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THE ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF THE EXTERIOR DIRICHLET PROBLEM FOR BROWNIAN MOTION PERTURBED BY A SMALL PARAMETER DRIFT[1]

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Let $L_\varepsilon = \frac{1}{2}\Delta + \varepsilon b \cdot \nabla$ in R^d, $d \geq 3$, generate a recurrent diffusion for each $\varepsilon > 0$, where $b \in C^\alpha(R^d)$, and let $D \subset R^d$ be an exterior domain. Then by the recurrence assumption, for each $\psi \in C(\partial D)$, there exists a unique solution in the class of bounded solutions to the Dirichlet problem $L_\varepsilon u_\varepsilon = 0$ in D and $u_\varepsilon = \psi$ on ∂D. On the other hand, by the transience of d-dimensional Brownian motion, there is no uniqueness in the class of bounded solutions for the Dirichlet problem $\frac{1}{2}\Delta u = 0$ in D and $u = \psi$ on ∂D. Since the Martin boundary at ∞ for Brownian motion consists of a single point, uniqueness is obtained by adding the condition $\lim_{|x|\to\infty} u(x) = c$. We show that $u_0(x) \equiv \lim_{\varepsilon\to 0} u_\varepsilon(x)$ exists and satisfies $\frac{1}{2}\Delta u_0 = 0$ in D, $u_0 = \psi$ on ∂D and $\lim_{|x|\to\infty} u_0(x) = c$, where c is given as follows. Let P_x^h denote the measure associated with Doob's conditioned Brownian motion conditioned to exit D at ∂D rather than at ∞. Let $\tau = \inf\{t \geq 0: X(t) \in \partial D\}$ and define the harmonic measure $u_x^h(dy) = P_x^h(X(\tau) \in dy)$. Then $\mu_\infty^h \equiv \lim_{|x|\to\infty} \mu_x^h$ exists and $c = \int_{\partial D} \psi(y) \mu_\infty^h(dy)$. We also show that the energy integral $\int_D |\nabla u|^2\, dx$, when varied over all bounded functions $u \in W_{\text{loc}}^{1,2}(D)$ which satisfy $u = \psi$ on ∂D, takes on its minimum uniquely at u_0.

1. Introduction. Let $L_\varepsilon = \frac{1}{2}\Delta + \varepsilon b \cdot \nabla$ generate a recurrent diffusion in R^d, $d \geq 3$, for each $\varepsilon > 0$, where $b \in C^\alpha(R^d)$, and let $D \subset R^d$ be an exterior domain, that is, the complement of a compact set. Assume that ∂D is a Lipschitz boundary. Then, for each $\psi \in C(\partial D)$ and each $\varepsilon \geq 0$, the Dirichlet problem

$$(1.1)\qquad \begin{aligned} L_\varepsilon u &= 0 \quad \text{in } D,\\ u &= \psi \quad \text{on } \partial D \end{aligned}$$

possesses a bounded solution $u_\varepsilon \in C^2(R^d)$. If $\varepsilon > 0$, then by the recurrence of the process, this solution is unique in the class of bounded solutions and obeys the maximum principle: $\sup_{x\in D} u_\varepsilon(x) = \sup_{x\in\partial D} \psi(x)$. A proof of these facts is sketched at the end of this section. On the other hand, when $\varepsilon = 0$, the resulting diffusion process, namely Brownian motion, is transient and thus there are infinitely many bounded solutions to (1.1). In order to obtain uniqueness in the transient case, one must also specify boundary data on the Martin boundary at ∞. In particular, the Martin boundary at ∞ for Brownian

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motion consists of a single point, thus it follows from the Martin representation [6] that for each $\psi \in C(\partial D)$ and each $c \in R$, there exists a unique solution to

$$(1.2)\qquad \begin{aligned} \tfrac{1}{2}\Delta u &= 0 \quad \text{in } D,\\ u &= \psi \quad \text{on } \partial D,\\ \lim_{|x|\to\infty} u(x) &= c. \end{aligned}$$

In this paper, we study the asymptotics of u_ε as $\varepsilon \to 0$. This problem is very different from the classical small parameter problem in which the small parameter appears in the diffusion coefficient rather than the drift and in which the domain D is bounded (see [3], Chapter 4, for a good exposition of the various guises of this problem). Before stating our results, we establish a bit of notation and introduce Doob's conditioned Brownian motion. Let $\Omega = C([0,\infty), R^d)$ with the topology of uniform convergence on compacts and let $X(\cdot)$ denote the generic point in Ω. Define $\tau = \inf\{t \geq 0\colon X(t) \in \partial D\}$. Denote by P_x^ε the measure on Ω corresponding to the diffusion generated by L_ε and starting from x and let P_x denote d-dimensional Wiener measure starting from x. The corresponding expectations will be denoted by E_x^ε and E_x. Let h be the unique solution to (1.2) in the case $\psi \equiv 1$ and $c = 0$. Of course, by Itô's formula, it follows that $h(x) = P_x(\tau < \infty)$. Then, following Doob [2], Brownian motion in D, conditioned on exiting D at ∂D rather than at ∞, may be realized as the Markov diffusion process on D generated by $\frac{1}{2}\Delta^h$, where Δ^h is defined by

$$\Delta^h f = \frac{1}{h}\Delta(hf) = \Delta f + \frac{2\nabla h}{h}\nabla f.$$

Denote by P_x^h the measure on Ω corresponding to this conditioned Brownian motion starting from x. Finally, let $\mu_x^h(dy) = P_x^h(X(\tau) \in dy)$ denote the harmonic measure on ∂D corresponding to the conditioned Brownian motion. We can now state our main theorem.

THEOREM 1. *Let $D \subset R^d$, $d \geq 3$, be an exterior domain with Lipschitz boundary and for $\varepsilon > 0$, let u_ε be the unique bounded solution to* (1.1), *where $\psi \in C(\partial D)$. Then $u_0(x) \equiv \lim_{\varepsilon\to 0} u_\varepsilon(x)$ exists and solves* (1.2) *with c given as follows. The harmonic measures $\{\mu_x^h\}_{x\in D}$ possess a unique weak limit as $|x| \to \infty$. Let*

$$(1.3)\qquad \mu_\infty^h = \operatorname*{w\text{-}lim}_{|x|\to\infty} \mu_x^h.$$

Then

$$(1.4)\qquad c = \int_{\partial D} \psi(y)\mu_\infty^h(dy).$$

The measure μ_∞^h can be represented in the following manner. Let $B_\gamma \subset R^d$

denote the ball of radius γ centered at the origin and let $l(dx)$ denote Lebesgue measure on ∂B_γ. Then if $D \subseteq B_\gamma$, the measure μ_∞^h satisfies

$$\mu_\infty^h = \frac{\int_{\partial B_\gamma} \mu_x^h h(x) l(dx)}{\int_{\partial B_\gamma} h(x) l(dx)}. \tag{1.5}$$

REMARK 1. An alternative formulation of the theorem is this: The harmonic measures $\mu_x^\varepsilon(dy) = P_x^\varepsilon(X(\tau) \in dy)$ satisfy

$$\text{w-}\lim_{\varepsilon \to 0} \mu_x^\varepsilon = P_x(\tau < \infty)\mu_x^h + P_x(\tau = \infty)\mu_\infty^h.$$

REMARK 2. In fact μ_∞^h possesses the density $\nabla h \cdot n(x)/\int_{\partial D} \nabla h \cdot n(x)\sigma(dx)$, where n is the unit outward normal to D at ∂D. See the remark following Theorem 2.

