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OPTIMAL SWITCHING BETWEEN A PAIR OF BROWNIAN MOTIONS

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Consider two Brownian motions $B^1_{s_1}$ and $B^2_{s_2}$, each taking values on an interval $[0, a_i]$, $i = 1, 2$, with absorption at the endpoints. The time evolution of the two processes can be controlled separately: i.e., we can alternate between letting $B^1_{s_1}$ run while freezing $B^2_{s_2}$ and letting $B^2_{s_2}$ run while freezing $B^1_{s_1}$. This results in a switched process that evolves in the rectangle, $D = [0, a_1] \times [0, a_2]$ like a horizontal Brownian motion when $B^2_{s_2}$ freezes and like a vertical Brownian motion when $B^1_{s_1}$ freezes. Let $f(x_1, x_2)$ be a nonnegative continuous payoff function defined on the boundary ∂D of D. A control consists of a switching strategy and a stopping time τ. We study the problem of finding an optimal control which maximizes the expected payoff obtained at time τ (stopping in the interior results in zero reward). In the interior of the rectangle, the optimal switching strategy is determined by a partition into three sets: a horizontal control set, a vertical control set and an indifference set. We give an explicit characterization of these sets in the case when the payoff function is either linear or *strongly concave* on each face.

1. Introduction. Consider a pair of Brownian motions $(B^i, \mathscr{F}^i, P^i)$, $i = 1, 2$. The process $B^i = \{B^i_{s_i}, s_i \geq 0\}$ evolves on the interval $[0, a_i]$ and is absorbed at the endpoints. For each i, B^i is adapted to the filtration $\mathscr{F}^i = \{\mathscr{F}^i_{s_i}, s_i \geq 0\}$ on the space Ω^i of continuous functions and has expectation operators $\mathbf{E}^i_{x_i}$ corresponding to the probability measures $P^i_{x_i}$, $x_i \in [0, a_i]$. Also, we assume that the filtration $\mathscr{F}^i$ is right-continuous and complete relative to every measure $P^i_{x_i}$, $x_i \in [0, a_i]$. The time evolution of the two processes can be controlled separately: i.e., we can let the B^1 process run and freeze B^2 or we can let the B^2 process run and freeze B^1. This results in a switched process whose state space is the rectangle $D = [0, a_1] \times [0, a_2]$. On the faces of D, only one of the Brownian motions can actually move because the other is at an absorbing state. Likewise, the corners are completely absorbing since both processes are in absorbing states. Suppose that we are free to choose both the

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switching strategy and the stopping rule. When we decide to stop, we collect a payoff depending on the location of the pair (B^1, B^2) within D at the time of stopping. Our goal is to maximize the expected payoff.

We consider a special form of this problem: We assume that the payoff for stopping in the interior D^0 of D is zero and that on the boundary ∂D of D the reward is specified by a nonnegative continuous payoff function $f(x_1, x_2)$. As mentioned above, once the process hits a face of D, it must remain on that face for all time. Therefore, on each face the problem reduces to the classical optimal stopping problem for a single Brownian motion (see, e.g., [6]). Thus the problem does not change essentially if on each face we replace f by its smallest concave majorant and simply stop as soon as we hit a face. Also, stopping before a face is hit is never advantageous since the reward will then be zero. Hence, the stopping rule part of the problem has now been trivialized: Stop at the first hitting time of ∂D. The problem then is to find the switching mechanism that maximizes the expected payoff at the first hitting time of the boundary.

To formulate the problem precisely, we use the notion of a switching strategy. A *switching strategy* T is a family of random time pairs,

$$T = \{T(t) = (T_1(t), T_2(t)), t \geq 0\}, \tag{1.1}$$

satisfying

$$T(0) = (0,0), \tag{1.2}$$

$$T_i(t) \text{ is increasing in } t \text{ for each } i, \tag{1.3}$$

$$T_1(t) + T_2(t) = t \tag{1.4}$$

and

$$\{T_1(t) \leq s_1, T_2(t) \leq s_2\} \in \mathscr{F}^1_{s_1} \times \mathscr{F}^2_{s_2}. \tag{1.5}$$

The random variable $T_i(t)$ represents the amount of time the ith Brownian motion has been used up to time t. The interpretation of (1.4) is that, at time t, the total allocation of time between the two processes must equal t. Condition (1.5) says that the switching strategy must be nonanticipating. This notion of two-parameter random time change was first introduced in the discrete case in [3]. It was independently proposed by two of the current authors in [13]. Walsh seems to have been the first to use this notion in continuous time in [19], where T is called an *optional increasing path*. The *switched process* X^T is defined as

$$X^T(t) = B_{T(t)} = \left(B^1_{T_1(t)}, B^2_{T_2(t)}\right). \tag{1.6}$$

The problem is:

PROBLEM. Find a switching strategy T^* that maximizes the expected payoff at the first hitting time of the boundary:

$$v(x) = \mathbf{E}_x f(B_{T^*(\tau^*)}) = \sup_T \mathbf{E}_x f(B_{T(\tau)}), \tag{1.7}$$

where

$$\tau = \inf\{t > 0 \colon B_{T(t)} \in \partial D\} \tag{1.8}$$

and $\mathbf{E}_x$ denotes expectation using the product measure $P_x = P^1_{x_1} \times P^2_{x_2}$, $x = (x_1, x_2) \in D$.

Note that $\mathbf{E}_x \tau < \infty$ for any strategy T and all $x \in D$ since τ does not exceed the sum of the absorption times of B^1 and B^2. The function v is called the *value function*.

This problem is interesting for several reasons. For example, the Hamilton–Jacobi–Bellman principle of dynamic programming suggests that the associated value function $v(x_1, x_2)$ is in some sense a solution of the nonlinear Dirichlet problem

$$\max\left(\frac{\partial^2 v}{\partial x_1^2}, \frac{\partial^2 v}{\partial x_2^2}\right) = 0, \quad \text{in } D,$$
$$v = f, \quad \text{on } \partial D.$$

(This is proved in the discrete case in [13] and assuming some regularity on v it is proved in the continuous case in [14].) Nonlinear Dirichlet problems of this type have been extensively studied (see, e.g., [7], [10], [9], [16] and [18]), but this particular equality corresponds to a degenerate case of the general theory and none of the general results seem to apply. The solution we propose is explicit and the behavior of the optimal solution is somewhat surprising. For example, on the boundary between the horizontal and vertical control sets, the switched process exhibits an interesting Brownian local-time behavior.

As mentioned earlier, it suffices to consider the case where f is concave on each of the four faces of ∂D. We restrict our attention mostly to two extremal cases: linear and strongly concave. By *strongly concave*, we mean twice continuously differentiable and strictly concave.

The existence of an optimal switching strategy has been considered in varying degrees of generality in [18], [14], [4] and [5]. Our contribution is that, in the specific setting described above, we give explicit analytical formulas for the value function. Consequently, while parts of the paper are probabilistic in nature, other parts are distinctly analytical. This is in the same spirit as [1] and [17].

2. The main results. In this section we present the main results. The proofs are deferred to Section 3. Sometimes it is convenient to write the arguments of a function as subscripts rather than in parentheses. We do this freely and hopefully without causing confusion.

PROPOSITION 1. *Suppose that f is linear on each of the four faces of ∂D. Let*

$$w(x_1, x_2) = f_{0,0} + \left(f_{a_1,0} - f_{0,0}\right)\frac{x_1}{a_1} + \left(f_{0,a_2} - f_{0,0}\right)\frac{x_2}{a_2}$$
$$+ \left(f_{a_1,a_2} - f_{a_1,0} - f_{0,a_2} + f_{0,0}\right)\frac{x_1 x_2}{a_1 a_2}$$

denote the bilinear interpolation of f. Then $v(x) = w(x)$ and every switching strategy is optimal.

