The Annals of Probability
1990, Vol. 18, No. 4, 1746–1758

A NOTE ON HYPERCONTRACTIVITY OF STABLE RANDOM VARIABLES[1]

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It is shown that every symmetric α-stable random variable X, $0 < \alpha \leq 2$, has the property:

For any p and q, $0 \leq h(\alpha) < q < p < \alpha$, there is a constant $s > 0$ such that

$$(E\|x + sXy\|^p)^{1/p} \leq (E\|x + Xy\|^q)^{1/q},$$

for all x and y from a normed space. The quantity $h(\alpha)$ is identically 0 if $\alpha \leq 1$. It is strictly less than 1 for every $\alpha < 2$ which reveals the contrast to the Gaussian case in which $q > h(2) = 1$.

1. Introduction. The classical "two-point inequality" [Bonami (1970) and Gross (1975)]

$$\left(\frac{|x + sy|^p + |x - sy|^p}{2}\right)^{1/p} \leq \left(\frac{|x + y|^q + |x - y|^q}{2}\right)^{1/q}, \qquad x, y \in \mathbb{R}, \tag{1.1}$$

where $1 < q < p < \infty$, $s = \sqrt{(q-1)/(p-1)}$, and its extensions play a fundamental role in the theory of hypercontractive operators [Bonami (1970), Nelson (1966) and Weissler (1980)], convolution inequalities [Ritter (1984)], logarithmic Sobolev inequalities [Gross (1975)], stochastic Ising models [Holley and Stroock (1987)] and related subjects in harmonic analysis, statistical mechanics and quantum physics, to name just a few areas of great importance nowadays.

Some more sophisticated counterparts of (1.1) were recently applied in the intensively developing theory of multiple stochastic integrals, random multilinear forms and stochastic chaoses, topics originating with Wiener (1938) in the late 1930s. Using some properties of hypercontractive operators, Borell (1984) showed that all pth norms, $p \geq 2$, of Hilbert space-valued polynomial chaoses in independent random variables are comparable, generalizing a fortiori classical results of Marcinkiewicz, Paley and Zygmund (1932, 1937). The latter is closely related to comparison of moments of sums of independent random variables studied in Banach spaces by Hoffmann-Jørgensen (1972/73). Hypercontraction allows us to extend his results into multilinear random forms with

Received September 1988; revised October 1989.

[1]This research was initiated at the Center for Stochastic Processes, Department of Statistics, University of North Carolina at Chapel Hill with support from AFOSR Contract F49620-85-C-0144, and completed with support from NSF Grant DMS-87-13103.

AMS 1980 *subject classifications*. Primary 60E07, 60B11; secondary 60E15, 43A15, 42C05.

Key words and phrases. Hypercontraction, stable random variables, domain of normal attraction, normed space.

vector coefficients. Moreover, even for sums of independent random variables, a precise estimate of the best constant in Rosenthal-type inequalities can be obtained in a short way [cf. Talagrand (1989) and Kwapień and Szulga (1988)]. Furthermore, this concept became a basic part of a construction of a stable multiple stochastic integral due to Krakowiak and Szulga (1988b). A notion of hypercontractive random variables was introduced for this purpose, having also an intrinsic interest.

Following what is the key feature of (1.1), a random variable Y is called *hypercontractive with indices* $p, q > 0$ *in a normed space* $\mathbb{B}$ if there exists a constant s such that

$$(1.2) \qquad \left(E\|x + sYy\|^p\right)^{1/p} \le \left(E\|x + Yy\|^q\right)^{1/q}, \qquad x, y \in \mathbb{B};$$

$Y \in \mathrm{HC}(p, q, \mathbb{B}; s)$, in short. We shall also write $Y \in \mathrm{HC}(p, q, \mathbb{B})$ if $Y \in \mathrm{HC}(p, q, \mathbb{B}; s)$ for some $s > 0$, and $\mathrm{HC}(p, q) = \mathrm{HC}(p, q, \mathbb{R})$.

For example, Borell's extension (1984) of (1.1) says that for a *Rademacher random variable* ε taking values ± 1 with probability $\frac{1}{2}$, $\varepsilon \in \mathrm{HC}(p, q, \mathbb{B}; s_{p,q})$ in any normed space $\mathbb{B}$ with $1 < q < \infty$. By a central limit theorem argument, a symmetric standard Gaussian random variable $Z \in \mathrm{HC}(p, q, \mathbb{B}; s_{p,q})$ with $1 < q \le p < \infty$ [cf. also Nelson (1966)].