REMARK 3. Let L be any strictly elliptic transient generator with coefficients bounded on compacts and let $h(x) = P_x(\tau_D < \infty)$, where P_x now corresponds to the process generated by L. Then the process conditioned to exit D at ∂D rather than at ∞ is generated by L^h, where $L^h f = (1/h)L(fh)$. As before, let $\mu_x^h(dy) = P_x^h(X(\tau_D) \in dy)$, where P_x^h corresponds to L^h. The proof of Theorem 1 will reveal that as long as $\mu_\infty^h(dy) \equiv \text{w-}\lim_{|x| \to \infty} \mu_x^h(dy)$ exists, then in fact the theorem is also valid with $\frac{1}{2}\Delta$ replaced by L. In fact, in [7] it was proved that this condition on the harmonic measure is equivalent to the condition that the Martin boundary at ∞ for $\tilde{L}$, the adjoint of L, be one point. It was pointed out that in the case that L generates a reversible diffusion, the Martin boundaries of L and $\tilde{L}$ coincide; thus, in the reversible case the above condition on the harmonic measure is equivalent to the Martin boundary at ∞ for L consisting of one point. It was conjectured that the Martin boundary at ∞ of L is one point if and only if the Martin boundary at ∞ of $\tilde{L}$ is one point. If this is indeed correct, then Theorem 1 holds for any such diffusion generator L with a one-point Martin boundary at ∞, or equivalently, for any such diffusion generator L possessing no nonconstant positive harmonic functions.

It is interesting that the solution u_0 in Theorem 1 also arises as the minimizer of a variational problem. Define the energy integral $J(u) = \int_D |\nabla u|^2\, dx$ and let

$$J = \inf_{\substack{u \in W_{\text{loc}}^{1,2}(D) \\ u = \psi \text{ on } \partial D \\ u \text{ bdd}}} J(u),$$

where ψ and D are as in Theorem 1. By varying $J(u)$ by a compactly supported C^∞-function, one easily concludes that, if the minimum above is indeed attained, say at $\hat{u}$, then $\hat{u}$ must satisfy $\frac{1}{2}\Delta\hat{u} = 0$ in D and $\hat{u} = \psi$ on ∂D. In fact then, $\hat{u}$ must satisfy (1.2) for some c. We will prove the following theorem.

THEOREM 2. *Let $D \subset R^d$, $d \geq 3$, be an exterior domain. Then*

$$J = \inf_{\substack{u \in W^{1,2}_{\mathrm{loc}}(D) \\ u = \psi \text{ on } \partial D \\ u \text{ bdd}}} \int_D |\nabla u|^2 \, dx$$

is attained at u_0, where u_0 is as in Theorem 1.

REMARK. From the previous paragraph, it follows that the minimum must be of the form $u = v + c(1 - h)$, where v satisfies $\frac{1}{2}\Delta v = 0$ in D, $v = \psi$ on ∂D and $\lim_{|x| \to \infty} v(x) = 0$. Plugging this into $J(u)$, minimizing over c and integrating by parts (it is easy to show that no contribution arises at ∞ in the integration by parts), we obtain

$$c = \frac{\int_{\partial D} \psi(x) \nabla h \cdot n(x) \sigma(dx)}{\int_{\partial D} \nabla h \cdot n(x) \sigma(dx)},$$

where n is the unit outward normal to D at ∂D. Now it seems to be known that μ^h_∞ possesses the density $\nabla h \cdot n(x) / \int_{\partial D} \nabla h \cdot n(x) \sigma(dx)$; however, we could not find the result in the literature. Thus, we give an alternative proof to Theorem 2 which, coupled with the above derivation, actually gives a proof that the density of μ^h_∞ is $\nabla h \cdot n(x) / \int_{\partial D} \nabla h \cdot n(x) \sigma(dx)$.

We now sketch the proof of the facts that were stated in the first paragraph of the paper. Existence of a solution to the exterior Dirichlet problem (1.1) for $\varepsilon \geq 0$ can be given as follows. Let $B \subset R^d$ denote the open ball of radius n centered at the origin. If n satisfies $D^c \subset B_n$, let $u_{\varepsilon,n} \in C^2(D \cap B_n) \cap C(\overline{D \cap B_n})$ denote the solution to $L_\varepsilon u_{\varepsilon,n} = 0$ in $B_n \cap D$, $u_{\varepsilon,n} = \psi$ on ∂D and $u_{\varepsilon,n} = 0$ on ∂B_n. Such a solution exists by Theorem 6.13 and problem 6.3 in [4]. By the maximum principle and standard Schauder interior estimates ([4], Theorem 6.2), it follows that $u_\varepsilon \equiv \lim_{n \to \infty} u_{\varepsilon,n}$ exists, is bounded, is in $C^{2,\alpha}(D)$ and satisfies $L_\varepsilon u_\varepsilon = 0$ in D and $u_\varepsilon = \psi$ on ∂D. In the case $\varepsilon > 0$, relying on the boundedness of u_ε and the fact that $\tau < \infty$ a.s. $[P^\varepsilon_x]$, a standard application of Itô's formula gives $u_\varepsilon(x) = E^\varepsilon_x \psi(X(\tau))$. This gives uniqueness and the maximum principle.

Theorem 1 is proved in the section that follows and Theorem 2 is proved in the final section.

2. Proof of Theorem 1. Let $\tau_n = \inf\{t \geq 0: |X(t)| = n\}$. We first note the following simple facts:

(2.1) $$P_x(\tau = t) = P_x(\tau_n = t) = 0 \quad \text{for all } t \geq 0.$$

(2.2) τ and τ_n are P_x a.s. continuous functionals on Ω.

Since P_x is Wiener measure, (2.1) is obvious. (2.2) is an easy consequence of the law of the iterated logarithm. We will prove (2.2) for τ, the proof for τ_n

being identical. Let $\hat{\tau} = \inf\{t \geq 0: X(t) \in \overline{D}^c\}$ be the first penetration time of $\overline{D}^c$. Then τ is continuous at all paths $X(\cdot) \in \Omega$ which satisfy $\tau = \hat{\tau}$. But, by the law of the iterated logarithm and the assumption that ∂D is Lipschitz, it follows that $P_x(\tau = \hat{\tau}) = 1$.

Now fix $x_0 \in D$. In fact (2.1) also holds for P_x^ε. This follows from the existence of a transition probability density ([8], Chapter 9). Actually in [8], it is assumed that b is bounded, but a localization argument shows that b bounded on compacts is sufficient. By Itô's formula and (2.1) for P^ε, for any $t > 0$, we have

$$\begin{aligned} u_\varepsilon(x_0) &= E_{x_0}^\varepsilon(\psi(X(\tau)); \tau < \tau_n \wedge t) + E_{x_0}^\varepsilon(u_\varepsilon(X(\tau_n)); \tau_n < \tau \wedge t) \\ &\quad + E_{x_0}^\varepsilon(u_\varepsilon(X(t)); t < \tau \wedge \tau_n) \\ &\equiv I_1(n,t,\varepsilon) + I_2(n,t,\varepsilon) + I_3(n,t,\varepsilon). \end{aligned} \tag{2.3}$$

Pick $n_\delta = n_\delta(x_0)$ and $t_\delta = t_\delta(x_0)$ so large that

$$P_{x_0}(\tau < \infty) - P_{x_0}(\tau < \tau_n \wedge t) < \delta \quad \text{if } n \geq n_\delta \text{ and } t \geq t_\delta. \tag{2.4}$$

From (2.4) we obtain

$$\left|E_{x_0}(\psi(X(\tau)); \tau < \infty) - E_{x_0}(\psi(X(\tau)); \tau < \tau_n \wedge t)\right| < \|\psi\|\delta \quad \text{for } n \geq n_\delta \text{ and } t \geq t_\delta. \tag{2.5}$$