REMARK. Whenever the value function is bilinear in some region, then in that region either the horizontal or the vertical Brownian motion can be chosen (or any combination). We call such a region an *indifference region*. The conclusion of the previous theorem is that all of D is an indifference region.

To discuss the remaining cases, we need a notation for the faces of ∂D. We will call them north, south, east and west and denote them by F_N, F_S, F_E and F_W.

PROPOSITION 2. *Suppose that f is concave on F_S and F_N and is linear on F_E and F_W. Then*

$$v(x_1, x_2) = \frac{x_2}{a_2} f(x_1, a_2) + \left(1 - \frac{x_2}{a_2}\right) f(x_1, 0)$$

and an optimal strategy is to run the vertical Brownian motion

$$T^*(t) = (0, t).$$

REMARKS. (i) If f is strictly concave on either F_S or F_N, then T^* is unique.
(ii) By taking f to be linear on F_N, we get the following heuristic: When near a face having concave data, it is best to choose the Brownian motion that runs perpendicular to this face.

2.1. *Two concave faces.* We now proceed to the case of two adjacent faces having concave data. Based on the above heuristic, we suspect that near the faces having concave data we should choose the Brownian motion that runs perpendicular to this concave face. Hence, there must be an interface where the control switches between horizontal and vertical. This is indeed what happens.

First, we briefly digress to discuss concave functions. We associate with any strongly concave function γ defined on an interval $[0, a]$ another function Γ defined on the same interval by

$$\Gamma(x) = \gamma(x) - x\gamma'(x) - \gamma(0) \tag{2.1}$$

$$= -\int_0^x u\gamma''(u)\,du. \tag{2.2}$$

From (2.2), we see that Γ is strictly increasing, continuously differentiable and that $\Gamma(0) = 0$. The function Γ has a simple geometric interpretation: $\Gamma(x)$ is the difference between the y intercept of the line tangent to γ at the point x and $\gamma(0)$.

Suppose that f is strongly concave on F_S and F_W and linear on F_E and F_N. Let

$$\gamma_1(x_1) = f(x_1, 0),$$
$$\gamma_2(x_2) = f(0, x_2)$$

denote the strongly concave restrictions of f to F_S and F_W, respectively.

We will call a strictly increasing parametric curve $\xi(u) = (\xi_1(u), \xi_2(u))$, $0 \le u \le \bar{u}$, a *switching curve* if $\xi(0) = (0, 0)$ and $\xi(\bar{u}) \in F_N \cup F_E$. Our aim is to show that for a certain switching curve, the optimal control is to run the horizontal Brownian motion above the curve and to run the vertical Brownian motion below. Hence, the value function should be linear in x_1 above the curve and linear in x_2 below it. If we knew the value function along the curve, then these linearity constraints would completely define the value function. This leads to the following definition. Let ξ be a switching curve and associate with it the rectangle R defined by the constraints $0 \le x_1 \le \xi_1(\bar{u})$ and $0 \le x_2 \le \xi_2(\bar{u})$. Given any real-valued function $\theta(u)$, $0 \le u \le \bar{u}$, that satisfies $\theta(0) = f(0, 0)$, let w denote the continuous function defined on R that

1. agrees with θ along the switching curve,
2. agrees with γ_1 along the x_1 axis,
3. agrees with γ_2 along the x_2 axis,
4. is linear in x_1 above the switching curve and
5. is linear in x_2 below the switching curve.

We call w the (ξ, θ) *sweep* of (γ_1, γ_2). The explicit formula for w is

$$w(x_1, x_2) = \begin{cases} x_2\phi(x_1) + \gamma_1(x_1), & \xi_2^{-1}(x_2) \le \xi_1^{-1}(x_1), \\ x_1\psi(x_2) + \gamma_2(x_2), & \xi_1^{-1}(x_1) \le \xi_2^{-1}(x_2), \end{cases} \tag{2.3}$$

where

$$\phi(x_1) = \frac{\theta\big(\xi_1^{-1}(x_1)\big) - \gamma_1(x_1)}{\xi_2\big(\xi_1^{-1}(x_1)\big)} \tag{2.4}$$

and

$$\psi(x_2) = \frac{\theta\big(\xi_2^{-1}(x_2)\big) - \gamma_2(x_2)}{\xi_1\big(\xi_2^{-1}(x_2)\big)}. \tag{2.5}$$

Let Γ_i denote the increasing function associated with γ_i as in (2.1) and for $0 \le u \le \Gamma_i(a_i)$, let

$$\xi_i(u) = \Gamma_i^{-1}(u), \qquad i = 1, 2. \tag{2.6}$$

If we put

$$\bar{u} = \min(\Gamma_1(a_1), \Gamma_2(a_2)),$$

then it is easy to see that $\xi(u) = (\xi_1(u), \xi_2(u))$, $0 \le u \le \bar{u}$, is a switching curve. It terminates on F_E (resp. F_N) if $\Gamma_1(a_1) \le \Gamma_2(a_2)$ [resp. $\Gamma_2(a_2) \le \Gamma_1(a_1)$].

Call ξ in (2.6) the (γ_1, γ_2) *switching curve*. We are now ready to state

THEOREM 3. *Suppose that f is strongly concave on F_S and F_W and is linear on F_E and F_N. Let $\xi(u)$, $0 \le u \le \bar{u}$, be the (γ_1, γ_2) switching curve and let R be its associated rectangle. Then, in R, the value function v is precisely the (ξ, θ) sweep of (γ_1, γ_2), where for $0 \le u \le \bar{u}$,*

(2.7)

$$\theta(u) = \xi_1(u)\xi_2(u)\left\{\frac{\theta(\bar{u})}{\xi_1(\bar{u})\xi_2(\bar{u})} + \int_u^{\bar{u}} \frac{\xi_1'\gamma_2(\xi_2)}{\xi_1^2\xi_2}\,ds + \int_u^{\bar{u}} \frac{\xi_2'\gamma_1(\xi_1)}{\xi_2^2\xi_1}\,ds\right\},$$

and $\theta(\bar{u}) = f(\xi(\bar{u}))$. Outside R, v is the linear interpolation of the boundary values on the two opposite faces. Furthermore, there exists an optimal strategy T^* such that $T_1^*(t)$ increases only when $X^{T^*}(t)$ is on or above the switching curve and $T_2^*(t)$ increases only when $X^{T^*}(t)$ is on or below the switching curve.

REMARKS. (i) The region above the switching curve is the horizontal control region and the region below it is the vertical control region. The behavior of the optimal switched process is quite interesting. Perhaps the simplest way to visualize what happens is to consider a near optimal strategy where the process is allowed to penetrate ε units past the switching curve before switching. For example, suppose that the process starts below the switching curve. Initially, the vertical Brownian motion is run until either it exits D^0 on F_S or it penetrates ε units above the switching curve. Suppose that, in fact, it penetrates by ε before hitting F_S. Now we switch to the horizontal Brownian motion, which is run until either it hits F_W or it penetrates by ε, the switching curve. Since it is starting close to the switching curve, the probability is high that it will penetrate by ε before it hits F_W. Assuming that this happens, the process is now located very close to the switching curve in the vertical control region. Continuing in this way, it is clear that the process tends to wander up the switching curve, making occasional excursions either into the horizontal or the vertical control region. If we let ε shrink to zero, we have a pretty clear idea of how the optimal switched process behaves. For example, this process can exit D^0 only on F_W, F_S or at $\xi(\bar{u})$. The special case where the switching curve is the diagonal, $x_1 = x_2$, is studied in detail in [12]. In this case, the *crawling up the diagonal* is described by the local time of a certain Brownian motion. Extending this detailed analysis to general switching curves is nontrivial and is the subject of current research.

