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SPECIAL INVITED PAPER

THE RATE OF ESCAPE OF RANDOM WALK[1]

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Let $\{X_k\}$ be an i.i.d. sequence and define $S_n = X_1 + \cdots + X_n$. The problem is to determine for a given sequence $\{\beta_n\}$ whether $P\{|S_n| \le \beta_n \text{ i.o.}\}$ is 0 or 1. A history of the problem is given along with two new results for the case when $P\{X_1 \ge 0\} = 1$: (a) An integral test that solves the problem in case the summands satisfy Feller's condition for stochastic compactness of the appropriately normalized sums and (b) necessary and sufficient conditions for a sequence $\{\beta_n\}$ to exist such that $\liminf S_n/\beta_n = 1$ a.s.

1. Introduction. Let $X, X_1, X_2, \ldots$ be a sequence of nondegenerate, independent, identically distributed random variables taking values in $\mathbb{R}^d$ and having distribution function F. The random walk $\{S_n\}$ is defined by $S_n = X_1 + \cdots + X_n$. The problem of interest here is the rate of escape of $|S_n|$ to ∞.

The first question is whether $|S_n| \to \infty$; the random walk is said to be transient if it does. This problem was solved by Chung and Fuchs (1951) who gave a criterion in terms of φ, the characteristic function of X. A slightly more elegant form was obtained by Spitzer (1964) and Ornstein (1969) [see also Kesten and Spitzer (1965)]; it reads:

S_n is transient iff the real part of $(1 - \varphi(\theta))^{-1}$ is locally integrable at 0.

Once the random walk is determined to be transient, there are different ways to measure the rate of escape. One possibility is to attempt to find a sequence $\{\beta_n\}$ such that

$$\liminf_{n\to\infty} \frac{|S_n|}{\beta_n} = 1 \quad \text{a.s.} \tag{1.1}$$

When this is possible it provides much of the information about the rate of escape. Even so, it does not answer the question of whether $|S_n|$ eventually stays above β_n almost surely. Furthermore, it is often the case that for all sequences $\{\beta_n\}$, $\liminf \beta_n^{-1}|S_n|$ is either 0 or ∞ almost surely. Thus the most desirable way of measuring the rate of escape is to have a criterion that determines, for a given sequence $\{\beta_n\}$, whether $P\{|S_n| \le \beta_n \text{ i.o.}\}$ is 0 or 1.

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(Here i.o. means infinitely often and the Hewitt–Savage zero–one law guarantees that the probability is either 0 or 1.) Such a criterion is usually called an integral test and is often written in the form of an integral although we will use sums instead. Thus the integral tests we will consider here will be of the form

$$P\{|S_n| \le \beta_n \text{ i.o.}\} = \begin{cases} 0 & \text{iff } \sum_n d_n < \infty, \\ 1 & \text{iff } \sum_n d_n = \infty, \end{cases}$$

where $d_n = d_n(\beta_n, F)$ should be a sequence that can be readily determined when $\{\beta_n\}$ and F are given. In particular, it should not depend explicitly on knowing the distribution of S_n. To give some feeling for this, we will now discuss some of the known results. In most of these, there are some regularity conditions imposed on the sequence $\{\beta_n\}$; in some cases, these are important, but we will shorten the present discussion by ignoring them.

The first result was an integral test for simple random walk in $\mathbb{R}^d$ with $d \ge 3$. (This makes the random walk transient.) This test was obtained by Dvoretzky and Erdös (1951) and is equivalent to taking

$$d_n = \left(\frac{\beta_n}{\sqrt{n}}\right)^{d-2} \frac{1}{n} = \left(\frac{\beta_n}{\sqrt{n}}\right)^{d} \frac{1}{\beta_n^2}.$$

Note that in this case the convergence of d_n is not affected if β_n is multiplied by a constant. This means that

$$\liminf_{n\to\infty} \frac{|S_n|}{\beta_n} = \begin{cases} 0 & \text{iff } \sum_n d_n = \infty, \\ \infty & \text{iff } \sum_n d_n < \infty. \end{cases} \tag{1.2}$$

Similar tests were obtained by Takeuchi (1964) and Taylor (1967) when X is a stable random variable of index $\alpha < d$ not supported on a half-space and by Erickson (1976) for some distributions in the domain of attraction of such stable laws. The most complete result of this type is due to Griffin (1983b). His result includes the following: If X is in the domain of attraction of a genuinely d-dimensional stable law of index $\alpha < d$, $\alpha \ne 1$, the stable law is not supported on a half-space, and $EX = 0$ if it exists, then (1.2) is true if

$$d_n = \left(\frac{\beta_n}{a_n}\right)^{d} Q(\beta_n),$$

where $Q(x) = E(x^{-1}|X| \wedge 1)^2$ and a_n is the solution of $Q(a_n) = n^{-1}$. Note that in case $EX = 0$ and $EX^2 < \infty$, this gives the same test as that obtained by Dvoretzky and Erdös for simple random walk. There is one general result that is related to the Dvoretzky–Erdös test. It is due to Kesten (1978) who proved Erickson's conjecture that if $d \ge 3$, any random walk escapes at least as fast as simple random walk.

The next class of results we will discuss is concerned with the case where $d = 1$ and $P\{X \geq 0\} = 1$. The first result here was an integral test for nonnegative random variables in the domain of attraction of a nonnegative stable law obtained by Lipschutz (1956) under a slight additional assumption. Also see Fristedt (1964) and Breiman (1968) for an alternative approach. In this case, there is a "correct" norming sequence which is of order $n^{1/\alpha}(\log\log n)^{-(1-\alpha)/\alpha}$, where α is the index of the stable law. The sequence d_n for the integral test is

$$d_n = \frac{1}{n}\left(\frac{n^{1/\alpha}}{\beta_n}\right)^{\alpha/2(1-\alpha)} \exp\left(-c\left[\frac{\beta_n}{n^{1/\alpha}}\right]^{-\alpha/(1-\alpha)}\right),$$

where the constant c depends on the scale parameter of the stable law. Note that in this case changing β_n by a multiplicative factor may affect convergence and this is the reason that it is possible to find a norming sequence $\{\beta_n\}$ so that (1.1) holds. Fristedt and Pruitt (1971) proved that under the rather mild restriction that $EX^{\varepsilon} < \infty$ for some $\varepsilon > 0$, there is a norming sequence $\{\beta_n\}$ such that (1.1) holds. The sequence is defined by

$$\beta_n = \frac{\log\log n}{\eta(\gamma n^{-1}\log\log n)},$$

where η is the inverse function of the log of the reciprocal of Ee^{-uX}, and γ is a constant larger than 1. This norming sequence makes the lim inf in (1.1) a positive, finite constant but the fairly crude methods used gave little information as to its value. Bounds on the value of the lim inf were obtained by Zhang (1986) for fairly general norming sequences. Fristedt and Pruitt also showed that if

$$P\{X > x\} = \frac{1}{\log x}, \qquad x > e, \tag{1.3}$$

then for any norming sequence, the lim inf must be 0 or ∞ so that some condition is needed to obtain (1.1) even when $P\{X \geq 0\} = 1$. Some results that give information on how large a negative tail the distribution might have with the lim inf still as in (1.1) were obtained by Feller (1946), Kesten (1970), Klass (1976, 1977, 1982), Klass and Teicher (1977) and Pruitt (1981). The relation of these results to the present ones is explained below for the case of Feller's paper.

There are distributions in one dimension in the domain of attraction of a nonnegative stable law that are not covered by either of the above classes of results. This will occur if the negative tail of the distribution is positive but small compared to the positive tail and the positive tail resembles a stable tail. The lim inf problem was recently solved for a class of examples of this type in Pruitt (1989). These are the first results that show how the transition takes place between the two radically different types of behavior described above. An integral test is not yet available in this case even for the examples. It has recently been shown by Cox (1982) that in infinite-dimensional spaces there may be "correct" norming sequences even in the symmetric case so that one may not get these two different types of behavior in these spaces.

One of the main ingredients that is needed for the solution of this problem is good estimates of $P\{|S_n| \le x\}$ when this probability is small. In the first class of results described above, this is a local limit theorem type of estimate. The recent generalizations in this area that were obtained in Griffin (1983a) and in Griffin, Jain and Pruitt (1984) were fundamental to the recent improvements in the rate of escape results. In the second class of results when the summands are nonnegative, we need estimates of $P\{S_n \le x\}$ when this is small and this is a large (or moderate) deviation problem; we are now interested in a tail estimate instead of a local limit estimate. The recent estimates in Jain and Pruitt (1987) will be used in the present paper to provide much more general integral tests for this class of random walks. In the third class of results described above where the distribution is asymmetric without being too asymmetric, the estimation of $P\{|S_n| \le x\}$ is a combination local limit, large deviation problem. The interval $[-x, x]$ is in the tail but does not go all the way to ∞. Probability estimates of this type were obtained in Pruitt (1989) for the class of examples discussed there but the problem of doing this for more general distributions for the summands still remains.

There are two major new results in the present paper. Both are for the case of nonnegative summands. The first is an integral test that we believe to be completely general although it has only been proved in the case that the sums can be normalized so as to be stochastically compact. This is a very general class that includes all laws in the domain of attraction of any stable law. An integral test was not previously available except that obtained by Lipschutz (1956) in the domain of attraction setting under some additional restrictions. In fact, not even the exact value of the lim inf was known outside the domain of attraction setting. The test sequence takes the form

$$d_n = n^{-1} p_n \log \frac{3}{p_n}, \tag{1.4}$$

where $p_n = P\{S_n \le \beta_n\}$. This does not appear to meet the criteria for a good test since it depends on the distribution of S_n. However, it actually does since the exact asymptotic behavior of p_n is available in this situation in Jain and Pruitt (1987). This test is obtained as Theorem 1 in Section 3. As evidence that the test is valid in general, we will show that the convergent part of the test does work in general and in Theorem 3 in Section 4 we will also show that the test is valid in the extreme case when the tail of the distribution is so fat that there is no correct norming sequence for the lim inf problem. Of course, the integral test is not as delicate in this latter case. In the intermediate situation between the cases covered by Theorems 1 and 3, we do not yet have adequate estimates of p_n.

To demonstrate the usefulness of this integral test, we will give some results here for a class of examples which are the analogues of the improvements in the law of the iterated logarithm that follow from the Kolmogorov–Erdös integral test. We consider the family of distributions for the summands given by

$$P\{X > x\} = \frac{1}{x^{\alpha}}, \qquad x \geq 1.$$

Define

$$\xi_n(k, \varepsilon) = lln + \tfrac{3}{2} l_3 n + l_4 n + \cdots + (1 + \varepsilon) l_k n,$$

where lln is $\log\log n$ and $l_k n$ is the logarithm iterated k times; then let

$$\beta_n(k, \varepsilon) = \begin{cases} \alpha(1-\alpha)^{(1-\alpha)/\alpha}(\Gamma(1-\alpha))^{1/\alpha} n^{1/\alpha}(\xi_n(k,\varepsilon))^{-(1-\alpha)/\alpha}, & 0 < \alpha < 1, \\ n \log \dfrac{n}{\xi_n(k,\varepsilon)} - Cn, & \alpha = 1, \\ n \dfrac{\alpha}{\alpha - 1} - \dfrac{\alpha}{\alpha - 1}(\Gamma(2-\alpha))^{1/\alpha} n^{1/\alpha}(\xi_n(k,\varepsilon))^{(\alpha-1)/\alpha}, & 1 < \alpha < 2, \\ 2n - (2n\xi_n(k,\varepsilon)\log n)^{1/2}, & \alpha = 2, \end{cases}$$

where C is Euler's constant. Then for every $k \geq 4$ and every $\varepsilon > 0$,

$$P\{S_n \leq \beta_n(k, 0) \text{ i.o.}\} = 1 \quad \text{and} \quad P\{S_n \leq \beta_n(k, \varepsilon) \text{ i.o.}\} = 0.$$

As k increases, the difference between $\beta_n(k, 0)$ and $\beta_n(k, \varepsilon)$ becomes smaller so these bounds improve. These results follow from the corollary to Theorem 1. In order to obtain them, one needs to obtain good asymptotic expansions of the functions g and R that are defined below. This introduces some errors but these can be made small compared to the absolute difference between $\beta_n(k, \varepsilon)$ and $\beta_n(k + 1, \varepsilon)$ for $\varepsilon \geq 0$. These examples are already included in the cases covered by the integral test of Lipschutz except for $\alpha = 1$ and 2. [Note, however, that for $1 < \alpha < 2$, the probability estimate and the integral test in Lipschutz (1956) are missing a factor of $\alpha^{-1/(\alpha-1)}$ in the exponent and thus would give an incorrect result.]

We will give one example which is not in a domain of attraction. For this, we choose the Petersburg game which is discussed in Feller (1968). We take

$$P\{X = 2^k\} = 2^{-k}, \qquad k = 1, 2, \dots .$$

This should be somewhat closely related to the example given above for $\alpha = 1$. But it is not in the domain of attraction of the Cauchy and it turns out that there are some rather surprising differences. Now we should take

$$\beta_n(k, \varepsilon) = n \operatorname{Log} \frac{n}{\xi_n(k, \varepsilon)} - C(\lambda_n) n,$$

where Log denotes the logarithm to the base 2. The log is to the base 2 due to putting mass 2^{-k} at 2^k; if we had used mass $(e - 1)e^{-k}$ at e^k, then it would have been a natural log. The interesting feature is the coefficient $C(\lambda_n)$. Here λ_n is a sequence which tends to 0; it is defined in the corollary to Theorem 1.

The function C satisfies $C(2\lambda) = C(\lambda)$ and

$$C(\lambda) = 0.803979xxx,$$

where the digits xxx vary from 632 to 973. Thus C is almost constant but not quite! Since it is essential to have this term in $\beta_n(k, \varepsilon)$, this means that it is more difficult to actually compute $\beta_n(k, \varepsilon)$ in this example than in the ones given above. It appears that this phenomenon of coefficients that vary a little bit but not very much is to be expected when the distribution for the summands is one that can be normalized so as to be stochastically compact but is not in any domain of attraction. This phenomenon can also occur with branching processes; see Section 3 in Barlow and Perkins (1988) and the references therein.

We also want to mention the connection of Theorem 1 with the work of Feller (1946). Feller was interested in whether

$$P\{S_n \geq \beta_n \text{ i.o.}\} = 0 \text{ or } 1,$$

in the context where the summands have mean 0 and variance 1. He showed that the Kolmogorov–Erdös test works if

$$EX^2 1\{|X| > x\} = O\big((llx)^{-1}\big), \tag{1.5}$$

i.e., if $\beta_n = n^{1/2}\varphi_n$ then the test is in terms of

$$d_n = n^{-1}\varphi_n \exp\big(-\varphi_n^2/2\big).$$

Feller also shows that when (1.5) fails, the test still works but one should use $\beta_n = B_n^{1/2}\varphi_n$, where

$$B_n = EX^2 1\big\{|X| \leq n^{1/2}/(lln)^2\big\}.$$

Feller's problem is converted to the one we consider here by considering the transformation $X \to -X$. Then our integral test reproduces the results of Feller for the case where the support of the distribution is bounded above. In fact, we can show that the definition of B_n may be changed to the slightly more elegant

$$B_n = EX^2 1\{|X| \leq n^{1/2}\}.$$

It should be noted that although Feller claims his result is general, he actually tacitly assumes that the summands have a symmetric distribution. The general case has recently been obtained by Bai (1989) and he gives even more information about the possible choices for B_n. There is also an interesting difference in the methods used. Feller uses two levels of truncation and only needs good probability estimates for the sums of the terms coming from the center of the distribution. We are able to obtain good probability estimates for S_n without using truncation by making use of the one-sided boundedness of the support of the distribution.

