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ON ORDERED STOPPING TIMES OF A MARKOV PROCESS[1]

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Let X be a strong Markov process with potential kernel U. We show that if (ν_n) and μ are measures on the state space of X such that $\nu_1 U \le \nu_2 U \le \cdots \le \mu U$, then there is a decreasing sequence (T_n) of randomized stopping times such that ν_n is the law of X_{T_n} when the initial distribution of X is μ.

1. Introduction. Let X be a transient strong Markov process with state space E. Fix an initial distribution μ and a stopping time T, and let ν denote the P^μ-law of X_T. Then $\nu U \le \mu U$, where U is the potential kernel of X. Rost (1971) has shown that this necessary condition is also sufficient for the existence of a (randomized) stopping time linking μ and ν as above.

Now suppose that $\{\nu_n\}_{n \ge 1}$ is a sequence of finite measures on E, *ordered* in the sense that

$$\nu_1 U \le \nu_2 U \le \cdots \le \nu_n U \le \nu_{n+1} U \le \cdots \le \mu U.$$

We show that there is a decreasing sequence (T_n) of randomized stopping times such that ν_n is the P^μ-law of X_{T_n}. This is easy to do for a *finite* sequence $\{\nu_n\}_{1 \le n \le N}$, and the general case is handled by a limiting argument, using the Baxter–Chacon compactness theorem for randomized stopping times, and a criterion for the ordering of same.

One consequence of the above representation is the following characterization of fine continuity: Given a bounded Borel function f on E, $t \mapsto f(X_t)$ is right continuous a.s. P^μ if and only if $\nu_n(f) \to \nu_\infty(f)$ whenever $\{\nu_n\}_{1 \le n \le \infty}$ is a sequence of measures on E such that $\nu_n U \uparrow \nu_\infty U \le \mu U$. In fact, this result [proved by other means in Fitzsimmons (1988b)] was the source of the problem solved in this note.

The problem considered here has also been solved by Shih (1990), using different methods. Shih obtains a more detailed description of the times T_n and provides several examples.

2. Ordered stopping. Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, P^x)$ be a right Markov process with state space $(E, \mathscr{E})$, in the sense of Sharpe (1988). Thus E is a universally measurable subset of a compact metric space (with the subspace topology) and $\mathscr{E}$ is the class of Borel sets in E. The semigroup (P_t) of X need only be sub-Markovian, so a cemetery state Δ is adjoined to E, and X is absorbed in Δ at its lifetime ζ. The potential kernel U is defined by $\int_0^\infty P_t\,dt$,

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and we assume that X is transient in the sense that there is a strictly positive, universally measurable function f on E such that $Uf \le 1$.

Given an initial distribution μ, recall that $(\mathscr{F}_t^\mu)$ denotes the natural filtration of X augmented by the P^μ-null sets in the P^μ-completion $\mathscr{F}^\mu$ of $\sigma\{X_t,\ t \ge 0\}$; $(\mathscr{F}_t^\mu)$ is a right continuous filtration. Let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ be an auxiliary probability space. A *randomized stopping time* (RST) over the system $(\Omega, \mathscr{F}^\mu, \mathscr{F}_t^\mu, P^\mu)$ [with randomization space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$] is an $\tilde{\mathscr{F}} \otimes \mathscr{F}^\mu$-measurable mapping $T\colon \tilde{\Omega} \times \Omega \to [0, \infty]$ such that

$$\{(z, \omega) \in \tilde{\Omega} \times \Omega\colon T(z, \omega) \le t\} \in \tilde{\mathscr{F}} \otimes \mathscr{F}_t^\mu, \qquad \forall\, t \ge 0.$$

If, in addition, $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}) = ([0,1], \mathscr{B}_{[0,1]}, \lambda)$, the unit interval equipped with Lebesgue measure, and

$$T(\,\cdot\,, \omega) \text{ is increasing and right continuous on } [0,1], \qquad \forall\, \omega \in \Omega,$$

then we speak of a *canonical* RST. We shall write $\bar{P}^\mu$ for $\tilde{P} \otimes P^\mu$; processes $Z_t(\omega)$ defined on $[0,\infty] \times \Omega$ are extended to $[0,\infty] \times \tilde{\Omega} \times \Omega$ in the obvious way.