The number $\sqrt{(q-1)/(p-1)}$ is the best possible constant for Z and ε. Moreover, there must be $q > 1$. This is obvious in the case of a Rademacher random variable because, for $t \in (0, 1)$,

$$E|1 + t\varepsilon|^q \begin{cases} \le 1 & \text{if } q \le 1, \\ > 1 & \text{if } q > 1. \end{cases}$$

In the case of a Gaussian random variable Z we have, for $0 < q \le 1 < p < \infty$ and $t \in (0, 1)$,

$$\begin{aligned} E|1 + tZ|^q &< 1 + \binom{q}{2} t^2 E Z^2 \mathbb{1}\{|Z| \le 1\} + \mathrm{const} \cdot e^{-1/t} \\ &< 1 - \mathrm{const} \cdot t^2, \\ E|1 + tZ|^p &> 1 + \binom{p}{2} E Z^2 \mathbb{1}\{|Z| \le 1\} t^2. \end{aligned}$$

Here, and in the sequel, "const" denotes a positive real number.

An α-stable symmetric random variable is hypercontractive in an arbitrary normed space if $1 < q < p < \alpha \le 2$ [Krakowiak and Szulga (1988a)]. Although this is an attribute of a more general class of probability distributions [Krakowiak and Szulga (1988a)] integrability and limitation to parameters $q > 1$ sounded like the prerequisite conditions. All symmetric random variables with finite variance mimic the Gaussian case, as it is seen from the central limit theorem (cf. Corollary 2.4 below).

In contrast to this, symmetric α-stable random variables show much more flexibility. We show in this paper that they are hypercontractive for all α, p, q such that $h(\alpha) < q < p < \alpha < 2$, in any normed space. The quantity $h(\alpha) = 0$

when $\alpha \le 1$ and takes values in the interval $(0, 1)$ when $1 < \alpha < 2$. Moreover, it increases with α and goes to 1 when α approaches 2.

Related processes, like stable multilinear forms and multiple stable integrals, exhibit similar properties.

2. Preliminaries.

For a $p > 0$ and a Banach space $\mathbb{B}$ we denote by $Lp(\mathbb{B})$ the Banach (Fréchet, if $p < 1$) space of $\mathbb{B}$-valued random variables Y such that $\|Y\|_p = (E\|Y\|^p)^{1/p} < \infty$. A function F_k: $\mathbb{N}^k \to \mathbb{B}$, $k \ge 1$, is called *tetrahedral* if F_k vanishes outside the "tetrahedron" $D_k = \{\mathbf{i}_k = (i_1, \dots, i_k) \in \mathbb{N}^k: 1 \le i_1 < \cdots < i_k\}$. We define a homogeneous polynomial of degree k on $\mathbb{R}^{\mathbb{N}}$ by

$$\langle F_k; \mathbf{t}^k \rangle = \sum_{\mathbf{i} \in D_k} F_k(\mathbf{i}_k) \cdot t_{i_1} \cdots t_{i_k}, \qquad \mathbf{t} = (t_i) \in \mathbb{R}^{\mathbb{N}}$$

(by convention, $F_0 \in \mathbb{B}$). We need a natural extension of the notion of hypercontractivity. Let $\mathscr{Y}$ be a nonvoid set. A family $\mathbf{Y} = \{Y_y: y \in \mathscr{Y}\}$ of random variables taking values in a normed space $\mathbb{B}$ is said to be *hypercontractive with parameters* p, q if there is a constant s such that

$$\|x + sY_y\|_p \le \|x + Y_y\|_q, \qquad x \in \mathbb{B}, y \in \mathscr{Y}.$$

We shall also write $\mathbf{Y} \in \mathrm{HC}(p, q, \mathbb{B}; s)$. Note that hypercontractivity of a single random variable Y, and of the stochastic process $\{Y \cdot y: y \in \mathbb{B}\}$, have the same meaning.

The class $\mathrm{HC}(p, q, \mathbb{B}; s)$ is closed under certain algebraic operations and under taking weak limits of its distributions. The latter statement follows immediately from the definition of weak convergence of probability measures [see also the proof of Proposition 2.2 in Krakowiak and Szulga (1988a) for more details]. The precise meaning of the first part of the above observation is contained in the following result [Krakowiak and Szulga (1988a)].

THEOREM 2.1. *Let $0 < q \le p < \infty$, $\mathbf{Y} = (Y_j)$ be a sequence of independent p-integrable random variables, and $F_0, \dots, F_n$ be finitely supported tetrahedral functions valued in $\mathbb{B}$. If $Y_j \in \mathrm{HC}(p, q, \mathbb{B}; s_j)$ for each $j = 1, 2, \dots,$ then*

$$\left\| \sum_{k=0}^{n} \langle F_k; (\mathbf{s}\mathbf{Y})^k \rangle \right\|_p \le \left\| \sum_{k=0}^{n} \langle F_k; \mathbf{Y}^k \rangle \right\|_q, \tag{2.1}$$

where $\mathbf{sY} = (s_j \cdot Y_j)$.

