The Annals of Probability
1991, Vol. 19, No. 2, 783–811

THE ASYMPTOTIC DISTRIBUTION OF EXTREME SUMS

By Sándor Csörgő,[1] Erich Haeusler and David M. Mason[2]

University of Szeged, University of Munich and University of Delaware

Let $X_{1,n} \le \cdots \le X_{n,n}$ be the order statistics of n independent random variables with a common distribution function F and let k_n be positive integers such that $k_n \to \infty$ and $k_n/n \to \alpha$ as $n \to \infty$, where $0 \le \alpha < 1$. We find necessary and sufficient conditions for the existence of normalizing and centering constants $A_n > 0$ and C_n such that the sequence

$$E_n = \frac{1}{A_n}\left\{\sum_{i=1}^{k_n} X_{n+1-i,n} - C_n\right\}$$

converges in distribution along subsequences of the integers $\{n\}$ to nondegenerate limits and completely describe the possible subsequential limiting distributions. We also give a necessary and sufficient condition for the existence of A_n and C_n such that E_n be asymptotically normal along a given subsequence, and with suitable A_n and C_n determine the limiting distributions of E_n along the whole sequence $\{n\}$ when F is in the domain of attraction of an extreme value distribution.

1. Introduction and statements of results. Let $X, X_1, X_2 \ldots$ be a sequence of independent nondegenerate random variables with a common distribution function $F(x) = P\{X \le x\}$, $x \in R$, and for each integer $n \ge 1$, let $X_{1,n} \le \cdots \le X_{n,n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. Throughout the paper k_n will be a sequence of integers such that

$$1 \le k_n \le n, \quad k_n \to \infty \quad \text{and} \quad k_n/n \to 0 \quad \text{as } n \to \infty; \tag{1.1}$$
$$\text{or} \quad k_n = [n\alpha] \quad \text{with } 0 < \alpha < 1,$$

where $[\cdot]$ denotes integer part. (We shall refer to the first case as the case $\alpha = 0$.) The study of the asymptotic distribution of the (properly normalized and centered) sums of extreme values

$$\sum_{i=1}^{k_n} X_{n+1-i,n} = \sum_{i=n-k_n+1}^{n} X_{i,n} \tag{1.2}$$

was initiated in [6] for the case when $\alpha = 0$ in (1.1) and under the restrictive

Received April 1988; revised December 1989.
[1]Partially supported by the Hungarian National Foundation for Scientific Research, Grants 1808/86 and 457/88.
[2]Partially supported by the Alexander von Humboldt Foundation, NSF Grant DMS-88-03209 and a Fulbright grant.
AMS 1980 *subject classification*. Primary 60F05.
Key words and phrases. Sums of extreme values, asymptotic distribution.

assumption that F belongs to the domain of attraction of a nonnormal stable law. Later the problem was solved in [7] assuming that F has a regularly varying upper tail and by Lo [14] for all F which are in the domain of attraction of a Gumbel distribution in the sense of extreme value theory. When put together, these results say that whenever F is in the domain of attraction of any one of the three possible limiting extreme value distributions for the maximum $X_{n,n}$ and $\alpha = 0$ in (1.1), then the sums in (1.2), with suitable centering and normalization, have a limiting distribution which is either nonnormal stable or normal (see Corollary 2 below). The first aim of the present paper is to give an exhaustive study of the problem of the asymptotic distribution of the sums in (1.2) for an arbitrary F.

All three papers ([6], [7] and [14]) mentioned employ a direct probabilistic approach based upon the asymptotic behavior of the uniform empirical distribution function in conjunction with the behavior of the inverse or quantile function of F defined as

$$Q(s) = \inf\{x: F(x) \geq s\}, \qquad 0 < s \leq 1, \qquad Q(0) = Q(0+).$$

This quantile-transform method for handling the whole sums $\sum_{j=1}^{n} X_j$ was first used in [2] and [3] to obtain probabilistic proofs of the sufficiency parts of the normal and stable convergence criteria, respectively. In [3], the effect of trimming off a finite number of the smallest and largest summands is also considered. A refined version of this method in [5] produces a complete asymptotic distribution theory of the finitely trimmed sums $T_n(m,k) = \sum_{i=m+1}^{n-k} X_{i,n}$, where $m \geq 0$ and $k \geq 0$ are arbitrarily fixed integers. Included in this paper is a new description in terms of the quantile function of the classical theory concerning domains of attraction, domains of partial attraction and stochastic compactness of the whole sum $T_n(0,0)$.

The paper ([6]) also initiated the study of trimmed sums of the form $T_n(m_n, k_n)$, where k_n is as in (1.1) with $\alpha = 0$ and m_n satisfies the same condition as k_n. This was done in [6] only in the domain of attraction case. A different refinement of the quantile-transform method in [4] has established the complete asymptotic distributional theory for the sums $T_n(m_n, k_n)$.

The second aim of this paper is to completely round off our study of sums of order statistics by means of the quantile-transform method, so that papers [4], [5] and the present one together constitute a complete and unified general theory of the asymptotic distribution of sums of order statistics of independent, identically distributed random variables.

We emphasize very strongly that the quantile-transform method itself is by no means new. It has been in wide use in nonparametric statistics for many decades and scattered applications of it can be found in probability as well. A good source for its earlier use is the book by Shorack and Wellner ([20]). It is the approximation result for the uniform empirical and quantile processes in weighted supremum metrics in [2] in combination with Poisson approximation techniques for extremes that has made this old method especially feasible for the treatment of problems of the asymptotic distribution of various ordered portions of the sums of independent, identically distributed random variables.

The method was augmented in [4] by a general pattern of necessity proofs which has already been applied in [5] and [16].

There are, of course, several other methods for studying sums of order statistics. Besides classical approaches, various new methods have been invented recently to deal with a number of different types of trimmed sums and the influence of extremes on the whole sum. For descriptions of these methodologies along with discussions of their advantages and disadvantages, including ours, see the monograph [12] where extensive lists of references can also be found.

Now we introduce some notation. The basic function for the present paper is the left-continuous function

$$H(s) = -Q((1-s)-), \qquad 0 \le s < 1.$$

Notice that if $U_{1,n} \le \cdots \le U_{n,n}$ are the order statistics of a sample of size n from the uniform distribution on $(0,1)$, then for each $n \ge 1$ we have the distributional equality

$$(1.3) \qquad (X_{1,n}, \dots, X_{n,n}) =_{\mathscr{D}} (-H(U_{n,n}), \dots, -H(U_{1,n})).$$

For $0 \le s \le t \le 1$, consider the truncated variance function

$$\sigma^2(s,t) = \int_s^t \int_s^t (u \wedge v - uv)\, dH(u)\, dH(v),$$

where $u \wedge v = \min(u,v)$, and for a given sequence k_n satisfying (1.1), set $b_n = \sigma(1/n, k_n/n)$ and

$$a_n = \begin{cases} b_n, & \text{if } b_n > 0, \\ 1, & \text{otherwise}. \end{cases}$$

Note that $\sigma^2(s,t)$ is the variance of $\int_s^t B(u)\, dH(u)$, where $B(\cdot)$ is a Brownian bridge. Such random variables will be seen to enter the picture quite naturally.

