denotes the Lebesgue measure of the set B.

To obtain a random space-change result for planar point processes, we need to characterize planar Poisson processes in terms of i-compensators. The following lemmas are proved in Brown, Ivanoff and Weber (1986) when the time domain is $[0, 1]^2$. Extension to $\mathbb{R}_+^2$ is straightforward, and a sketch of the proof will be given in Section 4, when the time domain is $\mathbb{R}_+^3$.

LEMMA 2.1. *Let N be a simple point process on $\mathbb{R}_+^2$ and $\mathbb{F}$ be a filtration such that N is* 1-*adapted* (2-*adapted*). *If the* 1-*compensator* (2-*compensator*) *of N is continuous then, with probability* 1, *N has at most one point on every horizontal* (*vertical*) *line.*

LEMMA 2.2. *Suppose N has deterministic, continuous* 1-*compensator* (2-*compensator*) *and has at most one point on every vertical* (*horizontal*) *line. Then N is a Poisson process whose mean coincides with the* 1-*compensator* (2-*compensator*).

REMARK 2.1. By Lemmas 2.1 and 2.2 it follows that if N has 1-compensator and 2-compensator continuous and one of them is deterministic, then N is a Poisson process.

Finally, we need a notion of convexity. A real-valued function f defined on $\mathbb{R}^n$ is said to be *i-convex*, for $i = 1, 2, \dots, n$ if f is convex in the ith coordinate. Later we will be using the following result from Hardy, Littlewood and Pólya (1934), Theorem 86: A continuous increasing function f on $\mathbb{R}$ is convex if and only if

$$f(x+h) - f(x) \le f(x+2h) - f(x+h) \quad \text{for every } x \in \mathbb{R} \text{ and } h > 0. \tag{2}$$

3. Stopping sets and random space change. In this section we define stopping sets and state and prove random space theorems for point processes using i-compensators. We need additional notation. For points $z = (s, t)$ and $z' = (s', t')$ we write $z \wedge z'$ if $s \le s'$ and $t \ge t'$. A connected subset Γ of $\mathbb{R}_+^2$ is called a *separating* set if it is closed and $z, z' \in \Gamma$ implies $z \wedge z'$ or $z' \wedge z$. Let

S denote the class of all separating subsets of $\mathbb{R}_+^2$. For $z \in \mathbb{R}_+^2$ let $R_z = \{z': z' \le z\}$ and for $\Gamma \in S$ let $R_\Gamma = \bigcup_{z \in \Gamma} R_z$.

Stopping sets are defined by several authors [e.g., Wong and Zakai (1977), Cairoli and Walsh (1978), Merzbach (1980) and Merzbach and Nualart (1986)]. We give one of these which is suitable for our purpose.

DEFINITION 3.1. A stopping set $D(\omega)$, with respect to a filtration $\mathbb{F}$, is a map from Ω to the subsets of $\mathbb{R}_+^2$ satisfying:

(i) For all $\omega \in \Omega$ such that $D(\omega)$ is nonempty, $D(\omega)$ is closed and $z \in D(\omega)$ implies $R_z \subseteq D(\omega)$.

(ii) For all $z \in \mathbb{R}_+^2$, $\{\omega: z \in D(\omega)\} \in \mathbb{F}_z$.

REMARK 3.1. By Proposition 2.1 of Cairoli and Walsh (1978), it follows that if D is a stopping set, then the process $X_z = I_D(z)$ is progressively measurable. Moreover, if D^1 and D^2 are two $\mathbb{F}$-stopping sets, then the stochastic interval $(D^1, D^2]$ defined by

$$(D^1, D^2] = \{(\omega, z): z \in D^2(\omega) \text{ and } z \notin D^1(\omega)\}$$

is a $\mathbb{F}$-predictable set. This is also true if we replace $\mathbb{F}$ by $\mathbb{F}^1$ or $\mathbb{F}^2$. A proof of this fact can be found in Merzbach (1980), Theorem 3.4 and the remark on page 60.