Therefore, the hypercontractivity (by means of families of random variables) is a hereditary feature under taking polynomials of several variables (not necessarily homogeneous). In particular, using just linear forms, we obtain immediately the following statements.

COROLLARY 2.2. *If $\mathbf{Y} = (Y_j)$ is a sequence of i.i.d. random variables and each $Y_j \in \mathrm{HC}(p, q, \mathbb{B}; s)$ then the space of all $\mathbb{B}$-rank linear combinations of*

Y_j's consists of hypercontractive random variables, i.e.,

$$\left\{\sum x_j Y_j : (x_j) \in \mathbb{B}^{\mathbb{N}}\right\} \subset \mathrm{HC}(p, q, \mathbb{B}; s).$$

COROLLARY 2.3. *If $Y \in \mathrm{HC}(p, q, \mathbb{B}; s)$ and Y belongs to the domain of normal attraction of a random variable X then $X \in \mathrm{HC}(p, q, \mathbb{B}; s)$.*

Therefore, by the central limit theorem, we obtain the following corollary.

COROLLARY 2.4. *If $Y \in \mathrm{HC}(p, q, \mathbb{B})$ and $EY^2 < \infty$ then $q \geq 1$.*

Notice that the hypercontraction constant $s \in (-1, 1)$ provided $q < p$. Moreover, hypercontractivity is determined by a local behavior of pth and qth moments of the process $(x + tYy : t \in \mathbb{R})$ in the vicinity of 0. A precise meaning of this assertion is given in the proposition following an auxiliary lemma.

LEMMA 2.5. *Let Y be a symmetric random variable, $q > 0$, $\mathbb{B}$ be a normed space and $d > 0$. Assume that Y is unbounded if $q \geq 1$, and that $E|1 + tY|^q > 1$ for all $|t| \geq d$ if $q < 1$. Then the quantity*

$$c_q(Y) = \inf\left\{\frac{\|x + tyY\|_q - 1}{t} : t \geq d, \|y\| = \|x\| = 1\right\}$$

is strictly positive.

PROOF. When $q \geq 1$ we proceed as follows. Fix $t > 0$, and $x, y \in \mathbb{B}$ such that $\|x\| = \|y\| = 1$. Then, by convexity,

$$\begin{aligned} E\|x + tyY\|^q &\geq 1 + E\left(\frac{\|x + tyY\|^q + \|x - tyY\|^q}{2} - 1\right)\mathbb{1}\{|Y| > a\} \\ &\geq 1 + \left((ta - 1)^q - 1\right)P\{|Y| > a\} \end{aligned}$$

for every $a > 0$. Choosing $a > 2/d$ we make the function

$$g(t) := \left(1 + \left((ta - 1)^q - 1\right)P\{|Y| > a\}\right)^{1/q}$$

convex on $[d, \infty)$. Hence

$$g(t) \geq 1 + mt, \qquad t > d,$$

where $m > 0$ is such that $g(d) \geq 1 + md$ and $g'(d) \geq m$. Therefore $c_q(Y) \geq m$.

The lack of convexity in case of $q < 1$ requires a different argument. It is enough to show that for $q < 1$ and $t \in \mathbb{R}$,

$$\rho_q(t) := \inf\{E\|x + tyY\|^q : \|x\| = \|y\| = 1\} = E|1 + tY|^q. \tag{2.2}$$

The inequality " $\le$ " is obvious. Let $x, y \in \mathbb{B}$ with $\|x\| = \|y\| = 1$ and x^*, y^* be a couple of linear functionals on $\mathbb{B}$ with norm 1 such that $\langle x^*, x\rangle = \langle y^*, y\rangle = 1$. Put $a = \langle x^*, y\rangle$ and $b = \langle y^*, x\rangle$. Then

$$\rho_q(t) \ge \inf\{E \max(|1 + atY|^q, |b + tY|^q) : 0 \le a \le 1, |b| \le 1\}.$$

Since for a Rademacher random variable ε independent of Y,

$$E\|x + tyY\|^q = E_Y E_\varepsilon \|x + t\varepsilon|Y|y\|^q,$$

it is enough to prove that for every $t \ge 0$,

$$\inf\{E \max(|1 + at\varepsilon|^q, |b + t\varepsilon|^q) : 0 \le a \le 1, |b| \le 1\} \ge E|1 + \varepsilon t|^q.$$

Observe that for $t \le 1$ and $0 \le a \le 1$,

$$E|1 + at\varepsilon|^q \ge E|1 + t\varepsilon|^q \ge \inf_{|b|\le 1} E|b + t\varepsilon|^q.$$