If necessary, choose $t_\delta = t_\delta(x_0)$ even larger so that

$$P_{x_0}(\tau \wedge \tau_{n_\delta} > t) < \delta \quad \text{for } t \geq t_\delta. \tag{2.6}$$

Now $P_{x_0}^\varepsilon$ converges weakly to P_{x_0} ([8], Theorem 11.4). Thus, by (2.1) and (2.2) along with the standard theorem which gives equivalent conditions for weak convergence ([1], Theorem 2.1), it follows that, for each $t > 0$ and each n,

$$\lim_{\varepsilon \to 0} P_{x_0}^\varepsilon(\tau \wedge \tau_n > t) = P_{x_0}(\tau \wedge \tau_n > t) \tag{2.7}$$

and

$$\lim_{\varepsilon \to 0} E_{x_0}^\varepsilon(\psi(X(\tau)); \tau < \tau_n \wedge t) = E_{x_0}(\psi(X(\tau)); \tau < \tau_n \wedge t). \tag{2.8}$$

From (2.7) and the maximum principle, we conclude that

$$\limsup_{\varepsilon \to 0} E_{x_0}^\varepsilon(u_\varepsilon(X(t)); t < \tau \wedge \tau_n) \leq \|\psi\| P_{x_0}(\tau \wedge \tau_n > t). \tag{2.9}$$

From (2.4)–(2.9), it follows that

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} I_1(n_\delta, t_\delta, \varepsilon) = E_{x_0}(\psi(X(\tau)); \tau < \infty) \tag{2.10}$$

and

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} I_3(n_\delta, t_\delta, \varepsilon) = 0. \tag{2.11}$$

The solution to (1.2) may be represented stochastically as

$$u_0(x_0) = E_{x_0}(\psi(X(\tau)); \tau < \infty) + cP_{x_0}(\tau = \infty).$$

Except for (1.5) (which is contained in Lemma 1 below), the theorem will follow from (2.3), (2.10) and (2.11) if we prove that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I_2(n_\delta, t_\delta, \varepsilon) = cP_{x_0}(\tau = \infty), \tag{2.12}$$

where c is as in (1.4).

We start with the following lemma concerning the harmonic measure on ∂D of the conditioned Brownian motion P^h. Henceforth, we will frequently employ the notation $x = (r, \phi)$, where $r = |x|$ and $\phi \in S^{d-1}$. Analogously, we will write $X(t) = (r(t), \phi(t))$.

LEMMA 1. *$\mu_\infty^h \equiv$ w-$\lim_{r\to\infty} \mu_{(r,\phi)}^h$ exists and the convergence is uniform over $\phi \in S^{d-1}$. Furthermore, μ_∞^h satisfies* (1.5).

PROOF. Note that, by the definition of $\mu_{(r,\phi)}^h$ and by the strong Markov property, it suffices to prove the lemma for some sequence $\{r_n\}_{n=1}^\infty$ with $\lim_{n\to\infty} r_n = \infty$. First consider the case in which ∂D is a sphere—say $\partial D = \{x\colon\ |x| = \gamma\}$. In this case $h(x) = \gamma^{d-2}/|x|^{d-2}$ and $\frac{1}{2}\Delta^h = \frac{1}{2}(\Delta - 2(d-2)/r\,\partial/\partial r)$. In (r, ϕ) coordinates, this becomes

$$\frac{1}{2}\Delta^h = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} - \frac{d-3}{r}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^{d-1}},$$

where $\Delta_{S^{d-1}}$ is the Laplace–Beltrami operator on S^{d-1}. The exit time is given by $\tau = \inf\{t \ge 0\colon |X(t)| = \gamma\}$. It will be convenient to represent the process $X(t) = (r(t), \phi(t))$ as a skew product in the following manner. On a probability space $(\hat{\Omega}, \mathscr{F}, \mathscr{P})$, for each $r > \gamma$, let $r^h(t) = r^h(t; r) = r^h(t; r, \omega)$ be a one-dimensional diffusion generated by $\frac{1}{2}(d^2/dr^2 - (d-3)/r\,d/dr)$ starting from r, and define

$$\rho(t; r) = \int_0^t \frac{ds}{\left(r^h(s; r)\right)^2}.$$

[Actually, $r^h(t)$ is only defined until it hits 0 which it will do with probability 1, so we ought to specify that the process is killed at 0. However, in the sequel we will only be considering $r^h(t)$ up to the time it hits γ.] Let $\theta(t; \phi) = \theta(t, \omega; \phi)$, also defined on $(\hat{\Omega}, \mathscr{F}, \mathscr{P})$, be a standard Brownian motion on S^{d-1}, starting from $\phi \in S^{d-1}$, generated by $\frac{1}{2}\Delta_{S^{d-1}}$ and independent of $r^h(t)$. Now define $\phi^h(t) = \phi^h(t; \phi, r) = \theta(\rho(t; r); \phi)$ and $\tau^h = \tau^h(r) = \inf\{t \ge 0\colon\ r^h(t) = \gamma\}$. Then, by the skew product decomposition [5], P_x^h restricted to $\mathscr{F}_\tau$ [where $\mathscr{F}_t = \sigma(X(s), 0 \le s \le t)$] is the measure induced by the measure $\mathscr{P}$ under the map $\omega \to \{(r^h(s), \phi^h(s)),\ 0 \le s \le \tau^h\})$. We will work with $(r^h(\cdot), \phi^h(\cdot))$ on $(\hat{\Omega}, \mathscr{F}, \mathscr{P})$. Note that the exit time, τ^h, is independent of $\theta(\cdot)$. The harmonic measure $\mu_{(r,\phi)}^h$ is given by

$$\begin{aligned}&\mu_{(r,\phi)}^h(dy)\\ &\quad = \mathscr{P}\big(\phi^h(\tau^h(r); \phi, r) \in dy\big) = \mathscr{P}\big(\theta\big(\rho(\tau^h(r); r); \phi\big) \in dy\big).\end{aligned} \tag{2.13}$$

Since $r^h(\cdot)$ and τ^h are independent of $\theta(\cdot)$, it follows that, conditioned on $r^h(\cdot)$, the distribution of $\theta(\rho(\tau^h(r); r); \phi)$ is equal to that of $\theta(\hat{\rho}; \phi)|_{\hat{\rho}=\rho(\tau^h(r); r)}$. But $\theta(t; \phi)$ is ergodic on S^{d-1} and $\mathscr{P}(\theta(t; \phi) \in dy)$ converges weakly as $t \to \infty$ to normalized Lebesgue measure on S^{d-1}. By symmetry, it is clear that this convergence is uniform in ϕ. Thus, if we show that for some sequence $\{r_n\}_{n=1}^{\infty}$ converging to ∞,

$$\lim_{n\to\infty} \rho(\tau^h(r_n), r_n) = \lim_{n\to\infty} \int_0^{\tau^h(r_n)} \frac{ds}{(r^h(s; r_n))^2} = \infty \quad \text{a.s. } [\mathscr{P}], \tag{2.14}$$

then it will follow from (2.13) and the first sentence in the proof of this lemma that $\mu^h_{r,\phi}$ converges weakly to normalized Lebesgue measure, uniformly over $\phi \in S^{d-1}$.