Choose and fix any sequence of positive constants δ_n such that $\delta_n < 1$ and $n\delta_n \to 0$ as $n \to \infty$. Then we have $P\{\delta_n \le U_{1,n} \le U_{n,n} \le 1 - \delta_n\} \to 1$, as $n \to \infty$. The following two sequences of functions will govern the asymptotic behavior:

$$\psi_n(x) = \begin{cases} \dfrac{k_n^{1/2}\{H(k_n/n + x(k_n^{1/2})/n) - H(k_n/n)\}}{n^{1/2} a_n}, & \text{if } -\dfrac{k_n^{1/2}}{2} \le x \le \dfrac{k_n^{1/2}}{2}, \\ \psi_n\left(-\dfrac{k_n^{1/2}}{2}\right), & \text{if } -\infty < x < -\dfrac{k_n^{1/2}}{2}, \\ \psi_n\left(\dfrac{k_n^{1/2}}{2}\right), & \text{if } \dfrac{k_n^{1/2}}{2} < x < \infty, \end{cases}$$

and

$$\varphi_n(y) = \begin{cases} \dfrac{H(y/n) - H(1/n)}{n^{1/2}a_n}, & \text{if } 0 < y \le n - n\delta_n, \\ \dfrac{H(1-\delta_n) - H(1/n)}{n^{1/2}a_n}, & \text{if } n - n\delta_n < y < \infty. \end{cases}$$

The crucial (necessary and sufficient) conditions these functions will have to satisfy are the following three conditions in which $\{n_1\}$ is a subsequence of the positive integers $\{n\}$, $\{A_{n_1}\}$ is a sequence of positive constants and $\{k_{n_1}\}$ is the corresponding subsequence of $\{k_n\}$ from (1.1).

CONDITION 1. There exists a nondecreasing, left-continuous function ψ defined on $(-\infty, \infty)$ with $\psi(0) \le 0$ and $\psi(0+) \ge 0$ necessarily holding such that

$$\psi_{n_1}^*(x) = \frac{n_1^{1/2}a_{n_1}}{A_{n_1}}\psi_{n_1}(x) \to \psi(x) \quad \text{as } n_1 \to \infty,$$

at every continuity point x of ψ.

CONDITION 2. There exists a nondecreasing, left-continuous function φ defined on $(0, \infty)$ with $\varphi(1) \le 0$ and $\varphi(1+) \ge 0$ necessarily holding such that

$$\varphi_{n_1}^*(y) = \frac{n_1^{1/2}a_{n_1}}{A_{n_1}}\varphi_{n_1}(y) \to \varphi(y) \quad \text{as } n_1 \to \infty,$$

at every continuity point y of φ.

CONDITION 3. There exists a constant $0 \le a < \infty$ such that

$$n_1^{1/2}b_{n_1}/A_{n_1} \to a \quad \text{as } n_1 \to \infty.$$

Conditions 1 and 2 are not directly related to each other. Condition 2 controls the largest extremes while Condition 1 governs the behavior of the smallest terms in the sum (1.2). Condition 3 will say that there are two qualitatively different ways to normalize this sum according to the two cases when $a > 0$ or $a = 0$. The exact probabilistic meaning of the conditions along with equivalent forms expressed through F are discussed in Section 3, where illustrative examples are also constructed.

It will be shown in Lemma 2.5 in the next section that if Conditions 2 and 3 both hold, then $\varphi(y) \le a$, for all $y \in (0, \infty)$. Therefore the finite limit $\varphi(\infty) := \lim_{y\to\infty}\varphi(y) \le a$ exists and, as Lemma 2.5 will also show, for the nondecreasing, left-continuous, nonpositive function $\varphi(\cdot) - \varphi(\infty)$ defined on $(0, \infty)$, we have

$$\int_{\varepsilon}^{\infty} (\varphi(y) - \varphi(\infty))^2 \, dy < \infty \quad \text{for all } \varepsilon > 0. \tag{1.4}$$

Consider now a standard (intensity one) right-continuous Poisson process $N(t)$, $0 \le t < \infty$, and two independent standard normal random variables Z_1 and Z_2 such that (Z_1, Z_2) is also independent of $N(\cdot)$. Given a function ψ as in Condition 1, a function φ as in Condition 2 satisfying (1.4) and constants $0 \le b < \infty$ and $0 \le r \le (1-\alpha)^{1/2}$, where α is the limit in (1.1), consider the random variable

$$V(\varphi, \psi, b, r, \alpha) := \int_1^{\infty} (N(t) - t) \, d\varphi(t) + \int_0^1 N(t) \, d\varphi(t) - \varphi(1) + bZ_1 + \int_{-Z(r,\alpha)}^{0} \psi(x) \, dx,$$

with $Z(r, \alpha) = -rZ_1 + (1 - \alpha - r^2)^{1/2} Z_2$, where the first integral exists almost surely as an improper Riemann integral by (1.4). Finally, a natural centering sequence for the extreme sums in (1.2) will be seen to be

$$\mu_n := -n \int_{1/n}^{k_n/n} H(u) \, du - H\left(\frac{1}{n}\right).$$

Our principle results are contained in the following two theorems, where $\to_{\mathscr{D}}$ denotes convergence in distribution.

THEOREM 1. *If Conditions* 1, 2 *and* 3 *are satisfied for a subsequence* $\{n_1\}$ *of* $\{n\}$ *and a sequence* $\{A_{n_1}\}$ *of positive constants, then there exist a subsequence* $\{n_2\} \subset \{n_1\}$ *and a sequence of positive numbers* l_{n_2} *satisfying* $l_{n_2} \to \infty$ *and* $l_{n_2}/k_{n_2} \to 0$, *as* $n_2 \to \infty$, *such that either* $\sigma(l_{n_2}/n_2, k_{n_2}/n_2) > 0$ *for all* n_2, *in which case for some* $0 \le b \le a$ *and* $0 \le r \le (1-\alpha)^{1/2}$,

$$n_2^{1/2} \sigma(l_{n_2}/n_2, k_{n_2}/n_2)/A_{n_2} \to b,$$

$$r_{n_2} = \left(\frac{n_2}{k_{n_2}}\right)^{1/2} \frac{1 - k_{n_2}/n_2}{\sigma(l_{n_2}/n_2, k_{n_2}/n_2)} \int_{l_{n_2}/n_2}^{k_{n_2}/n_2} s \, dH(s) \to r \quad \text{as } n_2 \to \infty, \tag{1.5}$$

or $\sigma(l_{n_2}/n_2, k_{n_2}/n_2) = 0$ *for all* n_2, *in which case we put* $b = r = 0$. *In either case*

$$\frac{1}{A_{n_2}} \left\{ \sum_{i=1}^{k_{n_2}} X_{n_2+1-i, n_2} - \mu_{n_2} \right\} \to_{\mathscr{D}} V(\varphi, \psi, b, r, \alpha) \tag{1.6}$$

as $n_2 \to \infty$, *where* α *is zero or positive according to the two cases in* (1.1), φ *necessarily satisfies* (1.4) *and* ψ *necessarily satisfies*

$$\psi(x) \ge -ar/(1-\alpha), \qquad -\infty < x < \infty. \tag{1.7}$$

Moreover, if $a > 0$ *and* $\varphi \equiv 0$, *then* $b = a$ *in* (1.5), *while if* $a = 0$, *then* $\varphi(y) = 0$, *for all* $y > 1$.