(ii) The exact form for ξ and θ was discovered using the *principal of smooth fit* (see, e.g., [8]). That is, we found ξ and θ as the unique (ξ, θ) sweep that is twice continuously differentiable in D^0. In fact, stipulating that first derivatives agree across the switching curve $\xi(u)$ reduces to

$$\theta' = \frac{\xi_1'}{\xi_1}(\theta - \gamma_2(\xi_2)) + \frac{\xi_2'}{\xi_2}(\theta - \gamma_1(\xi_1)). \tag{2.8}$$

[This is obtained by differentiating and then equating the two expressions on the right-hand side in (2.3).] Formula (2.8) arises no matter which first derivative is stipulated. Making second derivatives agree reduces to

$$\tag{2.9} \begin{aligned} &\frac{\xi_1}{\xi_1'}\left\{\theta' - \frac{\xi_2'}{\xi_2}(\theta - \gamma_1(\xi_1))\right\} - \xi_1\gamma_1'(\xi_1) \\ &\quad = \frac{\xi_2}{\xi_2'}\left\{\theta' - \frac{\xi_1'}{\xi_1}(\theta - \gamma_2(\xi_2))\right\} - \xi_2\gamma_2'(\xi_2). \end{aligned}$$

This formula arises no matter which of the three pairs of second derivatives are matched. Substituting (2.8) into (2.9) yields

$$\tag{2.10} \Gamma_1(\xi_1) = \Gamma_2(\xi_2).$$

Choosing the common value of Γ_i as the parameter, we get

$$\xi_i(u) = \Gamma_i^{-1}(u).$$

Now that the curve $\xi(u) = (\xi_1(u), \xi_2(u))$ is known, we get θ in (2.7) by integrating (2.8) using the integrating factor $1/(\xi_1\xi_2)$.

(iii) For future reference, consider the following scenario: Suppose that γ_1 attains its maximum in the interior of $[0, a_1]$ at say x_1^* and similarly suppose that γ_2 attains its maximum at x_2^*. If $\gamma_1(x_1^*) = \gamma_2(x_2^*)$, then it follows from (2.1) and (2.10) that the switching curve passes through the point (x_1^*, x_2^*).

EXAMPLE. Suppose that the two concave functions are in fact quadratic,

$$\gamma_i(x) = c_i x(a_i - x).$$

Then $\Gamma_i(x) = c_i x^2$ and, hence, the switching curve is a straight line,

$$x_2 = \xi_2(\xi_1^{-1}(x_1)) = \sqrt{c_1/c_2}\,x_1.$$

2.2. *Three concave faces.* Now suppose that f is strongly concave on F_S, F_E and F_W and that it is linear on F_N. First, we ignore the concave data on F_E and we construct, as above, the switching curve $\xi(u)$, $0 \le u \le \bar{u}$, emanating from (0, 0). Denote by $C_1 = \{(\xi_1(u), \xi_2(u)), 0 \le u \le \bar{u}\}$ the graph of the switching curve. Let $\theta(u)$ be given by (2.7). Instead of requiring that $\bar{u}$ represent the exit through the north or east face, we now allow it to be a parameter that will be chosen. Also, $\theta(\bar{u})$ remains to be chosen. There are thus two unknown parameters needed to completely specify a smooth (ξ, θ) sweep of the boundary data on F_S and F_W. We call any such (ξ, θ) sweep a *partial sweep* of (F_S, F_W). Similarly, by ignoring the concave data on F_W, one constructs a switching curve that emanates from $(a_1, 0)$. Let C_2 denote its point set. Associated with this second switching curve we have partial sweeps of (F_S, F_E) which also require two specification parameters—namely, how far the sweep extends to the northwest and the value associated with its northwest corner.

There are two cases to consider depending on whether C_1 and C_2 intersect in D. If they do not intersect, then they both must exit through F_N with C_1

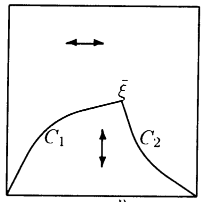

FIG. 1. *Three concave faces.*

hitting F_N to the left of where C_2 hits F_N. In this case, we can drop a vertical line V through D in such a way that C_1 lies entirely to the left of V and C_2 lies entirely to the right. By putting linear data along V we can reduce this case to a union of two smaller problems which have been solved in Theorem 3.

We now focus on the more interesting case where C_1 and C_2 intersect at a point, say $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2)$ (see Figure 1). Let

$$R_N = \{(x_1, x_2) \in D: x_2 \geq \bar{\xi}_2\}, \tag{2.11}$$

$$R_{SW} = \{(x_1, x_2) \in D: x_1 \leq \bar{\xi}_1, x_2 \leq \bar{\xi}_2\}, \tag{2.12}$$

$$R_{SE} = \{(x_1, x_2) \in D: x_1 \geq \bar{\xi}_1, x_2 \leq \bar{\xi}_2\}. \tag{2.13}$$

THEOREM 4. *On R_N, the value function v is the linear interpolation of f on F_E and F_W,*

$$v(x_1, x_2) = \frac{x_1}{a_1} f(a_1, x_2) + \frac{a_1 - x_1}{a_1} f(0, x_2), \qquad (x_1, x_2) \in R_N.$$

On R_{SW}, v is the partial sweep of (F_S, F_W) determined by the requirement that

$$v(\bar{\xi}_1, \bar{\xi}_2) = \frac{\bar{\xi}_1}{a_1} f(a_1, \bar{\xi}_2) + \frac{a_1 - \bar{\xi}_1}{a_1} f(0, \bar{\xi}_2). \tag{2.14}$$

Similarly, on R_{SE}, v is the partial sweep of (F_S, F_E) determined by (2.14). *An optimal strategy exits. It runs the vertical Brownian motion below the switching curves in R_{SW} and R_{SE} and it runs the horizontal Brownian motion everywhere else.*

2.3. *Four concave faces.* Now we put strongly concave data on all four faces. This time we expect switching curves to emanate from all four corners (see Figure 2). The solution depends on how these curves intersect. As Figure 2 shows, the intersections can look quite complicated. Hence, it is quite remarkable that even this case has an explicit solution. As before, let C_1 and C_2 denote the switching curves emanating from the southwest and southeast corners, respectively. Let C_0 and C_3 denote the ones emanating from the northwest and northeast. Even though C_1 and C_3 can intersect several times (and so can C_0 and C_2), the curves emanating from adjacent corners can intersect only once at most. Let ξ^i denote the intersection of C_i and C_{i+1} (we

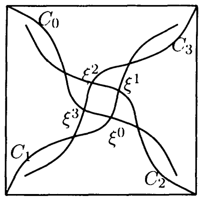

FIG. 2. *Intersections of switching curves.*

use the convention that index arithmetic is done mod 4). If any of these pairs of curves fail to intersect within D, simply choose ξ^i to be a point on the opposite face between the points where the curves exit D^0. It turns out that the solution depends on the following possibilities (see Figure 3):

1. Whether ξ^0 is to the left or right of ξ^2.
2. Whether ξ^1 is above or below ξ^3.

Based on this there are three cases to consider. Two of the cases reduce to cases already solved. We take care of these first.

CASE 1. ξ^0 is to the left of ξ^2. In this case, we can drop a vertical line V so that ξ^0 is to the left of V and ξ^2 is to the right. By putting linear data along V, we can reduce this case to a union of two smaller problems each of which has strongly concave data on three sides and linear data on the fourth.

CASE 2. ξ^1 is below ξ^3. For this case, we throw in a horizontal line H so that ξ^3 is above H and ξ^1 is below H. H separates the state space into two regions, thereby reducing the problem to a previously solved case.