DEFINITION 3.2 [Merzbach (1980)]. Let D be a stopping set and for $\Gamma \in S$ let $\mathbb{F}_\Gamma = \bigvee_{z \in \Gamma} \mathbb{F}_z$. The stopped σ-field $\mathbb{F}_D$ is defined by

$$\mathbb{F}_D = \sigma\{A: A \cap \{\omega: D(\omega) \subseteq R_\Gamma\} \in \mathbb{F}_\Gamma \text{ for every } \Gamma \in S\}.$$

The σ-field $\mathbb{F}_D$ has most of the properties of the corresponding σ-field of stopping time in the one-parameter case. Readers may refer to Merzbach (1980) for details. Readers may also refer to Merzbach (1988) for a survey of recent developments in the theory of planar point processes.

Now we are ready to state a random space-change theorem for planar point processes. The transformation in the following theorem involves changing only the y coordinates of points of a point process as in Merzbach and Nualart (1986).

THEOREM 3.3. *Let N be a simple point process on $\mathbb{R}_+^2$ which is* 1*-adapted to a filtration $\mathbb{F}$. Suppose that A, the* 1*-compensator of N, is* 1*-convex, continuous and satisfies for all $s, h > 0$,*

$$A((s,0),(s+h,t)] \to \infty \quad \text{as } t \to \infty. \tag{3}$$

If N has at most one point on every vertical line, then there exists a family of stopping sets $\{D_z, z \in \mathbb{R}_+^2\}$ such that $\hat{N} = \{\hat{N}_z, z \in \mathbb{R}_+^2\}$ is a unit rate Poisson process adapted to the filtration $\mathbb{F}^1$, where $\hat{N}_z = \int I_{D_z}(u)\, dN_u$.

PROOF. First we construct a family of stopping sets $\{D_z, z \in \mathbb{R}_+^2\}$ using A. Fix $z = (s,t) > (0,0)$. For any n such that $[s2^n] > 1$, and $i = 1, 2, \dots, [s2^{-n}]$ define $t_n(i2^{-n})$ by

$$t_n(i2^{-n}) = \inf\{u > 0: A(((i-1)2^{-n}, 0), (i2^{-n}, u)] > t2^{-n}\}$$

and let $D_n^i = ((i-1)2^{-n}, 0), (i2^{-n}, t_n(i2^{-n}))]$. By condition (3) it follows that $t_n(i2^{-n}) < \infty$, and by the continuity of A we have $A(D_n^i) = t2^{-n}$.

Since A is 1-convex, by using (2) we get

$$t_n((i+1)2^{-n})) \le t_n(i2^{-n}) \tag{4}$$

and

$$t_{n+1}(2i2^{-(n+1)}) = t_{n+1}(i2^{-n}) \le t_n(i2^{-n}). \tag{5}$$

The inequality in (5) follows because, if it is not true then by (4) we will have $D_n^i \subsetneqq D_{n+1}^{2i-1} \cup D_{n+1}^{2i}$. This will give $A(D_n^i) < t2^{-n}$, which is a contradiction.

Let $x \in (0, s]$ and $\{i_n\}$ be the sequence of integers such that $i_n 2^{-n} < x \le (i_n + 1)2^{-n}$. Then $i_n 2^{-n}$ increases to x as $n \to \infty$. Hence from (4) and (5) it follows that $\{t_n(i_n 2^{-n})\}$ is a decreasing sequence. Let $t(x) = \lim_{n\to\infty} t_n(i_n 2^{-n})$. It can be seen from (4) and (5) that $0 < t(x) < \infty$. To define $t(0)$, note that $\{t_n(2^{-n})\}$ is an increasing sequence and let $t(0) = \lim_{n\to\infty} t_n(2^{-n})$ which may take the value ∞.

By (4) it follows that, for fixed t, $t(x)$ is a decreasing function in x. And since, for $x > 0$, $t(x)$ is the limit of $t_n(i_n 2^{-n})$ for $i_n 2^{-n} \le x$ as $n \to \infty$ it also follows that $t(x)$ is left continuous in x for fixed t.