The other new result resolves the question raised by Fristedt and Pruitt (1971) as to exactly when there is a right norming sequence for the lim inf problem. [A partial answer to this question is in Klass (1982).] The result is stated as Theorem 2 in Section 4 but we will try to give a little feeling for it here. Define

$$u_j = P\{X \le e^{j+1} | X > e^j\}. \tag{1.6}$$

Then the condition is that

$$\limsup_{j \to \infty} u_j \log r_j = \infty, \tag{1.7}$$

where r_j is the rank of u_j when ranked in decreasing order. The first observation is that there may be a correct norming sequence for the lim inf even if the tail of the distribution of the summands is slowly varying. To get some idea of the "dividing line" as to when there is a correct norming sequence, we consider the class of examples with

$$P\{X > x\} = \exp\left(- \frac{\log x}{(\log\log x)^{\alpha}} \right) \tag{1.8}$$

for large x. Then there is a norming sequence that gives (1.1) iff $\alpha < 1$. Note that all these examples have much thinner tails than the example of (1.3) given in Fristedt and Pruitt.

In the case where the lim inf must be 0 or ∞, the test is given in Theorem 3. The lim inf is 0 iff

$$\sum_n \left(P\{X > \beta_n\} \vee n^{-1}\right) e^{-nP\{X > \beta_n\}} = \infty.$$

[By using Lemmas 2 and 3, one may check that in this case this is the same test given by (1.4), but the form of this test was suggested by Theorem 3 of Kesten (1970).] It is interesting to note that this is exactly the same integral test obtained by Klass (1985) to solve the question of whether $M_n \le \beta_n$ i.o., where $M_n = \max(X_1, X_2, \dots, X_n)$. This raises the question of whether M_n and S_n are essentially the same when the tail of the distribution of the summands is this fat. This question was answered in Pruitt (1987), where necessary and sufficient conditions were obtained for

$$\liminf_{n \to \infty} \frac{M_n}{S_n} > 0. \tag{1.9}$$

Since this ratio is bounded above by 1, M_n and S_n will be comparable when (1.9) holds. The condition for (1.9) to hold is that

$$\sum_n u_n^{r+1} < \infty$$

for some r, where u_n is as in (1.6). Then if r is chosen so that this series converges but it diverges if the power is changed to r, the lim inf in (1.9) is r^{-1}. If one takes

$$P\{X > x\} = \exp\left(-(\log x)^{\alpha}\right), \qquad x > 1,$$

with $0 < \alpha < 1$, then $\liminf M_n/S_n > 0$ a.s., but if the tail is as in (1.8) with $\alpha \ge 1$, then this lim inf is 0 so that S_n and M_n are not always comparable and yet

$$P\{S_n \le \beta_n \text{ i.o}\} = P\{M_n \le \beta_n \text{ i.o.}\}$$

no matter what the sequence $\{\beta_n\}$ is.

A simpler version of the proof given here shows that (1.7) is also necessary and sufficient for a sequence $\{\beta_n\}$ to exist such that

$$\liminf_{n\to\infty} \frac{M_n}{\beta_n} = 1.$$

This means that S_n is normalizable iff M_n is normalizable. The "if" part of this statement is in Klass (1982).

An interesting feature of both the necessary and sufficient condition for normalizability and the determination of the lim inf of M_n/S_n is that they only depend on the sequence of probabilities $\{u_j\}$ and not on how the probability is distributed within the intervals $(e^j, e^{j+1}]$. Moreover, the probability may be rearranged by permuting these intervals in any way without changing the criteria and also any number of these intervals may be assigned zero probability with the other probabilities being assigned to later intervals without affecting the criteria. I do not have an intuitive explanation for these facts but it seems that such an explanation would be of interest.

Some notation and preliminary results are presented in Section 2. The integral test is given in Section 3. The necessary and sufficient conditions for normalizability of the lim inf are presented in Section 4.

2. Preliminaries. We start by collecting the results that we need from Jain and Pruitt (1987) on the lower tail of the distribution of S_n. Recall that X is nonnegative and nondegenerate. For $u > 0$, let

$$\varphi(u) = Ee^{-uX}, \qquad g(u) = -\frac{\varphi'(u)}{\varphi(u)}, \tag{2.1}$$
$$R(u) = -\log\varphi(u) - ug(u).$$

The functions φ, g and R are continuous on $(0, \infty)$, g is strictly decreasing, R is strictly increasing and

$$g'(u) = -V(u), \qquad R'(u) = uV(u), \tag{2.2}$$

where

$$V(u) = \frac{\varphi(u)\varphi''(u) - (\varphi'(u))^2}{(\varphi(u))^2}. \tag{2.3}$$

V is strictly positive since X is nondegenerate. The limiting behavior of g and R is given by

$$g(0) = EX, \qquad g(\infty) = a, \qquad R(0) = 0, \qquad R(\infty) = -\log q, \tag{2.4}$$

where $EX \le \infty$ and

$$a = \inf\{x > 0: P\{X \le x\} > 0\}, \qquad q = P\{X = a\}. \tag{2.5}$$

If $q = 0$, then $R(\infty) = \infty$. Furthermore,

$$u^2V(u) = O(R(u)), \qquad u \to 0. \tag{2.6}$$

The proof given in Jain and Pruitt (1987) actually shows that $u^2V(u) \le 4R(u)$ for small u and we will use this now. (In fact, the 4 may be reduced to $2 + \varepsilon$ with a little more care.) Note that

$$\frac{d}{du}\frac{R(u)}{u} = \frac{u^2V(u) - R(u)}{u^2} \le \frac{3}{4}V(u).$$

If $EX < \infty$ then $u^{-1}R(u) \to 0$ as $u \to 0$ and so in this case this leads to

$$g(0) - g(\lambda) = \int_0^\lambda V(u)\,du \ge \int_0^\lambda \frac{4}{3}\frac{d}{du}\frac{R(u)}{u} = \frac{4}{3}\frac{R(\lambda)}{\lambda}. \tag{2.7}$$

Next, let $a < x_n < EX$ and define λ_n by $g(\lambda_n) = x_n$. Then

$$P\{S_n \le nx_n\} \le \exp(-nR(\lambda_n)). \tag{2.8}$$

Moreover,

$$P\{S_n \le nx_n\} \to 0 \quad \text{iff } nR(\lambda_n) \to \infty. \tag{2.9}$$

Finally, if $nR(\lambda_n) \to \infty$ and $\lambda_n \to 0$, then

$$\log P\{S_n \le nx_n\} \sim -nR(\lambda_n). \tag{2.10}$$

These facts are all in Lemmas 2.1, 2.2, 2.3 and Theorem 2.1 of Jain and Pruitt (1987).

Next, we will describe Feller's (1967) condition for stochastic compactness. For $x > 0$, define

$$\begin{gathered} G(x) = P\{X > x\}, \quad K(x) = x^{-2}EX^2 1\{X \le x\}, \\ M(x) = x^{-1}EX1\{X \le x\}, \\ Q(x) = G(x) + K(x) = E\{(x^{-1}X)^2 \wedge 1\}. \end{gathered} \tag{2.11}$$

Here $1\{\cdot\}$ denotes the indicator function of the event described in the braces. Q is continuous and strictly decreasing for $x > a$. The analytic form of the stochastic compactness condition is

$$\limsup_{x \to \infty} \frac{G(x)}{K(x)} < \infty. \tag{2.12}$$

Since $G(x)/K(x)$ actually converges to a finite limit when X is in the domain of attraction of any stable law, this condition is more general. Assuming (2.12), it follows that

$$u^2V(u) \approx R(u) \approx Q(u^{-1}), \qquad 0 < u \le 1, \tag{2.13}$$

where $\approx$ means that the ratio is bounded above and below by positive, finite constants. The comparison of R and Q in (2.13) is valid even without (2.12).

Another bound that we will need follows from Schwarz inequality and (2.13):

$$\begin{aligned} ug(u) &\sim -u\varphi'(u) = EuXe^{-uX} \le \left(E(uX)^2e^{-uX}\right)^{1/2}(\varphi(u))^{1/2} \\ &\sim \left(E(uX)^2e^{-uX}\right)^{1/2} \le \left(Q(u^{-1})\right)^{1/2} \approx (R(u))^{1/2}. \end{aligned} \tag{2.14}$$

Furthermore, when $EX < \infty$ and (2.12) holds, we can show that

$$u^{\varepsilon}(g(0) - g(u))/R(u) \text{ is decreasing for small } u, \tag{2.15}$$

provided that ε is small. To see this, we have by (2.2), (2.13) and (2.7):

$$\begin{aligned} \frac{d}{du}\frac{u^{\varepsilon}(g(0) - g(u))}{R(u)} &= \frac{[\varepsilon R(u) - u^2V(u)](g(0) - g(u)) + R(u)uV(u)}{u^{1-\varepsilon}R^2(u)} \\ &\le \frac{[\varepsilon C - 1 + \frac{3}{4}]u^2V(u)(g(0) - g(u))}{u^{1-\varepsilon}R^2(u)}. \end{aligned}$$

The constant C comes from (2.13) and depends only on the underlying distribution of X. Thus we may make the derivative negative by taking ε small, say $\varepsilon = 1/8C$. Finally, if (2.12) holds, we have a stronger version of (2.10): If $a < x_n < EX$, λ_n is defined by $g(\lambda_n) = x_n$, and $\lambda_n \to 0$, $nR(\lambda_n) \to \infty$, then

$$P\{S_n \le nx_n\} \approx (nR(\lambda_n))^{-1/2}\exp(-nR(\lambda_n)). \tag{2.16}$$

These last facts are in Lemma 4.1, Theorem 4.1 and Remark 4.1 of Jain and Pruitt (1987). [Even the exact asymptotic behavior is available in Theorem 4.1 of Jain and Pruitt (1987), but the form given in (2.16) is more useful here.]

3. The stochastically compact case. In this section we will assume that the sums can be normalized so as to be stochastically compact, i.e.,

$$G(x) \le CK(x) \quad \text{for large } x, \tag{3.1}$$

where G and K are defined in (2.11). The main result is the integral test for the rate of escape problem. We suspect that it is true in general [i.e., without (3.1)] in the form stated but we do not have adequate estimates of $P\{S_n \le \beta_n\}$ in order to prove it. Nevertheless, we will be able to show that the convergent part of the test works in general and in Section 4 we will show that the test is valid whenever the tails of the distribution of X are so fat that there is no exact rate of escape function.

THEOREM 1. *Assume X is nonnegative and nondegenerate,* (3.1) *is satisfied and $n^{-1}\beta_n$ is nondecreasing. Then*

$$P\{S_n \le \beta_n\ i.o.\} = \begin{cases} 0 & \text{iff} \sum_n n^{-1}p_n \log(3p_n^{-1}) < \infty, \\ 1 & \text{iff} \sum_n n^{-1}p_n \log(3p_n^{-1}) = \infty, \end{cases}$$

where $p_n = P\{S_n \le \beta_n\}$. [Note that the asymptotic behavior of p_n is available in (2.16) *under assumption* (3.1).]

As an immediate consequence, we obtain the following results which are analogous to the usual improvements of the law of the iterated logarithm that follow from the Kolmogorov–Erdös integral test. A further simplification of these results was given in the introduction for a class of examples. A related result that gives the correct lim inf behavior under assumptions somewhat weaker than (3.1) is in Theorem 4 in Section 4.

COROLLARY. *Define*

$$\beta_n(k, \varepsilon) = ng(\lambda_n(k, \varepsilon)),$$

where $\lambda_n(k, \varepsilon)$ *is defined by*

$$R(\lambda_n(k, \varepsilon)) = \frac{\xi_n(k, \varepsilon)}{n}$$

and

$$\xi_n(k, \varepsilon) = lln + \tfrac{3}{2}l_3 n + l_4 n + \cdots + (1 + \varepsilon) l_k n.$$

Then, for any $k \geq 4$ *and any* $\varepsilon > 0$,

$$P\{S_n \leq \beta_n(k, 0) \text{ i.o.}\} = 1 \quad \text{and} \quad P\{S_n \leq \beta_n(k, \varepsilon) \text{ i.o.}\} = 0.$$

PROOF OF THEOREM 1. We will proceed as far as possible without using (3.1). Since the function $x \log(3x^{-1})$ is increasing on $(0, 1)$, we note that if we have two sequences $\{\beta_n\}$ and $\{\tilde{\beta}_n\}$ with the property that $\beta_n \leq \tilde{\beta}_n$ for all $n \geq n^*$ and we can prove the divergent part of the test for $\{\beta_n\}$, then we automatically have it for $\{\tilde{\beta}_n\}$. Similarly, the convergent part of the test for $\{\tilde{\beta}_n\}$ implies the convergent part of the test for $\{\beta_n\}$. We will use this observation to make a few simplifications. First, note that if we let $\beta_n = \beta_n(c) = ng(\lambda_n)$, where λ_n is defined by $R(\lambda_n) = n^{-1} c \log\log n$, then the series in the statement of the theorem converges if $c > 1$ and diverges if $c \leq 1$. [Use the estimate in (2.16) for p_n.] Thus by using the above comparison and recalling (2.4), we see that we may assume that $a < n^{-1}\beta_n < EX$ for all n. This means that there exists λ_n such that $g(\lambda_n) = n^{-1}\beta_n$. Furthermore, if λ_n does not approach 0, then β_n will eventually be smaller than $\beta_n(2)$ and so we may use the comparison again provided, of course, that we can prove the theorem for $\beta_n(2)$. Finally, if $\liminf nR(\lambda_n) < \infty$, then $\limsup p_n > 0$ and so $P\{S_n \leq \beta_n \text{ i.o.}\} = 1$ by the zero–one law. The series diverges in this case since λ_n and hence $R(\lambda_n)$ are nonincreasing and so for $n_k < n \leq 2n_k$,

$$nR(\lambda_n) \leq 2n_k R(\lambda_{n_k}) \leq 2C \quad \text{if } n_k R(\lambda_{n_k}) \leq C.$$

Thus $p_n \geq c > 0$ and so $n^{-1} p_n \log(3p_n^{-1}) \geq c_1 n^{-1}$ for $n_k \leq n \leq 2n_k$. Thus we have seen that we may suppose that

$$(3.2) \qquad nR(\lambda_n) \to \infty \quad \text{and} \quad \lambda_n \to 0, \quad \text{where } g(\lambda_n) = n^{-1}\beta_n.$$

This means that we may simplify the form of the series somewhat. Note that by (2.10)

$$n^{-1}\log(3p_n^{-1}) = n^{-1}(\log 3 - \log p_n) \sim n^{-1}(\log 3 + nR(\lambda_n)) \sim R(\lambda_n)$$

and so we may consider the series $\Sigma p_n R(\lambda_n)$ instead. Next we define the subsequence that will be used in the proof. Let

$$n_k = \min\left\{n\colon \sum_{j=1}^{n} R(\lambda_j) \ge k\right\}.$$

Note that by (3.2), $\Sigma R(\lambda_j) = \infty$ so n_k is well defined for all k, and since $R(\lambda_j) \to 0$, it follows that

$$\sum_{j=n_{k-1}+1}^{n_k} R(\lambda_j) \sim 1. \tag{3.3}$$

Next we will see that we can essentially bound p_n on the interval $n_{k-1} < n < n_k$ by its values at the endpoints. Suppose that $n_{k-1} \le m < n \le n_k$ and consider

$$q_{mn} = P\{S_n - S_m \le \beta_n - \beta_m\}.$$

We want to show that q_{mn} is bounded below by a positive constant for large k. Since

$$g(\mu_{mn}) = \frac{\beta_n - \beta_m}{n - m} \ge \frac{\beta_n}{n} = g(\lambda_n)$$

by the monotonicity of $n^{-1}\beta_n$, we have by (3.3)

$$(n - m)R(\mu_{mn}) \le (n - m)R(\lambda_n) \le \sum_{j=m+1}^{n} R(\lambda_j) < 2.$$