Given an RST, T, consider the measure M on $[0,\infty] \times \Omega$ defined by $M(Z) = \bar{P}^\mu(Z_T)$. Evidently the second marginal of M is P^μ, so there is a disintegration

$$M(dt, d\omega) = dA_t(\omega) P^\mu(d\omega),$$

where A_t, $t \in [0,\infty]$, is a positive, increasing, right continuous $(\mathscr{F}_t^\mu)$-adapted process with $A_\infty = 1$. The right continuous inverse,

$$T'(z, \omega) := \inf\{t\colon A_t(\omega) > z\}, \qquad 0 \le z \le 1,$$

is a canonical RST, and by a standard change-of-variable formula,

$$\bar{P}^\mu(Z_T) = P^\mu\left(\int_{[0,\infty]} Z_t\, dA_t\right) = \bar{P}^\mu(Z_{T'}).$$

In particular, taking Z of the form $\int_t^\infty g(X_s)\, ds$ and using the strong Markov property, we see that $\bar{P}^\mu(Ug(X_T)) = \bar{P}^\mu(Ug(X_{T'}))$, hence

$$\bar{P}^\mu(X_T \in \cdot\,) = \bar{P}^\mu(X_{T'} \in \cdot\,),$$

since a potential uniquely determines its charge [cf. (1.1) in Getoor and Glover (1983)]. We shall refer to T' as the canonical (monotone) rearrangement of T. For a complete discussion of these matters, see Meyer (1978).

Let ν be a second measure on E such that $\nu U \le \mu U$; i.e., $\nu U(A) \le \mu U(A)$ for all $A \in \mathscr{E}$. According to a theorem of Rost (1971), there is a canonical RST, T, such that $\nu = \bar{P}^\mu(X_T \in \cdot\,)$; more precisely,

$$(1) \qquad \nu(f) = \int_E f\, d\nu = \int_0^1 \int_\Omega f\big(X_{T(z,\omega)}(\omega)\big) 1_{\{T(z,\cdot) < \zeta\}}(\omega) P^\mu(d\omega)\, dz,$$

for all positive Borel functions f on E. [For a proof that works in the present setting of general right processes see Fitzsimmons (1988a).] Conversely, the existence of an RST, T, such that (1) holds implies that $\nu U \le \mu U$, by the strong Markov property. We follow the usual convention that a function f defined on E is extended to $E \cup \{\Delta\}$ by setting $f(\Delta) = 0$; thus, the condition $\{T < \zeta\}$ can be omitted in (1).

Now consider an infinite sequence $\{\nu_n\}$ of finite measures on E such that

$$\nu_n U \le \nu_{n+1} U \le \mu U, \qquad \forall\, n \ge 1.$$

THEOREM. *There is a sequence* $\{T_n\}$ *of canonical randomized stopping times such that*

(i) $T_n(z,\omega) \ge T_{n+1}(z,\omega)$, *for all* $z \in [0,1]$, $\omega \in \Omega$, *and* $n \ge 1$;
(ii) $\bar{P}^\mu(X_{T_n} \in \cdot) = \nu_n$, *for all* $n \ge 1$;
(iii) $T_\infty := \downarrow \lim_n T_n$ *is a canonical RST, and* $\bar{P}^\mu(X_{T_\infty} \in \cdot) = \nu_\infty$, *where* ν_∞ *is the weak limit of the sequence* $\{\nu_n\}$. *Moreover,* $\nu_n U \uparrow \nu_\infty U$ *as* $n \to \infty$.

For the proof of this theorem we require two lemmas. The first of these is the sequential compactness theorem of Baxter and Chacon (1977), as formulated by Meyer (1978), while the second is Lemma 2.14 in Baxter and Chacon (1977). Let $\mathscr{C}$ denote the class of bounded processes Z over $(\Omega, \mathscr{F}^\mu, P^\mu)$ such that $t \mapsto Z_t(\omega)$ is continuous on $[0,\infty]$ for all $\omega \in \Omega$. [Note that $Z \in \mathscr{C}$ need not be adapted to $(\mathscr{F}_t^\mu)$.] In the sequel the initial distribution μ remains fixed, and all statements are relative to the system $(\Omega, \mathscr{F}^\mu, \mathscr{F}_t^\mu, P^\mu)$.