THEOREM 5. *Cases* 1 *and* 2 *are disjoint.*

CASE 3. ξ^0 is to the right of ξ^2 and ξ^1 is above ξ^3. This is the most interesting configuration. Without loss of generality, assume further that ξ^0 is below ξ^2. It is easy to see that this implies the relative positions of each of the ξ^i's are as shown in Case 3 of Figure 3. If ξ and η are two points on a simple

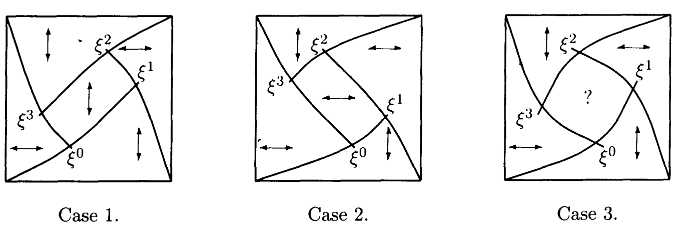

FIG. 3. *The three cases.*

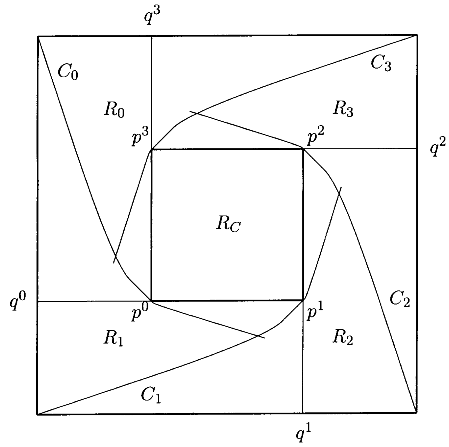

FIG. 4. *The most interesting case.*

arc C, let $C(\xi, \eta)$ denote the portion of C lying between ξ and η (including the endpoints). Let $\mathscr{R}_C$ denote the region bounded by $C^i(\xi^{i-1}, \xi^i)$, $i = 0, 1, 2, 3$.

THEOREM 6. *There exists a unique rectangle R_C inscribed in $\mathscr{R}_C$ having vertical and horizontal sides such that each of the faces of $\mathscr{R}_C$ is touched by one of the vertices of R_C.*

Let p^i denote the vertex of R_C that lies on $C_i(\xi^{i-1}, \xi^i)$ (see Figure 4). Let

$$R_0 = \{(x_1, x_2) \in D: x_1 \le p_1^0, x_2 \ge p_2^0\},$$

$$R_1 = \{(x_1, x_2) \in D: x_1 \le p_1^1, x_2 \le p_2^1\},$$

$$R_2 = \{(x_1, x_2) \in D: x_1 \ge p_1^2, x_2 \le p_2^2\},$$

$$R_3 = \{(x_1, x_2) \in D: x_1 \ge p_1^3, x_2 \ge p_2^3\}.$$

Let $M = D^0 \setminus (R_C^0 \cup R_0^0 \cup R_1^0 \cup R_2^0 \cup R_3^0)$.

THEOREM 7. *On M, the value function v is the unique continuous function that* (i) *agrees with the boundary data and* (ii) *is linear along each of the line segments comprising M. In R_C, v is the bilinear interpolation of the values on $M \cap R_C$. On each R_i, v is the partial sweep of the boundary data determined by the requirement that v is continuous at $R_i \cap M$. A switching strategy is optimal if it runs the vertical Brownian motion*

(i) *above C_0 in R_0,*
(ii) *below C_1 in R_1,*
(iii) *below C_2 in R_2,*
(iv) *above C_3 in R_3*

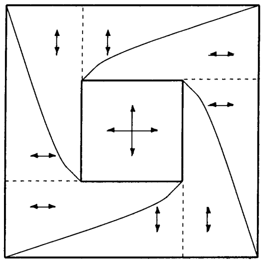

FIG. 5. *The most interesting case and its solution.*

and the horizontal Brownian motion

(i) *below* C_0 *in* R_0,
(ii) *above* C_1 *in* R_1,
(iii) *above* C_2 *in* R_2,
(iv) *below* C_3 *in* R_3.

R_C *is an indifference zone.*

Figure 5 shows the control regions.

Although in general it is hard to determine analytically whether a specific problem falls into Case 1, 2 or 3, there is one special case described in the following example for which this can be determined.

EXAMPLE. Suppose that the boundary function f has the following properties:

(i) The maximum value on each of the four faces occurs in the interior of the face.
(ii) All four maxima have the same value.
(iii) The maximum on F_S occurs to the right of the maximum on F_N.
(iv) The maximum on F_E occurs above the maximum on F_W.

Then from Remark 3 following Theorem 3, Case 3 prevails and the points q^i shown in Figure 4 are the places where the maxima occur.

The general case where each face is assumed to have merely concave data presents further challenges that cannot be easily overcome by our methods. For example, instead of Itô calculus, we probably need to use results from a general bipotential theory (as in [14] or [15]) some of which have yet to be developed.

3. Proofs. For the proofs, we use the general theory of two-parameter processes. The reader unfamiliar with the basic definitions and results is referred to either [14], [13] or [19]. Throughout this section, if we say that a two-parameter process is a martingale, we mean that it is a strong martingale relative to the product filtration $\mathscr{F} = \{\mathscr{F}^1_{s_1} \times \mathscr{F}^2_{s_2},\ s_1 \geq 0,\ s_2 \geq 0\}$. (A two-parameter process M_{s_1, s_2} is a strong martingale if it is adapted, integrable and satisfies the strong martingale property: $E\{M_{t_1, t_2} | \mathscr{F}^1_{s_1} \times \mathscr{F}^2_{s_2}\} = M_{s_1, s_2}$ for all $s_1 \le t_1$, $s_2 \le t_2$.) If we say that a one-parameter process [derived from a two-parameter process by following along a switching strategy $T(t)$] is a martingale, we mean that it is a martingale relative to the filtration $\mathscr{F}^T = \{\mathscr{F}_{T(t)}, t \ge 0\}$, where $\mathscr{F}_{T(t)}$ is defined as the σ-algebra containing all measurable sets A for which $A \cap \{T_1(t) \le s_1, T_2(t) \le s_2\} \in \mathscr{F}^1_{s_1} \times \mathscr{F}^2_{s_2}$ for all s_1, s_2. We begin by quoting a few well-known results.

PROPOSITION 8. (i) *If* $M_s = (M^1_{s_1}, M^2_{s_2})$ *is a two-parameter (super)-martingale and* $T(t)$ *is a switching strategy, then* $M_{T(t)}$ *is a (super)martingale.*

(ii) *If* w *is bilinear, then* $w(X^T_t)$ *is a martingale for any strategy* $T(t)$.

(iii) *If* w *is biconcave, then* $w(X^T_t)$ *is a supermartingale for any strategy* $T(t)$.

These results follow from Propositions 2.4 and 3.1 in [19].

All the proofs to follow depend on Lemma 9.

LEMMA 9. *Let* w *be a continuous, biconcave function on* D *that agrees with* f *on* ∂D. *If there exists a switching strategy* $\tilde{T}(t)$ *such that* $w(B_{\tilde{T}(t)})$ *is a martingale, then* w *is the value function defined in* (1.7) *and* $\tilde{T}(t)$ *is an optimal switching strategy.*

PROOF. Appealing to part (iii) of Proposition 8 and the optional sampling theorem, we conclude that

$$w(x) \ge \mathbf{E}_x w(B_{T(\tau)}) = \mathbf{E}_x f(B_{T(\tau)}) \tag{3.1}$$

for any switching strategy $T(t)$. Since $w(B_{\tilde{T}(t)})$ is a martingale we see that

$$w(x) = \mathbf{E}_x w(B_{\tilde{T}(\tau)}) = \mathbf{E}_x f(B_{\tilde{T}(\tau)}). \tag{3.2}$$

From (3.1) and (3.2), we conclude that w is the value function and that $\tilde{T}(t)$ is an optimal control. □

In the remaining proofs we simply exhibit a function w that is continuous, biconcave and agrees with the boundary data, and we describe a switching strategy $\tilde{T}(t)$ for which $w(B_{\tilde{T}(t)})$ is a martingale.