Define, for $z = (s,t) > (0,0)$,

$$D_z = \{(x,y): 0 \le y \le t(x), 0 \le x \le s\},$$

and, for $z = (s, 0)$ and $z = (0, s)$, $s \ge 0$, define $D_z = \{(u, 0)\}: 0 \le u \le s\}$ and $D_z = \{(0,0)\}$, respectively.

Now we show that D_z is a $\mathbb{F}^1$-stopping set. For each ω, D_z is a closed set by the left continuity of $t(x)$. The definition of D_z and the decreasing property of $t(x)$ implies that if $z' \in D_z$, then $R_{z'} \subseteq D_z$.

To show that D_z satisfies condition (ii) in Definition (3.1), let $(x,y) \in \mathbb{R}_+^2$ and $x > 0$. Since $D_z \subseteq [0,s] \times R_+$, $\{\omega: (x,y) \in D_z(\omega)\} = \varnothing$ if $x > s$. If $x \le s$, then $\{\omega: (x,y) \in D_z(\omega)\} = \{\omega: y \le t(x)\}$, which is equal to $\bigcap_{n=1}^{\infty}\{\omega: y \le t_n(i_n 2^{-n})\}$, where $\{i_n 2^{-n}\}$ is the approximating dyadic rationals for x from below. But

$$\{\omega: y \le t_n(i_n 2^{-n})\} = \{A(((i-1)2^{-n}, 0), (i_n 2^{-n}, y)] \le t2^{-n}\},$$

which is in $\mathbb{F}^1_{(i_n 2^{-n})} \subseteq \mathbb{F}^1_{(x,y)}$. Hence $\{\omega: (x,y) \in D_z(\omega)\} \in \mathbb{F}^1_{(x,y)}$. A similar argument holds for $x = 0$. Hence D_z is a $\mathbb{F}^1$-stopping set.

Let $\hat{N}_z = \int I_{D_z}\, dN$, $\hat{A}_z = \int I_{D_z}\, dA$ and $\hat{\mathbb{F}}_z^1 = \mathbb{F}_{D_z}^1$. If $z = (s,t)$, then $D_z \subseteq [0,s] \times R_+$. Hence by the definition of $\hat{\mathbb{F}}_z^1$ we have $\hat{\mathbb{F}}_z^1 \subseteq \mathbb{F}_z^1$. Again by the definition of $\mathbb{F}_z^1$ and since $(0,t) \in D_z$, $\mathbb{F}_z^1 \subseteq \mathbb{F}_z^1$. Hence $\mathbb{F}_z^1 = \hat{\mathbb{F}}_z^1$. This clearly implies that $\hat{N}_z$ is $\mathbb{F}_z^1$-measurable.

If $z' = (s', t') \geq z = (s, t)$, then by definition $t(i2^{-n}) \leq t'(i2^{-n})$. This implies that $t(x) \leq t'(x)$ for every $x \leq s$. Hence $D_z \subseteq D_{z'}$, which implies that $\hat{N}$ is an increasing process. The right continuity of $\hat{N}$ follows from the continuity of A. Since $\Delta\hat{N}(s, t) = \Delta N(s, t(s))$ it follows that $\hat{N}$ is a $\mathbb{F}^1$-adapted point process.

By the continuity of A we have

$$\hat{A}_z = \int I_{D_z}\, dA = \lim_{n\to\infty} \sum_{i=1}^{[s2^n]} A(((i-1)2^{-n}, 0), (i2^{-n}, t_n(i2^{-n}))]$$

$$= \lim_{n\to\infty} [s2^n] \cdot t2^{-n} = st.$$

Hence, to complete the proof of the theorem, by Lemma 2.2 it is enough to prove that $\hat{N} - \hat{A}$ is a 1-martingale and that $\hat{N}$ has at most one point on every vertical line. The second part is clear from the assumption that N has at most one point on every vertical line and that $\hat{N}$ has a jump at (s, t) if and only if N has a jump at $(s, t(s))$. Let $z < z'$. Then by Remark 3.1 the stochastic interval $(D_z, D_{z'}]$ is a 1-predictable set and so is $F \times \mathbb{R}_+^2 \cap (D_z, D_{z'}]$ for $F \in \mathbb{F}_z^1$. Hence by (1) we have

$$E\left[\int I_{F\times\mathbb{R}_+^2} I_{(D_z, D_{z'}]}\, dN\right] = E\left[\int I_{F\times\mathbb{R}_+^2} I_{(D_z, D_{z'}]}\, dA\right],$$

that is, $E[\hat{N}_{z'} - \hat{N}_z\colon F] = E[\hat{A}_{z'} - \hat{A}_z\colon F]$, which completes the proof. □