THEOREM 2. *If there exist a subsequence* $\{n_1\} \subset \{n\}$ *and two sequences* $A_{n_1} > 0$ *and* C_{n_1} *along it such that*

$$\frac{1}{A_{n_1}}\left\{\sum_{i=1}^{k_{n_1}} X_{n_1+1-i,n_1} - C_{n_1}\right\} \to_{\mathscr{D}} V, \tag{1.8}$$

where V *is a nondegenerate random variable, then there exists a subsequence* $\{n_2\} \subset \{n_1\}$ *such that Conditions* 1, 2 *and* 3 *hold along the subsequence* $\{n_2\}$ *for* A_{n_2} *in* (1.8) *and for appropriate functions* ψ *and* φ *and some constant* $0 \le a < \infty$, *with* φ *satisfying* (1.4) *and* ψ *satisfying* (1.7). *The random variable* V *in* (1.8) *is of the form* $V(\varphi, \psi, b, r, \alpha) + c$ *for appropriate constants* $0 \le b \le a$, $0 \le r \le (1-\alpha)^{1/2}$ *and* $-\infty < c < \infty$. *Moreover, either* $\varphi \not\equiv 0$ *or* $\psi \not\equiv 0$ *or* $b > 0$.

Just as in the case of full sums in [5], the method of proof makes it possible to see the effect on the limiting distribution of deleting a finite number of the largest summands from the extreme sums $T_n(n - k_n, 0) = \sum_{i=1}^{k_n} X_{n+1-i,n}$ at each stage n, both in the sufficiency and the necessity directions. Let $k \ge 0$ be any fixed integer. Then, replacing $\sum_{i=1}^{k_n} X_{n+1-i,n}$ by

$$T_n(n - k_n, k) = \sum_{i=k+1}^{k_n} X_{n+1-i,n} = \sum_{i=n-k_n+1}^{n-k} X_{i,n},$$

μ_n by

$$\mu_n(k) := -n\int_{(k+1)/n}^{k_n/n} H(u)\,du - H\left(\frac{k+1}{n}\right)$$

and $V(\varphi, \psi, b, r, \alpha)$ by

$$\begin{aligned} V_k(\varphi, \psi, b, r, \alpha) := &\int_{S_{k+1}}^{\infty} (N(t) - t)\,d\varphi(t) - \int_1^{S_{k+1}} t\,d\varphi(t) + k\varphi(S_{k+1}) \\ &- \int_1^{k+1} \varphi(t)\,dt - \varphi(1) + bZ_1 + \int_{-Z(r,\alpha)}^0 \psi(x)\,dx, \end{aligned}$$

where S_{k+1} is the $(k+1)$st jump-point of the Poisson process $N(\cdot)$, both Theorem 1 and Theorem 2 remain true word for word. This can be seen by simple adjustments of the proofs presented in Section 2 for the case $k = 0$ (cf. [5] for details). That the case $k = 0$ formally agrees with the previous theorems, that is, that $V_0(\varphi, \psi, b, r, \alpha) = V(\varphi, \psi, b, r, \alpha)$, is easily seen by noting

$$\begin{aligned} &\int_{S_1}^{\infty} (N(t) - t)\,d\varphi(t) - \int_1^{S_1} t\,d\varphi(t) \\ &\quad = \int_1^{\infty} (N(t) - t)\,d\varphi(t) + \int_0^1 N(t)\,d\varphi(t). \end{aligned} \tag{1.9}$$

It is also straightforward to formulate Theorems 1 and 2 (or their generalized versions just described) for the sum of the lower extremes $\sum_{i=1}^{m_n} X_{i,n}$, where $m_n \to \infty$ and $m_n/n \to 0$ as $n \to \infty$, or $m_n = [\beta n]$ with $0 < \beta < 1$. The limiting random variable is of the form $-V(\varphi, \psi, b, r, \beta)$ with appropriate ingredients. In fact, if at least one of α and β is zero, then the two convergence statements hold jointly with the limiting random variables being independent.

Now we come back to the principal results and formulate some consequences of them.

COROLLARY 1. *Let $\{n_1\}$ be any subsequence of the positive integers. There exist sequences of constants $A^*_{n_1} > 0$ and C_{n_1} such that*

$$\frac{1}{A^*_{n_1}}\left\{\sum_{i=1}^{k_{n_1}} X_{n_1+1-i, n_1} - C_{n_1}\right\} \to_{\mathscr{D}} Z \tag{1.10}$$

holds as $n_1 \to \infty$ for a nondegenerate normal random variable Z if and only if Conditions 1 and 2 are satisfied with $A_{n_1} \equiv n_1^{1/2} a_{n_1}$, $\psi \equiv 0$ and $\varphi \equiv 0$, in which case (1.10) is true with the choice $A^*_{n_1} \equiv n_1^{1/2} a_{n_1}$ and $C_{n_1} \equiv \mu_{n_1}$ with Z being standard normal.

On heuristic grounds one expects that the existence of normalizing and centering constants $d_n > 0$ and c_n such that

$$d_n^{-1}(X_{n,n} - c_n) \to_{\mathscr{D}} Y \quad \text{with } Y \text{ nondegenerate} \tag{1.11}$$

implies that the suitably centered and normalized extreme sums $T_n(n - k_n, 0)$ or $T_n(n - k_n, k)$ also converge in distribution to a nondegenerate variable along the whole sequence $\{n\}$. This is the content of Corollary 2.

As was pointed out in [8], results from de Haan [9] imply that (1.11) holds if and only if

$$\lim_{s \downarrow 0} \frac{H(xs) - H(ys)}{H(vs) - H(ws)} = \frac{x^{-c} - y^{-c}}{v^{-c} - w^{-c}} \quad \text{for some } -\infty < c < \infty, \tag{1.12}$$

for all distinct $0 < x, y, v, w < \infty$, where for $c = 0$, the limit is understood as $(\log x - \log y)/(\log v - \log w)$. (The case $c = 0$, going back to Mejzler [17], is explicitly stated as Theorem 2.4.1 in [9].) If (1.12) holds, the constants d_n and c_n can be chosen so that when $c > 0$, then $P\{Y \le y\} = \exp(-y^{-1/c})$, $y > 0$; when $c = 0$, then $P\{Y \le y\} = \exp(-\exp(-y))$, $-\infty < y < \infty$; and when $c < 0$, then $P\{Y \le y\} = \exp(-(-y)^{-1/c})$, $y < 0$, and by Gnedenko's classic theorem these are the only possible limiting types. Whenever (1.12) holds, we write $F \in \Delta(c)$.

For $c > \frac{1}{2}$, set

$$D(c) := c\left(\frac{2c-1}{2c}\right)^{1/2}\left\{\int_1^\infty (N(t) - t)t^{-c-1}\,dt + \int_0^1 N(t)t^{-c-1}\,dt\right\}.$$

According to Corollary 3 in [5], the random variable $D(c)$ is stable with index

$1/c$. This fact can also be derived after some calculations from the representations given by Ferguson and Klass [11] and LePage, Woodroofe and Zinn [13]. For the case when $\alpha > 0$ in (1.1), consider

$$\psi_\alpha(x) := \begin{cases} 0, & \text{if } x \le 0, \\ \alpha^{1/2}(H(\alpha+) - H(\alpha))/\sigma(0,\alpha), & \text{if } x > 0, \end{cases} \tag{1.13}$$

and note that when (1.12) holds, then

$$0 < \sigma(0,\alpha) \begin{cases} < \infty, & \text{if } c < \frac{1}{2}, \\ = \infty, & \text{if } c > \frac{1}{2}. \end{cases} \tag{1.14}$$

If $c = \frac{1}{2}$, then $\sigma(0, \alpha)$ can be finite or infinite. Define $\sigma(0,0) = \sigma(0, 0+)$ and, finally, put

$$M(\alpha) := \begin{cases} 0, & \text{if } \alpha = 0, \\ \int_{-Z(\alpha)}^0 \psi_\alpha(x)\,dx, & \text{if } \alpha > 0, \end{cases}$$

where $Z(\alpha) = -r_\alpha Z_1 + (1 - \alpha - r_\alpha^2)^{1/2} Z_2$, with

$$r_\alpha := \frac{1-\alpha}{\alpha^{1/2}\sigma(0,\alpha)} \int_0^\alpha s\,dH(s) \le (1-\alpha)^{1/2}.$$

Note that for $\alpha > 0$,

$$M(\alpha) = \alpha^{1/2}(H(\alpha+) - H(\alpha))\min(0, Z(\alpha))/\sigma(0,\alpha).$$

Versions of the three subcases $c > \frac{1}{2}$; $-\infty < c \le \frac{1}{2}$, $c \ne 0$; and $c = 0$ of the case $\alpha = 0$ in Corollary 2 were proven in [6], [7] and [14], respectively. The full form of the corollary follows presently from Theorem 1.