To prove (2.14), we will utilize scaling. Define $\tilde{r}^h(t; r) = (1/k)r^h(k^2t; kr)$. From the homogeneity of the generator $\frac{1}{2}(d^2/dr^2 - (d-3)/r\, d/dr)$, one can check readily that $\tilde{r}^h(\cdot; r) =_d r^h(\cdot; r)$. For $a < r$, define $\sigma_a^r = \inf\{t \geq 0: r^h(t; r) = a\}$ and $\tilde{\sigma}_a^r = \inf\{t \geq 0: \tilde{r}^h(t; r) = a\}$. Then we have

$$\begin{aligned} \int_0^{\sigma_a^r} \frac{ds}{(r^h(s; r))^2} &=_d \int_0^{\tilde{\sigma}_a^r} \frac{ds}{(\tilde{r}^h(s; r))^2} \\ &= \int_0^{\tilde{\sigma}_a^r} \frac{k^2\, ds}{(r^h(k^2 s; kr))^2} \\ &= \int_0^{k^2\tilde{\sigma}_a^r} \frac{dt}{(r^h(t; kr))^2}. \end{aligned} \tag{2.15}$$

However,

$$k^2\tilde{\sigma}_a^r = k^2 \inf\left\{t \geq 0: \frac{1}{k} r^h(k^2 t; kr) = a\right\} = \inf\{t \geq 0: r^h(t; kr) = ka\} = \sigma_{ka}^{kr}.$$

Thus, from (2.15), we conclude that

$$\int_0^{\sigma_a^r} \frac{ds}{(r^h(s; r))^2} =_d \int_0^{\sigma_{ka}^{kr}} \frac{ds}{(r^h(s; kr))^2}. \tag{2.16}$$

Now, if $2^n\gamma \leq r < 2^{n+1}\gamma$, define

$$\sigma^n = \inf\{t \geq 0: r^h(t; r) = 2^n\gamma\},$$

$$\sigma_j = \inf\{t \geq \sigma_{j+1}: r^h(t; r) = 2^j\gamma\} \quad \text{for } j = n-1, n-2, \dots 0.$$

Thus $\tau^h = \sigma_0$. We have

$$\int_0^{\tau^h} \frac{ds}{(r^h(s; r))^2} = \int_0^{\sigma_n} \frac{ds}{(r^h(s; r))^2} + \sum_{j=0}^{n-1} \int_{\sigma_{n-j}}^{\sigma_{n-j-1}} \frac{ds}{(r^h(s; r))^2}. \tag{2.17}$$

From (2.16), (2.17) and the strong Markov property, we conclude that

$$\int_0^{\tau^h} \frac{ds}{\left(r^h(s;r)\right)^2} = Y + \sum_{j=1}^{n} X_j, \tag{2.18}$$

where the X_j's are positive i.i.d. random variables and Y is nonnegative and independent of the X_i's. Since, by Borel–Cantelli, the sum of n positive i.i.d. random variables converges to ∞ almost surely as $n \to \infty$, we obtain from (2.18) that

$$\lim_{r\to\infty} \int_0^{\tau^h} \frac{ds}{\left(r^h(s;r)\right)^2} = \infty$$

in probability and (2.14) follows. This proves the lemma in the special case that ∂D is a sphere.

We now turn to the general case. We abandon $(\hat{\Omega}, \mathscr{F}, \mathscr{P})$ and return to our original notation. Pick γ so that the γ-ball B_γ encloses ∂D. Recall that $\tau_\gamma = \inf\{t \geq 0\colon |X(t)| = \gamma\}$. Assume $X(0) = x = (r, \phi)$ with $r > \gamma$. Then, by the strong Markov property, for any $g \in C(\partial D)$,

$$\int_{\partial D} g(y)\mu_x^h(dy) = E_x^h g(X(\tau)) = E_x^h E_{X(\tau_\gamma)}^h g(X(\tau)) = E_x^h H(X(\tau_\gamma)),$$

where $H(y) = E_y^h g(X(\tau))$. Since $X(t)$ is Feller under $P_\cdot^h$, it follows from (2.2) that $H(y)$ is continuous. [Alternatively, one could appeal to elliptic regularity theory since $H(y)$ is harmonic for Δ^h and Δ^h has C^∞-coefficients.] Thus, to prove the lemma in the general case, it suffices to show that as $r \to \infty$ the measures $P_x^h(X(\tau_\gamma) \in dy) = P_{(r,\phi)}^h(X(\tau_\gamma) \in dy)$ converge weakly, uniformly in ϕ, to $h(y)l(dy)/\int_{\partial B_\gamma} h(z)l(dz)$.

Let $f \in C(\partial B_\gamma)$ and recall that $h(y) = P_y(\tau < \infty)$. Of course, h is continuous for the same reason H is. Now P^h is, up to normalization, nothing but P restricted to those paths which eventually reach ∂D. Since $\tau_\gamma < \tau$ on $\{\tau_\gamma < \infty\}$ a.s. P_x, we have, by the strong Markov property,

$$\begin{aligned} E_x^h f(X(\tau_\gamma)) &= \frac{E_x\left(f(X(\tau_\gamma))I_{\tau<\infty}\right)}{P_x(\tau<\infty)} = \frac{E_x\left(f(X(\tau_\gamma))I_{\tau_\gamma<\infty}I_{\tau-\tau_\gamma<\infty}\right)}{P_x(\tau<\infty)} \\ &= \frac{E_x\left(f(X(\tau_\gamma))I_{\tau_\gamma<\infty}E_{X(\tau_\gamma)}I_{\tau<\infty}\right)}{P_x(\tau<\infty)} \\ &= \frac{E_x\left(f(X(\tau_\gamma))h(X(\tau_\gamma))I_{\tau_\gamma<\infty}\right)}{P_x(\tau_\gamma<\infty)} \frac{P_x(\tau_\gamma<\infty)}{P_x(\tau<\infty)} \\ &= \frac{E_x\left(f(X(\tau_\gamma))h(X(\tau_\gamma));\tau_\gamma<\infty\right)}{P_x(\tau_\gamma<\infty)} \left[\frac{E_x\left(h(X(\tau_\gamma));\tau_\gamma<\infty\right)}{P_x(\tau_\gamma<\infty)}\right]^{-1}. \end{aligned} \tag{2.19}$$

But the distribution of $X(\cdot)$ under $P_x(\cdot\,;\tau_\gamma < \infty)/P_x(\tau_\gamma < \infty)$ is that of Brownian motion conditioned to reach ∂B_γ. By the first part of the proof, we know that for Brownian motion conditioned to reach ∂B_γ, the distribution of $X(\tau_\gamma)$ converges weakly as $r \to \infty$, uniformly in ϕ, to normalized Lebesgue measure on ∂B_γ. Thus, as $r \to \infty$, the rightmost expression in (2.19) converges, uniformly in ϕ, to

$$\frac{\int_{\partial B_\gamma} f(y)h(y)l(dy)}{\int_{\partial B_\gamma} h(y)l(dy)}$$

and the same is therefore true of the left-hand side of (2.19). This completes the proof of the lemma.