This gives the lower bound for q_{mn} provided that $n - m \ge n^*$ for some n^* by (2.9). The bound is trivial for $n - m < n^*$ and large k since

$$\begin{aligned} P\{S_n - S_m \le \beta_n - \beta_m\} &\ge P\left\{S_n - S_m \le \frac{\beta_n}{n}(n - m)\right\} \\ &\ge \left(P\left\{X \le \frac{\beta_n}{n}\right\}\right)^{n-m} \ge \left(P\left\{X \le \frac{\beta_n}{n}\right\}\right)^{n^*} \\ &\to (P\{X < EX\})^{n^*} > 0, \end{aligned}$$

where the convergence follows from $n^{-1}\beta_n = g(\lambda_n) \uparrow EX$ under (3.2) and $P\{X \ge EX\} < 1$ since X is nondegenerate. Then we have for $n_{k-1} \le m < n \le n_k$,

$$p_n = P\{S_n \le \beta_n\} \ge P\{S_m \le \beta_m, S_n - S_m \le \beta_n - \beta_m\} = p_m q_{mn} \ge c p_m.$$

In particular, this means that $cp_{n_{k-1}} \le p_n \le c^{-1} p_{n_k}$ for $n_{k-1} < n < n_k$. Thus by (3.3),

$$\sum_{n=n_{k-1}+1}^{n_k} p_n R(\lambda_n) \ge c p_{n_{k-1}} \sum_{n=n_{k-1}+1}^{n_k} R(\lambda_n) \sim c p_{n_{k-1}},$$

and essentially the same argument gives an upper bound of $c^{-1}p_{n_k}$. This means that the convergence of the series $\Sigma p_n R(\lambda_n)$ is equivalent to that of the series Σp_{n_k}. To complete the proof of the convergent case, we need an improvement of the above upper bound for p_n which is essentially Skorokhod's inequality. Define

$$A_n = \left\{ \frac{S_n}{\beta_n} \le 1, \frac{S_j}{\beta_j} > 1 \text{ for } n_{k-1} < j < n \right\}.$$

Then

$$\begin{aligned} P\left\{ \min_{n_{k-1} < n \le n_k} \frac{S_n}{\beta_n} \le 1 \right\} &= \sum_{n=n_{k-1}+1}^{n_k} P(A_n) \\ &= \sum_{n=n_{k-1}+1}^{n_k} q_{nn_k}^{-1} P\{A_n; S_{n_k} - S_n \le \beta_{n_k} - \beta_n\} \\ &\le c^{-1} \sum_{n=n_{k-1}+1}^{n_k} P\{A_n; S_{n_k} \le \beta_{n_k}\} \le c^{-1} p_{n_k}. \end{aligned}$$

This maximal inequality is crucial here even though S_n is monotone as filling in between the members of the subsequence by monotonicity is too crude in general. We have now completed the proof of the convergent case and shown that Σp_{n_k} diverges in the divergent case. In order to complete the proof we only need to show that the supplementary condition holds for the generalized Borel–Cantelli lemma [see Kochen and Stone (1964)]. But showing this is technical, quite delicate and somewhat long. This is where we will use (3.1). Since we will be dealing exclusively with the subsequence from now on, we will abuse the notation by letting $\beta_j = \beta_{n_j}$, $p_j = p_{n_j}$ and $\lambda_j = \lambda_{n_j}$. Thus by (2.16)

$$p_j = P\{S_{n_j} \le \beta_j\} \approx \left(n_j R(\lambda_j)\right)^{-1/2} \exp\left(-n_j R(\lambda_j)\right)$$

and we will let

$$p_{jk} = P\{S_{n_j} \le \beta_j, S_{n_k} \le \beta_k\}.$$

We will prove that for $j < k$,

$$p_{jk} \le C_1(k-j)^{-2}(p_j + p_k) + C_2 p_j p_k \tag{3.4}$$

which easily implies that

$$\limsup_{N\to\infty} \frac{\Sigma_{j=1}^N \Sigma_{k=1}^N p_{jk}}{\left(\Sigma_{k=1}^N p_k\right)^2} \le C_2.$$

The proof of (3.4) will be broken down into three cases. First, assume

$$|n_j R(\lambda_j) - n_k R(\lambda_k)| \ge 2\log(k-j). \tag{A}$$

If $n_j R(\lambda_j) \ge n_k R(\lambda_k) + 2\log(k-j)$,

$$\begin{aligned} p_{jk} \le p_j &\approx \left(n_j R(\lambda_j)\right)^{-1/2} \exp\left(-n_j R(\lambda_j)\right) \\ &\le \left(n_k R(\lambda_k)\right)^{-1/2} \exp\left(-n_k R(\lambda_k) - 2\log(k-j)\right) \approx p_k (k-j)^{-2}. \end{aligned}$$

In the other case, a similar argument shows that $p_{jk} \le p_k \le Cp_j(k-j)^{-2}$.

$$\text{(B)}\qquad \begin{aligned} &|n_j R(\lambda_j) - n_k R(\lambda_k)| < 2\log(k-j) \\ &\text{and}\quad (n_k - n_j)R(\lambda_k) \ge C_3 \log(k-j). \end{aligned}$$

First, we will specify the value of C_3. Take $\rho \in (\frac{1}{2}, 1)$ such that

$$R(\rho\lambda) \ge \tfrac{1}{2}R(\lambda) \quad \text{for all } \lambda \in (0,1). \tag{3.5}$$

To see that this is possible, we have by (2.2) and (2.6)

$$R(\lambda) - R(\rho\lambda) = \int_{\rho\lambda}^{\lambda} uV(u)\,du \le C\int_{\rho\lambda}^{\lambda} u^{-1}R(u)\,du \le CR(\lambda)\log\rho^{-1}$$

and so

$$R(\rho\lambda) \ge R(\lambda)(1 - C\log\rho^{-1}).$$

Here C depends only on the distribution of X and so we may choose ρ close enough to 1 to make (3.5) hold. Then we take

$$C_3 = 32\rho(1-\rho)^{-1}.$$

Now we define μ, α and ν by

$$n_j R(\mu) = n_j R(\lambda_j) + 2\log(k-j), \qquad \alpha = n_j g(\mu), \tag{3.6}$$

$$\begin{aligned} g(\nu) &= \frac{\beta_k - \alpha}{n_k - n_j} = \frac{\beta_k}{n_k} + \frac{n_j}{n_k - n_j}\left(\frac{\beta_k}{n_k} - \frac{\alpha}{n_j}\right) \\ &= g(\lambda_k) + \frac{n_j}{n_k - n_j}(g(\lambda_k) - g(\mu)). \end{aligned} \tag{3.7}$$

There is the question of whether μ and ν exist. For μ, we observe that by case (B),

$$C_3\log(k-j) \le (n_k - n_j)R(\lambda_k) \le n_k R(\lambda_k) \le n_j R(\lambda_j) + 2\log(k-j),$$

which implies that $\log(k-j) = O(n_j R(\lambda_j))$ since $C_3 \ge 32$. This means that μ will even go to 0. ν presents a problem only when $EX < \infty$ since then the value given for $g(\nu)$ might exceed EX. To see that this cannot happen (for large k), first note that by case (B)

$$\begin{aligned} n_j(R(\mu) - R(\lambda_k)) &= n_j\big(R(\lambda_j) - R(\lambda_k)\big) + 2\log(k-j) \\ &\le (n_k - n_j)R(\lambda_k) + 4\log(k-j) \le \tfrac{9}{8}(n_k - n_j)R(\lambda_k). \end{aligned}$$

Now we use the generalized mean value theorem. Recalling (2.2), there exists $\xi \in (\lambda_k, \mu)$ such that $R(\mu) - R(\lambda_k) = \xi(g(\lambda_k) - g(\mu))$, which in conjunction with (2.7) leads to (for large k)

$$n_j(g(\lambda_k) - g(\mu)) \le \tfrac{9}{8}(n_k - n_j)R(\lambda_k)/\lambda_k \le \tfrac{27}{32}(n_k - n_j)(g(0) - g(\lambda_k)).$$

Thus

$$g(\lambda_k) + \frac{n_j}{n_k - n_j}(g(\lambda_k) - g(\mu)) \le g(\lambda_k) + \tfrac{27}{32}(g(0) - g(\lambda_k)) < g(0).$$

This means that ν also exists. Since $\mu > \lambda_j$, we have $\alpha < n_j g(\lambda_j) = \beta_j$. Now we will use

$$\begin{aligned} p_{jk} &\le P\{S_{n_j} \le \alpha\} + P\{\alpha < S_{n_j} \le \beta_j;\, S_{n_k} \le \beta_k\} \\ &\le P\{S_{n_j} \le \alpha\} + p_j P\{S_{n_k - n_j} \le \beta_k - \alpha\}. \end{aligned} \tag{3.8}$$

Since

$$\begin{aligned} P\{S_{n_j} \le \alpha\} &\approx (n_j R(\mu))^{-1/2} \exp(-n_j R(\mu)) \\ &\le (n_j R(\lambda_j))^{-1/2} \exp(-n_j R(\lambda_j) - 2\log(k-j)) \approx p_j (k-j)^{-2}, \end{aligned}$$

the first term in (3.8) satisfies (3.4). For the second term, we will use the bound [from (2.8)]

$$P\{S_{n_k - n_j} \le \beta_k - \alpha\} \le \exp(-(n_k - n_j) R(\nu))$$

and so it will suffice to show that

$$(n_k - n_j) R(\nu) \ge 2\log(k-j). \tag{3.9}$$

Take $\delta = (1-\rho)/4\rho$ and note that $\delta < \frac{1}{4}$ since $\rho > \frac{1}{2}$. If $R(\nu) \ge \delta R(\lambda_k)$, then

$$(n_k - n_j) R(\nu) \ge \delta (n_k - n_j) R(\lambda_k) \ge \delta C_3 \log(k-j) = 8\log(k-j)$$

so we may assume that

$$R(\nu) < \delta R(\lambda_k) < \tfrac{1}{4} R(\lambda_k) < R(\rho\lambda_k). \tag{3.10}$$

Thus $\nu < \rho\lambda_k$. We define η by

$$(n_k - n_j)(g(\rho\lambda_k) - g(\lambda_k)) = n_j(g(\eta) - g(\mu)). \tag{3.11}$$

(It will be clear in a moment that such an η exists.) Then by (3.7) and (3.11)

$$\begin{aligned} (n_k - n_j)(g(\nu) - g(\rho\lambda_k)) &= n_j((g(\lambda_k) - g(\mu)) \\ &\qquad - (g(\eta) - g(\mu))) \\ &= n_j(g(\lambda_k) - g(\eta)). \end{aligned} \tag{3.12}$$

Thus we have

$$\nu < \rho\lambda_k < \lambda_k < \eta < \mu.$$

Now we are going to use the generalized mean value theorem four times. We have

$$R(\rho\lambda_k) - R(\nu) = \xi_1(g(\nu) - g(\rho\lambda_k)), \qquad R(\lambda_k) - R(\rho\lambda_k) = \xi_2(g(\rho\lambda_k) - g(\lambda_k)),$$
$$R(\eta) - R(\lambda_k) = \xi_3(g(\lambda_k) - g(\eta)), \qquad R(\mu) - R(\eta) = \xi_4(g(\eta) - g(\mu)),$$

where

$$\nu < \xi_1 < \rho\lambda_k < \xi_2 < \lambda_k < \xi_3 < \eta < \xi_4 < \mu.$$

Using this in conjunction with (3.12) and (3.11) leads to

$$(3.13)\qquad \begin{aligned} (n_k - n_j)(R(\rho\lambda_k) - R(\nu)) &= \xi_1\xi_3^{-1} n_j(R(\eta) - R(\lambda_k)) \\ &\le \rho n_j(R(\eta) - R(\lambda_k)), \\ (n_k - n_j)(R(\lambda_k) - R(\rho\lambda_k)) &= \xi_2\xi_4^{-1} n_j(R(\mu) - R(\eta)) \\ &\le n_j(R(\mu) - R(\eta)). \end{aligned}$$

This gives [using (3.10) and case (B)]

$$\begin{aligned} (1-\delta)(n_k - n_j)R(\lambda_k) &\le (n_k - n_j)(R(\lambda_k) - R(\nu)) \\ &\le n_j\big(R(\mu) - R(\eta) + \rho(R(\eta) - R(\lambda_k))\big) \\ &= n_j\big(R(\mu) - R(\lambda_k) - (1-\rho)(R(\eta) - R(\lambda_k))\big) \\ &= n_j\big(R(\lambda_j) - R(\lambda_k) - (1-\rho)(R(\eta) - R(\lambda_k))\big) \\ &\quad + 2\log(k-j) \\ &\le (n_k - n_j)R(\lambda_k) - (1-\rho)n_j(R(\eta) - R(\lambda_k)) \\ &\quad + 4\log(k-j). \end{aligned}$$

Therefore

$$(1-\rho)n_j(R(\eta) - R(\lambda_k)) \le \delta(n_k - n_j)R(\lambda_k) + 4\log(k-j).$$

Using (3.13) again,

$$\begin{aligned} (n_k - n_j)(R(\rho\lambda_k) - R(\nu)) &\le \rho n_j(R(\eta) - R(\lambda_k)) \\ &\le \rho(1-\rho)^{-1}\big(\delta(n_k - n_j)R(\lambda_k) + 4\log(k-j)\big). \end{aligned}$$

Finally, this gives by (3.5) and the definitions of δ and C_3,

$$\begin{aligned} &(n_k - n_j)R(\nu) \\ &\quad \ge (n_k - n_j)\big(R(\rho\lambda_k) - \delta\rho(1-\rho)^{-1}R(\lambda_k)\big) - 4\rho(1-\rho)^{-1}\log(k-j) \\ &\quad \ge (n_k - n_j)R(\lambda_k)\big(\tfrac{1}{2} - \delta\rho(1-\rho)^{-1}\big) - 4\rho(1-\rho)^{-1}\log(k-j) \\ &\quad \ge \log(k-j)\big(\tfrac{1}{4}C_3 - 4\rho(1-\rho)^{-1}\big) = C_3\log(k-j)/8 \ge 4\log(k-j). \end{aligned}$$

This proves (3.9) and hence finishes the proof in case (B).

$$\text{(C)}\qquad \begin{aligned} &|n_jR(\lambda_j) - n_kR(\lambda_k)| < 2\log(k-j) \\ &\text{and}\quad (n_k - n_j)R(\lambda_k) < C_3\log(k-j). \end{aligned}$$

First, note that by (3.3) and case (C) we have for large j,

$$\begin{aligned}\frac{k-j}{2} &\le (n_k - n_j)R(\lambda_j) = (n_k - n_j)\big(R(\lambda_j) - R(\lambda_k)\big) + (n_k - n_j)R(\lambda_k)\\ &\le (n_k - n_j)\left[R(\lambda_k)\frac{n_k - n_j}{n_j} + 2\frac{\log(k-j)}{n_j}\right] + C_3\log(k-j)\\ &\le \frac{C_3(C_3+2)\log^2(k-j)}{n_jR(\lambda_k)} + C_3\log(k-j).\end{aligned}$$

Thus we have

$$n_jR(\lambda_k) = O\big((k-j)^{-1}\log^2(k-j)\big). \tag{3.14}$$

Since $x^2Q(x)$ is increasing and $R(u) \approx Q(u^{-1})$ by (2.13), we have

$$R(\rho\lambda) \le C_4\rho^2R(\lambda) \tag{3.15}$$

for $\rho > 1$. Note that for large j, using (3.2), (3.15) and (3.14),

$$1 \le n_jR(\lambda_j) \le C_4\left(\frac{\lambda_j}{\lambda_k}\right)^2 n_jR(\lambda_k) \le C_5\left(\frac{\lambda_j}{\lambda_k}\right)^2\frac{\log^2(k-j)}{k-j}. \tag{3.16}$$