COROLLARY 2. *Let $\{k_n\}$ satisfy (1.1). Whenever $F \in \Delta(c)$ for some $-\infty < c < \infty$,*

$$\left\{\sum_{i=1}^{k_n} X_{n+1-i,n} - \mu_n\right\}\Big/\left(n^{1/2}a_n\right) \to_{\mathscr{D}} \begin{cases} D(c), & \text{if } c > \frac{1}{2}, \\ Z_1, & \text{if } c = \frac{1}{2} \text{ and } \sigma(0,\alpha) = \infty, \\ Z_1 + M(\alpha), & \text{if } c = \frac{1}{2} \text{ and } \sigma(0,\alpha) < \infty \\ & \text{or } c < \frac{1}{2}. \end{cases}$$

If we replace the sum $T_n(n - k_n, 0)$ by $T_n(n - k_n, k)$ and μ_n by $\mu_n(k)$ in Corollary 2, where $k \ge 0$ is a fixed integer, then the limits remain exactly the same in the cases of $c \le \frac{1}{2}$, while if $c > \frac{1}{2}$, then $D(c)$ should be replaced by

$$D_k(c) = \left(\frac{2c-1}{2c}\right)^{1/2}\left\{c\int_{S_{k+1}}^\infty (N(t) - t)t^{-c-1}\,dt - c\int_1^{S_{k+1}} t^{-c}\,dt \right.$$
$$\left. + k(1 - S_{k+1}^{-c}) - \int_1^{k+1}(1 - t^{-c})\,dt\right\}.$$

Again, $D_0(c) = D(c)$.

Of course, if the quantile function Q is continuous at $1 - \alpha$, then $M(\alpha) = 0$ also in the case when $\alpha > 0$. It was Stigler [21] who discovered that such terms generally enter in the limiting distribution of the classical trimmed mean. So the third subcase of the case $\alpha > 0$ in Corollary 2 may be looked upon as a Stigler phenomenon for sums of extreme values. In this regard we also note that in the case when $\alpha > 0$ in (1.1), we could have worked with a sequence k_n more general than $k_n = [\alpha n]$. Namely, if k_n is a sequence of integers such that $1 \le k_n \le n$ and $n^{1/2}(k_n/n - \alpha) \to 0$ as $n \to \infty$ with some $0 < \alpha < 1$, then all the results so far stated remain valid without change *provided* that we modify the definition of ψ_n by replacing k_n/n by α in the arguments of H in its numerator. However, then the proofs would have to be separated for the cases $\alpha = 0$ and $\alpha > 0$. (For the slight modifications needed for the case $\alpha > 0$, we refer to [16].) In order to keep a unified proof for the two cases we have chosen to work with the special sequence $k_n = [\alpha n]$ here. A similar remark applies to our version of Stigler's theorem, formulated as Theorem 5 in [4].

Consider now linear combinations of k_n extreme values of the form

$$L_n = \sum_{i=1}^{k_n} c_{i,n} X_{n+1-i,n},$$

where $c_{i,n}$, $i = 1,\dots,n$, $n \ge 1$, is a triangular array of weights. Under the same regularity conditions as in [16], one can easily formulate and prove the analogues for L_n of the results in [16] on the asymptotic distribution of linear combinations of the middle order statistics. The proofs consist of a straightforward technical extension of those in the present paper combined with details from [16]. To keep, however, the main ideas easily accessible to a wider audience we decided to only consider sums of extremes in the present paper.

We take this opportunity to correct some minor oversights and misprints in paper [4]. First, Lemma 2.6 is not correct as stated. The specification that $f(0) = 0$ must be changed to $f(0) \le 0$ and $f(0+) \ge 0$. For this reason, whenever in the statements of results or in the proofs a function is specified to be zero at zero, this must be changed to the requirement that it be nonpositive on $(-\infty, 0]$ and nonnegative on $(0,\infty)$. In particular, in Theorem 1, the requirement that $\Psi_1(0) = \Psi_2(0) = 0$ should be weakened to read $\Psi_i(0) \le 0$, $\Psi_i(0+) \ge 0$, $i = 1, 2$, with analogous changes needed for Theorem 2. Also, all integrals of the form $\int_0^{-z}(z + x)\,dg(x)$ should be read as

$$\int_0^{-z} x\,dg(x) + zg(-z) = \int_{-z}^{0} g(x)\,dx,$$

whenever $(z, g) = (Z_i, \Psi_i), (Z_i, \Psi_i^*), (Z_i, \varphi_i)$, $i = 1, 2$. With these corrections the proofs proceed as before. Finally, the $X_{1,n}$ appearing in Theorems 2, 3 and 5 should be $X_{i,n}$, the subscript 1 in formulae (1.12), (1.13) and (1.15) should be i and the expressions for r_1 and r_2 (on page 678) should be divided by $\sigma(\alpha, \beta)$.

2. Proofs.

Introducing the empirical distribution function

$$G_n(u) = \frac{1}{n}\sum_{i=1}^{n} I(U_{i,n} \le u), \qquad 0 \le u \le 1, \tag{2.1}$$

where I is the indicator function, using (1.3), and integrating by parts yields

$$\begin{aligned}\sum_{i=1}^{k_n} X_{n+1-i,n} - \mu_n &=_{\mathscr{D}} -\sum_{i=1}^{k_n} H(U_{i,n}) - \mu_n \\ &= -\int_0^{U_{k_n,n}} nH(u)\,dG_n(u) + \int_{1/n}^{k_n/n} nH(u)\,du + H(1/n) \\ &= -nG_n(U_{k_n,n})H(U_{k_n,n}) + \int_0^{U_{k_n,n}} nG_n(u)\,dH(u) \\ &\quad + k_nH\left(\frac{k_n}{n}\right) - \int_{1/n}^{k_n/n} nu\,dH(u) \\ &= \int_0^{1/n} nG_n(u)\,dH(u) + \int_{1/n}^{k_n/n} n(G_n(u)-u)\,dH(u) \\ &\quad + \int_{k_n/n}^{U_{k_n,n}} n\left(G_n(u) - \frac{k_n}{n}\right)dH(u).\end{aligned}$$