We now prove (2.12). Recall that

$$\begin{aligned} I_2(n,t,\epsilon) &= E^\varepsilon_{x_0}\big(u_\varepsilon(X(\tau_n)); \tau_n < \tau \wedge t\big) \\ &= E^\varepsilon_{x_0}\big(E^\varepsilon_{X(\tau_n)}\psi(X(\tau)); \tau_n < \tau \wedge t\big). \end{aligned} \tag{2.20}$$

From Lemma 1, we know that for large r, uniformly in ϕ, the distribution of $X(\tau)$ under P^h_x, $x = (r,\phi)$, is close to μ^h_∞. The main step in the proof of (2.12) is to show that for small ε, the above statement is also true with P^ε_x replacing P^h_x. Let $n_\delta = n_\delta(x_0)$ and $t_\delta = t_\delta(x_0)$ be as defined in (2.4) and (2.6). Pick $m_\delta > n_\delta$ so large that

$$\frac{P_x(\tau_m < \tau < \infty)}{P_x(\tau < \infty)} < \frac{\delta}{4\|\psi\|}$$

for all x with $|x| = n_\delta$, if $m \ge m_\delta$. This is clearly possible since $P.$ is Feller. [Note that $m_\delta = m_\delta(x_0)$ since $n_\delta = n_\delta(x_0)$.] Now increase $t_\delta = t_\delta(x_0)$ if necessary so that

$$\frac{P_x(\tau_m \wedge t_\delta < \tau < \infty)}{P_x(\tau < \infty)} < \frac{\delta}{2\|\psi\|} \quad \text{for all } x \text{ with } |x| = n_\delta \text{ and } m \ge m_\delta. \tag{2.21}$$

Increase $t_\delta = t_\delta(x_0)$ again if necessary so that

$$\sup_{|x|=n_\delta} \frac{P^\varepsilon_x\big(t_\delta < \tau < \tau_{m_\delta}\big)}{P^\varepsilon_x(\tau < \tau_{m_\delta})} < \frac{\delta}{2\|\psi\|} \quad \text{for all small } \varepsilon. \tag{2.22}$$

A proof of the existence of such a t_δ can be given as follows. Since one can certainly guarantee (2.22) for P in place of P^ε, it suffices to show that

$$\begin{aligned} \lim_{\varepsilon\to 0} P^\varepsilon_x(\tau < \tau_{m_\delta}) &= P_x(\tau < \tau_{m_\delta}), \\ \lim_{\varepsilon\to 0} P^\varepsilon_x\big(t_\delta < \tau < \tau_{m_\delta}\big) &= P_x\big(t_\delta < \tau < \tau_{m_\delta}\big). \end{aligned}$$

But $P^\varepsilon_x(\tau < \tau_{m_\delta}) = P^\varepsilon_x(\tau < \tau_{m_\delta} \wedge T) + P^\varepsilon_x(T < \tau < \tau_{m_\delta})$ and, by weak convergence along with (2.1) and (2.2), the limit of the first term is $P_x(\tau < \tau_{m_\delta} \wedge T)$ which converges to $P_x(\tau < \tau_{m_\delta})$ as $T \to \infty$. This and a similar calculation for $P^\varepsilon_x(t_\delta < \tau < \tau_{m_\delta})$ reveal that it suffices to show that for some $\varepsilon_0 > 0$,

$$\lim_{T\to\infty} \sup_{0<\varepsilon\le\varepsilon_0} \sup_{|x|=n_\delta} P^\varepsilon_x\big(\tau \wedge \tau_{m_\delta} > T\big) = 0. \tag{2.23}$$

To this end, pick $N > m_\delta$, let ϕ_0, chosen to be positive, denote the lead eigenfunction of $-\frac{1}{2}\Delta$ on B_N, the ball of radius N centered at the origin, with the Dirichlet boundary condition on ∂B_N and let $\lambda > 0$ denote the corresponding eigenvalue. Then

$$\sup_{|x|\le m_\delta} \frac{L_\varepsilon \phi_0}{\phi_0} = \sup_{|x|\le m_\delta} \frac{\frac{1}{2}\Delta\phi_0 + \varepsilon b \cdot \nabla\phi_0}{\phi_0} \le -\lambda + \varepsilon M,$$

where $M = \sup_{|x|\le m_\delta} |b \cdot \nabla\phi_0|/\phi_0$. Thus, for $\varepsilon \le \varepsilon_0 \equiv \lambda/2M$, we have

$$\sup_{|x|\le m_\delta} \frac{L_\varepsilon \phi_0}{\phi_0} \le \frac{-\lambda}{2}.$$

Now define $\phi \in C^2(R^d)$ such that $\phi(x) = \phi_0(x)$ for $|x| \le m_\delta$, $\inf_{x\in R^d} \phi(x) > 0$ and

$$\inf_{x\in R^d} \frac{L_\varepsilon \phi}{\phi}(x) > -\infty \quad \text{for } \varepsilon \le \varepsilon_0. \tag{2.24}$$

(For example, let $\phi \equiv c > 0$ for large $|x|$.) We have

$$\sup_{|x|\le m_\delta} \frac{L_\varepsilon \phi}{\phi}(x) \le \frac{-\lambda}{2}. \tag{2.25}$$

The Feynman–Kac formula and (2.24) yield

$$E_x^\varepsilon \exp\left(-\int_0^t \frac{L_\varepsilon\phi}{\phi}(X(s))\,ds\right)\phi(X(t)) = \phi(x) \quad \text{for } |x| \le m_\delta. \tag{2.26}$$

Using (2.25) and (2.26), we obtain for $\varepsilon \le \varepsilon_0$ the Chebyshev estimate

$$\begin{aligned} P_x^\varepsilon(\tau \wedge \tau_{m_\delta} > T) &\le \exp\left(T \sup_{|y|\le m_\delta} \frac{L_\varepsilon\phi}{\phi}\right) E_x^\varepsilon \exp\left(-\int_0^T \frac{L_\varepsilon\phi}{\phi}(X(s))\,ds\right) \\ &\le \frac{e^{-\lambda T/2}}{\inf_{y\in R^d}\phi(y)} E_x^\varepsilon(\exp\left(-\int_0^T \frac{L_\varepsilon\phi}{\phi}(X(s))\,ds\right)\phi(X(T)) \\ &= \frac{\phi(x)}{\inf_{y\in R^d}\phi(y)} e^{-\lambda T/2}. \end{aligned}$$

This proves (2.23) and consequently justifies (2.22).

The weak convergence of P_x^ε to P_x is in fact uniform over $\{|x| = n_\delta\}$. To prove this, assume to the contrary that there exists a continuous bounded function $f\colon \Omega \to R$, an $\alpha > 0$, a sequence $\{\phi_m\}_{m=1}^\infty \subset S^{d-1}$ and a sequence $\varepsilon_m \downarrow 0$ such that $|\int f\,dP^{\varepsilon_m}_{n_\delta,\phi_m} - \int f\,dP_{n_\delta,\phi_m}| > \alpha$ for all m. We may assume that ϕ_m converges to some $\phi_0 \in S^{d-1}$. But by the Feller property, $P_{n_\delta,\phi_m} \Rightarrow_w P_{n_\delta,\phi_0}$ as $m \to \infty$ and by [8], Theorem 11.4, $P^{\varepsilon_m}_{n_\delta,\phi_m} \Rightarrow_w P_{n_\delta,\phi_0}$ as $m \to \infty$. This contradicts the above inequality and proves the claim. From (2.1) and (2.2) along with the uniform weak convergence of P_x^ε to P_x, we obtain

$$\lim_{\varepsilon\to 0}\sup_{|x|=n_\delta}\left|\frac{E_x^\varepsilon\big(\psi(X(\tau));\tau<\tau_{m_\delta}\wedge t_\delta\big)}{P_x^\varepsilon\big(\tau<\tau_{m_\delta}\wedge t_\delta\big)} - \frac{E_x\big(\psi(X(\tau));\tau<\tau_{m_\delta}\wedge t_\delta\big)}{P_x\big(\tau<\tau_{m_\delta}\wedge t_\delta\big)}\right| = 0. \tag{2.27}$$