Thus

$$\lambda_k\log(k-j) = O\left(\lambda_j\frac{\log^2(k-j)}{(k-j)^{1/2}}\right) = o(\lambda_j)$$

for $k-j$ large and so we may assume that $\lambda_j > \lambda_k\log(k-j)$. Now define μ and ν as in (3.6) and (3.7). This time μ may not exist but we defer this possibility until later. In case $EX < \infty$ we must show that ν exists. Observe that by case (C) and (3.14), if $k-j$ is large,

$$\begin{aligned}n_jR(\mu) &= n_jR(\lambda_j) + 2\log(k-j) \le n_kR(\lambda_k) + 4\log(k-j)\\ &= (n_k - n_j)R(\lambda_k) + n_jR(\lambda_k) + 4\log(k-j) \le (C_3+5)\log(k-j).\end{aligned}$$

Thus, for any $\varepsilon > 0$, we have by (3.16) and (3.2)

$$\frac{n_jR(\mu)}{n_kR(\lambda_k)}\left(\frac{\lambda_k}{\lambda_j}\right)^{\varepsilon} \le (C_3+5)\log(k-j)\left(\frac{\log^2(k-j)}{k-j}\right)^{\varepsilon/2} \to 0$$

as $k-j\to\infty$. By (2.15) we have

$$\frac{n_j(g(0)-g(\mu))}{n_k(g(0)-g(\lambda_k))} \le \left(\frac{\lambda_k}{\lambda_j}\right)^{\varepsilon}\frac{n_jR(\mu)}{n_kR(\lambda_k)} \to 0.$$

Thus, if $k-j$ is not too small,

$$n_j(g(0)-g(\mu)) < n_k(g(0)-g(\lambda_k)),$$

which is equivalent to $g(\nu)$ as defined in (3.7) being less than EX. To continue with the proof, we will now use $\rho = \log(k-j)$ (assuming that $k-j \geq 3$) and then define ζ by

$$g(\zeta) = g(\nu) - \frac{n_j}{n_k - n_j}(g(\lambda_k) - g(\rho\lambda_k))$$
$$= g(\lambda_k) + \frac{n_j}{n_k - n_j}(g(\rho\lambda_k) - g(\mu)).$$

This means that ζ is defined and lies in the interval (ν, λ_k). Now we are going to use the generalized mean value theorem four times again:

$$R(\zeta) - R(\nu) = \xi_1(g(\nu) - g(\zeta)), \qquad R(\lambda_k) - R(\zeta) = \xi_2(g(\zeta) - g(\lambda_k)),$$
$$R(\rho\lambda_k) - R(\lambda_k) = \xi_3(g(\lambda_k) - g(\rho\lambda_k)),$$
$$R(\mu) - R(\rho\lambda_k) = \xi_4(g(\rho\lambda_k) - g(\mu)),$$

where $\nu < \xi_1 < \zeta < \xi_2 < \lambda_k < \xi_3 < \rho\lambda_k < \xi_4 < \mu$. Using this in conjunction with the definition of ζ, (3.15) and (3.14) leads to [recall that $\rho = \log(k-j)$]

$$(n_k - n_j)(R(\zeta) - R(\nu)) = \xi_1\xi_3^{-1}n_j(R(\rho\lambda_k) - R(\lambda_k)) \leq C_4 n_j \rho^2 R(\lambda_k)$$
$$= O\big(\log^4(k-j)/(k-j)\big) = O(1),$$

and also using case (C)

$$\begin{aligned}
&(n_k - n_j)(R(\lambda_k) - R(\zeta)) \\
&\quad = \xi_2\xi_4^{-1}n_j(R(\mu) - R(\rho\lambda_k)) \leq \rho^{-1}n_j(R(\mu) - R(\lambda_k)) \\
&\quad = \rho^{-1}\big(n_j R(\lambda_j) + 2\log(k-j) - n_j R(\lambda_k)\big) \\
&\quad \leq \rho^{-1}\big((n_k - n_j)R(\lambda_k) + 4\log(k-j)\big) \\
&\quad \leq \rho^{-1}(C_3 + 4)\log(k-j) = C_3 + 4.
\end{aligned}$$

Combining these last two facts shows that $(n_k - n_j)(R(\lambda_k) - R(\nu))$ is bounded. Thus, recalling (3.14),

$$(n_k - n_j)R(\nu) = n_k R(\lambda_k) + O(1).$$

Thus

$$P\{S_{n_k - n_j} \leq \beta_k - \alpha\} \approx \big((n_k - n_j)R(\nu)\big)^{-1/2}\exp\big(-(n_k - n_j)R(\nu)\big)$$
$$\approx \big(n_k R(\lambda_k)\big)^{-1/2}\exp(-n_k R(\lambda_k)) \approx p_k.$$

Recalling (3.8), this completes the proof of case (C) when μ exists. We must still consider the case when μ does not exist. In this case, we will ignore μ and use

$$p_{jk} \leq P\{S_{n_j} \leq \beta_j\}P\{S_{n_k - n_j} \leq \beta_k\},$$

and so it will be enough to show that

$$P\{S_{n_k - n_j} \leq \beta_k\} = O(p_k). \tag{3.17}$$

Thus we define v by $g(v) = (n_k - n_j)^{-1}\beta_k$. Once more, if $EX < \infty$, we must show that v exists. First, note that the fact that μ does not exist means that

$$n_j R(\lambda_j) + 2\log(k-j) > n_j R(1),$$

which implies that

$$n_j = O(\log(k-j)). \tag{3.18}$$

By (3.14) we have

$$R(\lambda_k) = \frac{n_j R(\lambda_k)}{n_j} = O\left(\frac{\log^2(k-j)}{k-j}\right) \tag{3.19}$$

so that

$$n_j(R(\lambda_k))^{1/2} \to 0 \tag{3.20}$$

as $k - j \to \infty$. Since $x^2Q(x)$ is increasing and $R(\lambda) \approx Q(\lambda^{-1})$, it follows that $\lambda^{-2}R(\lambda) \geq c > 0$ for small λ. Using (2.7), (3.2) and (3.20) leads to

$$\frac{n_k(g(0) - g(\lambda_k))}{n_j g(0)} \geq \frac{4}{3}\frac{n_k}{n_j}\frac{R(\lambda_k)}{\lambda_k g(0)} \geq c_1 \frac{n_k R(\lambda_k)}{n_j(R(\lambda_k))^{1/2}} \to \infty.$$

This means that

$$n_k(g(0) - g(\lambda_k)) > n_j g(0),$$

when $k - j$ is large and this is equivalent to

$$g(v) = (n_k - n_j)^{-1} n_k g(\lambda_k) < EX$$

so that v exists. Next we use the generalized mean value theorem:

$$R(\lambda_k) - R(v) = \xi(g(v) - g(\lambda_k)) = \xi(n_k - n_j)^{-1} n_j g(\lambda_k),$$

where $v < \xi < \lambda_k$. Thus by (2.14) and (3.20)

$$(n_k - n_j)(R(\lambda_k) - R(v)) \leq n_j \lambda_k g(\lambda_k) \leq C n_j (R(\lambda_k))^{1/2} \to 0.$$

Thus we have

$$(n_k - n_j)R(v) = n_k R(\lambda_k) + o(1)$$

and by (2.16) this implies (3.17), which completes the proof. □

4. Necessary and sufficient conditions for normalization. We let $G(x) = P\{X > x\}$ as above and define

$$u_j = 1 - \frac{G(e^{j+1})}{G(e^j)} = P\{X \leq e^{j+1} | X > e^j\}, \qquad j = 1, 2, \dots.$$

If $G(e^j) = 0$, we define $u_j = 1$ by convention. Next, let r_j be the rank of u_j when ranked in decreasing order,

$$r_j = \operatorname{card}\{k: u_k > u_j \text{ or } u_k = u_j \text{ and } k \leq j\}.$$

If $u_j = 0$, then $r_j = \infty$ since $\{j: u_j > 0\}$ is always infinite. When G is slowly varying, $u_j \to 0$ so r_j will be finite for all j such that $u_j > 0$. But $r_j = \infty$ for any j such that $0 < u_j < \limsup u_k$. The main result of this section is Theorem 2.

THEOREM 2. *Assume X is nonnegative and nondegenerate. Then it is possible to find β_n, increasing, such that*

$$\liminf_{n\to\infty} \frac{S_n}{\beta_n} = 1 \quad a.s.$$

iff

$$\limsup_{j\to\infty} u_j \log r_j = \infty. \tag{4.1}$$

(We use the convention that $u_j \log r_j = 0$ when $u_j = 0$.)

We start with the necessity of (4.1). We will prove some lemmas which culminate in an integral test (Theorem 3) which shows that if (4.1) fails, then for any given sequence $\{\beta_n\}$, the lim inf is either 0 or ∞. The first lemma gives the estimate for the lower tail of the distribution of S_n that will be used here since (2.16) is not available.

LEMMA 1. *Assume that G is slowly varying and*

$$nG^2(x) \le C_1, \tag{4.2}$$

$$n(G(y) - G(ey)) \le C_2 \quad \textit{for all } y \in \left[\frac{x}{nG(x)}, \frac{x}{e}\right]. \tag{4.3}$$

Then there exist C_3 and x_0 (depending on C_3) such that for all $x \ge x_0$,

$$C_3 e^{-nG(x)} \le P\{S_n \le x\} \le e^{-nG(x)}. \tag{4.4}$$

Furthermore, C_3 depends only on C_1 and C_2 and may be made arbitrarily close to 1 by making C_1 and C_2 small. If (4.2) *and* (4.3) *are replaced by $nG(x) \le C$, then* (4.4) *holds for any $C_3 < 1$.*

PROOF. The upper bound in (4.4) is trivial and completely general:

$$P\{S_n \le x\} \le P\{X_i \le x, i = 1, 2, \dots, n\} = (1 - G(x))^n \le e^{-nG(x)}. \tag{4.5}$$

For the lower bound, we first consider the case where $nG(x) \le C$. Define the events

$$A = \{X_i \le x, i = 1, 2, \dots, n\}, \qquad D = \left\{\sum_{i\le n} X_i 1\{X_i \le x\} \le x\right\}.$$

Then $AD \subset \{S_n \le x\}$ so that

$$P\{S_n \le x\} \ge P(AD) \ge P(A) - P(D^c),$$

and since $nG^2(x) \le CG(x) < \varepsilon$ for large x, we have

$$(4.6) \quad P(A) = (1 - G(x))^n \ge \left(e^{-G(x)-G^2(x)}\right)^n > e^{-nG(x)-\varepsilon} \ge e^{-C-\varepsilon}.$$

Next, we use a first moment estimate for D^c. Recall that

$$M(x) = x^{-1}EX1\{X \le x\}.$$

Then

$$P(D^c) \le x^{-1}nxM(x) = nM(x) \le CM(x)/G(x) \le \varepsilon e^{-C-\varepsilon} \le \varepsilon P(A),$$

for large x, where we have used the fact that when G is slowly varying

$$\frac{M(x)}{G(x)} \to 0 \quad \text{as } x \to \infty;$$

see Lemma 2.5 of Pruitt (1981). Thus we have

$$P\{S_n \le x\} \ge e^{-nG(x)}e^{-\varepsilon}(1 - \varepsilon)$$

and $e^{-\varepsilon}(1 - \varepsilon)$ may be made close to 1 by taking ε small. For the remainder of the proof, we will fix a small positive number η and then take δ small enough that

$$\delta \le \tfrac{1}{2}, \quad 2C_4\delta \le \tfrac{9}{10}, \quad \delta(C_2 + 1) < \eta \quad \text{and} \quad 3\delta < \eta,$$

where $C_4 = e^3C_2 \vee 1$ and let $C = e^{2/\delta}$; from the above we may assume that $nG(x) > C$. Next, define

$$k_0 = \min\{k: (l_k x)^2 < nG(x)\}, \qquad k_1 = \min\{k: (l_k x)^2 < C\},$$

where l_k is the logarithm iterated k times, and the events (we abuse the notation a little by letting A and D now stand for slightly different events)

$$A = \left\{X_i \le \frac{x}{C}, i = 1, 2, \dots, n\right\}, \qquad D = \left\{\sum_{i \le n} X_i 1\left\{X_i \le \frac{x}{nG(x)}\right\} \le \frac{x}{10}\right\},$$

$$B_k = \left\{\sum_{i \le n} 1\left\{\frac{x}{(l_k x)^2} < X_i \le \frac{x}{(l_{k+1}x)^2}\right\} \le C_4 l_{k+1} x\right\}, \qquad k = k_0, \dots, k_1 - 1,$$

$$B_{k_0-1} = \left\{\sum_{i \le n} 1\left\{\frac{x}{nG(x)} < X_i \le \frac{x}{(l_{k_0}x)^2}\right\} \le C_4 l_{k_0} x\right\}.$$

Now, on the event $AD \cap \bigcap_k B_k$, we have

$$\begin{aligned} S_n &= \sum_{i=1}^{n} X_i 1\left\{X_i \le \frac{x}{C}\right\} \\ &\le \sum_{i=1}^{n} X_i 1\left\{X_i \le \frac{x}{nG(x)}\right\} + \sum_{i=1}^{n} X_i 1\left\{\frac{x}{nG(x)} < X_i \le \frac{x}{(l_{k_0}x)^2}\right\} \\ &\quad + \sum_{i=1}^{n} \sum_{k=k_0}^{k_1-1} X_i 1\left\{\frac{x}{(l_k x)^2} < X_i \le \frac{x}{(l_{k+1}x)^2}\right\} \\ &\le \frac{x}{10} + C_4 x \sum_{k=k_0}^{k_1} \frac{1}{l_k x}. \end{aligned}$$

Now, for $k < k_1$, we have

$$l_k x = \exp(l_{k+1}x) \ge e l_{k+1} x$$

since $e^u \ge eu$ for $u \ge 1$, and

$$l_{k+1}x \ge l_{k_1}x = \log(l_{k_1-1}x) \ge \log C^{1/2} = \delta^{-1} \ge 2. \tag{4.7}$$

Thus we may compare the above sum with a geometric series:

$$\sum_{k=k_0}^{k_1} \frac{1}{l_k x} \le \frac{1}{l_{k_1}x}(1 + e^{-1} + e^{-2} + \cdots) \le \delta \frac{e}{e-1} < 2\delta. \tag{4.8}$$

Thus by our choice of C_4 we have

$$AD \cap \bigcap_k B_k \subset \{S_n \le x\}.$$

We will use the estimate

$$P\{S_n \le x\} \ge P(A) - P(AD^c) - \sum_k P(AB_k^c). \tag{4.9}$$

Next, for $x/nG(x) \le y \le x$, we have by (4.3)

$$\begin{aligned} n(G(y) - G(x)) &= n \sum_{k=1}^{j} \left(G(ye^{k-1}) - G(ye^k)\right) \\ &\quad + n\left(G(ye^j) - G(x)\right) \\ &\le C_2(j+1) \le C_2\left(1 + \log\frac{x}{y}\right), \end{aligned} \tag{4.10}$$

where $j = [\log(x/y)]$. Thus

$$nG(x/nG(x)) \le nG(x) + C_2(1 + \log(nG(x))) \le (C_2 + 1)nG(x)$$

so that

$$G(x/nG(x)) \approx G(x). \tag{4.11}$$

In particular, this means that $x/nG(x)$ is large for large x. Now it is time to estimate the probabilities. First, we have

$$P(A) = \left(1 - G\left(\frac{x}{C}\right)\right)^n \geq e^{-nG(x/C) - nG^2(x/C)},$$

and by (4.10)

$$n(G(x/C) - G(x)) \leq C_2(1 + \log C)$$

and

$$\begin{aligned} nG^2(x/C) &\leq n\big(G(x) + C_2(1 + \log C)n^{-1}\big)^2 \\ &\leq nG^2(x) + 2C_2(1 + \log C)G(x) + C_2^2(1 + \log C)^2 n^{-1} \\ &\leq C_1 + 2C_2(1 + \log C) + C_2^2(1 + \log C)^2. \end{aligned}$$