Here the sum of the first two terms can be written as

$$\begin{aligned}&\int_{U_{1,n}}^{1/n}(nG_n(u)-1)\,dH(u) + H(1/n) - H(U_{1,n}) \\ &\qquad + \int_{1/n}^{U_{1,n}} n(G_n(u)-u)\,dH(u) + \int_{U_{1,n}}^{k_n/n} n(G_n(u)-u)\,dH(u) \\ &= \int_{U_{1,n}}^{1/n}(nu-1)\,dH(u) + H(1/n) - H(U_{1,n}) \\ &\qquad + \int_{U_{1,n}}^{k_n/n} n(G_n(u)-u)\,dH(u) \\ &= \int_{U_{1,n}}^{1/n}(nu-1)\,dH(u) + H(1/n) - H(U_{1,n}) \\ &\qquad + \int_{U_{1,n}}^{m_n/n} n(G_n(u)-u)\,dH(u) \\ &\qquad + \int_{m_n/n}^{l_n/n} n(G_n(u)-u)\,dH(u) + \int_{l_n/n}^{k_n/n} n(G_n(u)-u)\,dH(u),\end{aligned}$$

where, for the time being, m_n and l_n are any real numbers such that $1 \le m_n \le l_n \le k_n$. Hence

$$\frac{1}{A_n}\left\{\sum_{i=1}^{k_n} X_{n+1-i,n} - \mu_n\right\} =_{\mathscr{D}} \Delta_n^{(1)}(m_n) + \Delta_n^{(2)}(m_n, l_n) + \Delta_n^{(3)}(l_n, k_n), \tag{2.2}$$

where

$$\Delta_n^{(1)}(m_n) = \int_{nU_{1,n}}^{m_n}\left(nG_n\left(\frac{u}{n}\right) - u\right) d\frac{H(u/n) - H(1/n)}{A_n} + \int_{nU_{1,n}}^{1}(u-1)\, d\frac{H(u/n) - H(1/n)}{A_n} - \frac{H(nU_{1,n}/n) - H(1/n)}{A_n},$$

$$\Delta_n^{(2)}(m_n, l_n) = \int_{m_n}^{l_n}\left(nG_n\left(\frac{u}{n}\right) - u\right) d\frac{H(u/n) - H(1/n)}{A_n}$$

and

$$\Delta_n^{(3)}(l_n, k_n) = \int_{l_n/n}^{k_n/n} n(G_n(u) - u)\, d\frac{H(u)}{A_n} + \int_{k_n/n}^{U_{k_n,n}} n\left(G_n(u) - \frac{k_n}{n}\right) d\frac{H(u)}{A_n}.$$

This distributional equality is of course true without regard to the underlying probability space where the order statistics $U_{1,n}, \dots, U_{n,n}$ are defined. In the proof of Theorem 1 we shall be working on a specially constructed space $(\Omega, \mathscr{A}, P)$ described in [2], [3] and [5]. It carries two independent sequences $\{Y_n^{(j)}, n \geq 1\}$, $j = 1, 2$, of independent, exponentially distributed random variables with mean one and a sequence $\{B_n(t), 0 \leq t \leq 1; n \geq 1\}$ of Brownian bridges with the following property: For the G_n in (2.1) and the uniform quantile function $U_n(s) = U_{i,n}$ for $(i-1)/n < s \leq i/n$, $i = 1, \dots, n$, $U_n(0) = U_{1,n}$, determined by the order statistics $U_{k,n} = \tilde{S}_k(n)/\tilde{S}_{n+1}(n)$, $k = 1, \dots, n$, given by $\tilde{S}_k(n) = \tilde{Y}_1(n) + \cdots + \tilde{Y}_n(n)$, where $\tilde{Y}_j(n) = Y_j^{(1)}$ for $j = 1, \dots, [n/2]$, and $\tilde{Y}_j(n) = Y_{n+2-j}^{(2)}$, for $j = [n/2] + 1, \dots, n+1$, we have

$$\Delta_n(\nu) = \sup_{n^{-1} \leq s \leq 1 - n^{-1}} |n^{1/2}\{G_n(s) - s\} - B_n(s)|/(s(1-s))^{1/2-\nu} = O_P(n^{-\nu}) \tag{2.3}$$

and

$$\sup_{n^{-1} \leq s \leq 1 - n^{-1}} |n^{1/2}\{s - U_n(s)\} - B_n(s)|/(s(1-s))^{1/2-\nu} = O_P(n^{-\nu}), \tag{2.4}$$

for any fixed $0 \leq \nu < \frac{1}{4}$. Note that it is justified to call the above $U_{k,n}$ "order statistics" since it is well known that $(\tilde{S}_1(n)/\tilde{S}_{n+1}(n), \dots, \tilde{S}_n(n)/\tilde{S}_{n+1}(n))$ equals in distribution to the vector of order statistics of n independent random variables uniformly distributed on (0, 1). We will also need the Poisson process $N(\cdot)$ with jump-points $S_n = S_n^{(1)} = Y_1^{(1)} + \cdots + Y_n^{(1)}$, $n \geq 1$, defined to be right-continuous by setting

$$N(t) = \sum_{k=1}^{\infty} I(S_k \le t), \qquad 0 \le t < \infty. \tag{2.5}$$

The behavior of the term $\Delta_n^{(3)}(l_n, k_n)$ is described in Lemma 2.4, which requires three preparatory lemmas, the first of which is crucial at many other places, too.

Lemma 2.1. *If $0 < s < t \le 1 - \varepsilon$, where $0 < \varepsilon < 1$, then*

$$s(H(t) - H(s))^2/\sigma^2(s,t) \le 1/\varepsilon,$$

where $0/0 := 1$.

Proof. If $s \le u \le v < t$ or $s \le v \le u < t$, then $u \wedge v - uv \ge \varepsilon s$. Hence

$$\sigma^2(s,t) = \int_s^t \int_s^t (u \wedge v - uv)\, dH(u)\, dH(v) \ge \varepsilon s (H(t) - H(s))^2,$$

from which the result follows upon noting also that $\sigma(s,t) = 0$ if and only if $H(s) = H(t)$, which justifies the definition $0/0 := 1$. □

Lemma 2.2. *For all $\beta \ne \frac{1}{2}$ and $0 < s < t \le 1 - \varepsilon$, where $0 < \varepsilon < 1$,*

$$\frac{\int_s^t u^\beta\, dH(u)}{\sigma(s,t)} \le \frac{1}{\varepsilon^{1/2}} \left\{ \frac{\beta t^{\beta - 1/2}}{\beta - \frac{1}{2}} + \frac{s^{\beta - 1/2}}{1 - 2\beta} \right\}.$$

Proof. Integrating by parts we see that

$$\int_s^t u^\beta\, dH(u) = s^\beta \{H(t) - H(s)\} + \beta \int_s^t \{H(t) - H(u)\} u^{\beta - 1}\, du.$$

Thus the lemma follows from Lemma 2.1. □

In the following two lemmas k_n is as in (1.1) and l_n is any sequence of positive numbers such that

$$0 \le l_n \le n, \qquad l_n \to \infty \quad \text{and} \quad k_n/l_n \to \infty \quad \text{as} \quad n \to \infty. \tag{2.6}$$

Lemma 2.3. *Suppose there exists a subsequence $\{n_2\} \subset \{n\}$ such that $\sigma^2(l_{n_2}/n_2, k_{n_2}/n_2) > 0$ for all $n_2 \in \{n_2\}$ and consider the two-dimensional random vectors*

$$\left(Z_{n_2}^{(1)}, Z_{n_2}^{(2)}\right) := \left(\frac{\int_{l_{n_2}/n_2}^{k_{n_2}/n_2} B_{n_2}(s)\, dH(s)}{\sigma(l_{n_2}/n_2, k_{n_2}/n_2)}, -\left(\frac{n_2}{k_{n_2}}\right)^{1/2} B_{n_2}\!\left(\frac{k_{n_2}}{n_2}\right) \right).$$