From (2.21) and the fact that P_x^h is P_x conditioned on $\tau<\infty$, we obtain

$$\begin{aligned}
&\left|E_x^h\psi(X(\tau)) - \frac{E_x\big(\psi(X(\tau));\tau<\tau_{m_\delta}\wedge t_\delta\big)}{P_x\big(\tau<\tau_{m_\delta}\wedge t_\delta\big)}\right| \\
&\quad=\left|\frac{E_x\big(\psi(X(\tau));\tau<\infty\big)}{P_x(\tau<\infty)} - \frac{E_x\big(\psi(X(\tau));\tau<\tau_{m_\delta}\wedge t_\delta\big)}{P_x\big(\tau<\tau_{m_\delta}\wedge t_\delta\big)}\right| \\
&\quad=\left|\frac{E_x\big(\psi(X(\tau));\tau_{m_\delta}\wedge t_\delta<\tau<\infty\big)P_x\big(\tau<\tau_{m_\delta}\wedge t_\delta\big)}{P_x(\tau<\infty)P_x\big(\tau<\tau_{m_\delta}\wedge t_\delta\big)}\right. \\
&\qquad\left.-\frac{P_x\big(\tau_{m_\delta}\wedge t_\delta<\tau<\infty\big)E_x\big(\psi(X(\tau));\tau<\tau_{m_\delta}\wedge t_\delta\big)}{P_x(\tau<\infty)P_x\big(\tau<\tau_{m_\delta}\wedge t_\delta\big)}\right| \\
&\quad\le\frac{2\|\psi\|P_x\big(\tau_{m_\delta}\wedge t_\delta<\tau<\infty\big)}{P_x(\tau<\infty)}<\delta\quad\text{if } |x|=n_\delta.
\end{aligned}\tag{2.28}$$

Similarly, by (2.22), we have

$$\left|\frac{E_x^\varepsilon\big(\psi(X(\tau));\tau<\tau_{m_\delta}\wedge t_\delta\big)}{P_x^\varepsilon\big(\tau<\tau_{m_\delta}\wedge t_\delta\big)} - \frac{E_x^\varepsilon\big(\psi(X(\tau));\tau<\tau_{m_\delta}\big)}{P_x^\varepsilon\big(\tau<\tau_{m_\delta}\big)}\right|<\delta. \tag{2.29}$$

Now the key observation is this. There exists a probability measure $\mu(dy) = \mu(dy;\varepsilon,m_\delta,n_\delta,x)$ on ∂B_{n_δ} such that, for $|x|=n_\delta$,

$$E_x^\varepsilon\psi(X(\tau)) = \int_{\partial B_{n_\delta}}\frac{E_y^\varepsilon\big(\psi(X(\tau));\tau<\tau_{m_\delta}\big)}{P_y^\varepsilon\big(\tau<\tau_{m_\delta}\big)}\mu(dy;\varepsilon,m_\delta,n_\delta,x). \tag{2.30}$$

This follows from the strong Markov property and the fact that P_x^ε is recurrent.

Now, using (2.30), (2.29), (2.27), (2.28) and Lemma 1 in that order, we obtain for $x=(n_\delta,\phi)$,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\sup_{\phi\in S^{\phi-1}}\left|E_{(n_\delta,\phi)}^\varepsilon\psi(X(\tau)) - \int_{\partial D}\psi(y)\mu_\infty^h(dy)\right| = 0. \tag{2.31}$$

From (2.4) and (2.1), $P_{x_0}(\tau > \tau_{n_\delta} \wedge t_\delta) - P_{x_0}(\tau = \infty) < \delta$ and thus $P_{x_0}(\tau > \tau_{n_\delta}) - P_{x_0}(\tau = \infty) < \delta$. Hence, by picking t_δ larger if necessary, we may assume that $|P_{x_0}(\tau \wedge t_\delta > \tau_{n_\delta}) - P_{x_0}(\tau = \infty)| < 2\delta$. This fact, and weak convergence along with (2.1) and (2.2) give

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left| P^{\varepsilon}_{x_0}(\tau \wedge t_\delta > \tau_{n_\delta}) - P_{x_0}(\tau = \infty) \right| = 0. \tag{2.32}$$

Using (2.31) and (2.32) in (2.20), we conclude that

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} I_2(n_\delta, t_\delta, \varepsilon) = P_{x_0}(\tau = \infty) \int_{\partial D} \psi(y) \mu^h_\infty(dy).$$

This completes the proof of Theorem 1. □

3. Proof of Theorem 2. We introduce the following notation. Let

$$J^M = \inf_{\substack{u \in W^{1,2}_{\mathrm{loc}}(D) \\ u = \psi \text{ on } \partial D \\ |u| \le M}} \int_D |\nabla u|^2 \, dx \quad \text{for } M \ge \|\psi\|.$$

Also, if $D^c \subset B_n$, where B_n is the ball of radius n centered at the origin, let

$$J_n = \inf_{\substack{u \in W^{1,2}_{\mathrm{loc}}(D \cap B_n) \\ u = \psi \text{ on } \partial D}} \int_{D \cap B_n} |\nabla u|^2 \, dx.$$

Since, for all n, $J_n \le \int_D |\nabla \tilde{u}|^2 \, dx$, for any function $\tilde{u} \in W^{1,2}_{\mathrm{loc}}(D)$ satisfying $\tilde{u} = \psi$ on ∂D, it follows that $\limsup_{n \to \infty} J_n < \infty$. As is well known, J_n is attained at u_n, where u_n solves $\Delta u_n = 0$ in $D \cap B_n$, $u_n = \psi$ on ∂D and $\nabla u_n \cdot \nu = 0$ on ∂B_n, where ν is the outward unit normal on ∂B_n. By the maximum principle, $\|u_n\|_\infty = \|\psi\|_\infty$. To prove Theorem 2, we utilize the following lemmas.

LEMMA 2. *To prove Theorem* 2, *it suffices to show that J^M is attained at u_0 for every $M \ge \|\psi\|_\infty$.*

PROOF. Assume that Theorem 2 does not hold. Then there exists a bounded $\tilde{u} \not\equiv u_0$ satisfying $\tilde{u} = \psi$ on ∂D and such that $\int_D |\nabla \tilde{u}|^2 \, dx < \int_D |\nabla u_0|^2 \, dx$. With $M = \|\tilde{u}\|_\infty$, this contradicts the assumption that J^M is attained at u_0. □

LEMMA 3. $J^M \ge \limsup_{n \to \infty} J_n$, *for all $M \ge \|\psi\|_\infty$.*

PROOF. For $\varepsilon > 0$, pick $\tilde{u}_\varepsilon \in W^{1,2}_{\mathrm{loc}}(D)$ with $\tilde{u}_\varepsilon = \psi$ on ∂D, $\|\tilde{u}_\varepsilon\|_\infty \le M$ and such that $J^M \ge \int_D |\nabla \tilde{u}_\varepsilon|^2 \, dx - \varepsilon$. Then for any n satisfying $D^c \subset B_n$,

$$J^M \ge \int_D |\nabla \tilde{u}_\varepsilon|^2 \, dx - \varepsilon \ge \int_{D \cap B_n} |\nabla \tilde{u}_\varepsilon|^2 \, dx - \varepsilon \ge J_n - \varepsilon.$$

Thus $\limsup_{n \to \infty} J_n \le J^M + \varepsilon$ and the lemma follows since ε is arbitrary. □

LEMMA 4. $\lim_{n\to\infty} u_n = u_0$, *where* u_0 *is as in Theorem* 1.