The random time-change theorem can be given in terms of the 2-compensator of N, with similar conditions. Also it follows from Remark 2.1 that if N has 1-compensator and 2-compensator continuous, then $\hat{N}$ is a unit rate Poisson process.

REMARK 3.2. Merzbach and Nualart (1986) have proved a similar result in the case of martingales, when N has an intensity, that is, when the compensator A is given by $\int_{[(0,0),z]} \lambda(u, v)\, du\, dv$, for some predictable process λ, and the filtration $\mathbb{F}$ is generated by a Poisson process. Their results can be stated in terms of i-martingales and i-compensators. In this case the conditions of their result, the function $s \to \int_0^t \lambda(s, u)\, du$ is nondecreasing for all $t > 0$ and tends to ∞ as $t \to \infty$, clearly implies (3) and 1-convexity of A, by (2).

As in Theorem 5 of Merzbach and Nualart (1986) we can replace the requirement of 1-convexity of A by a weaker condition.

THEOREM 3.4. *Let N be a simple point process which is* 1*-adapted and has continuous* 1*-compensator A. Suppose N has at most one point on every vertical line, and A satisfies* (3) *and that there exists a positive, strictly* *increasing, function defined on* $[0,\infty)$ *with* $\alpha(0)=0$ *satisfying*

$$\frac{A((s,0),(s+h,t)]}{\alpha(s+h)-\alpha(s)} \le \frac{A((s+h,0),(s+2h,t)]}{\alpha(s+2h)-\alpha(s+h)},$$

for every $s \ge 0$, $t > 0$ *and* $h > 0$. *Then there exists a family of stopping sets* $\{D_z,\ z \in \mathbb{R}_+^2\}$ *such that* $\hat{N} = \{\hat{N}_z,\ z \in \mathbb{R}_+^2\}$ *is a unit rate Poisson process adapted to the filtration* $\hat{\mathbb{F}}^1$, *where* $\hat{N}_z = \int I_{D_z}(u)\, dN_u$, *and* $\hat{\mathbb{F}}_z^1 = \mathbb{F}_{D_z}^1$.

PROOF. The proof is similar to that of Theorem 3.3, except that the definition of t_n and D_z are slightly different and the time-changed process need not be $\mathbb{F}^1$-adapted. We will outline the changes.

For $z = (s,t) > (0,0)$ define

$$t_n(i2^{-n}) = \inf\{u > 0\colon A(((i-1)2^{-n},0),(i2^{-n},u)] \ge (\alpha(i2^{-n}) - \alpha((i-1)2^{-n}))t\}.$$

And for $x \in [0,s]$, we define $t(x)$ as the limit of t_n as in Theorem 3.3. Let $\hat{\alpha}(s) = \inf\{u > 0\colon \alpha(u) > s\}$ and

$$D_z = \{(x,y)\colon 0 \le y \le t(x),\ 0 \le x \le \hat{\alpha}(s)\}.$$

Define $\hat{N}$ and $\hat{\mathbb{F}}^1$ as before. Since the transformation involves changing both coordinates, the process $\hat{N}$ need not be $\mathbb{F}^1$-adapted. However, since D_z is a $\mathbb{F}^1$-stopping set, $\hat{N}$ is $\hat{\mathbb{F}}^1$-adapted. Also, $\hat{N}$ has a point at (s,t) if and only if N has a point at $(\hat{\alpha}(s), t(\hat{\alpha}(s))$. Hence, to prove that $\hat{N}$ is a Poisson process, it is enough to show that $\hat{N} - \hat{A}$ is a $\hat{\mathbb{F}}^1$-martingale. This follows as in the proof of the Theorem 3.3, by noting that if $z \le z'$ and $F \in \hat{\mathbb{F}}_z^1$, then $F \times \mathbb{R}_+^2 \cap (D_z, D_{z'}]$ is a $\mathbb{F}^1$-predictable set. □

4. Extension to *n*-parameters. The results of the previous section can be extended to the case when the time domain is $\mathbb{R}_+^n$. In this section we outline how this can be done in $\mathbb{R}_+^3$. Extension to R_+^n for $n > 3$ is straightforward, but notationally complicated.