Then there exist a subsequence $\{n_3\} \subset \{n_2\}$ and a number $0 \le r \le (1 - \alpha)^{1/2}$

such that, as $n_3 \to \infty$,

$$\begin{aligned}\left(Z_{n_3}^{(1)}, Z_{n_3}^{(2)}\right) &\to_{\mathscr{D}} \left(Z_1, -rZ_1 + (1-\alpha-r^2)^{1/2} Z_2\right)\\ &=_{\mathscr{D}} N\left((0,0), \begin{pmatrix} 1 & -r \\ -r & 1-\alpha \end{pmatrix}\right).\end{aligned}$$

PROOF. First notice that $(Z_{n_2}^{(1)}, Z_{n_2}^{(2)})$ is, for each n_2, a bivariate normal random variable with mean vector zero and covariance matrix

$$\begin{pmatrix} 1 & -r_{n_2} \\ -r_{n_2} & 1 - k_{n_2}/n_2 \end{pmatrix},$$

where

$$0 \le r_{n_2} = \left(\frac{n_2}{k_{n_2}}\right)^{1/2} \frac{(1 - k_{n_2}/n_2)}{\sigma(l_{n_2}/n_2, k_{n_2}/n_2)} \int_{l_{n_2}/n_2}^{k_{n_2}/n_2} s\, dH(s) \le \left(1 - \frac{k_{n_2}}{n_2}\right)^{1/2}.$$

Thus by (1.1), there exist a subsequence $\{n_3\} \subset \{n_2\}$ and a number $0 \le r \le (1-\alpha)^{1/2}$ such that $r_{n_3} \to r$ as $n_3 \to \infty$, which implies the lemma. □

LEMMA 2.4. *Suppose that Conditions* 1 *and* 3 *hold along some* $\{n_1\}$ *and let* k_n *and* l_n *be as in* (1.1) *and* (2.6). *Then there exist a subsequence* $\{n_3\} \subset \{n_1\}$ *and numbers* $0 \le b \le a$ *and* $0 \le r \le (1-\alpha)^{1/2}$ *such that*

$$\Delta_{n_3}^{(3)}(l_{n_3}, k_{n_3}) \to_{\mathscr{D}} V_3(\psi, b, r, \alpha) := bZ_1 + \int_{-Z(r,\alpha)}^{0} \psi(x)\, dx,$$

where $Z(r,\alpha) = -rZ_1 + (1-\alpha-r^2)^{1/2} Z_2$.

PROOF. There are two cases.

CASE 1. There exists a subsequence $\{n_2\} \subset \{n_1\}$ such that

$$\sigma(l_{n_2}/n_2, k_{n_2}/n_2) > 0 \quad \text{for all } n_2 \in \{n_2\}.$$

Now let $\{n_3\} \subset \{n_2\}$ be the subsequence given by the proof of Lemma 2.3.

To see the behavior of the first term in $\Delta_{n_3}^{(3)}(l_{n_3}, k_{n_3})$ in (2.2), fix any $0 < \nu < \frac{1}{4}$. Then for the random variable $\tilde{Z}_{n_3}$ defined by the equation

$$\int_{l_{n_3}/n_3}^{k_{n_3}/n_3} n_3\left(G_{n_3}(u) - u\right) d\frac{H(u)}{A_{n_3}} = \frac{n_3^{1/2}\sigma(l_{n_3}/n_3, k_{n_3}/n_3)}{A_{n_3}} \tilde{Z}_{n_3},$$

we have by Lemma 2.2 and (2.3),

$$\begin{aligned}|\tilde{Z}_{n_3} - Z_{n_3}^{(1)}| &\le \Delta_{n_3}(\nu) \int_{l_{n_3}/n_3}^{k_{n_3}/n_3} s^{1/2-\nu}\, dH(s) / \sigma(l_{n_3}/n_3, k_{n_3}/n_3)\\ &= O_P(n_3^{-\nu}) O\left((l_{n_3}/n_3)^{-\nu}\right)\\ &= O_P\left(l_{n_3}^{-\nu}\right) = o_P(1),\end{aligned}$$

by (2.6) as $n_3 \to \infty$. By Condition 3 we can assume without loss of generality that $\{n_3\}$ is chosen in such a way that for some $0 \le b \le a$, we also have

$$n_3^{1/2}\sigma(l_{n_3}/n_3, k_{n_3}/n_3)/A_{n_3} \to b \quad \text{as } n_3 \to \infty$$

and hence we have

$$\Delta_{n_3}^{(3)}(l_{n_3}, k_{n_3}) = bZ_{n_3}^{(1)} + \int_{k_{n_3}/n_3}^{U_{k_{n_3},n_3}} n_3\left(G_{n_3}(u) - \frac{k_{n_3}}{n_3}\right) d\frac{H(u)}{A_{n_3}} + o_P(1). \tag{2.7}$$

Note that this step corresponds to Lemma 2.2 in [4].

Let $0 < M < \infty$ be fixed. Adapting the proof of (2.6) in Lemma 2.3 in [4] to the present situation (i.e., replacing the function $\psi_{1,n}$ there by the function ψ_n^* of Condition 1 but otherwise proceeding line by line in exactly the same way), by (2.3) and (2.4) we obtain

$$\begin{aligned} &\int_{k_{n_3}/n_3}^{U_{k_{n_3},n_3}} n_3\left(G_{n_3}(u) - \frac{k_{n_3}}{n_3}\right) d\frac{H(u)}{A_{n_3}} I\left(|Z_{n_3}^{(2)}| < M\right) \\ &\quad = \int_0^{-Z_{n_3}^{(2)}} \left(Z_{n_3}^{(2)} + x\right) d\psi_{n_3}^*(x) I\left(|Z_{n_3}^{(2)}| < M\right) + o_P(1), \end{aligned} \tag{2.8}$$

as $n_3 \to \infty$.

Now using Lemma 2.3, we obtain exactly as in the proof of Lemma 2.4 in [4] that

$$\begin{aligned} &bZ_{n_3}^{(1)} + \int_0^{-Z_{n_3}^{(2)}} \left(Z_{n_3}^{(2)} + x\right) d\psi_{n_3}^*(x) I\left(|Z_{n_3}^{(2)}| < M\right) \\ &\quad \to_{\mathscr{D}} bZ_1 + \int_{-Z(r,\alpha)}^{0} \psi(x)\, dx\, I(|Z(r,\alpha)| < M). \end{aligned} \tag{2.9}$$

Now (2.7), (2.8), (2.9), the simple little argument finishing the proof of the first part of Theorem 1 in [4] and Theorem 4.2 in Billingsley [1] give the lemma in the first case.

CASE 2. For all $n_1 \in \{n_1\}$ sufficiently large, $\sigma(l_{n_1}/n_1, k_{n_1}/n_1) = 0$. Then the first term of $\Delta_{n_1}^{(3)}(l_{n_1}, k_{n_1})$ is almost surely zero for all these n_1 and $a = 0$ in Condition 3 in this case. Thus, with $b = r = 0$, a simplified version of the argument in Case 1 gives

$$\Delta_{n_1}^{(3)}(l_{n_1}, k_{n_1}) \to_{\mathscr{D}} \int_{-Z(0,\alpha)}^{0} \psi(x)\, dx. \qquad \square$$

Now we turn to the behavior of the terms $\Delta_n^{(1)}$ and $\Delta_n^{(2)}$ in (2.2). First we need the following.