As the proof of Lemma 4 is a bit involved, we postpone it until after the completion of the proof of Theorem 2, which goes as follows: Fix m such that $D^c \subset B_m$ and let $n > m$. Then

$$J_n = \int_{D\cap B_n} |\nabla u_n|^2\,dx \ge \int_{D\cap B_m} |\nabla u_n|^2\,dx, \qquad n > m. \tag{3.1}$$

Since $\limsup_{n\to\infty} J_n < \infty$ and since $\|u_n\|_\infty = \|\psi\|_\infty$ for all n, it follows that $\{u_n\}_{n=1}^\infty$ is weakly compact in $W^{1,2}(D \cap B_m)$. On the other hand, by Lemma 4, u_n converges pointwise to u_0. Thus, in fact, u_n converges weakly to u_0 in $W^{1,2}(D \cap B_m)$ as $n \to \infty$. Since the norm cannot increase in the limit under weak convergence, we obtain from (3.1)

$$\int_{D\cap B_m} |\nabla u_0|^2\,dx \le \liminf_{n\to\infty} J_n. \tag{3.2}$$

[Of course, one could use Schauder estimates to conclude that $u_n \to u_0$ strongly in $W^{1,2}(D \cap B_m)$ but we do not need this.] Since m is arbitrary, we conclude that

$$\int_D |\nabla u_0|^2\,dx \le \liminf_{n\to\infty} J_n. \tag{3.3}$$

From Lemma 3 and (3.3) we conclude that in fact

$$J^M = \int_D |\nabla u_0|^2\,dx \quad \text{for } M \ge \|\psi\|_\infty. \tag{3.4}$$

Theorem 2, except for the uniqueness, now follows from (3.4) and Lemma 2.

The proof of uniqueness is standard. Any minimizer $\tilde{u}$ must satisfy

$$\int_D \nabla\tilde{u}\,\nabla q\,dx = 0 \tag{3.5}$$

for all bounded $q \in W^{1,2}_{\mathrm{loc}}(D)$ which satisfy $q = 0$ on ∂D. Thus, if u_0 and, say, $\tilde{u}$ are both minimizers, then $\int_D(\nabla\tilde{u} - \nabla u_0)\nabla q\,dx = 0$ for all q as in (3.5). In particular picking $q = \tilde{u} - u_0$, we obtain $\int_D|\nabla\tilde{u} - \nabla u_0|^2\,dx = 0$. Since $\tilde{u} = u_0 = \psi$ on ∂D, we conclude that $\tilde{u} = u_0$ a.s. This completes the proof of Theorem 2. □

We now give the proof of Lemma 4.

PROOF OF LEMMA 4. Let Ω, $X(\cdot) = (r(\cdot), \phi(\cdot))$ and τ be as defined in the paragraph following (1.2) of Section 1. Also let $\tau_\gamma = \inf\{t \ge 0\colon X(t) \in \partial B_\gamma\}$, where $B_\gamma = \{|x| \le \gamma\}$. Let P_x^n be the measure induced by d-dimensional Brownian motion in B_n starting from $x \in B_n$ and reflected at ∂B_n and let P_x

denote d-dimensional Wiener measure. Then u_n is given by $u_n(x) = E_x^n\psi(X(\tau))$. By the strong Markov property, we have

(3.6) $$u_n(x) = E_x^n(\psi(X(\tau)); \tau < \tau_n) + E_x^n(E_{X(\tau_n)}^n\psi(X(\tau)); \tau_n < \tau).$$

Clearly,

(3.7) $$\lim_{n\to\infty} E_x^n(\psi(X_n(\tau)); \tau < \tau_n) = E_x(\psi(X(\tau)); \tau < \infty)$$

and

(3.8) $$\lim_{n\to\infty} P_x^n(\tau < \tau_n) = P_x(\tau < \infty) = h(x),$$

where $h(x)$ is in the notation of the first two sections. In light of the formula for u_0 which appears between formulas (2.11) and (2.12), the proof of Lemma 4 will follow from (3.6)–(3.8) if we show that

(3.9) $$\operatorname*{w-lim}_{n\to\infty} P_{(n,\phi)}^n(X(\tau) \in dy) = \mu_\infty^h(dy), \quad \text{uniformly over } \phi \in S^{d-1}.$$

Assume that (3.9) holds in the case that ∂D is a sphere, that is, $D = \{|x| > \gamma\}$. Recall that if $\partial D = \partial B_\gamma$, then $\mu_\infty^h(dy) = \mu_\infty^h(dr, d\phi) = \delta_\gamma(dr)\bar{l}(d\phi)$, where $\delta_\gamma(dr)$ is the atomic probability measure at γ and $\bar{l}(d\phi)$ is normalized Lebesgue measure on S^{d-1}. Under this assumption, we will prove (3.9) for general D; then we will return to prove (3.9) in the case that ∂D is a sphere.

We introduce some notation. Fix γ such that $D^c \subset B_\gamma$. Let

$$\mu_{r,\phi}^n(dy) = P_{(r,\phi)}^n(X(\tau) \in dy) \quad \text{for } \gamma \le r \le n.$$

For $n > \gamma$, let $\mu_{n,\phi;\gamma}^n(d\phi) = P_{(n,\phi)}^n(\phi(\tau_\gamma) \in \partial\phi)$. Now (3.9) reads

(3.9)′ $$\operatorname*{w-lim}_{n\to\infty} \mu_{n,\phi_0}^n(d\phi) = \mu_\infty^h(d\phi), \quad \text{uniformly over } \phi_0 \in S^{d-1}.$$

The assumption that (3.9) holds in the case that D is a ball reads

(3.10) $$\operatorname*{w-lim}_{n\to\infty} \mu_{n,\phi_0;\gamma}^n(d\phi) = \bar{l}(d\phi), \quad \text{uniformly over } \phi_0 \in S^{d-1}.$$

Let $h_n(\gamma,\phi) = P_{\gamma,\phi}^n(\tau < \tau_n)$ and let $\bar{\mu}_{\gamma,\phi_0}^n(d\phi) = P_{\gamma,\phi_0}^n(X(\tau) \in d\phi | \tau < \tau_n)$. Then, as in (3.7) and (3.8),

(3.11) $$\operatorname*{w-lim}_{n\to\infty} \bar{\mu}_{\gamma,\phi_0}^n(d\phi) = \mu_{\gamma,\phi_0}^h(d\phi)$$

and

(3.12) $$\lim_{n\to\infty} h_n(\gamma,\phi) = h(\gamma,\phi).$$

Finally, let

$$\bar{\nu}_{\gamma,\phi_0}^n(\partial\phi) = P_{(\gamma,\phi_0)}^n(\phi(\tau_n) \in d\phi | \tau_n < \tau).$$

Let $f \in C(\partial D)$. By the strong Markov property,

(3.13) $$\int_{\partial D} f(\phi)\mu_{n,\phi_0}^n(d\phi) = \int_{S^{d-1}} \mu_{n,\phi_0;\gamma}^n(ds)\int_{\partial D} f(\phi)\mu_{\gamma,s}^n(d\phi).$$

But, again by the strong Markov property,

$$\int_{\partial D} f(\phi)\mu^n_{\gamma,s}(d\phi) = h_n(\gamma,s)\int_{\partial D} f(\phi)\bar{\mu}^n_{\gamma,s}(d\phi) + (1-h_n(\gamma,s))\int_{S^{d-1}}\int_{\partial D} f(\phi)\mu^n_{n,t}(d\phi)\bar{\nu}^n_{\gamma,s}(dt). \tag{3.14}$$

Substituting (3.14) into (3.13) gives

$$\begin{aligned}\int_{\partial D} f(\phi)\mu^n_{n,\phi_0}(d\phi) &= \int_{S^{d-1}}\mu^n_{n,\phi_0;\gamma}(ds)h_n(\gamma,s)\int_{\partial D} f(\phi)\bar{\mu}^n_{\gamma,s}(d\phi)\\ &\quad+\int_{S^{d-1}}\mu^n_{n,\phi_0;\gamma}(ds)(1-h_n(\gamma,s))\\ &\quad\times\int_{S^{d-1}}\int_{\partial D} f(\phi)\mu^n_{n,t}(d\phi)\bar{\nu}^n_{\gamma,s}(dt)\\ &\equiv A^1_n + R^1_n.\end{aligned} \tag{3.15}$$