PROOF OF PROPOSITION 1. Let w denote the bilinear interpolation of f. Clearly, w is continuous, bilinear and agrees with f on the boundary. Since w is bilinear, it follows from part (ii) of Proposition 8 that $w(B_{T(t)})$ is a martingale for any strategy $T(t)$. □

PROOF OF PROPOSITION 2. Let

$$w(x_1, x_2) = \frac{x_2}{a_2} f(x_1, a_2) + \left(1 - \frac{x_2}{a_2}\right) f(x_1, 0).$$

Clearly w is continuous, biconcave and agrees with f on the boundary. Consider the strategy $\tilde{T}(t) = (0, t)$. Since w is linear in its second variable, it follows that $w(B_{\tilde{T}(t)})$ is a martingale. This completes the proof. □

3.1. *Two concave faces.*

PROOF OF THEOREM 3. Let $\xi(u)$ denote the (γ_1, γ_2)-switching curve, let $\theta(u)$ be defined by (2.7) and let w denote the (ξ, θ) sweep of (γ_1, γ_2). It is easy to check [using (2.3)] that w is twice continuously differentiable in D^0 and

$$\frac{\partial^2 w}{\partial x_i^2}(\xi_1(u), \xi_2(u)) = 0, \qquad i = 1, 2, 0 \le u \le \bar{u}. \tag{3.3}$$

It follows from (2.3) and (3.3) that w is biconcave. Also, w is clearly continuous and agrees with f on the boundary.

Let $T(t)$ be any switching strategy. Since the functions x_i, $i = 1, 2$, and $x_1 x_2$ are bilinear, it follows from part (ii) of Proposition 8 that $X_i^T(t)$, $i = 1, 2$, and $X_1^T(t) X_2^T(t)$ are martingales. Hence, for $t \ge 0$, the quadratic covariation between $X_1^T(t)$ and $X_2^T(t)$ is

$$\langle X_1^T, X_2^T \rangle_t = 0. \tag{3.4}$$

For $i = 1, 2$, $(B_{s_i}^i)^2 - s_i$ is a two-parameter martingale (a rather trivial one at that) and so, by part (i) of Proposition 8, $(X_i^T(t))^2 - T_i(t)$, $i = 1, 2$, are martingales. Hence, we see that, for $t \ge 0$, the quadratic variation of $X_i^T(t)$ is

$$\langle X_i^T \rangle_t = T_i(t). \tag{3.5}$$

Since w is twice continuously differentiable, we can apply Itô's formula together with (3.4) and (3.5) to see that

$$\begin{aligned} w(X^T(t)) - w(X^T(0)) &= \sum_i \int_0^t \frac{\partial w}{\partial x_i}(X^T(s))\, dX_i^T(s) \\ &\quad + \frac{1}{2} \sum_i \int_0^t \frac{\partial^2 w}{\partial x_i^2}(X^T(s))\, dT_i(s). \end{aligned} \tag{3.6}$$

Suppose there exists a switching strategy $\tilde{T}(t)$ for which

$$\int_0^t \frac{\partial^2 w}{\partial x_i^2}(X^{\tilde{T}}(s))\, d\tilde{T}_i(s) = 0 \tag{3.7}$$

(a.s. P_x for all $x \in D$) for $i = 1, 2$ and for all $t \ge 0$. Then from (3.6) it follows that $w(X^{\tilde{T}}(t))$ is a martingale and we are done.

All that remains then is to construct a switching strategy $\tilde{T}(t)$ that satisfies (3.7). If $\Gamma_1(B_0^1) < \Gamma_2(B_0^2)$, then B^1 should be run either until $\Gamma_1(B_t^1) = \Gamma_2(B_0^2)$

or until it reaches zero. Similarly, if $\Gamma_1(B_0^1) > \Gamma_2(B_0^2)$, then B^2 should be run either until $\Gamma_1(B_0^1) = \Gamma_2(B_t^2)$ or until it reaches zero. Hence, it suffices to assume that the process starts on the switching curve; i.e., $\Gamma_1(B_0^1) = \Gamma_2(B_0^2)$. Put

$$U_i(t) = \inf\{u \geq 0 \colon \Gamma_i(B_u^i) \geq \Gamma_i(B_0^i) + t\}, \qquad i = 1, 2.$$

Note that $U_i(t)$ is left continuous, strictly increasing and $U_i(0) = 0$. The continuity of the paths of Brownian motion implies that

$$\Gamma_i\left(B_{U_i(t)}^i\right) = \max_{0 \leq u \leq U_i(t)} \Gamma_i\left(B_u^i\right) = \Gamma_i\left(B_0^i\right) + t. \tag{3.8}$$

Let

$$\mathscr{S} = \left\{(s_1, s_2) \colon \max_{0 \leq u \leq s_1} \Gamma_1\left(B_u^1\right) \geq \max_{0 \leq u \leq s_2} \Gamma_2\left(B_u^2\right)\right\}, \tag{3.9}$$

$$\partial\mathscr{S} = \left\{(s_1, s_2) \colon \max_{0 \leq u \leq s_1} \Gamma_1\left(B_u^1\right) = \max_{0 \leq u \leq s_2} \Gamma_2\left(B_u^2\right)\right\}. \tag{3.10}$$

$\partial\mathscr{S}$ is merely a notation and does not refer to the topological boundary of S. It follows from (3.8) that $U(t) = (U_1(t), U_2(t)) \in \partial\mathscr{S}$ for all t. At this point we need a lemma which is taken from Proposition 5 in [11].

LEMMA 10. *U_1 and U_2 have no simultaneous jumps a.s. P_x for all $x \in D$.*

It follows from this lemma that almost surely, for each t, $\partial\mathscr{S} \cap \{(s_1, s_2) \colon s_1 + s_2 = t\}$ consists of exactly one point and hence that $\partial\mathscr{S}$ is the upper left boundary of $\mathscr{S}$. Call this point of intersection $\tilde{T}(t)$. On the set of measure zero where simultaneous jumps do occur, $\partial\mathscr{S}$ will contain rectangles corresponding to each simultaneous jump. In this case the intersection consists of either points or closed line segments. On the line segments, $\tilde{T}(t)$ can be chosen in any predetermined manner. By Theorem 2.7 in [19], it follows that $\tilde{T}(t)$ is a switching strategy. Put $\tau(t) = U_1(t) + U_2(t)$. The switching strategy $\tilde{T}(t)$ and the increasing process $U(t) = (U_1(t), U_2(t))$ are related by the time change $\tau(t)$:

$$\tilde{T}(\tau(t)) = U(t) \tag{3.11}$$

(see Figure 6).

Consider a point t of continuity for $\tau(t)$,

$$u := \tau(t) = \tau(t+).$$

In this case we see from (3.11) that $\tilde{T}(u) = U(t)$ and hence from (3.8) it follows that

$$\Gamma_1\left(B_{\tilde{T}_1(u)}^1\right) = \Gamma_2\left(B_{\tilde{T}_2(u)}^2\right).$$

Now consider a point t of discontinuity of $\tau(t)$: $\tau(t) < \tau(t+)$. By Lemma 10, it follows that either $U_1(t) = U_1(t+)$ or $U_2(t) = U_2(t+)$. Suppose without loss of generality that the first condition holds and consider a point $u \in [\tau(t), \tau(t+))$.