LEMMA 2.5. *Suppose that Conditions 2 and 3 hold along some subsequence $\{n_1\}$. Then $\varphi(y) \le a$, for all $0 < y < \infty$, and (1.4) holds true.*

PROOF. Since $a \geq 0$ and $\varphi(y) \leq 0$ for $0 < y \leq 1$, it is sufficient to consider $y > 1$. Then for all sufficiently large n_1, whenever $\sigma(1/n_1, y/n_1) > 0$,

$$\varphi_{n_1}^*(y) = \frac{n_1^{1/2} b_{n_1}}{A_{n_1}} \frac{\sigma(1/n_1, y/n_1)}{\sigma(1/n_1, k_{n_1}/n_1)} \frac{H(y/n_1) - H(1/n_1)}{n_1^{1/2}\sigma(1/n_1, y/n_1)} \leq (1+\varepsilon) n_1^{1/2} \frac{b_{n_1}}{A_{n_1}}$$

by an application of Lemma 2.1, where $\varepsilon > 0$ is any preassigned number. [Of course, when $\sigma(1/n_1, y/n_1) = 0$, then $\varphi_{n_1}^*(y) = 0$.] The first statement is therefore clear.

Since φ is a nondecreasing, the limit $\varphi(\infty) := \lim_{y \to \infty} \varphi(y) \leq a$ exists. Thus $\tilde{\varphi}(y) := \varphi(y) - \varphi(\infty)$ is a nonpositive, left-continuous, nondecreasing function on $(0, \infty)$, and the fact that Conditions 2 and 3 imply

$$\int_s^\infty \int_s^\infty (u \wedge v)\, d\tilde{\varphi}(u)\, d\tilde{\varphi}(v) < \infty, \quad \text{for all } 1 < s < \infty,$$

can be shown exactly as in the proof of Lemma 2.5 in [5]. Since $\tilde{\varphi}(y) \to 0$ as $y \to \infty$, this is sufficient to conclude that (1.4) indeed follows. □

LEMMA 2.6. *Suppose that Conditions* 2 *and* 3 *hold along some subsequence* $\{n_1\}$ *and* k_n *is as in* (1.1). *Then there exist sequences of positive constants* m_{n_1} *and* l_{n_1} *such that* $1 \leq m_{n_1} \leq l_{n_1} \leq k_{n_1}$ *and* $m_{n_1} \to \infty$, $l_{n_1}/m_{n_1} \to \infty$, $k_{n_1}/l_{n_1} \to \infty$,

$$\Delta_{n_1}^{(1)}(m_{n_1}) \to_{\mathscr{D}} V_1(\varphi) := \int_{S_1}^\infty (N(t) - t)\, d\varphi(t) - \int_1^{S_1} t\, d\varphi(t) - \varphi(1),$$

with φ *satisfying* (1.4) *and* $\Delta_{n_1}^{(2)}(m_{n_1}, l_{n_1}) \to_P 0$ *as* $n_1 \to \infty$. *Moreover, if* $\varphi \equiv 0$, *then* $\{l_{n_1}\}$ *can be chosen so that*

$$n_1^{1/2}\sigma(1/n_1, l_{n_1}/n_1)/A_{n_1} \to 0. \tag{2.10}$$

PROOF. Let $1 < m < l$ be arbitrarily fixed continuity points of φ. Then, noting that in [5] the proofs of Lemmas 2.1, 2.2 and 2.3 require only the assumption of the weak convergence of functions corresponding to the present sequence $\{\varphi_{n_1}^*\}$, Condition 2 and these lemmas together with (2.10) in the proof of Lemma 2.2, all in [5], directly imply that

$$\begin{aligned}\Delta_{n_1}^{(1)}(m) &\to_{\mathscr{D}} \int_{S_1}^m (N(t) - t)\, d\varphi(t) + \int_{S_1}^1 (t-1)\, d\varphi(t) - \varphi(S_1)\\ &= \int_{S_1}^m (N(t) - t)\, d\varphi(t) - \int_1^{S_1} t\, d\varphi(t) - \varphi(1)\end{aligned}$$

and

$$\Delta_{n_1}^{(2)}(m, l) \to_{\mathscr{D}} V_{m,l} := \int_m^l (N(t) - t)\, d\varphi(t).$$

Since we have (1.4) by Lemma 2.5,

$$\limsup_{l,\, m\to\infty,\, l>m} EV_{m,l}^2 \le \lim_{m\to\infty} \int_m^\infty \int_m^\infty (u \wedge v - uv)\, d\varphi(u)\, d\varphi(v) = 0.$$

Thus $V_{m,l} \to_P 0$ as $m, l \to \infty$, $l \ge m$.

Using these findings, the construction of the sequences $\{m_{n_1}\}$ and $\{l_{n_1}\}$ is accomplished by a routine diagonal selection procedure.

Finally, choose any number $d \ge 1$ and let c_1 and c_2 be continuity points of φ such that $c_1 \le 1$, $d \le c_2$. Then, using Condition 2 and a weak convergence argument, we obtain

$$\begin{aligned}
&\limsup_{n_1\to\infty} n_1 \sigma^2(1/n_1, d/n_1)/A_{n_1}^2 \\
&\quad \le \lim_{n_1\to\infty} n_1 \int_{c_1/n_1}^{c_2/n_1} \int_{c_1/n_1}^{c_2/n_1} (u \wedge v - uv)\, dH(u)\, dH(v)/A_{n_1}^2 \\
&\quad = \int_{c_1}^{c_2} \int_{c_1}^{c_2} (s \wedge t)\, d\varphi(s)\, d\varphi(t),
\end{aligned}$$

so that if $\varphi \equiv 0$, then for each number $d \ge 1$,

$$n_1^{1/2} \sigma(1/n_1, d/n_1)/A_{n_1} \to 0 \quad \text{as } n_1 \to \infty.$$

From this, the last statement of the lemma also follows. □

Proof of Theorem 1. Choose $\{m_{n_1}\}$ and $\{l_{n_1}\}$ according to Lemma 2.6. Since $\sigma(l_{n_1}/n_1, k_{n_1}/n_1) \le b_{n_1}$, by Condition 3 and Lemmas 2.4 and 2.6 there exists a subsequence $\{n_2\} \subset \{n_1\}$ such that we have (1.5),

$$\Delta_{n_2}^{(1)}(m_{n_2}) \to_{\mathscr{D}} V_1(\varphi), \qquad \Delta_{n_2}^{(2)}(m_{n_2}, l_{n_2}) \to_P 0$$

and

$$\Delta_{n_2}^{(3)}(l_{n_2}, k_{n_2}) \to_{\mathscr{D}} V_3(\psi, b, r, \alpha)$$

as $n_2 \to \infty$, where φ satisfies (1.4). Moreover, $\{n_2\}$ can clearly be chosen so that in addition either $\sigma(l_{n_2}/n_2, k_{n_2}/n_2) > 0$ for all n_2, or $\sigma(l_{n_2}/n_2, k_{n_2}/n_2) = 0$ for all n_2.

Since $l_{n_1}/m_{n_1} \to \infty$, an elementary argument similar to the one used in the proof of Theorem 2 of Mason [15] based on Satz 4 of Rossberg [19] shows that $\Delta_{n_2}^{(1)}(m_{n_2})$ and $\Delta_{n_2}^{(3)}(l_{n_2}, k_{n_2})$ are asymptotically independent. Therefore, on account of (2.2) and the above convergence relations, (1.6) follows since by (1.9),

$$V(\varphi, \psi, b, r, \alpha) = V_1(\varphi) + V_3(\psi, b, r, \alpha). \tag{2.11}$$

If $\varphi \equiv 0$ and $0 < a < \infty$ in Condition 3, then by Lemma 2.6, the sequence $\{l_{n_1}\}$ can be chosen so that (2.10) holds. It is easy to see that this implies (1.5) with $b = a$. Also, since $\varphi(1+) \geq 0$, if $a = 0$, then $\varphi(y) = 0$ for all $y > 1$, by Lemma 2.5.