Therefore, for $t \le 1$,

$$\begin{aligned}\inf_a \inf_b E \max(|1 + at\varepsilon|^q, |b + t\varepsilon|^q) &\ge \inf_a \inf_b \max(E|1 + at\varepsilon|^q, E|b + t\varepsilon|^q)\\ &\ge \max\Big(\inf_a E|1 + at\varepsilon|^q, \inf_b E|b + t\varepsilon|^q\Big)\\ &\ge \max\Big(E|1 + t\varepsilon|^q, \inf_b E|b + t\epsilon|^q\Big)\\ &= E|1 + t\varepsilon|^q.\end{aligned}$$

Finally, for $t > 1$, we have

$$\rho(t) \ge \max\Big(\inf_a E|1 + at\varepsilon|^q, \inf_b E|b + t\varepsilon|^q\Big) = 2^{q-1},$$

which shows (2.2). This completes the proof. □

PROPOSITION 2.6. *Let Y be a symmetric nondegenerated random variable, $0 < q < p < \infty$, and $\mathbb{B}$ be a normed space. Then $Y \in \mathrm{HC}(p, q, \mathbb{B})$ iff*

$$E|1 + tY|^q > 1 \tag{2.3}$$

for all $t \ne 0$, and there are $d > 0$ and $s > 0$ such that

$$\left(E\|x + sYy\|^p\right)^{1/p} \le \left(E\|x + Yy\|^q\right)^{1/q} \tag{2.4}$$

for all $x, y \in \mathbb{B}$ such that $\|x\| = 1$, $\|y\| \le d$.

PROOF. For necessity, it is enough to show that the hypercontractivity yields (2.3). Suppose that $Y \in \mathrm{HC}(p, q, \mathbb{B}, s)$ for some $s > 0$ and $q < p$. If the inequality (2.3) fails then for some $t_0 > 0$,

$$1 \ge \|1 + t_0 Y\|_q \ge \|1 + st_0 Y\|_p \ge \|1 + st_0 Y\|_q \ge \cdots \ge \|1 + s^n t_0 Y\|_p \searrow 1,$$

by the hypercontractivity of Y. Hence qth and pth norms of a random variable $|1 + st_0 Y|$ are equal, so it must be degenerated. Since Y is symmetric, $Y = 0$ a.s., a contradiction.

Now, suppose that (2.3) holds and (2.4) is fulfilled with some $s \in (0,1)$ and $d > 0$. Since $c_q(Y) > 0$ by Lemma 2.5 then, for every $x, y \in \mathbb{B}$, $\|x\| = \|y\| = 1$,

$$\|x + tyY\|_q \geq 1 + c_q(Y)t, \qquad t \geq d.$$

Let $r = \min(1, p)$. We solve the second of the following inequalities with respect to s, uniformly for $t \geq d$,

$$\frac{\|x + syY\|_p}{\|x + yY\|_q} \leq \frac{\left(1 + s^r t^r \|Y\|_p^r\right)^{1/r}}{1 + c_q(Y)t} \leq 1.$$

Define a positive number s_1 by

$$s_1 = \inf_{t \geq d} \frac{\left(\left(1 + c_q(Y)t\right)^r - 1\right)^{1/r}}{t\|Y\|_p}$$

$$= \begin{cases} \dfrac{c_q(Y)}{\|Y\|_p} & \text{if } p \geq 1, \\[2ex] \dfrac{\left(\left(1 + c_q(Y)d\right)^r - 1\right)^{1/r}}{d\|Y\|_p} & \text{if } p \leq 1. \end{cases}$$

We observe that if s is a hypercontractivity constant, or it is a number appearing in (2.4), then any power s^n, $n \in \mathbb{N}$, has the same property. Recall also that $|s| < 1$. Then, for a suitable integer n, we can find a hypercontractivity constant $s^n \leq s_1$. This completes the proof. □

If $q \geq 1$ then the quantity $\rho_q(t)$ depends essentially on the geometry of the normed space $\mathbb{B}$ [cf., e.g., Lindenstrauss and Tzafriri (1979)].

3. Hypercontractivity of stable random variables. For a random variable X define

$$H(X) = \inf \bigcup \{[q, p]: 0 < q < p < \infty, X \in \mathrm{HC}(p, q)\}. \tag{3.1}$$

The aim of this section is to evaluate the function

$$(0, 2) \ni \alpha \mapsto H(X(\alpha)),$$

where $X(\alpha)$ is a symmetric α-stable random variable. Clearly, our task depends on an estimation of absolute qth moments of a transformed α-stable random variable $x + yX(\alpha)$. Explicit integral formulas, as one can find, e.g., in Zolotarev (1980), page 63, are too complex for our purposes, so we rather switch $X(\alpha)$ to an appropriate random variable from its domain of normal attraction. Let $Y(\alpha)$ be such a random variable with the density

$$f(x) = \begin{cases} 0 & \text{if } |x| < 1, \\ \dfrac{\alpha}{2}|x|^{-1-\alpha} & \text{otherwise.} \end{cases} \tag{3.2}$$

Put

$$h(\alpha) = H(Y(\alpha)).$$

THEOREM 3.1. *Let* $0 < \alpha < 2$ *and* $Y = Y(\alpha)$ *be a random variable with the density* (3.2).