Thus

$$P(A) \geq C_5 e^{-nG(x)}, \tag{4.12}$$

where

$$C_5 = \exp\big(-C_1 - 3C_2(1 + \log C) - C_2^2(1 + \log C)^2\big)$$

and since C is fixed (note that C is independent of C_2 when $C_2 < e^{-3}$), C_5 will be close to 1 when C_1 and C_2 are small. Next, letting

$$A_1 = \{X_i \leq xC^{-1}, i = 2, \dots, n\},$$

we have

$$\begin{aligned} P(AD^c) &= P\left\{A; \sum X_i 1\left\{X_i \leq \frac{x}{nG(x)}\right\} > \frac{x}{10}\right\} \\ &\leq \frac{10}{x}\int_{AD^c} \sum X_i 1\left\{X_i \leq \frac{x}{nG(x)}\right\} \\ &\leq \frac{10}{x} n \int_{A_1} X_1 1\left\{X_1 \leq \frac{x}{nG(x)}\right\} \\ &= \frac{10}{x} n \frac{x}{nG(x)} M\left(\frac{x}{nG(x)}\right) \frac{P(A)}{1 - G(xC^{-1})} \\ &\leq \delta \frac{G(x/nG(x))}{G(x)} P(A) \leq \delta(C_2 + 1)P(A) \end{aligned}$$

since $M(y) = o(G(y))$, when G is slowly varying as mentioned above and since we have already seen that $x/nG(x)$ is large. Finally, for the B_k, we let

$$p_k = \begin{cases} P\left\{\dfrac{x}{(l_k x)^2} < X \le \dfrac{x}{C}\right\}, & k = k_1 - 1, \\ P\left\{\dfrac{x}{(l_k x)^2} < X \le \dfrac{x}{(l_{k+1} x)^2}\right\}, & k_0 \le k < k_1 - 1, \\ P\left\{\dfrac{x}{nG(x)} < X \le \dfrac{x}{(l_{k_0} x)^2}\right\}, & k = k_0 - 1. \end{cases}$$

Then, using (4.12) and the elementary inequality $j! \ge j^j e^{-j}$, $j \ge 1$, we have

$$P(AB_k^c) \le \sum_{j=C_4 l_{k+1} x}^{n} \binom{n}{j} p_k^j (1 - G(xC^{-1}) - p_k)^{n-j} \le \sum \binom{n}{j} p_k^j (1 - G(x))^{n-j}$$

$$\le e^{-nG(x)} \sum \left(\frac{np_k e}{j(1 - G(x))}\right)^j \le C_5^{-1} P(A) \sum \left(\frac{np_k e}{j(1 - G(x))}\right)^j.$$

We will be able to compare the series with a geometric series since

$$\left(\frac{np_k e}{(j+1)(1 - G(x))}\right)^{j+1} \left(\frac{np_k e}{j(1 - G(x))}\right)^{-j} = \frac{np_k e}{1 - G(x)} \left(\frac{j}{j+1}\right)^{j+1} \frac{1}{j}$$

$$\le \frac{np_k}{j(1 - G(x))} \sim \frac{np_k}{j}.$$

Now, by (4.10)

$$np_k \le n\left(G\left(\frac{x}{(l_k x)^2}\right) - G(x)\right) \le C_2(1 + 2l_{k+1} x),$$

and so

$$\frac{np_k}{j} \le \frac{C_2(1 + 2l_{k+1} x)}{C_4 l_{k+1} x} \le \frac{5}{2e^3} \tag{4.13}$$

since $C_4 = e^3 C_2 \vee 1$ and $l_{k+1} x \ge 2$ for all $k < k_1$ by (4.7). [For $k = k_0 - 1$, replace $G(x/(l_k x)^2)$ by $G(x/nG(x))$. The bounds given above are then still valid.] Thus the sum will be comparable to the first term and by (4.13) we have for large x

$$\left(\frac{np_k e}{C_4 l_{k+1} x(1 - G(x))}\right)^{C_4 l_{k+1} x} \le \left(\frac{5e}{2e^3(1 - G(x))}\right)^{C_4 l_{k+1} x} \le e^{-C_4 l_{k+1} x} \le \frac{1}{l_k x}.$$

Thus we have

$$P(AB_k^c) \le C_5^{-1} P(A) \frac{1}{l_k x} \frac{1}{1 - e^{-2}}$$

and so by (4.8),

$$\sum_{k=k_0}^{k_1} P(AB_{k-1}^c) \le C_5^{-1} P(A) \frac{2\delta}{1-e^{-2}}.$$

Putting the pieces together using (4.9), (4.12) and the conditions on δ, we have

$$\begin{aligned} P\{S_n \le x\} &\ge P(A)\left(1 - \delta(C_2+1) - C_5^{-1}\frac{2\delta}{1-e^{-2}}\right) \\ &\ge e^{-nG(x)}(C_5(1-\eta) - \eta). \end{aligned}$$

Since η is small and C_5 is close to 1 when C_1 and C_2 are small, this is enough. □

The next step is to show that when (4.1) fails, Lemma 1 applies in what turns out to be the relevant range. Define

$$I_n = \left\{ j\colon e^j \le \beta_n nG(\beta_n) \text{ and } e^{j+1} \ge \frac{\beta_n}{(nG(\beta_n))^2} \right\},$$

and let

$$\nu_n = \min\{r_j\colon j \in I_n\}.$$

Lemma 2. *Suppose that* $\beta_n \to \infty$,

$$\limsup_{j\to\infty} u_j \log r_j < \infty$$

and

$$nG(\beta_n) \le 3lln \wedge 3\log \nu_n. \tag{4.14}$$

Then for all $c > 0$, *we have*

$$P\{S_n \le c\beta_n\} \approx e^{-nG(c\beta_n)} \approx e^{-nG(\beta_n)}. \tag{4.15}$$

Furthermore, if $u_j \log r_j \to 0$, *then* $\approx$ *may be replaced by* $\sim$ *in* (4.15).

Proof. We fix C such that $u_j \log r_j \le C$ for all j. We also note that this implies that G is slowly varying. Next, suppose that $nG(\beta_n) \le C_6$ for n in a subsequence. Then, along the subsequence, $nG(c\beta_n) \sim nG(\beta_n) \le C_6$ and so by Lemma 1,

$$P\{S_n \le c\beta_n\} \sim e^{-nG(c\beta_n)} \sim e^{-nG(\beta_n)}.$$

The second $\sim$ follows from

$$n|G(c\beta_n) - G(\beta_n)| = nG(\beta_n)\left|\frac{G(c\beta_n)}{G(\beta_n)} - 1\right| \le C_6 \left|\frac{G(c\beta_n)}{G(\beta_n)} - 1\right| \to 0.$$

Thus we may assume that $nG(\beta_n) \to \infty$. Next,

$$nG^2(c\beta_n) \sim nG^2(\beta_n) = \frac{(nG(\beta_n))^2}{n} \le \frac{9(lln)^2}{n}$$

so we may assume that (4.2) holds with C_1 small. If

$$[y, ey] \subset \left[\frac{\beta_n}{(nG(\beta_n))^2}, \beta_n nG(\beta_n) \right],$$

then we take m so that $e^m \le y < e^{m+1}$ and then we have

$$\frac{\beta_n}{(nG(\beta_n))^2} \le y < e^{m+1} \quad \text{and} \quad e^m \le y \le \frac{\beta_n nG(\beta_n)}{e}$$

so that $m, m+1 \in I_n$. Therefore

$$u_m \le \frac{C}{\log r_m} \le \frac{C}{\log \nu_n} \quad \text{and} \quad u_{m+1} \le \frac{C}{\log r_{m+1}} \le \frac{C}{\log \nu_n}$$

and so

$$\begin{aligned} n(G(y) - G(ey)) &\le n\big(G(e^m) - G(e^{m+1}) + G(e^{m+1}) - G(e^{m+2})\big) \\ &= n\big(G(e^m)u_m + G(e^{m+1})u_{m+1}\big) \qquad (4.16) \\ &\le 2nG(e^m)C(\log \nu_n)^{-1}. \end{aligned}$$

Adding over the relevant values of y gives (for large n)

$$\frac{G\big(\beta_n(nG(\beta_n))^{-2}\big) - G(\beta_n nG(\beta_n))}{G\big(\beta_n(nG(\beta_n))^{-2}\big)} \le 3C\frac{1 + 3\log(nG(\beta_n))}{\log \nu_n}. \qquad (4.17)$$

Since $nG(\beta_n) \le 3\log \nu_n$, we must have $\nu_n \to \infty$ and

$$3\log(nG(\beta_n)) \le 3ll\nu_n + 3\log 3$$

so the right-hand side of (4.17) tends to 0. Thus

$$G\big(\beta_n e^{-1}(nG(\beta_n))^{-2}\big) \sim G\big(\beta_n(nG(\beta_n))^{-2}\big) \sim G(\beta_n nG(\beta_n))$$

and then with m as in (4.16)

$$G(e^m) \le G\big(\beta_n e^{-1}(nG(\beta_n))^{-2}\big) \sim G(\beta_n nG(\beta_n)) \le G(\beta_n).$$

Using this bound in (4.16) leads to a bound of $7C$ for large n. Thus we have (4.3) satisfied for large n with $C_2 = 7C$ when $x = c\beta_n$. [We are using the fact that $nG(\beta_n) \to \infty$ to imply

$$\frac{\beta_n}{(nG(\beta_n))^2} \le \frac{c\beta_n}{nG(c\beta_n)} \quad \text{and} \quad c\beta_n \le \beta_n nG(\beta_n)$$

for large n.] Furthermore, if $u_j \log r_j \to 0$, then we may assume that C and hence C_2 is small. Thus Lemma 1 applies and gives

$$P\{S_n \le c\beta_n\} \approx e^{-nG(c\beta_n)}.$$

This is enough since by (4.10)

$$n|G(c\beta_n) - G(\beta_n)| \le C_2(1 + |\log c|). \qquad \square$$

Next, we must deal with the terms where (4.14) fails. Let $E = E_1 \cup E_2$, where

$$E_1 = \{n: nG(\beta_n) > 3lln\} \quad \text{and} \quad E_2 = \{n: nG(\beta_n) > 3\log \nu_n\}.$$

Then we have Lemma 3.

LEMMA 3. *Suppose that $\beta_n \to \infty$ and that G is slowly varying. Then*

$$\sum_{n\in E} G(c\beta_n)e^{-nG(c\beta_n)} < \infty \quad \textit{for all } c > 0.$$

PROOF. Fix $c > 0$. Since $G(c\beta_n) \sim G(\beta_n)$, we have for large $n \in E_1$, $nG(c\beta_n) > 2lln$. Then

$$\sum_{n\in E_1} G(c\beta_n)e^{-nG(c\beta_n)} = \sum_{n\in E_1} n^{-1}(nG(c\beta_n))e^{-nG(c\beta_n)}$$
$$\le \sum_n 2n^{-1}llne^{-2lln} = \sum_n 2lln(n\log^2 n)^{-1} < \infty.$$

The situation with E_2 is more complicated. Let

$$E_{ijk} = \{n: \nu_n = k, i\log k < nG(c\beta_n) \le (i+1)\log k, e^j < c\beta_n \le e^{j+1}\} \setminus E_1,$$

and note that except for some small values of n which may be neglected

$$E_2 \setminus E_1 \subset \{n: \nu_n = 1\} \setminus E_1 \cup \bigcup_{k=2}^{\infty} \bigcup_{i=2}^{\infty} \bigcup_{j=-\infty}^{\infty} E_{ijk}.$$

Suppose that $n \notin E_1$. Then

$$\frac{\beta_n}{(nG(\beta_n))^2} = \frac{n\beta_n G(\beta_n)}{(nG(\beta_n))^3} \ge \frac{n}{(3lln)^3}\beta_n G(\beta_n) \to \infty$$

since G is slowly varying and $\beta_n \to \infty$. Recalling the definition of I_n, this means that the minimum integer in I_n tends to ∞ and this implies that $\nu_n \to \infty$. Thus $\{n: \nu_n = 1\} \cap E_1^c$ is a finite set and so we may ignore it in the summation. Now, if $E_{ijk} \neq \varnothing$, define $n_{ijk} = \min\{n: n \in E_{ijk}\}$. Since we are assuming that $\beta_n \to \infty$, we only need consider $j \to \infty$. Then we have for $\delta > 0$ and $n \in E_{ijk}$,

$$nG(e^{j+1}) \sim nG(e^j) \ge nG(c\beta_n) \ge i\log k$$

so that

$$n_{ijk}G(e^{j+1}) \ge (1-\delta)i\log k$$

and then

$$\sum_{n\in E_{ijk}} G(c\beta_n)e^{-nG(c\beta_n)} \le \sum_{n\in E_{ijk}} G(e^j)e^{-nG(e^{j+1})} \le G(e^j)\frac{e^{-n_{ijk}G(e^{j+1})}}{1-e^{-G(e^{j+1})}}$$
$$\sim e^{-n_{ijk}G(e^{j+1})} \le k^{-(1-\delta)i}.$$

Now we want to fix i and k and sum over j. As observed above, for $n \notin E_1$, $\nu_n \to \infty$, and so if $n \in E_2$ also, then $nG(\beta_n) \to \infty$. This means that for large n, I_n will be nonempty. If $E_{ijk} \neq \varnothing$, take $n \in E_{ijk}$ and $m \in I_n$. Then we must have

$$e^j < c\beta_n \le c(nG(\beta_n))^2 e^{m+1} \sim c(nG(c\beta_n))^2 e^{m+1} \le c(i+1)^2 \log^2 k e^{m+1}$$

and

$$e^{j+1} \ge c\beta_n \ge \frac{ce^m}{nG(\beta_n)} \sim \frac{ce^m}{nG(c\beta_n)} \ge \frac{ce^m}{(i+1)\log k}.$$

Thus j must satisfy

$$m + \log c - \log(i+1) - llk - 1 \le j < m + 1 + \log c + 2\log(i+1) + 2llk$$

so the number of j with $E_{ijk} \neq \varnothing$ is at most

$$3\log(i+1) + 3llk + 3.$$

Thus

$$\sum_j \sum_{n \in E_{ijk}} G(c\beta_n) e^{-nG(c\beta_n)} \le 3(\log(i+1) + llk + 1) k^{-(1-\delta)i}.$$

Now we sum over $i \ge 2$. This leads to a bound of order

$$(1 + llk) k^{-2(1-\delta)}.$$

Now this is summable on k so long as we take $\delta < \frac{1}{2}$. □

Now we are ready to prove the necessity of (4.1). This will follow immediately from Theorem 3.