Now $h_n(\gamma,\phi)$ is increasing in n and $h(\gamma,\phi)$ is continuous. Thus, by Dini's theorem, the convergence in (3.12) is uniform over $\phi\in S^{d-1}$. The convergence in (3.11) is also uniform over $\phi\in S^{d-1}$. Indeed, a straightforward calculation reveals that the convergence in (3.11) will be uniform as long as $P_{\gamma,\phi}(\tau_n<\tau<\infty)$ converges to 0 as $n\to\infty$ uniformly in ϕ. But, by the strong Markov property,

$$P_{\gamma,\phi}(\tau_n<\tau<\infty)\le \sup_{s\in S^{d-1}} P_{n,s}(\tau<\infty)\le \sup_{s\in S^{d-1}} P_{n,s}(\tau_r<\infty),$$

for any r satisfying $D^c\subset B_r\subset B_n$. But $P_{n,s}(\tau_r<\infty)=r^{d-2}/n^{d-2}$. The uniform convergence in (3.11) and (3.12) along with (3.10) give

$$A\equiv\lim_{n\to\infty}A^1_n=\int_{S^{d-1}} f(\phi)\mu^h_{\gamma,s}(d\phi)h(\gamma,s)\bar{l}(ds). \tag{3.16}$$

Now R^1_n satisfies

$$|R^1_n|\le\|f\|\int_{S^{d-1}}(1-h_n(\gamma,s))\mu^n_{n,\phi_0;\gamma}(ds).$$

Since $h_n(\gamma,s)$ is increasing and $h_n(\gamma,s)>0$ for all $n>\gamma$, it follows that there exists a $\delta>0$ such that

$$|R^1_n|\le\|f\|(1-\delta)\quad\text{if } n\ge\gamma+1. \tag{3.17}$$

Now, returning to (3.15), note that R^1_n contains the term $\int_{\partial D} f(\phi)\mu^n_{n,t}(d\phi)$ which is equal to the left-hand side of (3.15) with ϕ_0 replaced by t and thus is in fact equal to the entire right-hand side of (3.15) with ϕ_0 replaced by t and t replaced by another dummy variable. Substituting this for $\int_{\partial D} f(\phi)\mu^n_{n,t}(d\phi)$ in the expression for R^1_n gives

$$R^1_n = A^2_n + R^2_n, \tag{3.18}$$

where

$$\lim_{n\to\infty} A_n^2 = \int_{S^{d-1}} (1 - h(\gamma, s))\bar{l}(ds) \cdot A = (1-p)A, \tag{3.19}$$

with $p = \int_{S^{d-1}} h(\gamma, s)\bar{l}(ds)$ and

$$R_n^2 \le (1-\delta)^2 \|f\| \quad \text{for } n \ge \gamma + 1. \tag{3.20}$$

Iterating this procedure, we conclude from (3.15)–(3.20) that

$$\lim_{n\to\infty} \int_{\partial D} f(\phi)\mu_{n,\phi_0}^n(d\phi) = \frac{1}{p}A = \frac{\int_{S^{d-1}} f(\phi)\mu_{\gamma,s}^h(d\phi)h(\gamma,s)\bar{l}(ds)}{\int_{S^{d-1}} h(\gamma,s)\bar{l}(ds)}.$$

From the representation of $\mu_\infty^h(dy)$ given in (1.5), it follows that (3.9)′ holds. This completes the proof of the lemma under the assumption that the lemma is true when ∂D is a sphere.

We now return to prove the lemma in the case that $\partial D = \{|x| = \gamma\}$. It will be convenient to represent the reflected Brownian motion corresponding to the measure P^n in skew product form as follows: On a probability space $(\hat{\Omega}, \mathscr{F}, \mathscr{P})$, let $r_n(t) = r_n(t, \omega)$ be a one-dimensional diffusion on $[0, n]$ starting from n, with reflection at n and generated by $\frac{1}{2}d^2/dr^2 + ((d-1)/2r)\,d/dr$. Define $\rho_n(t) = \int_0^t ds/r_n^2(s)$. Let $\theta(t;\phi) = \theta(t,\omega;\phi)$ be a Brownian motion on S^{d-1} starting from $\phi \in S^{d-1}$ and independent of $r_n(t)$ and define $\phi_n(t) = \phi_n(t;\phi) = \theta(\rho_n(t);\phi)$. Then, by the skew product decomposition, the measure $P_{(n,\phi)}^n$ is the measure induced by the measure $\mathscr{P}$ under the map $\omega \to \{(r_n(t), \phi_n(t)), 0 \le t < \infty\}$. Let $\tau_\gamma^n = \inf\{t \ge 0: r_n(t) = \gamma\}$. Then, almost exactly as in the proof of Lemma 1, (3.9) will follow if we show that $\rho_n(\tau_\gamma^n)$ converges in probability to ∞ as $n \to \infty$. [We say "almost" exactly because in Lemma 1 we proved almost sure convergence; however, it is clear that convergence in probability is enough to guarantee (3.9).] The scaling argument used in the proof of Lemma 1 does not work here; because of the reflection, we do not end up with identically distributed random variables. We mention in passing that, on the other hand, the type of argument we give here could also be used in Lemma 1. It is enough to show that $\lim_{n\to\infty} \mathscr{E} e^{-\rho_n(\tau_\gamma^n)} = 0$, where $\mathscr{E}$ denotes expectation with respect to the measure $\mathscr{P}$. By the Feynman–Kac formula, $\mathscr{E} e^{-\rho_n(\tau_\gamma^n)} = u_n(n)$, where $u_n(r)$, $\gamma \le r \le n$, solves

$$\left(\frac{1}{2}\frac{d^2}{dr^2} + \frac{d-1}{2r}\frac{d}{dr} - \frac{1}{r^2}\right)u_n(r) = 0,$$

$\gamma \le r \le n$, $u_n(\gamma) = 1$ and $u_n'(n) = 0$. We must show that $\limsup_{n\to\infty} u_n(n) = 0$. We have $(r^{d-1}u_n'(r))' = 2r^{d-3}u_n(r)$ and, upon integrating and using $u_n'(n) = 0$, we obtain $r^{d-1}u_n'(r) = -\int_r^n 2s^{d-3}u_n(s)\,ds$. Integrating again and using $u_n(\gamma) = 1$ gives

$$u_n(r) = 1 - \int_1^r s^{1-d} \int_s^n 2z^{d-3}u_n(z)\,dz. \tag{3.21}$$

We now assume that $\limsup_{n\to\infty} u_n(n) > 0$ and will arrive at a contradiction.

Without loss of generality, assume that $\lim_{n \to \infty} u_n(n) > 0$. By the probabilistic representation of $u_n(x)$ and the strong Markov property, $u_n(x)$ is decreasing for $x \in [\gamma, n]$. Thus, our assumption implies that there exists a $\delta > 0$ such that $u_n(x) \geq \delta$ for all $\gamma \leq x \leq n$ and $n > \gamma$. Thus, from (3.21),

$$u_n(n) \leq 1 - \delta \int_1^n ds\, s^{1-d} \int_s^n 2z^{d-3}\, dz. \tag{3.22}$$

A simple calculation reveals that the right-hand side of (3.22) converges to $-\infty$ as $n \to \infty$, which contradicts the positivity of $u_n(n)$. This completes the proof of Lemma 4. □

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