(i) *If* $\alpha \leq 1$ *then* $Y \in HC(p, q, \mathbb{R})$ *for every* q *and* p *such that* $0 < q < p < \alpha$.
(ii) h *is an increasing continuous function from* $[1, 2)$ *onto* $[0, 1)$ [*i.e.*, $h(1) = 0$, $h(2-) = 1$], *and*

$$\text{(ii)}' \quad Y \in \mathrm{HC}(p, q, \mathbb{R}) \quad \text{if } h(\alpha) < q < p < \alpha,$$

$$\text{(ii)}'' \quad Y \notin \mathrm{HC}(p, q, \mathbb{R}) \quad \text{if } p > q \text{ and } q \leq h(\alpha).$$

PROOF. Since L_q-norms are homogeneous and Y is symmetric, the proof will follow by investigating the function $t \mapsto E|1 + tY|^q$, $t \geq 0$.

We shall show that for $q > h(\alpha)$,

$$\inf_{t \geq 1} E|1 + tY|^q > 1; \tag{3.3}$$

and that there are positive constants $a = a_{q,\alpha}$ and $b = b_{q,\alpha}$ with the property

$$1 + at^\alpha \leq E|1 + tY|^q \leq 1 + bt^\alpha, \qquad 0 \leq t \leq 1. \tag{3.4}$$

The conditions (ii)′ will follow then from Proposition 2.6, where $d = 1$. To deduce (i), we shall see that $h \equiv 0$ on $(0, 1]$.

Define a real function

$$\Phi_q(u) = \frac{|1 + u|^q + |1 - u|^q}{2}.$$

Since

$$E|1 + tY|^q = 1 + \alpha t^\alpha \int_t^\infty \frac{\Phi_q(u) - 1}{u^{1+\alpha}}\, du, \tag{3.5}$$

then (3.3) and (3.4) follow trivially in case $\alpha > 1$ and $q \geq 1$. Therefore, for the rest of the proof we may assume that $q < 1$. Write

$$E|1 + tY|^q = 1 + a_{q,\alpha} t^\alpha + g_{q,\alpha}(t), \tag{3.6}$$

where

$$a_{q,\alpha} = \alpha \int_0^\infty \frac{\Phi_q(u) - 1}{u^{1+\alpha}}\, du,$$

$$g_{q,\alpha}(t) = \alpha t^\alpha \int_0^t \frac{1 - \Phi_q(u)}{u^{1+\alpha}}\, du.$$

It is easily seen that

$$h(\alpha) = \inf\{q \leq \min(1, \alpha) : a_{q,\alpha} > 0\}.$$

Now, (3.4) is obvious for $q > h(\alpha)$ with $a_{q,\alpha}$ as defined and

$$b_{q,\alpha} = a_{q,\alpha} + \alpha \int_0^1 \frac{1 - \Phi_q(u)}{u^{1+\alpha}}\, du$$
$$= \alpha \int_1^\infty \frac{\Phi_q(u) - 1}{u^{1+\alpha}}\, du.$$

On the other hand, (3.3) follows since

$$E|1 + tY|^q \geq E|1 + Y|^q \geq 1 + a_{q,\alpha},$$

whenever $t \geq 1$ [for instance, by computing the derivative of the right-hand side in (3.5) and then estimating integrals appropriately].

We evaluate the quantity $a_{q,\alpha}$. Notice that

$$a_{q,\alpha} = \alpha \int_0^1 \left(\frac{\Phi_q(u) - 1}{u^{1+\alpha}} + \frac{\Phi_q(u) u^{-q} - 1}{u^{1-\alpha}} \right) du$$
$$= \frac{q(1 - A(q,\alpha))}{\alpha - q}.$$

The two-parameter function A is given by the formula

$$A(q,\alpha) = \frac{\alpha(\alpha - q)}{q} \int_0^1 (1 - \Phi_q(u)) \left(\frac{1}{u^{1+\alpha}} + \frac{1}{u^{1+q-\alpha}} \right) du.$$

Therefore

$$h(\alpha) = \inf\{q \leq \min(1,\alpha) \colon A(q,\alpha) < 1\}.$$

By integrating the series

$$1 - \Phi_q(u) = -\sum_{k=1}^{\infty} \binom{q}{2k} u^{2k}, \qquad 0 \leq u < 1,$$

term by term, we obtain the representation of the function A,

$$A(q,\alpha) = \alpha(\alpha - q) \sum_{k=1}^{\infty} \frac{(1-q)\cdots(2k-1-q)}{(2k)!} \left(\frac{1}{2k - \alpha} + \frac{1}{2k + \alpha - q} \right).$$

A routine calculation proves that $A(q,\alpha)$ has the following properties:

(a) $q \mapsto A(q,\alpha)/(1-q)$ is a decreasing function on the interval $(0, \min(1,\alpha))$ for every fixed $\alpha > 0$ [hence $q \mapsto A(q,\alpha)$ is such].