THEOREM 3. *Assume X is nonnegative, nondegenerate, $\beta_n \uparrow$ and*

$$\limsup_{j \to \infty} u_j \log r_j < \infty.$$

Then

$$\liminf_{n \to \infty} \frac{S_n}{\beta_n} = 0 \quad a.s. \text{ iff } \sum_n (G(\beta_n) \vee n^{-1}) e^{-nG(\beta_n)} = \infty,$$

$$\lim_{n \to \infty} \frac{S_n}{\beta_n} = \infty \quad a.s. \text{ iff } \sum_n (G(\beta_n) \vee n^{-1}) e^{-nG(\beta_n)} < \infty.$$

PROOF. First, we must take care of two special cases. If $\beta_n \le C$ for all n, then $G(\beta_n) \ge G(C) = C_1 > 0$. (Recall that $u_j \log r_j \le C$ implies that G is slowly varying and so is positive everywhere.) Then the terms of the series are bounded by $e^{-C_1 n}$ and the series converges. But

$$\frac{S_n}{\beta_n} \ge \frac{S_n}{C} \to \infty \quad \text{a.s.}$$

since even $n^{-1}S_n \to \infty$ by the strong law. Thus the theorem is true in this case. The second special case is when

$$\liminf nG(\beta_n) < \infty.$$

Suppose that $n_k G(\beta_{n_k}) \le C$. Then for $n_k \le n \le 2n_k$,

$$nG(\beta_n) \le 2n_k G(\beta_{n_k}) \le 2C$$

so that

$$\sum_{n=n_k+1}^{2n_k} \left(G(\beta_n) \vee n^{-1}\right) e^{-nG(\beta_n)} \ge e^{-2C} \sum_{n=n_k+1}^{2n_k} \frac{1}{n} \sim e^{-2C}\log 2$$

and so the series diverges. By Lemma 1, we have

$$P\{S_{n_k} \le \varepsilon\beta_{n_k}\} \ge C_3 e^{-n_k G(\varepsilon\beta_{n_k})} \ge C_3 e^{-C-1}$$

for large k. Thus $S_n \le \varepsilon\beta_n$ i.o. with positive probability and this probability must then be 1 by the Hewitt–Savage zero–one law. Thus the test works in these special cases so in the rest of the proof we may assume that

$$\beta_n \to \infty \quad \text{and} \quad nG(\beta_n) \to \infty.$$

Thus the criterion becomes the convergence or divergence of the series

$$\sum G(\beta_n) e^{-nG(\beta_n)}. \tag{4.18}$$

By Lemma 2, we note that for $n \notin E$, $G(c\beta_n)e^{-nG(c\beta_n)} \approx G(\beta_n)e^{-nG(\beta_n)}$, so that by Lemma 3, the series $\sum G(c\beta_n)e^{-nG(c\beta_n)}$ either converges for all c or diverges for all c. Thus suppose that the series in (4.18) converges. We will show that $S_n > \beta_n$ eventually with probability 1. Since we could have used $C\beta_n$ with C large in place of β_n, this will prove the convergent part of the test. Now define a sequence $\{n_k\}$ by

$$n_k = \max\left\{ j \colon \sum_{n=1}^{j} G(\beta_n) \le k \right\}. \tag{4.19}$$

These will be well defined since we are assuming that $nG(\beta_n) \to \infty$. The assumption that $\beta_n \to \infty$ means that $G(\beta_n) \to 0$ and so as $k \to \infty$,

$$\sum_{n=1}^{n_k} G(\beta_n) = k + o(1), \qquad \sum_{n=n_k+1}^{n_{k+1}} G(\beta_n) = 1 + o(1).$$

Now suppose that $n_k < n \le n_{k+1}$. Then

$$1 + o(1) \ge \sum_{i=n_k+1}^{n} G(\beta_i) \ge (n - n_k)G(\beta_n) \tag{4.20}$$

and so using (4.20) twice,

$$\begin{aligned} nG(\beta_n) &= (n - n_k)G(\beta_n) + n_k G(\beta_n) \\ &\le 1 + o(1) + (n_k - n_{k-1})G(\beta_{n_k}) + n_{k-1}G(\beta_{n_k}) \\ &\le 2 + o(1) + n_{k-1}G(\beta_{n_k}). \end{aligned}$$

Thus for large k,

$$\begin{aligned}\sum_{n=n_k+1}^{n_{k+1}} G(\beta_n)e^{-nG(\beta_n)} &\geq e^{-n_{k-1}G(\beta_{n_k})-3} \sum_{n=n_{k+1}}^{n_{k+1}} G(\beta_n) \\ &= (1+o(1))e^{-3}e^{-n_{k-1}G(\beta_{n_k})} \\ &\geq (1+o(1))e^{-3}P\{S_{n_{k-1}} \leq \beta_{n_k}\}\end{aligned}$$

by (4.5). This means that

$$\sum_k P\{S_{n_{k-1}} \leq \beta_{n_k}\} < \infty$$

so that for large k and $n_{k-1} < n \leq n_k$,

$$S_n \geq S_{n_{k-1}} > \beta_{n_k} \geq \beta_n.$$

It remains to prove the divergent part of the test. By Lemma 3, we know that $\sum_{n \notin E} G(\beta_n)e^{-nG(\beta_n)}$ diverges. From now on we will restrict our attention to those $n \notin E$, i.e., we will assume that $n \notin E$ for all n. We define $\{n_k\}$ as in (4.19) but it may be different now since we have omitted all those n's in E. For $n_k < n \leq n_{k+1}$ we have by (4.20)

$$\begin{aligned}nG(\beta_n) &\geq n_k G(\beta_{n_{k+1}}) = (n_k - n_{k+1})G(\beta_{n_{k+1}}) + n_{k+1}G(\beta_{n_{k+1}}) \\ &\geq -1 + o(1) + n_{k+1}G(\beta_{n_{k+1}})\end{aligned} \tag{4.21}$$

and so

$$\begin{aligned}\sum_{n=n_k+1}^{n_{k+1}} G(\beta_n)e^{-nG(\beta_n)} &\leq e^2 e^{-n_{k+1}G(\beta_{n_{k+1}})} \sum_{n=n_k+1}^{n_{k+1}} G(\beta_n) \sim e^2 e^{-n_{k+1}G(\beta_{n_{k+1}})} \\ &\approx P\{S_{n_{k+1}} \leq \beta_{n_{k+1}}\}\end{aligned}$$

by Lemma 2 since we know $n_{k+1} \notin E$. Thus

$$\sum_k P\{S_{n_k} \leq \beta_{n_k}\} = \infty.$$

Thus once we show the supplementary condition for Borel–Cantelli [Kochen and Stone (1964)], this will prove the divergent case since we will have $S_n \leq \beta_n$ i.o. with probability 1 and we could have used $\varepsilon\beta_n$ in place of β_n, where ε is small. As in Section 3, this verification of the supplementary condition is quite lengthy. We will use the basic estimate for $j < k$,

$$\begin{aligned}P\{S_{n_j} \leq \beta_{n_j}, S_{n_k} \leq \beta_{n_k}\} &\leq P\{S_{n_j} \leq \beta_{n_j}; S_{n_k} - S_{n_j} \leq \beta_{n_k}\} \\ &\leq e^{-n_j G(\beta_{n_j})}e^{-(n_k - n_j)G(\beta_{n_k})} \\ &\approx P\{S_{n_j} \leq \beta_{n_j}\}P\{S_{n_k} \leq \beta_{n_k}\}e^{n_j G(\beta_{n_k})}.\end{aligned} \tag{4.22}$$

This follows from (4.5) and (4.15) since we know that $n_j, n_k \notin E$. [Note that we do not need to know that $n_k - n_j \notin E$ since (4.5) is general.] Now (4.22) is sufficient for those j and k with $n_j G(\beta_{n_k}) \le 1$. We define

$$H = \{(j,k): j < k, n_j G(\beta_{n_k}) > 1\} = H_1 \cup H_2,$$

where

$$H_1 = \{(j,k) \in H: n_k G(\beta_{n_k}) \ge n_j G(\beta_{n_j})\},$$

$$H_2 = \{(j,k) \in H: n_k G(\beta_{n_k}) < n_j G(\beta_{n_j})\}.$$

We will prove that for j fixed

$$(4.23) \qquad \sum_{\{k:(j,k)\in H_1\}} P\{S_{n_j} \le \beta_{n_j}, S_{n_k} \le \beta_{n_k}\} \le CP\{S_{n_j} \le \beta_{n_j}\}$$

and that for k fixed

$$(4.24) \qquad \sum_{\{j:(j,k)\in H_2\}} P\{S_{n_j} \le \beta_{n_j}, S_{n_k} \le \beta_{n_k}\} \le CP\{S_{n_k} \le \beta_{n_k}\}.$$

This will be sufficient for the supplementary condition. The two bounds are similar but there are a few differences. We will start with (4.23). We will first consider those j and k for which

$$L < n_j G(\beta_{n_j}) \le L + 1, \qquad M < n_k G(\beta_{n_k}) \le M + 1,$$

where L and M are nonnegative integers. In proving (4.23), L is fixed and M will vary with $M \ge L$ since $(j,k) \in H_1$. Next, we let $\delta_0 = j$ and define for $i \ge 1$,

$$\gamma_i = \min\{m \ge \delta_{i-1}: n_m G(\beta_{n_m}) \in (M, M+1]\},$$

$$\delta_i = \min\{m \ge \gamma_i: n_m G(\beta_{n_m}) \notin (M-1, M+2]\}.$$

γ_1 is well defined since k satisfies the condition. But since we are assuming that $nG(\beta_n) \to \infty$, there will be only finitely many γ_i which are well defined. We must have $k \in [\gamma_i, \delta_i)$ for some i. We will let i denote this index. We will sum on k by letting k vary in this interval, then letting i vary and finally letting M vary. Now suppose that

$$n_m < n \le n_{m+1} \quad \text{with } \gamma_i \le m < k.$$

Then by (4.20)

$$nG(\beta_n) \le 2 + n_m G(\beta_n) \le 2 + n_m G(\beta_{n_m}) \le M + 4 \quad \text{since } m < k < \delta_i,$$

and by (4.21)

$$nG(\beta_n) \ge -2 + n_{m+1} G(\beta_{n_{m+1}}) \ge M - 3 \quad \text{since } m + 1 \le k < \delta_i.$$

Therefore

$$k - \gamma_i + o(1) = \sum_{n=n_{\gamma_i}+1}^{n_k} G(\beta_n) = \sum_{n=n_{\gamma_i}+1}^{n_k} \frac{1}{n} nG(\beta_n)$$

$$= (M + O(1)) \sum_{n=n_{\gamma_i}+1}^{n_k} \frac{1}{n} = (M + O(1)) \left(\log \frac{n_k}{n_{\gamma_i}} + O\left(\frac{1}{n_{\gamma_i}}\right)\right).$$

Now

$$\frac{1}{n_{\gamma_i}} = \frac{G(\beta_{n_{\gamma_i}})}{n_{\gamma_i} G(\beta_{n_{\gamma_i}})} \le \frac{G(\beta_{n_{\gamma_i}})}{M} = o\left(\frac{1}{M}\right)$$

so the $O(1/n_{\gamma_i})$ error term may be incorporated with the $o(1)$ term on the other side. We have

$$\frac{n_k}{n_{\gamma_i}} = \exp\left(\frac{k - \gamma_i + o(1)}{M + O(1)}\right). \tag{4.25}$$

Next, we need to estimate $n_{\gamma_m}/n_{\gamma_{m-1}}$ for $m \le i$. If $n_{\delta_{m-1}} G(\beta_{n_{\delta_{m-1}}}) \le M - 1$, then

$$\frac{n_{\gamma_{m-1}}}{n_{\gamma_m}} < \frac{n_{\delta_{m-1}}}{n_{\gamma_m}} \le \frac{M-1}{n_{\gamma_m} G(\beta_{n_{\delta_{m-1}}})} \le \frac{M-1}{n_{\gamma_m} G(\beta_{n_{\gamma_m}})} < \frac{M-1}{M},$$

while if $n_{\delta_{m-1}} G(\beta_{n_{\delta_{m-1}}}) > M + 2$, then

$$\frac{n_{\gamma_{m-1}}}{n_{\gamma_m}} < \frac{n_{\gamma_{m-1}}}{n_{\delta_{m-1}}} < \frac{n_{\gamma_{m-1}} G(\beta_{n_{\delta_{m-1}}})}{M+2} \le \frac{n_{\gamma_{m-1}} G(\beta_{n_{\gamma_{m-1}}})}{M+2} \le \frac{M+1}{M+2}.$$

Thus, in either case,

$$\frac{n_{\gamma_m}}{n_{\gamma_{m-1}}} > 1 + \frac{1}{M+1}. \tag{4.26}$$

Finally, we have

$$M < n_{\gamma_1} G(\beta_{n_{\gamma_1}}) \le n_{\gamma_1} G(\beta_{n_j}) \le (L+1)\frac{n_{\gamma_1}}{n_j}. \tag{4.27}$$

Putting (4.25), (4.26) and (4.27) together yields

$$\begin{aligned} \frac{n_j}{n_k} &< \frac{L+1}{M}\left(\frac{M+1}{M+2}\right)^{i-1} \exp\left(-\frac{k-\gamma_i+o(1)}{M+O(1)}\right) \\ &\le \exp\left(-\frac{M-L-1}{M} - \frac{i-1}{M+2} - \frac{k-\gamma_i+o(1)}{M+O(1)}\right). \end{aligned} \tag{4.28}$$

We consider first the possibility that $i + k - \gamma_i \le M$. Then

$$\frac{M-L-1}{M} + \frac{i-1}{M+2} + \frac{k-\gamma_i+o(1)}{M+O(1)} \le 1 + \frac{M}{M+O(1)} \le 3$$

for large M. Using the inequality $1 - e^{-x} \geq cx$ for $x \leq 3$ leads to

$$(n_k - n_j)G(\beta_{n_k}) = n_k G(\beta_{n_k})\left(1 - \frac{n_j}{n_k}\right)$$
$$\geq Mc\left(\frac{M-L-1}{M} + \frac{i-1}{M+2} + \frac{k-\gamma_i+o(1)}{M+O(1)}\right)$$
$$\geq c_1(M - L + i + k - \gamma_i - 3).$$

Recalling (4.22), we have

$$P\{S_{n_j} \leq \beta_{n_j}, S_{n_k} \leq \beta_{n_k}\} \leq CP\{S_{n_j} \leq \beta_{n_j}\}e^{-(n_k-n_j)G(\beta_{n_k})}$$
$$\leq CP\{S_{n_j} \leq \beta_{n_j}\}e^{-c_1(M-L+i+k-\gamma_i-3)}$$

and we may sum the series on $k \geq \gamma_i$, then $i \geq 1$, and finally on $M \geq L$. We must still deal with those values of i, k and M such that $i + k - \gamma_i > M$. In this case,

$$\frac{M-L-1}{M} + \frac{i-1}{M+2} + \frac{k-\gamma_i+o(1)}{M+O(1)} \geq \frac{M-3}{M+O(1)} \sim 1$$

for large M, so that

$$(n_k - n_j)G(\beta_{n_k}) > M\left(1 - \frac{n_j}{n_k}\right) > M(1 - e^{-1/2}) = cM,$$

say. Thus

$$P\{S_{n_j} \leq \beta_{n_j}, S_{n_k} \leq \beta_{n_k}\} \leq Ce^{-cM}P\{S_{n_j} \leq \beta_{n_j}\}, \tag{4.29}$$

and it remains to count the number of possible values of i and k that can arise for a given value of M. For this we will use

$$\frac{n_j}{n_k} = \frac{n_j G(\beta_{n_k})}{n_k G(\beta_{n_k})} > \frac{1}{M+1}$$

since $(j, k) \in H$. Thus by (4.28)

$$\frac{1}{M+1} < \exp\left(-\frac{M-L-1}{M} - \frac{i-1}{M+2} - \frac{k-\gamma_i+o(1)}{M+O(1)}\right)$$

or

$$\log(M+1) > \frac{M-L-1}{M} + \frac{i-1}{M+2} + \frac{k-\gamma_i+o(1)}{M+O(1)}$$

so that

$$i = O(M \log M) \quad \text{and} \quad k - \gamma_i = O(M \log M).$$

Thus by (4.29), when we sum over $k \geq \gamma_i$ and $i \geq 1$ we will obtain a bound of

$$C(M \log M)^2 e^{-cM} P\{S_{n_j} \leq \beta_{n_j}\},$$

which is still summable on M. Note that there was no harm in assuming that M was large as there are only a finite number of k corresponding to any value of M since we are assuming here that $nG(\beta_n) \to \infty$. Thus we have proved (4.23). For (4.24), M will be fixed and L will vary with $L \geq M$ since $(j, k) \in H_2$. Let $\delta_0 = k$ and define for $i \geq 1$,

$$\gamma_i = \max\{m \leq \delta_{i-1} : n_m G(\beta_{n_m}) \in (L, L+1]\},$$

$$\delta_i = \max\{m \leq \gamma_i : n_m G(\beta_{n_m}) \notin (L-1, L+2]\}.$$

(We are abusing the notation by changing the meaning of γ_i and δ_i.) Since $\gamma_i \leq k$, there are only finitely many γ_i which are well defined. We must have $j \in (\delta_i, \gamma_i]$ for some i; this will define i as before. Now if

$$n_m < n \leq n_{m+1} \quad \text{with } j \leq m < \gamma_i$$

we have

$$nG(\beta_n) \leq 2 + n_m G(\beta_n) \leq 2 + n_m G(\beta_{n_m}) \leq L + 4 \quad \text{since } \delta_i < j \leq m < \gamma_i,$$

$$nG(\beta_n) \geq -2 + n_{m+1} G(\beta_{n_{m+1}}) \geq L - 3 \quad \text{since } \delta_i < j < m + 1 \leq \gamma_i.$$