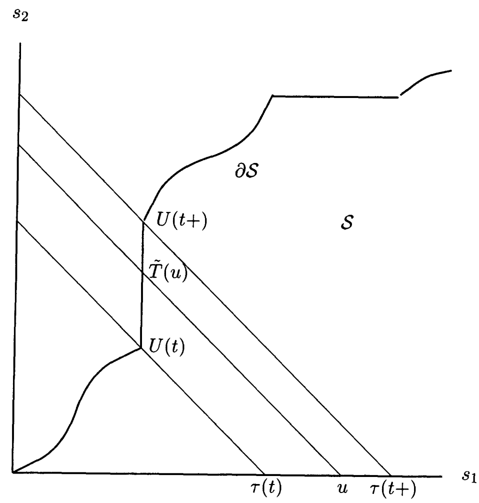

FIG. 6. *The optimal switching strategy.*

Then $\tilde{T}_1(u) = U_1(t)$ and hence (3.8) implies that

$$\Gamma_1\left(B^1_{\tilde{T}_1(u)}\right) = \max_{0 \le r \le \tilde{T}_1(u)} \Gamma_1\left(B^1_r\right). \tag{3.12}$$

By the continuity of $\Gamma_i(B^i_u)$ and the definition of $\tilde{T}(t)$, we get

$$\max_{0 \le r \le \tilde{T}_1(u)} \Gamma_1\left(B^1_r\right) = \max_{0 \le r \le \tilde{T}_2(u)} \Gamma_2\left(B^2_r\right). \tag{3.13}$$

Combining (3.12) and (3.13), we see that

$$\Gamma_2\left(B^2_{\tilde{T}_2(u)}\right) \le \Gamma_1\left(B^1_{\tilde{T}_1(u)}\right). \tag{3.14}$$

Also, note that u is a point of increase for $\tilde{T}_2$, but $\tilde{T}_1$ is constant on $[\tau(t), \tau(t+))$. [We say that u is a *point of increase* for a nondecreasing function f if $f(r) > f(u)$ for all $r > u$.]

From the analysis of the continuity and discontinuity points of $\tau(t)$, we see that

$$\tilde{T}_1(t) \text{ increases only when } \Gamma_1\left(B^1_{\tilde{T}_1(t)}\right) \le \Gamma_2\left(B^2_{\tilde{T}_2(t)}\right),$$

$$\tilde{T}_2(t) \text{ increases only when } \Gamma_2\left(B^2_{\tilde{T}_2(t)}\right) \le \Gamma_1\left(B^1_{\tilde{T}_1(t)}\right).$$

It now follows that (3.7) holds since

$$\frac{\partial^2 w}{\partial x_1^2}(x) = 0 \quad \text{on } \{x : \Gamma_1(x_1) \le \Gamma_2(x_2)\}, \tag{3.15}$$

$$\frac{\partial^2 w}{\partial x_2^2}(x) = 0 \quad \text{on } \{x : \Gamma_1(x_1) \ge \Gamma_2(x_2)\}. \tag{3.16}$$

□

REMARKS. (i) It is shown in [11] that Lemma 10 is a necessary and sufficient condition for the uniqueness (almost surely) of a switching strategy that satisfies (3.15), (3.16). Hence, our $\tilde{T}$ is almost surely unique.

(ii) From (2.3) and (3.3), we see that the value function v is biconcave and has the property that at every point it is linear in at least one of the two coordinate directions. Hence, v is a solution to the nonlinear Dirichlet problem

$$\max\left(\frac{\partial^2 v}{\partial x_1^2}, \frac{\partial^2 v}{\partial x_2^2}\right) = 0, \quad \text{in } D, \tag{3.17}$$

$$v = f, \quad \text{on } \partial D. \tag{3.18}$$

This differential equation is interesting in its own right. It is an open problem whether this Dirichlet problem has a unique solution in any appropriate space.

It is interesting that this nonlinear Dirichlet problem also arises in the derivation of sharp inequalities for martingale transforms (see [2]).

3.2. *Three concave faces.* Let M be a subset of D^0 consisting of a *finite* union of open horizontal and vertical line segments having the property that every endpoint of a horizontal (vertical) line segment lies either on ∂D or in the *interior* of some vertical (horizontal) segment. We call any such set a *maze* in D. Given a maze M, we say that a function defined on D is *linear* on M if it is continuous on M, linear in x_1 on all horizontal segments of M and linear in x_2 on all vertical segments of M.

LEMMA 11. *If w is linear on a maze M, then there exists a switching strategy $T(t)$ such that $w(B_{T(t)})$ is a martingale up until the first exit time from D^0 under every measure P_x, $x \in M$.*

PROOF. The idea is quite simple: Run the horizontal Brownian motion whenever the process is on a horizontal line segment and run the vertical Brownian motion whenever the process is on a vertical line segment. Clearly, if the process starts on the maze, it must stay on it up until the first exit time from D^0. Note that the result depends on excluding the possibility of a horizontal and a vertical line segment meeting to form a corner. □

PROOF OF THEOREM 4. Let w denote the function that is the horizontal linear interpolation of f on R_N, is the partial sweep of (F_S, F_W) determined by (2.14) on R_{SW} and is the partial sweep of (F_S, F_E) determined by (2.14) on R_{SE}. Clearly, w is well-defined (that is, there is no contradiction at the intersection of these three regions). Also, w is biconcave in each region separately, is continuous throughout D and agrees with f on ∂D. We must show that w is biconcave even across the boundaries between the three regions.

Recall that $\bar{\xi}$ is the northeast corner of R_{SW} and also the northwest corner of R_{SE}. Let p denote the southeast corner of R_{SW} (and hence it is also the southwest corner of R_{SE}). Derivatives in x_1 evaluated along the intersection between R_{SW} and R_{SE} using the formulas for w as given either in R_{SW} or R_{SE} are simply linear interpolations of the corresponding derivatives at $\bar{\xi}$ and p. Since the derivatives in x_1 up to order 2 are continuous both at $\bar{\xi}$ and at p, it follows that they exist and are continuous all along the intersection of R_{SW} and R_{SE}.

A similar interpolation argument reduces checking concavity in x_2 across the intersection of R_N and either R_{SW} or R_{SE} to simply checking it at the point $\bar{\xi}$. Let q denote the northwest corner of R_{SW} and let r denote the northeast corner of R_{SE}. Using the formula for w as given in R_N, we see that

$$\frac{\partial^+ w}{\partial x_2}(\bar{\xi}) = \left(1 - \frac{\bar{\xi}_1}{a_1}\right)\frac{\partial w}{\partial x_2}(q) + \frac{\bar{\xi}_1}{a_1}\frac{\partial w}{\partial x_2}(r). \tag{3.19}$$

Using the fact that $\Gamma_1(\bar{\xi}_1) = \Gamma_2(\bar{\xi}_2)$, together with the definition (2.1) of Γ_i and the continuity condition $\gamma_1(0) = \gamma_2(0)$, we see that

$$\frac{\partial w}{\partial x_2}(q) = \frac{w(q) - w(p) + \bar{\xi}_1(\partial w/\partial x_1)(p)}{\bar{\xi}_2}.$$

Similarly,

$$\frac{\partial w}{\partial x_2}(r) = \frac{w(r) - w(p) - (a_1 - \bar{\xi}_1)(\partial w/\partial x_1)(p)}{\bar{\xi}_2}.$$

Substituting these last two expressions into (3.19), we get

$$\begin{aligned}\frac{\partial^+ w}{\partial x_2}(\bar{\xi}) &= \frac{(1 - \bar{\xi}_1/a_1)w(q) + (\bar{\xi}_1/a_1)w(r) - w(p)}{\bar{\xi}_2}\\ &= \frac{w(\bar{\xi}) - w(p)}{\bar{\xi}_2}\\ &= \frac{\partial^- w}{\partial x_2}(\bar{\xi}).\end{aligned}$$

This is sufficient to establish the biconcavity of w throughout D.

If the process starts in R_N^0, then $T(t) = (t, 0)$ makes $w(B_{T(t)})$ into a martingale. If it starts in R_{SW}^0 (or R_{SE}^0), then the construction in Theorem 3 yields a switching strategy that ensures $w(B_{T(t)})$ is a martingale up to the first exit time from R_{SW}^0 (R_{SE}^0, respectively). At this time, the process is located on the maze $M = D^0 \setminus (R_N^0 \cup R_{SW}^0 \cup R_{SE}^0)$. Hence, we can use the optional sampling theorem and Lemma 11 to extend the strategy to one that makes $w(B_{T(t)})$ into a martingale all the way up to the first exit time from D^0. (This is a special case of the operation called *continuing* in [18].) In light of Lemma 9, we now recognize that w is the value function and the switching strategy we have described is optimal. □

REMARK. It is interesting to note that $\partial^2 v/\partial x_2^2$ has a jump discontinuity across the south face of R_N. For example, at $\bar{\xi}$, $\partial^2 v/\partial x_2^2$ evaluated from above may be strictly negative whereas from below it is zero. It follows that $C^2(D^0) \cap C(D)$ is not an appropriate space in which to attempt to prove existence and uniqueness theorems for (3.17)–(3.18).