(b) $\alpha \mapsto A(q,\alpha)$ is an increasing function on $[1,2)$ for every fixed $q \in (0,1)$.

Therefore $q = h(\alpha)$ is a continuous solution of the equation $A(q,\alpha) = 1$, and it is an increasing function on $[1,2)$ since $dq/d\alpha = -(\partial A/\partial\alpha)/(\partial A/\partial q) > 0$ by (a) and (b).

We infer from (a) that

$$A(q,\alpha) \leq (1-q)A(0,\alpha),$$

or, refining the latter inequality, we obtain the following estimate:

$$
\begin{aligned}
A(q,\alpha)& \\
&= \alpha(1-q)\sum_{k=1}^{\infty}\frac{(\alpha-q)(2-q)\cdots(2k-1-q)}{(2k)!}\left(\frac{1}{2k-\alpha}+\frac{1}{2k+\alpha-q}\right)\\
&\le \alpha(1-q)\sum_{k=1}^{\infty}\frac{\alpha\cdot 2\cdot 3\cdot\cdots\cdot(2k-1)}{(2k)!}\left(\frac{1}{2k-\alpha}+\frac{1}{2k+\alpha}\right)\\
&= (1-q)\sum_{k=1}^{\infty}\frac{2\alpha^2}{4k^2-\alpha^2}\\
&= (1-q)\left(1-\frac{\alpha\pi}{2}\cot\frac{\alpha\pi}{2}\right)
\end{aligned}
$$

[cf., e.g., Gradshteyn and Ryzhik (1980)]. This shows that

$$h(\alpha)\le 1-\frac{1}{1-\frac{\alpha\pi}{2}\cot\frac{\alpha\pi}{2}},$$

which implies that $h\equiv 0$ on $(0,1]$, and that $h(\alpha)<1$ for every $\alpha\in(0,2)$.

Next, for α close to 2, we employ just the first term from the series representation of $A(q,\alpha)$ to derive a lower estimate

$$
\begin{aligned}
A(q,\alpha)&\ge\frac{2\alpha(\alpha-q)(1-q)}{4-\alpha^2}\\
&\ge\frac{2\alpha(1-q)^2}{4-\alpha^2}.
\end{aligned}
$$

Hence

$$h(\alpha)\ge 1-\left(4-\alpha^2/2\alpha\right)^{1/2},$$

which gives $h(2-)=1$.

It is enough to prove the statement (ii)″ for $\alpha>1$. Suppose that $Y\in \mathrm{HC}(p,q,\mathbb{R})$ for some $q\le h(\alpha)$ and $q<p<\alpha$. Recall quantities $a_{q,\alpha}$ and $g_{q,\alpha}(t)$ appearing in formula (3.6). Note that

$$\lim_{t\to 0}\frac{g_{q,\alpha}(t)}{t^2}=\frac{\alpha q(1-q)}{2(2-\alpha)}.$$

If $q<h(\alpha)$ then $a_{q,\alpha}<0$, and thus $E|1+tY|^q<1$ for small t which, by Proposition 2.6, violates the necessary condition for the hypercontractivity. In the remaining case $q=h(\alpha)$ we see that $(E|1+tY|^q)^{1/q}=1+g_{q,\alpha}(t)\approx 1+\mathrm{const}\cdot t^2$ and $(E|1+stY|^p)^{1/p}\approx 1+\mathrm{const}\cdot s^\alpha t^\alpha$ for small t which is impossible. This completes the proof. □

COROLLARY 3.2. *Let* $0 < \alpha < 2$. *Then* $q > h(\alpha)$, *or equivalently* $Y(\alpha) \in \mathrm{HC}(p, q)$ *for some* $p > q$, *if and only if*

$$\|1 + tY(\alpha)\|_q \approx (1 + \mathrm{const} \cdot |t|^\alpha)^{1/\alpha}, \qquad t \to 0.$$

We notice another consequence of Theorem 3.1.