Thus

$$\gamma_i - j + o(1) = \sum_{n=n_j+1}^{n_{\gamma_i}} G(\beta_n) = \sum_{n=n_j+1}^{n_{\gamma_i}} \frac{1}{n} nG(\beta_n)$$

$$= (L + O(1)) \left(\log \frac{n_{\gamma_i}}{n_j} + O\left(\frac{1}{n_j}\right) \right)$$

and

$$\frac{1}{n_j} = \frac{G(\beta_{n_j})}{n_j G(\beta_{n_j})} \leq \frac{G(\beta_{n_j})}{L} = o\left(\frac{1}{L}\right).$$

Therefore

$$\frac{n_{\gamma_i}}{n_j} = \exp\left(\frac{\gamma_i - j + o(1)}{L + O(1)} \right). \tag{4.30}$$

For $n_{\gamma_{m-1}}/n_{\gamma_m}$, if $n_{\delta_{m-1}} G(\beta_{n_{\delta_{m-1}}}) \leq L - 1$, then

$$\frac{n_{\gamma_{m-1}}}{n_{\gamma_m}} \geq \frac{n_{\gamma_{m-1}}}{n_{\delta_{m-1}}} \geq \frac{n_{\gamma_{m-1}} G(\beta_{n_{\delta_{m-1}}})}{L-1} \geq \frac{n_{\gamma_{m-1}} G(\beta_{n_{\gamma_{m-1}}})}{L-1} > \frac{L}{L-1},$$

while if $n_{\delta_{m-1}} G(\beta_{n_{\delta_{m-1}}}) > L + 2$, then

$$\frac{n_{\gamma_{m-1}}}{n_{\gamma_m}} \geq \frac{n_{\delta_{m-1}}}{n_{\gamma_m}} \geq \frac{L+2}{n_{\gamma_m} G(\beta_{n_{\delta_{m-1}}})} \geq \frac{L+2}{n_{\gamma_m} G(\beta_{n_{\gamma_m}})} \geq \frac{L+2}{L+1}.$$

Now

$$\begin{aligned}\frac{G(\beta_{n_k})}{G(\beta_{n_j})} &= \frac{n_k G(\beta_{n_k})}{n_j G(\beta_{n_j})} \frac{n_j}{n_k} \leq \frac{M+1}{L} \frac{n_j}{n_{\gamma_1}} \\ &\leq \frac{M+1}{L} \left(\frac{L+1}{L+2} \right)^{i-1} \exp\left(-\frac{\gamma_i - j + o(1)}{L + O(1)} \right) \\ &\leq \exp\left(-\frac{L-M-1}{L} - \frac{i-1}{L+2} - \frac{\gamma_i - j + o(1)}{L + O(1)} \right).\end{aligned}$$

If $i + \gamma_i - j \leq L$, then

$$\frac{L-M-1}{L} + \frac{i-1}{L+2} + \frac{\gamma_i - j + o(1)}{L + O(1)} \leq 1 + \frac{L}{L + O(1)} \leq 3$$

for large L. Then using $1 - e^{-x} \geq cx$ for $x \leq 3$ again yields

$$\begin{aligned}n_j\big(G(\beta_{n_j}) - G(\beta_{n_k})\big) &\geq Lc\left(\frac{L-M-1}{L} + \frac{i-1}{L+2} + \frac{\gamma_i - j + o(1)}{L + O(1)} \right) \\ &\geq c_1(L - M + i + \gamma_i - j - 3).\end{aligned}$$

Recalling (4.22),

$$\begin{aligned}P\{S_{n_j} \leq \beta_{n_j}, S_{n_k} \leq \beta_{n_k}\} &\leq CP\{S_{n_k} \leq \beta_{n_k}\} e^{-n_j(G(\beta_{n_j}) - G(\beta_{n_k}))} \\ &\leq CP\{S_{n_k} \leq \beta_{n_k}\} e^{-c_1(L-M+i+\gamma_i-j-3)}\end{aligned}$$

and so we may sum this estimate on $j \leq \gamma_i$, then $i \geq 1$, and finally on $L \geq M$. It remains to consider $i + \gamma_i - j > L$. Then

$$\frac{L-M-1}{L} + \frac{i-1}{L+2} + \frac{\gamma_i - j + o(1)}{L + O(1)} \geq \frac{L-3}{L + O(1)} \sim 1$$

for large L, so that

$$n_j\big(G(\beta_{n_j}) - G(\beta_{n_k})\big) > L(1 - e^{-1/2}) = cL,$$

say, and then

$$P\{S_{n_j} \leq \beta_{n_j}, S_{n_k} \leq \beta_{n_k}\} \leq Ce^{-cL} P\{S_{n_k} \leq \beta_{n_k}\}.$$

The number of possible values of i and j for fixed L is counted as before:

$$\frac{1}{L+1} < \frac{n_j G(\beta_{n_k})}{n_j G(\beta_{n_j})} \leq \exp\left(-\frac{L-M-1}{L} - \frac{i-1}{L+2} - \frac{\gamma_i - j + o(1)}{L + O(1)} \right),$$

$$\log(L+1) > \frac{L-M-1}{L} + \frac{i-1}{L+2} + \frac{\gamma_i - j + o(1)}{L + O(1)}$$

so that $i = O(L \log L)$, $\gamma_i - j = O(L \log L)$ and then $(L \log L)^2 e^{-cL}$ is summable. □

We will now prove the sufficiency of (4.1) in Theorem 2. This will be done in two parts. First, we will take care of the case when G is slowly varying which is the harder part. Then we will construct a nicer normalizing sequence when G is not slowly varying.

PROOF OF SUFFICIENCY OF (4.1). We start by defining some sequences. Take $j_1 = 1$ and suppose that $j_1, \dots, j_{m-1}$ have been defined. Since we are assuming that G is slowly varying we have $u_n \to 0$. Take N_m such that

$$u_n < (\log m)^{-1} \quad \text{for all } n \geq N_m.$$

Then choose k_m such that

$$k_m > N_m, \qquad r_{k_m} > 2(N_m \vee j_{m-1}), \qquad u_{k_m} \log r_{k_m} \geq m^2.$$

This is possible by (4.1). Next, define

$$E_m = \{\nu: u_\nu \geq u_{k_m}\}, \qquad j_m = \max\{\nu: \nu \in E_m\}, \qquad F_m = E_m \cap \left(\tfrac{1}{2} r_{k_m}, j_m\right].$$

We list some properties for future reference:

$$j_m \geq \operatorname{card}(E_m) \geq r_{k_m} \quad \text{and so} \quad r_{k_m} > r_{k_{m-1}}; \tag{4.31}$$

$$\nu \in E_m \quad \text{implies} \quad u_\nu \geq u_{k_m} \geq m^2/\log r_{k_m}; \tag{4.32}$$

$$\nu \in F_m \quad \text{implies} \quad u_\nu < (\log m)^{-1}; \tag{4.33}$$

$$\operatorname{card}(F_m) \geq \operatorname{card}(E_m) - \tfrac{1}{2} r_{k_m} \geq \tfrac{1}{2} r_{k_m}. \tag{4.34}$$

Now suppose that $\nu \in E_m$ and let $\nu_0 = \nu$ and then define

$$\nu_k = \max\{i \leq \nu_{k-1}: u_i > e u_{\nu_{k-1}}\}$$

so long as this set is nonempty. Since

$$1 \geq u_{\nu_i} > e^i u_\nu \geq e^i m^2/\log r_{k_m}$$

there can be no more than llr_{k_m} of these defined. Thus for any $\nu \in F_m$, we can find in $(\nu - 3(llr_{k_m})^2, \nu)$ a block of $2[llr_{k_m}]$ consecutive u's with all of them no larger than e times the last one; furthermore, the last one will be in E_m. We let $\mu_m = [\frac{1}{2}(r_{k_m})^{1/2}]$ and split the members of F_m into μ_m blocks of at least $2\mu_m$ members each. Since $3(llr_{k_m})^2$ is small compared with μ_m we can find in the right half of each block $2[llr_{k_m}]$ u's with the above property. (These need not be in F_m but the last one will be.) We will suppress the dependence on m for now and denote the u's at the right end of these "good runs" as u_{i_k}. There will be μ_m of them. Furthermore, we have $i_k \in F_m$ and

$$u_{i_k - j} \leq e u_{i_k}, \qquad j = 1, \dots, 2\left[llr_{k_m}\right]; \tag{4.35}$$

(4.36) there are at least μ_m members of E_m in (i_k, i_{k+1})

for all k. Next, we define λ_k and n_k by

$$\frac{R(\lambda_k)}{g(\lambda_k)} = \frac{m^2}{3u_{i_k}e^{i_k+2}}, \qquad n_k = \left[\frac{m^2}{3u_{i_k}R(\lambda_k)}\right] = \left[\frac{e^{i_k+2}}{g(\lambda_k)}\right].$$

Since

$$\frac{m^2}{u_{i_k}e^{i_k}} \le \frac{\log r_{k_m}}{e^{i_k}} \le \frac{\log r_{k_m}}{e^{r_{k_m}/2}} \to 0$$

by (4.32), λ_k will be defined, at least for large m, and will approach 0 as $m \to \infty$. Now by (2.10) we have

$$\begin{aligned} -\log P\{S_{n_k} \le e^{i_k+2}\} &\le -\log P\{S_{n_k} \le n_k g(\lambda_k)\} \sim n_k R(\lambda_k) \\ &\le \frac{m^2}{3u_{i_k}} \le \frac{\log r_{k_m}}{3} \end{aligned} \tag{4.37}$$

so that for large m,

$$P\{S_{n_k} \le e^{i_k+2}\} \ge 3r_{k_m}^{-1/2}.$$

Since there are at least μ_m of these we can select just enough so that

$$1 \le \sum P\{S_{n_k} \le e^{i_k+2}\} \le 2, \tag{4.38}$$

where the sum is for fixed m. We discard any remaining u_{i_k}'s. We do this for each m and arrange the u_{i_k}'s that we have kept in order. We will now think of this sequence and the accompanying n_k and λ_k sequences as determined without changing the notation. Note that the m is still being suppressed. Next, we will show that

$$n_k G(e^{i_k+2}) \approx \frac{m^2}{u_{i_k}}. \tag{4.39}$$

For the upper bound, use (4.5) to obtain

$$P\{S_{n_k} \le e^{i_k+2}\} \le \exp(-n_k G(e^{i_k+2}))$$

and use this in conjunction with (4.37). The lower bound takes more work. We will need the following estimate for g [recall the definitions in (2.1) and (2.11)]:

$$\begin{aligned} g(\lambda) &\sim \int xe^{-\lambda x}\,dF(x) \le \lambda^{-1}M(\lambda^{-1}) + \lambda^{-1}G(\lambda^{-1}) \sim \lambda^{-1}G(\lambda^{-1}) \\ &\sim \lambda^{-1}Q(\lambda^{-1}) \approx \lambda^{-1}R(\lambda) \quad \text{as } \lambda \to 0. \end{aligned}$$

We have used (2.13) as well as the fact that when G is slowly varying, then $M(x) = o(G(x))$ and $K(x) = o(G(x))$ as $x \to \infty$; see Lemma 2.5 of Pruitt (1981). Thus we have for an appropriate constant C,

$$e^{i_k+2} \sim n_k g(\lambda_k) \le C n_k \lambda_k^{-1} R(\lambda_k) \sim \frac{Cm^2}{\lambda_k 3u_{i_k}} \tag{4.40}$$

by the definitions of λ_k and n_k. Now if we take $N \le 2llr_{k_m}$, then by (4.35)

$$G(e^{i_k+2}) = \frac{G(e^{i_k+2})}{G(e^{i_k+1})} \prod_{j=0}^{N} \frac{G(e^{i_k-j+1})}{G(e^{i_k-j})} G(e^{i_k-N}) \sim \prod_{j=0}^{N} (1 - u_{i_k-j}) G(e^{i_k-N})$$
$$\ge (1 - eu_{i_k})^{N+1} G(e^{i_k-N}).$$

Now if we take

$$N = \left[\log\left(\frac{Cm^2}{3u_{i_k}}\right)\right] + 1,$$

then by (4.40)

$$e^{i_k-N} \le e^{i_k} \frac{3u_{i_k}}{Cm^2} < \frac{1}{\lambda_k},$$

and so $G(e^{i_k-N}) \ge G(\lambda_k^{-1})$. This N is small enough since by (4.32)

$$\log\left(\frac{Cm^2}{3u_{i_k}}\right) \le \log\left(\frac{C \log r_{k_m}}{3}\right) \sim llr_{k_m}.$$

Finally, for large k,

$$(1 - eu_{i_k})^{N+1} \sim (1 - eu_{i_k})^{N} \ge \exp(-Neu_{i_k} - Ne^2u_{i_k}^2) \ge e^{-2e}(1 + o(1))$$

since $u_{i_k} \log u_{i_k} \to 0$ and $u_{i_k} \log m < 1$ by (4.33). Thus

$$G(e^{i_k+2}) \ge cG(\lambda_k^{-1})$$

for an appropriate c and then using

$$G(\lambda^{-1}) \sim Q(\lambda^{-1}) \approx R(\lambda)$$

and the definition of n_k and λ_k completes the proof of (4.39). Next we need to compare $G(e^{i_k})$ and $G(e^{i_{k-1}})$. First, for $\nu \in E_m$, we have

$$\frac{G(e^{\nu+1})}{G(e^{\nu})} = 1 - u_\nu \le 1 - \frac{m^2}{\log r_{k_m}}.$$