In $\mathbb{R}_+^3$ the partial order $\le$ is given by $(r,s,t) \le (r',s',t')$ if $r \le r'$, $s \le s'$ and $t \le t'$, and $(r,s,t) < (r',s',t')$ if $r < r'$, $s < s'$ and $t < t'$. If $\mathbb{F} = \{\mathbb{F}_z,\ z \in \mathbb{R}_+^3\}$ is a filtration, then the filtration $\mathbb{F}^1 = \{\mathbb{F}_z^1,\ z \in \mathbb{R}_+^3\}$ is defined by $\mathbb{F}_z^1 = \mathbb{F}_{(r,\infty,\infty)}$, when $z = (r,s,t)$. Similarly filtrations $\mathbb{F}^2$ and F^3 are defined. The notions of i-martingale, i-predictability and stopping sets are extended in a similar way. Readers are referred to Cairoli and Walsh (1975) and Merzbach (1980) for details.

To obtain a random space-change theorem in $\mathbb{R}_+^3$, we need to extend Lemmas 2.1 and 2.2 to $\mathbb{R}_+^3$. This is done in the following two propositions. Proofs of the propositions are outlined, which are extensions of the proofs in Brown, Ivanoff and Weber (1986).

PROPOSITION 4.1. *Let N be a simple point process on $\mathbb{R}_+^3$ and $\mathbb{F}$ be a filtration such that N is i-adapted for $i = 1, 2, 3$. If N has continuous 2-compensator and 3-compensator then, with probability 1, N has at most one point on every plane perpendicular to the x axis.*

PROOF. It is enough to show that N has at most one point on every plane perpendicular to the x axis contained in $[0, n]^3$ for every n. For, if E_n denotes the above event and E denotes the event to be proved, then $E = \bigcap_{n \geq 1} E_n$ and if $P(E_n) = 1$, then $P(E) = 1$. Define S_i, T_i, R_{1i} and R_{2i} as follows, for $i \geq 1$:

$$S_i = \inf\{s: N([0,n] \times [0,s] \times [0,n]) \geq i\},$$

$$T_i = \inf\{t: N([0,n] \times [0,n] \times [0,t]) \geq i\},$$

$$R_{1i} = \inf\{r: N([0,n] \times [0,S_i) \times [0,n]) + N([0,r] \times \{S_i\} \times [0,n]) \geq i\},$$

$$R_{2i} = \inf\{r: N([0,n] \times [0,n] \times [0,T_i)) + N([0,r] \times [0,n] \times \{T_i\}) \geq i\}.$$

Then (R_{1i}, S_i) $[(R_{2i}, T_i)]$ are the coordinates of the points on the xy (xz) plane, which are the projections of the points, in $[0, n]^3$, of the point process onto the xy (xz) plane.

Since $N([0,n] \times [0,S_i) \times [0,n]) + N([0,r] \times \{S_i\} \times [0,n])$ is $F_{S_i}^2$-measurable, it follows that R_{1i} is $F_{S_i}^2$-measurable. Similarly, R_{2i} is $\mathbb{F}_{T_i}^3$-measurable. Hence the processes

$$X(r,s,t) = I_{[R_{1i}=r]} I_{[S_i < s \leq n]} I_{[0 \leq t \leq n]}$$

and

$$Y(r,s,t) = I_{[R_{2i}=r]} I_{[0 \leq s \leq n]} I_{[T_i < t \leq n]}$$

are $\mathbb{F}^2$-predictable and $\mathbb{F}^3$-predictable, respectively. Also, N has at most one point on a plane perpendicular to the x axis if for each i, there are no points in the sets