3.3. *Four concave faces.* Now we embark on the proofs of the theorems pertaining to the case where all four sides have strongly concave data. We need to write down explicitly the parametric equations defining each of the switching curves C_i. Let

$$\begin{aligned} \gamma_0(x_2) &= f(0, a_2 - x_2), \\ \gamma_1(x_1) &= f(x_1, 0), \\ \gamma_2(x_2) &= f(a_1, x_2), \\ \gamma_3(x_1) &= f(a_1 - x_1, a_2) \end{aligned}$$

and

$$\begin{aligned} \lambda_0(x_2) &= f(0, x_2), \\ \lambda_1(x_1) &= f(a_1 - x_1, 0), \\ \lambda_2(x_2) &= f(a_1, a_2 - x_2), \\ \lambda_3(x_1) &= f(x_1, a_2). \end{aligned}$$

Let $\tilde{a}_i$ be equal to a_1 when i is odd and a_2 when i is even. Then the γ_i's and the λ_i's are related in a simple way:

$$\lambda_i(z) = \gamma_i(\tilde{a}_i - z).$$

Since f is continuous, we see that $\lambda_0(0) = \gamma_1(0)$, $\gamma_1(a_1) = \gamma_2(0)$, $\gamma_2(a_2) = \lambda_3(a_1)$ and $\lambda_3(0) = \lambda_0(a_2)$. Put

$$\Gamma_i(z) = -\int_0^z u\gamma_i''(u)\,du,$$

$$\Lambda_i(z) = -\int_0^z u\lambda_i''(u)\,du.$$

The switching curves C_i, $i = 0, 1, 2, 3$, are given by the equations

$$\begin{aligned} \Gamma_0(a_2 - x_2) &= \Lambda_3(x_1), \\ \Gamma_1(x_1) &= \Lambda_0(x_2), \\ \Gamma_2(x_2) &= \Lambda_1(a_1 - x_1), \\ \Gamma_3(a_1 - x_1) &= \Lambda_2(a_2 - x_2). \end{aligned}$$

Given a point $x = (x_1, x_2)$ on $C_0(\xi^3, \xi^0)$, let $\rho(x)$ denote its u parameter,

$$\rho(x) = \Gamma_0(a_2 - x_2) = \Lambda_3(x_1).$$

We are now ready for the proofs.

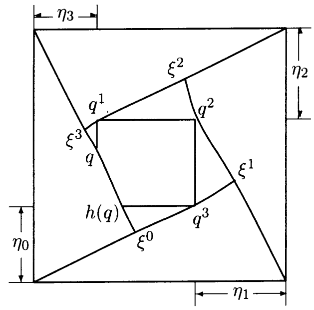

FIG. 7. *Relative positions.*

PROOF OF THEOREM 5. The proof is by contradiction. Let us start by assuming that ξ^0 is to the left of ξ^2 and that ξ^1 is below ξ^3. Without loss of generality, we may assume that ξ^0 is below ξ^2. It is easy to see that this implies the relative positions of each of the ξ^i's are as shown in Figure 7. Starting from any point q on $C_0(\xi^3, \xi^0)$, find the point q^1 on $C_3(\xi^2, \xi^3)$ that is directly above q. Next, find the point q^2 on $C_2(\xi^1, \xi^2)$ that is directly to the right of q^1. Then find the point q^3 on $C_1(\xi^0, \xi^1)$ that is below q^2 and, finally, find the point on $C_0(\xi^3, \xi^0)$ directly to the left of q^3. Denote this last point by $h(q)$. The function h is a continuous map of $C_0(\xi^3, \xi^0)$ into itself. Let $\tilde{h}$ denote the function on u space induced by h,

$$\tilde{h}(u) = \rho(h(q(u))),$$

where $q(u)$ is the parametric representation of $C_0(\xi^3, \xi^0)$. To arrive at a contradiction, we show that $\tilde{h}'(u) > 1$ for all u. This is impossible for a continuously differentiable function that maps a closed interval into itself.

Writing $\tilde{h}$ in terms of Γ_i's and Λ_i's, we get

$$\tilde{h}(u) = \Gamma_0\big(a_2 - \Lambda_0^{-1} \circ \Gamma_1\big(a_1 - \Lambda_1^{-1} \circ \Gamma_2\big(a_2 - \Lambda_2^{-1} \circ \Gamma_3\big(a_1 - \Lambda_3^{-1}(u)\big)\big)\big)\big).$$

Put

$$g_i(u) = \Gamma_i\big(\tilde{a}_i - \Lambda_i^{-1}(u)\big)$$

and then let $u_0 = u$ and $u_{i+1} = g_{3-i}(u_i)$, $i = 0, 1, 2$. For each i, the number u_i is the u parameter of the point q^i on the curve C_{3-i}. Differentiating $\tilde{h}$ yields

$$\tilde{h}'(u) = g_0'(u_3) g_1'(u_2) g_2'(u_1) g_3'(u_0).$$

Temporarily dropping the subscript i, we see that

$$g'(u) = -\frac{\Gamma'(\tilde{a} - \Lambda^{-1}(u))}{\Lambda'(\Lambda^{-1}(u))}.$$

Now $\Gamma'(\tilde{a} - z) = -(\tilde{a} - z)\gamma''(\tilde{a} - z)$ and $\Lambda'(z) = -z\lambda''(z)$. Therefore,

$$g'(u) = -\frac{\tilde{a} - \Lambda^{-1}(u)}{\Lambda^{-1}(u)}.$$

Put $\eta_i = \Lambda_i^{-1}(u_{3-i})$, $i = 0, 1, 2, 3$. The geometric interpretation of η_i is shown in Figure 7. The important observation is that

$$\eta_0 + \eta_2 < a_2,$$
$$\eta_1 + \eta_3 < a_1.$$

Consequently,

$$\begin{aligned}\tilde{h}'(u) &= \frac{a_2 - \eta_0}{\eta_0}\frac{a_1 - \eta_1}{\eta_1}\frac{a_2 - \eta_2}{\eta_2}\frac{a_1 - \eta_3}{\eta_3}\\ &= \frac{a_2 - \eta_2}{\eta_0}\frac{a_1 - \eta_3}{\eta_1}\frac{a_2 - \eta_0}{\eta_2}\frac{a_1 - \eta_1}{\eta_3} > 1.\end{aligned}$$

This contradiction completes the proof of Theorem 5. □

PROOF OF THEOREM 6. This proof is similar to the previous one (see Figure 8). Starting from any point q on $C_0(\xi^3, \xi^0)$, find the point q^1 on $C_1(\xi^0, \xi^1)$ that is directly to the right of q. Next, find the point q^2 on $C_2(\xi^1, \xi^2)$ that is directly above q^1. Then find the point q^3 on $C_3(\xi^2, \xi^3)$ that is to the left of q^2 and, finally, find the point on $C_0(\xi^3, \xi^0)$ directly below q^3. Denote this last point by $h(q)$. The function h maps $C_0(\xi^3, \xi^0)$ into itself. Let $\tilde{h}$ denote the function on u space induced by h,

$$\tilde{h}(u) = \rho(h(q(u))),$$

where, as in the previous proof, $q(u)$ is the parametric representation of $C_0(\xi^3, \xi^0)$. The domain of $\tilde{h}$ is an interval. We will show that $\tilde{h}$ is a continuously differentiable function of this interval into itself and that $\tilde{h}'(u) < 1$ for all u. Hence, $\tilde{h}$ is a strict contraction and so by Banach's theorem there is a unique fixed point u^0. Let $q^0 = q(u^0)$ and let q^i, $i = 1, 2, 3$, be the intermediate points in the definition of $h(q)$ obtained starting from q^0. These points

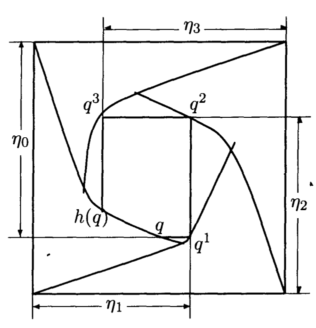

FIG. 8. *Geometric interpretation of* η_i*'s.*

then form the vertices of the unique inscribed rectangle. We call this rectangle R_C.