Since we have at least μ_m of these between i_{k-1} and i_k by (4.36)

$$\text{(4.41)} \qquad \frac{G(e^{i_k})}{G(e^{i_{k-1}})} \le \left(1 - \frac{m^2}{\log r_{k_m}}\right)^{\mu_m+1} \le \exp\left(-\frac{m^2 r_{k_m}^{1/2}}{2\log r_{k_m}}\right),$$

and so these ratios go to 0 as $m \to \infty$. This also gives us information about the n_k sequence. By (4.39), (4.32) and (4.41)

$$\frac{n_k}{n_{k-1}} \approx \frac{u_{i_{k-1}}}{u_{i_k}} \frac{G(e^{i_{k-1}})}{G(e^{i_k})} \ge \frac{m^2}{\log r_{k_m}} \exp\left(\frac{m^2 r_{k_m}^{1/2}}{2\log r_{k_m}}\right) \to \infty$$

as $m \to \infty$. If u_{i_k} happens to be the first of the u's in F_m, then the first factor in the above estimate should be $(m-1)^2/\log r_{k_{m-1}}$, but this is asymptotically even larger by (4.31). Now we are ready to define our β_n sequence. We let

$$\beta_n = e^{i_k}, \qquad n_k \le n < n_{k+1}.$$

This is valid, at least for large k, since we have shown that the n_k sequence increases and the i_k sequence increases by definition. Now we can complete the proof. We have

$$\begin{aligned} P\{S_{n_k} \le e^{i_k+2}\} &\ge n_k P\{e^{i_k} < X \le e^{i_k+1}\} P\{S_{n_k-1} \le e^{i_k}\} \\ &\ge n_k G(e^{i_k}) u_{i_k} P\{S_{n_k} \le e^{i_k}\} \approx m^2 P\{S_{n_k} \le e^{i_k}\} \end{aligned}$$

by (4.39). Thus by (4.38)

$$\sum_k P\{S_{n_k} \le e^{i_k}\} \le Cm^{-2} \sum_k P\{S_{n_k} \le e^{i_k+2}\} \le 2Cm^{-2},$$

where the sums are over those k with a fixed m. Since the bound is still summable on m, we have, with probability 1, for all k sufficiently large and $n_k \le n < n_{k+1}$,

$$S_n \ge S_{n_k} \ge \beta_{n_k} = \beta_n.$$

For the upper bound, we use

$$\sum_k P\{S_{n_k} - S_{n_{k-1}} \le e^{i_k+2}\} \ge \sum_k P\{S_{n_k} \le e^{i_k+2}\} \ge 1$$

by (4.38), where again the sums are for m fixed. Now summing on m will give a divergent series so that we have with probability 1,

$$S_{n_k} - S_{n_{k-1}} \le e^{i_k+2} \quad \text{i.o.}$$

By truncating at e^{i_k} and using a first moment estimate, we obtain

$$\begin{aligned} P\{S_{n_{k-1}} \ge e^{i_k}\} &\le n_{k-1} G(e^{i_k}) + \frac{n_{k-1} e^{i_k} M(e^{i_k})}{e^{i_k}} \sim n_{k-1} G(e^{i_k}) \\ &= n_{k-1} G(e^{i_{k-1}}) \frac{G(e^{i_k})}{G(e^{i_{k-1}})} \approx \frac{m^2}{u_{i_{k-1}}} \frac{G(e^{i_k})}{G(e^{i_{k-1}})} \\ &\le \log r_{k_m} \exp\left(- \frac{m^2 r_{k_m}^{1/2}}{2 \log r_{k_m}} \right) \le 4 \frac{(\log r_{k_m})^3}{r_{k_m}} \frac{1}{m^4} \end{aligned}$$

by (4.39), (4.32) and (4.41). As before, the bound is even better in the case of the first term in the mth block. Now when we sum on k there are at most μ_m terms so the sum over k for m fixed is bounded by

$$\frac{(\log r_{k_m})^3}{r_{k_m}^{1/2}} \frac{2}{m^4} \le \frac{C}{m^4}$$

for an appropriate constant C, and so we can still sum on m. This means that with probability 1, $S_{n_{k-1}} \le e^{i_k}$ for all large k. Thus we have

$$S_{n_k} \le e^{i_k+2} + e^{i_k} = \beta_{n_k}(e^2 + 1) \quad \text{i.o.} \quad \text{a.s.}$$

and this proves that

$$1 \le \liminf \frac{S_n}{\beta_n} \le e^2 + 1 \quad \text{a.s.}$$

This is enough since the lim inf is constant a.s. by the Hewitt–Savage zero–one law and then the constant value of the lim inf may be incorporated into the normalizing sequence.

It remains to consider the case where G is not slowly varying. This is equivalent to

$$\liminf_{x \to \infty} \frac{G(x)}{M(x)} < \infty$$

by Lemma 2.5 of Pruitt (1981) and it is easy to see that this is in turn equivalent to

$$\liminf_{\lambda \to 0} \frac{R(\lambda)}{\lambda g(\lambda)} < \infty.$$

Then we choose a sequence λ_k so that

$$R(\lambda_k) \le \frac{1}{k^2} R(\lambda_{k-1}) \quad \text{and} \quad \frac{R(\lambda_k)}{\lambda_k g(\lambda_k)} \le C. \tag{4.42}$$

Then we define n_k by

$$n_k = \max\{n \colon -\log P\{S_n \le ng(\lambda_k)\} \le \log k\}.$$

Since $g(\lambda_k) < EX$, the probability will approach 0 by the weak law and so n_k is well defined. Also since λ_k decreases, n_k increases. We define the norming sequence β_n by

$$\beta_n = n_k g(\lambda_k) \quad \text{for } n_k \le n < n_{k+1}.$$

We will prove that

$$\liminf_{n \to \infty} \frac{S_n}{\beta_n} = 1 \quad \text{a.s.} \tag{4.43}$$

For the lower bound, define γ_k by $g(\gamma_k) = (1 - \varepsilon)g(\lambda_k)$. Then by the generalized mean value theorem and (4.42),

$$\begin{aligned}
\varepsilon &= \frac{g(\lambda_k) - g(\gamma_k)}{g(\lambda_k)} = \frac{g(\lambda_k) - g(\gamma_k)}{R(\gamma_k) - R(\lambda_k)} \frac{R(\gamma_k) - R(\lambda_k)}{g(\lambda_k)} \\
&= \frac{V(\xi_k)}{\xi_k V(\xi_k)} \frac{R(\gamma_k) - R(\lambda_k)}{g(\lambda_k)} \\
&\le C \frac{R(\gamma_k) - R(\lambda_k)}{R(\lambda_k)},
\end{aligned}$$

where $\lambda_k < \xi_k < \gamma_k$. Thus

$$R(\gamma_k) \geq (1 + \varepsilon C^{-1}) R(\lambda_k).$$

Next we observe that by the definition of n_k,

$$P\{S_{n_k+1} \leq (n_k + 1) g(\lambda_k)\} < k^{-1}$$

and so tends to 0. By (2.9) we have

$$n_k R(\lambda_k) \sim (n_k + 1) R(\lambda_k) \to \infty.$$

Then by (2.10)

$$n_k R(\lambda_k) \sim -\log P\{S_{n_k+1} \leq (n_k + 1) g(\lambda_k)\} > \log k. \tag{4.44}$$

This means that for large k and $\delta < \varepsilon C^{-1}$,

$$n_k R(\gamma_k) \geq (1 + \varepsilon C^{-1}) n_k R(\lambda_k) > (1 + \delta) \log k.$$

Then by (2.8)

$$\begin{aligned} P\{S_{n_k} \leq (1 - \varepsilon)\beta_{n_k}\} &= P\{S_{n_k} \leq (1 - \varepsilon) n_k g(\lambda_k)\} = P\{S_{n_k} \leq n_k g(\gamma_k)\} \\ &\leq \exp\{-n_k R(\gamma_k)\} \leq k^{-(1+\delta)}. \end{aligned}$$

This means that for all sufficiently large k and $n_k \leq n < n_{k+1}$,

$$S_n \geq S_{n_k} > (1 - \varepsilon)\beta_{n_k} = (1 - \varepsilon)\beta_n.$$

For the upper bound, first note that

$$P\{S_{n_k} - S_{n_{k-1}} \leq \beta_{n_k}\} \geq P\{S_{n_k} \leq \beta_{n_k}\} \geq k^{-1},$$

and so $S_{n_k} - S_{n_{k-1}} \leq \beta_{n_k}$ i.o. a.s. by Borel–Cantelli. Thus we will be finished if we can show that $S_{n_{k-1}}/\beta_{n_k} \to 0$ a.s. We truncate at λ_k^{-1} and let $T_{n_{k-1}}$ denote the sum of the truncated variables. Then

$$ET_{n_{k-1}} = n_{k-1}\lambda_k^{-1} M(\lambda_k^{-1}) \leq n_{k-1} e g(\lambda_k) = (n_{k-1}/n_k) e \beta_{n_k}.$$

Since

$$n_k R(\lambda_k) \sim -\log P\{S_{n_k} \leq n_k g(\lambda_k)\} \leq \log k,$$

we have by (4.44) that

$$n_k R(\lambda_k) \sim \log k. \tag{4.45}$$

This means that by (4.42)

$$\frac{n_{k-1}}{n_k} = \frac{n_{k-1} R(\lambda_{k-1})}{n_k R(\lambda_k)} \frac{R(\lambda_k)}{R(\lambda_{k-1})} \sim \frac{R(\lambda_k)}{R(\lambda_{k-1})} \leq \frac{1}{k^2} \tag{4.46}$$

and so $ET_{n_{k-1}} = o(\beta_{n_k})$. Then, for large k,

$$\begin{aligned}
P\{S_{n_{k-1}} \geq 2\varepsilon\beta_{n_k}\} &\leq n_{k-1}G(\lambda_k^{-1}) + P\{T_{n_{k-1}} - ET_{n_{k-1}} \geq \varepsilon\beta_{n_k}\} \\
&\leq n_{k-1}G(\lambda_k^{-1}) + \frac{n_{k-1}\lambda_k^{-2}K(\lambda_k^{-1})}{\varepsilon^2\beta_{n_k}^2} \\
&\leq n_{k-1}G(\lambda_k^{-1}) + \frac{n_{k-1}K(\lambda_k^{-1})}{\varepsilon^2 n_k^2(\lambda_k g(\lambda_k))^2} \\
&= O(n_{k-1}R(\lambda_k)) + O\left(\frac{n_{k-1}}{n_k^2 R(\lambda_k)}\right) \\
&= O\left(\frac{n_{k-1}\log k}{n_k}\right) = O\left(\frac{\log k}{k^2}\right),
\end{aligned}$$

where we have used (2.13), (4.42), (4.45) and (4.46) at the last three steps. Since this gives a convergent series, we have $S_{n_{k-1}}/\beta_{n_k} \to 0$ a.s. This is what we needed to complete the proof. □

There is a nicer version of this last result if one assumes somewhat more than just that G is not slowly varying. Instead of assuming that $\liminf G(x)/M(x) < \infty$ as above, we assume that the lim sup is finite. This assumption is valid in the stochastically compact case but is more general. Thus we obtain a result that is less precise than the corollary to Theorem 1 but under weaker assumptions. Since the proof is fairly short, we will give it. Note that it is possible to prove this lim inf result even though we do not necessarily know that the strong probability estimate in (2.16) is valid here. The proof is similar to the classical proofs of the law of the iterated logarithm such as that in Feller (1968).

THEOREM 4. *Assume X is nonnegative and nondegenerate. Suppose $G(x) = O(M(x))$ and define $\beta_n = ng(\lambda_n)$, where λ_n is the solution of $R(\lambda_n) = n^{-1}lln$. Then*

$$\liminf_{n\to\infty} \frac{S_n}{\beta_n} = 1 \quad a.s.$$

PROOF. Since

$$\lambda g(\lambda) \sim E\lambda X e^{-\lambda X} \geq e^{-1}M(\lambda^{-1}), \tag{4.47}$$

the assumption and (2.13) yield

$$R(\lambda) = O(\lambda g(\lambda)). \tag{4.48}$$

(Actually, this is equivalent to the assumption relating G and M.) Take $C \geq 1$ so that $R(\lambda) \leq C\lambda g(\lambda)$ for all $\lambda \leq 1$, and define γ_n by

$R(\gamma_n) = (1 + \varepsilon C^{-1})R(\lambda_n)$. Then by the generalized mean value theorem,

$$\begin{aligned}\frac{g(\lambda_n) - g(\gamma_n)}{g(\lambda_n)} &= \frac{g(\lambda_n) - g(\gamma_n)}{R(\gamma_n) - R(\lambda_n)} \frac{R(\gamma_n) - R(\lambda_n)}{g(\lambda_n)} \\ &= \frac{V(\xi_n)}{\xi_n V(\xi_n)} \frac{R(\gamma_n) - R(\lambda_n)}{g(\lambda_n)} \\ &\le C\frac{R(\gamma_n) - R(\lambda_n)}{R(\lambda_n)} = \varepsilon.\end{aligned}$$

Thus $g(\gamma_n) \ge (1 - \varepsilon)g(\lambda_n)$. Now take $\rho \in (1, 1 + \varepsilon C^{-1})$ and let $n_k = [\rho^k]$. Then $n_{k+1} \sim \rho n_k$ so that

$$R(\lambda_{n_{k+1}}) = \frac{lln_{k+1}}{n_{k+1}} \sim \frac{1}{\rho}\frac{lln_k}{n_k} = \frac{1}{\rho}R(\lambda_{n_k}).$$

This means that $\lambda_{n_k} < \gamma_{n_{k+1}}$ and so

$$\beta_{n_k} = n_k g(\lambda_{n_k}) > n_k g(\gamma_{n_{k+1}}) > \frac{1 - \varepsilon}{1 + \varepsilon} n_{k+1} g(\lambda_{n_{k+1}}) = \frac{1 - \varepsilon}{1 + \varepsilon}\beta_{n_{k+1}}.$$

Also,

$$(1 - \varepsilon)g(\lambda_{n_k}) \le g(\gamma_{n_k}) < g(\lambda_{n_{k-1}})$$

so that by (2.8),

$$\begin{aligned}P\{S_{n_k} \le (1 - \varepsilon)^3\beta_{n_{k+1}}\} &\le P\{S_{n_k} \le (1 - \varepsilon)^2(1 + \varepsilon)\beta_{n_k}\} \\ &\le P\{S_{n_k} \le (1 - \varepsilon^2)n_k g(\lambda_{n_{k-1}})\} \\ &\le P\{S_{n_k} \le n_k g(\lambda_{n_{k-1}})\} \\ &\le \exp\{-n_k R(\lambda_{n_{k-1}})\}.\end{aligned}$$

Then since

$$n_k R(\lambda_{n_{k-1}}) \sim \rho n_{k-1} R(\lambda_{n_{k-1}}) = \rho lln_{k-1} \sim \rho \log k,$$

we will have for all sufficiently large k and $n_k \le n < n_{k+1}$,

$$S_n \ge S_{n_k} \ge (1 - \varepsilon)^3\beta_{n_{k+1}} \ge (1 - \varepsilon)^3\beta_n.$$

Since ε is arbitrary, this proves that the lim inf is at least 1. For the other bound we need to space the subsequence more. Take $\alpha \in (1, \rho)$ and let $m_k = [\exp(k^\alpha)]$ and $\nu_k = [\rho m_k]$. Then

$$R(\lambda_{m_k}) = \frac{llm_k}{m_k} \sim \rho\frac{ll\nu_k}{\nu_k} = \rho R(\lambda_{\nu_k}) < R(\gamma_{\nu_k})$$

and so $\lambda_{m_k} < \gamma_{\nu_k}$. This means that

$$g(\lambda_{m_k}) > g(\gamma_{\nu_k}) \ge (1 - \varepsilon)g(\lambda_{\nu_k}),$$

which leads to

$$P\{S_{m_k} - S_{m_{k-1}} \le (1-\varepsilon)^{-1}\beta_{m_k}\} \ge P\{S_{m_k} \le m_k g(\lambda_{\nu_k})\},$$

and since by (2.10)

$$-\log P\{S_{m_k} \le m_k g(\lambda_{\nu_k})\} \sim m_k R(\lambda_{\nu_k}) = m_k \frac{ll\nu_k}{\nu_k} \sim \frac{\alpha \log k}{\rho},$$

we will have $S_{m_k} - S_{m_{k-1}} \le (1-\varepsilon)^{-1}\beta_{m_k}$ i.o. a.s. Thus we will be finished if we show that $S_{m_{k-1}}/\beta_{m_k} \to 0$ a.s. As above, we truncate at $\lambda_{m_k}^{-1}$ and let $T_{m_{k-1}}$ denote the sum of the truncated terms. Then by (4.47)

$$ET_{m_{k-1}} = m_{k-1}\lambda_{m_k}^{-1} M(\lambda_{m_k}^{-1}) = O(m_{k-1} g(\lambda_{m_k})) = o(\beta_{m_k})$$

since $m_{k-1}/m_k \to 0$. By (2.13) and (4.48), this leads to

$$\begin{aligned} P\{S_{m_{k-1}} \ge 2\varepsilon\beta_{m_k}\} &\le m_{k-1} G(\lambda_{m_k}^{-1}) + P\{T_{m_{k-1}} - ET_{m_{k-1}} \ge \varepsilon\beta_{m_k}\} \\ &\le m_{k-1} Q(\lambda_{m_k}^{-1}) + \frac{m_{k-1}\lambda_{m_k}^{-2} K(\lambda_{m_k}^{-1})}{\varepsilon^2 \beta_{m_k}^2} \\ &\le C_1 m_{k-1} R(\lambda_{m_k}) + C_2 \frac{m_{k-1}}{m_k^2 R(\lambda_{m_k})} \\ &\sim C_1 \frac{m_{k-1} llm_k}{m_k}, \end{aligned}$$

which is summable. □

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