LEMMA 1. *Let* $\{T_n\}$ *be a sequence of canonical randomized stopping times. Then there is a subsequence* $\{n(k)\}$ *and a canonical randomized stopping time* T *such that*

$$\lim_k \bar{P}^\mu\big(Z_{T_{n(k)}}\big) = \bar{P}^\mu(Z_T), \qquad \forall\, Z \in \mathscr{C}.$$

LEMMA 2. *Let* S *and* T *be canonical randomized stopping times. Then* $S(z,\omega) \le T(z,\omega)$ *for all* $z \in [0,1]$, *for* P^μ*-a.e.* $\omega \in \Omega$, *if and only if*

$$\bar{P}^\mu(Z_S) \le \bar{P}^\mu(Z_T), \quad \text{for all positive increasing } Z \in \mathscr{C}. \tag{2}$$

In particular, if S and T are RST's, with the same auxiliary space $\tilde{\Omega}$, such that

$$S(z,\cdot) \le T(z,\cdot), \qquad \forall\, z \in \tilde{\Omega}, \text{ a.e. } P^\mu,$$

then the canonical rearrangements of S and T are likewise ordered.

PROOF OF THE THEOREM. Fix $k \ge 1$. Since $\nu_1 U \le \cdots \le \nu_k U \le \mu U$, it is easy to use Rost's theorem and the strong Markov property to produce RST's $\{T_{kn}^*\}_{1 \le n \le k}$, with auxiliary space the product of k copies of the unit interval, such that

$$T_{km}^*(z,\cdot) \ge T_{kn}^*(z,\cdot), \qquad \forall\, z \in [0,1]^k, 1 \le m < n \le k, \text{ a.e. } P^\mu, \tag{3}$$

$$\bar{P}^\mu\big(X_{T_{kn}^*} \in \cdot\big) = \nu_n, \qquad 1 \le n \le k. \tag{4}$$

See, for example, Section 2 in Shih (1990). In view of Lemma 2, the sequence $\{T_{kn}\}_{1 \le n \le k}$ of canonical rearrangements satisfies (3) and (4) (with the *'s deleted and $[0,1]^k$ replaced by $[0,1]$). By Lemma 1 and the Cantor diagonal procedure, there is a sequence $\{k(j)\}_{j \ge 1}$ and canonical RST's $\{T_n\}$ such that

$$\lim_j \bar{P}^\mu\big(Z_{T_{k(j),n}}\big) = \bar{P}^\mu\big(Z_{T_n}\big), \qquad \forall\, Z \in \mathscr{C}, n \ge 1. \tag{5}$$

Thus, if $m < n$, then $\bar{P}^{\mu}(Z_{T_m}) \geq \bar{P}^{\mu}(Z_{T_n})$ for all positive increasing $Z \in \mathscr{C}$. By Lemma 2,

$$T_m(z, \cdot) \geq T_n(z, \cdot), \qquad \forall\, z \in [0,1],\, 1 \leq m < n, \text{a.e. } P^{\mu},$$

from which point (i) follows easily.

To prove (ii) let g be a positive bounded Borel function on E such that $\mu Ug < \infty$; the transience of X implies that μU is σ-finite so there are many such g's. The process

$$Z_t = \int_{t \wedge \zeta}^{\zeta} g(X_s)\, ds = \left(\int_0^{\zeta} g(X_u)\, du \right) \circ \theta_t$$

is positive, decreasing and continuous in t, and $P^{\mu}(Z_0) = \mu Ug < \infty$. It follows that Z is of the class (D), so by Théorème 8 of Meyer (1978), (5) holds for this (unbounded) choice of Z. By the strong Markov property,

$$\begin{aligned} \nu_n Ug &= \lim_j \bar{P}^{\mu}\big(Ug\big(X_{T_{k(j),n}}\big)\big) = \lim_j \bar{P}^{\mu}\big(Z_{T_{k(j),n}}\big) \\ &= \bar{P}^{\mu}\big(Z_{T_n}\big) = \bar{P}^{\mu}\big(Ug\big(X_{T_n}\big)\big) = \tilde{\nu}_n Ug, \end{aligned}$$

where $\tilde{\nu}_n = \bar{P}^{\mu}(X_{T_n} \in \cdot)$. Varying g, we conclude that $\nu_n U = \tilde{\nu}_n U$; by the uniqueness of charges, $\nu_n = \tilde{\nu}_n$, and point (ii) is proved.

It is easy to check that T_∞ is a canonical RST, and the other assertions in the first sentence of (iii) follow from the right continuity of X. The final assertion can be proved using the argument of the previous paragraph. □

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