PROPOSITION 3.3. *Let* $X(\alpha)$ *be an* α*-stable random variable and* $h(\alpha) < q < \alpha < 2$. *Then there is a constant* $a > 0$ *such that*

$$(1 + a|t|^\alpha)^{1/\alpha} \le \|1 + tX(\alpha)\|_q.$$

PROOF. Combining the relation (3.4) and Lemma 2.5, we infer that

$$\|1 + tY\|_q \ge (1 + a|t|^\alpha)^{1/\alpha}, \tag{3.7}$$

for some positive constant a. By homogeneity

$$(|x|^\alpha + a|y|^\alpha)^{1/\alpha} \le \|x + yY(\alpha)\|_q, \qquad x, y \in \mathbb{R}.$$

Then, for independent copies Y_i of $Y(\alpha)$, we have

$$\left(|x|^\alpha + a\sum_{i=1}^{n} |y_i|^\alpha\right)^{1/\alpha} \le \left\| x + \sum_{i=1}^{n} y_i Y_i \right\|_q, \qquad x, y_1, \dots, y_n \in \mathbb{R}, \tag{3.8}$$

which, in the limit, yields

$$(1 + a|t|^\alpha)^{1/\alpha} \le \|1 + tX(\alpha)\|_q, \qquad t \in \mathbb{R}.$$

The inequality (3.8) routinely follows by a hypercontraction argument. We demonstrate it for $n = 2$. We may assume that Y_1 and Y_2 are defined on a product probability space and denote the corresponding expectations by E_1 and E_2. By Fubini's theorem and a Jensen-type inequality we obtain the following chain of estimates:

$$\begin{aligned}
\|x + y_1X_1 + y_2X_2\|_q &= \left(E_1E_2|(x + y_1Y_1) + y_2Y_2|^q\right)^{1/q} \\
&\ge \left(E_1(|x + y_1Y_1|^\alpha + a|y_2|^\alpha)^{q/\alpha}\right)^{1/q} \\
&\ge \left((E_1|x + y_1Y_1|^q)^{\alpha/q} + a|y_2|^\alpha\right)^{1/\alpha} \\
&\ge (|x| + a|y_1|^\alpha + a|y_2|^\alpha)^{1/\alpha}.
\end{aligned}$$

By repeating this procedure n times we prove (3.8) and the proposition. □

Observe that no hypercontractive arguments work for a similar upper estimate of $\|1 + tX(\alpha)\|_q$. However, one case is trivial (and it will be used later).

COROLLARY 3.4. *For a symmetric Cauchy random variable* $X(1)$ *and* $q < 1$ *we have*

$$\|1 + tX(1)\|_q \approx 1 + \mathrm{const} \cdot |t|.$$

THEOREM 3.5. *Let* $0 < \alpha < 2$ *and* $X = X(\alpha)$ *be an* α*-stable symmetric random variable.*

(i) *If* $\alpha \le 1$ *then* $X \in \mathrm{HC}(p, q, \mathbb{R})$ *for every* q *and* p *such that* $0 < q < p < \alpha$.
(ii) *The function* h *is such that*

(ii)′ $X \in \mathrm{HC}(p, q, \mathbb{R})$ *if* $h(\alpha) < q < p < \alpha$,

(ii)″ $X \notin \mathrm{HC}(p, q, \mathbb{R})$ *if* $p > q$ *and* $q \le h(\alpha)$.

PROOF. Let $g(x)$ denote the density of $X(\alpha)$ and $0 < q < \alpha$. Since $\lim_{u \to \infty} u^{1+\alpha} g(u) = c$ then, by dominated convergence,

$$\begin{aligned}\lim_{t \to 0_+} t^{-\alpha}(E|1 + tX(\alpha)|^q - 1) &= 2 \lim_{t \to 0_+} t^{-\alpha} \int_0^\infty (\Phi_q(tx) - 1) g(x)\, dx \\ &= 2 \lim_{t \to 0_+} t^{-1-\alpha} \int_0^\infty (\Phi_q(u) - 1) g(u/t)\, du \\ &= 2\int_0^\infty \frac{\Phi_q(u) - 1}{u^{1+\alpha}} \lim_{t \to 0_+} (u/t)^{1+\alpha} g(u/t)\, du \\ &= 2ca_{q,\alpha},\end{aligned}$$

where $a_{q,\alpha}$ has been defined in the proof of Theorem 3.1. The value of the constant c can be found in, e.g., Feller (1966), Section 17.6. The aforementioned asymptotic behavior about 0 and Proposition 3.3 determine hypercontractivity by virtue of Proposition 2.6. Thus $X(\alpha) \in \mathrm{HC}(p, q)$ if $h(\alpha) < q < p < \alpha$ since, in this case, $2ca_{q,\alpha} > 0$.

Similarly, $X(\alpha) \notin \mathrm{HC}(p, q)$ if $q < h(\alpha)$, $q < p < \alpha$. If $q = h(\alpha)$ then $E|1 + tX(\alpha)|^q - 1 = o(t^\alpha)$ and $E|1 + tX(\alpha)|^p - 1 \ge \mathrm{const} \cdot t^\alpha$ by Proposition 3.3 which contradicts hypercontractivity. This proves the theorem. □

COROLLARY 3.6. *Let* $0 < \alpha < 2$. *Then* $q > h(\alpha)$, *or, equivalently,* $X(\alpha) \in \mathrm{HC}(p, q)$ *for some* $p > q$ *if and only if*

$$\|1 + tX(\alpha)\|_q \approx (1 + \mathrm{const} \cdot |t|^\alpha)^{1/\alpha}.$$

Now we can provide examples of nonsymmetric hypercontractive random variables.