$$\{(r,s,t): r = R_{1i}, S_i < s \leq n, 0 \leq t \leq n\}$$

and

$$\{(r,s,t): r = R_{2i}, 0 \leq s \leq n, T_i < t \leq n\}.$$

For, suppose (r, s_1, t_1) and (r, s_2, t_2) are two points of the point process on the same plane, for $s_1 \leq s_2$. Then $s_1 < s_2$ or $t_1 < t_2$ or $t_1 > t_2$. Suppose $s_1 < s_2$. Then there exists an i such that $R_{1i} = r$ and $S_i = s_1$. Since $s_1 < s_2$, the point (r, s_2, t_2) lies in the first of the above two sets. We get similar conclusions in other cases.

Now, using an equation similar to (4.1) in the case $\mathbb{R}_+^3$, as in Lemma 4.1 of Brown, Ivanoff and Weber (1986), we get the result. □

PROPOSITION 4.2. *Suppose N has continuous i-compensator for $i = 1, 2, 3$ and one of them is deterministic. Then N is a three-dimensional Poisson process whose mean coincides with the deterministic compensator.*

PROOF. Suppose N has the 1-compensator continuous. By Proposition 4.1 N has at most one point on every plane perpendicular to the x axis. Hence N can be considered as a marked point process as in the case of $\mathbb{R}^2_+$. The rest of the proof follows as in Theorem 3.3 of Brown, Ivanoff and Weber (1986). □

Now we extend the Theorem 3.3 to $\mathbb{R}^3_+$.

THEOREM 4.3. *Let N be a simple point process on $\mathbb{R}^3_+$ which is i-adapted to the filtration $\mathbb{F}$ and has continuous i-compensator for $i = 1, 2, 3$. Suppose A, the 1-compensator of N is 1 and 2 convex, and satisfies,*

$$A[(r,s,0),(r+h,s+h,t)] \to \infty \quad \text{as } t \to \infty, \forall\ r,s \geq 0 \text{ and } h > 0.$$

Then there exists a family of stopping sets $\{D_z,\ z \in \mathbb{R}^3_+\}$ such that $\hat{N} = \{\hat{N}_z,\ z \in \mathbb{R}^3_+\}$ is a unit rate Poisson process which is 1- and 2-adapted to the filtration $\mathbb{F}$, where $\hat{N}_z = \int I_{D_z}(u)\, dN_u$.

PROOF. This proof is almost similar to that of the two-dimensional case except that we define t_n for each dyadic point in $\mathbb{R}^2_+$. That is, for fixed $z = (r,s,t)$ and n such that $[r2^{-n}] > 1$ and $[s2^{-n}] > 1$ we define $t_n(i2^{-n}, j2^{-n})$ by

$$t_n(i2^{-n}, j2^{-n}) = \inf\{u \colon A[((i-1)2^{-n},(j-1)2^{-n},0),(i2^{-n},j2^{-n},u)] > t2^{-2n}\}.$$

As in the two-dimensional case for $(x,y) \leq (r,s)$ the sequence $\{t_n(i_n2^{-n}, j_n2^{-n})\}$ is decreasing and converges to say $t(x,y)$, where i_n and j_n are such that $i_n2^{-n} < x \leq (i_n+1)2^{-n}$ and $j_n2^{-n} < y \leq (j_n+1)2^{-n}$. Then define

$$D_z = \{(x,y,u) \colon 0 \leq x \leq r, 0 \leq y \leq s, 0 \leq u \leq t(x,y)\},$$

and $\hat{N}_z = \int I_{D_z}\, dN$, $\hat{A}_z = \int I_{D_z}\, dA$. It follows that $\hat{N}$ has a point at (r,s,t) if and only if N has a point at $(r,s,t(r,s))$. And the rest of the proof follows as in the case of $\mathbb{R}^2_+$ by using Proposition 4.2. □

An extension of Theorem 3.4 can be also obtained in the general case, which is easy to guess and we leave out the details.

Acknowledgment. The author is grateful to Tim Brown for helpful suggestions and comments.

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