We must show that the map $\tilde{h}$ is a strict contraction. Writing $\tilde{h}$ in terms of Γ_i's and Λ_i's, we get

$$\tilde{h}(u) = \Lambda_3\big(a_1 - \Gamma_3^{-1} \circ \Lambda_2\big(a_2 - \Gamma_2^{-1} \circ \Lambda_1\big(a_1 - \Gamma_1^{-1} \circ \Lambda_0\big(a_2 - \Gamma_0^{-1}(u)\big)\big)\big)\big).$$

Put

$$g_i(u) = \Lambda_i\big(\tilde{a}_i - \Gamma_i^{-1}(u)\big)$$

and then let $u_0 = u$ and $u_{i+1} = g_i(u_i)$, $i = 0, 1, 2$. For each i, the number u_i is the u parameter of the point q^i on the curve C_i. Differentiating $\tilde{h}$, we get

$$\tilde{h}'(u) = g_3'(u_3) g_2'(u_2) g_1'(u_1) g_0'(u_0).$$

Temporarily dropping the subscript i, we see that

$$g'(u) = -\frac{\Lambda'(\tilde{a} - \Gamma^{-1}(u))}{\Gamma'(\Gamma^{-1}(u))}.$$

Now $\Gamma'(z) = -z\gamma''(z)$ and $\Lambda'(\tilde{a} - z) = -(\tilde{a} - z)\lambda''(\tilde{a} - z)$. Therefore,

$$g'(u) = -\frac{\tilde{a} - \Gamma^{-1}(u)}{\Gamma^{-1}(u)}.$$

Put $\eta_i = \Gamma_i^{-1}(u_i)$, $i = 0, 1, 2, 3$. The geometric interpretation of η_i is shown in Figure 8. The important observation is that

$$\eta_0 + \eta_2 > a_2,$$
$$\eta_1 + \eta_3 > a_1.$$

Consequently,

$$\begin{aligned} \tilde{h}'(u) &= \frac{a_1 - \eta_3}{\eta_3} \frac{a_2 - \eta_2}{\eta_2} \frac{a_1 - \eta_1}{\eta_1} \frac{a_2 - \eta_0}{\eta_0} \\ &= \frac{a_1 - \eta_1}{\eta_3} \frac{a_2 - \eta_0}{\eta_2} \frac{a_1 - \eta_3}{\eta_1} \frac{a_2 - \eta_2}{\eta_0} < 1. \end{aligned}$$

This completes the uniqueness proof. □

Proof of Theorem 7. Let w be defined as follows. On M, w is the unique continuous function that agrees with the boundary data and is linear along each of the line segments comprising M. In R_C, w is the multilinear interpolation of the values on $M \cap R_C$. On each R_i, w is the partial sweep of the boundary data determined by the requirement that w is continuous at $R_i \cap M$. Clearly, w is well defined, continuous throughout D, biconcave in each of the five rectangles and agrees with f on ∂D. We must show that w is, in fact, biconcave even across the boundaries between R_C and R_i, $i = 0, 1, 2, 3$. Let q^i, $i = 0, 1, 2, 3$, denote the boundary points shown in Figure 9 and let s_0 and r_0 be the distances shown. Let s_i and r_i, $i = 1, 2, 3$, be the analogous distances

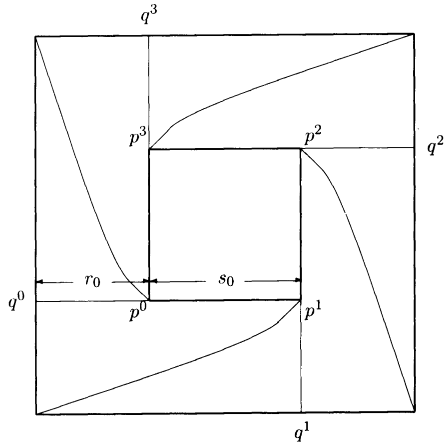

FIG. 9. *Some notation.*

and let $\eta_i = s_i + r_i$. Put

$$w_i = w(p^i),$$
$$\gamma_i = w(q^i) = \gamma_i(\eta_i),$$
$$\sigma_i = \frac{s_i}{s_i + r_i},$$
$$\rho_i = \frac{r_i}{s_i + r_i}.$$

Since w is linear along each segment of M, we see that

$$w_i = \sigma_i \gamma_i + \rho_i w_{i+1}, \qquad i = 0, 1, 2, 3. \tag{3.20}$$

Since the σ_i's, ρ_i's and γ_i's are known, (3.20) represents four equations in the four unknowns w_i, $i = 0, 1, 2, 3$. This system of equations yields the unique solution

$$w_i = \frac{\sigma_i \gamma_i + \rho_i \sigma_{i+1} \gamma_{i+1} + \rho_i \rho_{i+1} \sigma_{i+2} \gamma_{i+2} + \rho_i \rho_{i+1} \rho_{i+2} \sigma_{i+3} \gamma_{i+3}}{1 - \rho_0 \rho_1 \rho_2 \rho_3}.$$

(Remember that subscript addition is done mod 4.) To show that first derivatives are continuous across M, it suffices to check any one of the line segments comprising M and, by the usual interpolation argument, it suffices to check only the places where two such line segments meet. Hence, we only need to show that the derivative in x_2 is continuous at p^0. Equating the derivative as calculated from above and as calculated from below shows that we need to verify that

$$\frac{w_0 - w_3}{s_1} = \sigma_0 \gamma_0' + \rho_0 \frac{w_1 - w_2}{s_1}, \tag{3.21}$$

where $\gamma_i' = \gamma_i'(\eta_i)$. From the definitions of the switching curves, it follows that

$$\gamma_i' = \frac{\gamma_i - \gamma_{i-1} - r_i\gamma_{i-1}'}{r_{i-1} + s_{i-1}}. \tag{3.22}$$

Using (3.20) to express the γ_i's in terms of the w_i's, then substituting these expressions into (3.22) and finally solving for the γ_i''s in terms of the w_i's (using the fact that $s_0 = s_2$ and $s_1 = s_3$), we see that

$$\sigma_0\gamma_0' = \frac{w_0 - w_3}{s_1} + \rho_0\frac{w_1 - w_2}{s_1},$$

which is clearly the same as (3.21). Hence, we have now shown that $w \in C^1(D^0)$. This is sufficient to establish the biconcavity of w throughout D.

If the process starts in R_C, then any strategy $T(t)$ makes $w(B_{T(t)})$ into a martingale up to the first exit time from R_C. If it starts in one of the other rectangles, say R_i, then the construction in Theorem 3 yields a switching strategy $T(t)$ that makes $w(B_{T(t)})$ into a martingale up to the first exit time from R_i. At these first exit times, the process is located on the maze M. Hence, we can use the optional sampling theorem and the construction in Lemma 11 to extend the strategy to one that makes $w(B_{T(t)})$ into a martingale all the way up to the first exit time from D. In light of Lemma 9, we now recognize that w is the value function and the switching strategy we have described is optimal. □

REMARK. The value function v is not $C^2(D^0)$. Second derivatives have jump discontinuities as they cross M. To see this, note that in R_C, all second derivatives vanish but just on the other side of ∂R_C, they can jump to negative values.

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