PROPOSITION 3.7. $H(Z(\alpha)) = 0$ *for all positive* α*-stable random variables* $Z(\alpha)$ *with* $\alpha < 1$ (*sometimes called stable random variables totally skewed to the right*).

PROOF. Since for some constant c the product of independent random variables $cX(1)Z(\alpha)$ has the same distribution as $X(\alpha)$ [cf., e.g., Zolotarev (1986), Theorem 3.3.1], hence

$$\|1 + tX(\alpha)\|_q = \|1 + tcX(1)Z(\alpha)\|_q \approx \|1 + tZ(\alpha)\|_q,$$

by Fubini's theorem and Corollary 3.4. □

Theorem 3.5 shows in particular that the function $H(X)$ is not continuous with respect to the weak convergence of distributions (nor even with respect to much stronger means of convergence). For instance, truncated α-stable random variables $X_n = X\mathbb{1}_{\{|X|\le n\}}$ converge a.s. and in L_q, $q<\alpha$, to X but $H(X_n)\equiv 1$ and $H(X)=h(\alpha)<1$.

If $q\le 1$, then (p,q)-hypercontractivity, $q<p$, of any random variable X can be carried over to an arbitrary normed space $\mathbb{B}$ preserving the same constant s as in the real case [Kwapień and Szulga (1988), Proposition 2.3]. Then, by Jensen's inequalities, $\xi \in \mathrm{HC}(p', q', \mathbb{B}, s)$ for every p', q' such that $q \le q' < p' \le p$.

COROLLARY 3.8. *In the stable case one can always find* $q<1$ *such that* $X(\alpha)\in HC(p,q)$. *Therefore, in the formulations of Theorems* 3.1 *and* 3.5 *and Propositions* 3.3 *and* 3.7 *we can replace* $\mathbb{R}$ *by an arbitrary normed space* $\mathbb{B}$.

As mentioned in the introduction, one of the main probabilistic applications of hypercontraction methods is a construction of a multiple stochastic integral with respect to a symmetric Lévy stable process $Z(t)$ on $[0,T]$ (with further generalizations to the strictly stable case):

$$I_k(f) \stackrel{\text{df}}{=} \int\cdots\int_{[0,T]^k} f(t_1,\dots,t_k)\,dZ(t_1)\cdots dZ(t_k), \tag{3.9}$$

where f is a function on $[0,T]^k$ with values in a Banach space that is symmetric with respect to all permutations of its arguments and vanishes on diagonals [Krakowiak and Szulga (1988b)]. Formula (3.9) defines a product random measure M^k, and the hypercontraction property of α-stable variables yields a deterministic control measure μ^k [i.e., $\mu^k(A_n)\to 0$ iff $M^k(A_n)\to 0$]. However, this technique was quite simple only for $\alpha>1$. The case $\alpha\le 1$ required additional sophisticated results like decoupling inequalities. With hypercontraction of α-stable random variables, $0<\alpha\le 1$, at hand, the underlying construction of a control measure μ^k becomes as straightforward as in the case of integrable random variables.

In particular, a family of random variables of type (3.9) is hypercontractive, too. The latter statement can be deduced from Theorem 3.5 (the argument in Corollary 3.8 must be used in the context of a Banach space) by a standard technique. First, one considers simple functions and proves hypercontraction for multilinear α-stable forms. Next, since their limits in probability, or even in distribution, preserve this property, multiple stable integrals turn out to be hypercontractive. We refer to Krakowiak and Szulga (1988b) for details.

In some cases, even within multiple stochastic integration, applicability of hypercontraction is limited. For example, a single value $X(t)$ of a Poisson process is hypercontractive [Krakowiak and Szulga (1988a)] but the corresponding hypercontraction constant $s=s(t)\to 0$ if $t\to 0$. This explains why multiple Poisson integrals cannot be constructed by using the same technique as in the stable case. No surprise, a product Poisson measure does not have any deterministic control measure [Kallenberg and Szulga (1989)]!

Although the origins of the concept of hypercontractive random variables can be tracked to the theory of hypercontractive operators, further connection at this time are hardly visible. Theorem 2.1, for instance, can be interpreted by means of such operators but the point is that their domains are, in general, proper subspaces of L_p.

Acknowledgment. I am indebted to the referee for several valuable suggestions as well as for a careful reading of the manuscript and for pointing out a number of gaps.

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