It remains to establish the lower bound in (1.7). Since $\psi(0+) \geq 0$, it is enough to deal with $x < 0$. If $n_2 = n$ is large enough,

$$\begin{aligned}
\psi_n^*(x) &= -\frac{n^{1/2}a_n}{A_n}\left(\frac{k_n}{n}\right)^{1/2}\int_{k_n/n+xk_n^{1/2}/n}^{k_n/n}\frac{dH(u)}{a_n}\\
&\geq -\frac{n^{1/2}a_n}{A_n}\left(\frac{k_n}{n}\right)^{1/2}\left(\frac{k_n}{n}\left(1+\frac{x}{k_n^{1/2}}\right)\right)^{-1}\int_{k_n/n+xk_n^{1/2}/n}^{k_n/n}\frac{u\,dH(u)}{a_n}\\
&\geq -\frac{n^{1/2}a_n}{A_n}\frac{1}{1+x/k_n^{1/2}}\left(\frac{n}{k_n}\right)^{1/2}\int_{l_n/n}^{k_n/n}\frac{u\,dH(u)}{\sigma(l_n/n, k_n/n)}\\
&= -\frac{n^{1/2}a_n}{A_n}\frac{1}{1+x/k_n^{1/2}}\frac{1}{1-k_n/n}r_n,
\end{aligned}$$

where r_n is as in the proof of Lemma 2.3 and the formulation of the theorem. Hence (1.7) follows by (1.1), Condition 3 and the proof of Lemma 2.3. □

Proof of Theorem 2. Before starting the proof we note that by Theorem 3(i) in [5] the random variable $V_1(\varphi)$ is degenerate if and only if $\varphi \equiv 0$. Also, taking into account the bound (1.7), an application of the second part of Proposition 2 in [4] shows that $V_3(\psi, b, r, \alpha)$ is degenerate if and only if $b = 0$ and $\psi \equiv 0$. Since V_1 and V_3 are independent, by (2.11) we see that the limiting random variable $V(\varphi, \psi, b, r, \alpha)$ is degenerate if and only if $\varphi \equiv 0$, $\psi \equiv 0$, and $b \equiv 0$. Hence if we prove the first two statements of the theorem, the third will be automatically true.

We distinguish three cases.

Case 1. For all n_1 large enough $b_{n_1} > 0$,

$$\limsup_{n_1\to\infty}|\psi_{n_1}(x)| < \infty \quad \text{for all } -\infty < x < \infty,$$

$$\limsup_{n_1\to\infty}|\varphi_{n_1}(y)| < \infty \quad \text{for all } y > 0.$$

Then by the Helly–Bray theorem we can select a subsequence $\{n_2\} \subset \{n_1\}$ such that $\psi_{n_2} \to \overline{\psi}$ weakly and $\varphi_{n_2} \to \overline{\varphi}$ weakly as $n_2 \to \infty$, where $\overline{\psi}$ and $\overline{\varphi}$ have all the usual inherent properties. By Theorem 1, along a further subsequence $\{n_3\} \subset \{n_2\}$,

$$T_{n_3} := \frac{1}{n_3^{1/2}a_{n_3}}\left\{\sum_{i=1}^{k_{n_3}} X_{n_3+1-i,n_3} - \mu_{n_3}\right\} \tag{2.12}$$

converges in distribution to $\overline{V} := V(\overline{\varphi}, \overline{\psi}, b, r, \alpha)$, $0 \le b \le 1$, $0 \le r \le (1-\alpha)^{1/2}$. If $\overline{\varphi} \equiv 0$, then $b = 1$, so that $\overline{V}$ is a nondegenerate. Hence by the convergence of types theorem we have Condition 3 with some $a > 0$, and thus Conditions 1 and 2 with $\psi = a\overline{\psi}$ and $\varphi = a\overline{\varphi}$, and V is of the form stated with c being the limit of $(\mu_{n_3} - C_{n_3})/A_{n_3}$ as $n_3 \to \infty$.

CASE 2. There exists a subsequence $\{n_2\} \subset \{n_1\}$ such that $b_{n_2} > 0$ for all n_2 and

$$\lim_{n_2 \to \infty} |\psi_{n_2}(x)| = \infty \quad \text{for some } -\infty < x < \infty \tag{2.13}$$

or

$$\lim_{n_2 \to \infty} |\varphi_{n_2}(y)| = \infty \quad \text{for some } 0 < y < \infty. \tag{2.14}$$

First we note that by the argument at the end of the proof of Theorem 1,

$$\limsup_{n\to\infty} |\psi_n(x)| \le (1-\alpha)^{-1/2} \quad \text{for all } x \le 0, \tag{2.15}$$

and by the first part of the proof of Lemma 2.5,

$$\limsup_{n\to\infty} \varphi_n(y) \le 1 \quad \text{for all } 1 \le y < \infty. \tag{2.16}$$

At the beginning of the present section we saw that for T_n in (2.12),

$$\begin{aligned} T_n = R_n^{(1)} + W_n + R_n^{(2)} := \frac{n^{1/2}}{a_n} \int_0^{1/n} G_n(u)\, dH(u) \\ + \frac{n^{1/2}}{a_n} \int_{1/n}^{k_n/n} (G_n(u) - u)\, dH(u) \\ + \frac{n^{1/2}}{a_n} \int_{k_n/n}^{U_{k_n,n}} \left(G_n(u) - \frac{k_n}{n} \right) dH(u). \end{aligned}$$

We have

$$\lim_{M\to\infty} \liminf_{n\to\infty} P\{|R_n^{(i)}| < M\} > 0 \quad \text{for } i = 1, 2. \tag{2.17}$$

The case $i = 1$ is trivial because $R_n^{(1)} = 0$, if $U_{1,n} > n^{-1}$, and hence

$$P\{|R_n^{(1)}| < M\} \ge P\{nU_{1,n} > 1\} \to e^{-1} \quad \text{as } n \to \infty,$$

for all $M > 0$. The case $i = 2$ follows exactly as in the proof of Lemma 2.8 in [4], using (2.15) and noting only that in the present generality of having $\alpha \ge 0$ in (1.1), one has to replace c in the limit in relation (2.24) of [4] by $(1-\alpha)^{1/2}c$.

The next step is to derive from Satz 4 of Rossberg [19] that

$$|R_n^{(1)}| \quad \text{and} \quad |R_n^{(2)}| \quad \text{are asymptotically independent.} \tag{2.18}$$

Since $b_{n_2} > 0$, we have $\mathrm{Var}(W_{n_2}) = 1$, for each n_2, so that W_{n_2} is obviously stochastically bounded. Combining this fact, (2.17), (2.18) and the general Lemma 2.10 in [4], and proceeding exactly as in the proof of Lemma 2.11 in [4], we see that if the left-hand side of (1.8) is stochastically bounded then

$$n_2^{1/2} a_{n_2} R_{n_2}^{(i)} / \left(n_2^{1/2} a_{n_2} \vee A_{n_2} \right) = O_P(1), \qquad i = 1, 2, \tag{2.19}$$

where $x \vee y = \max(x, y)$. Thus assumption (1.8) implies (2.19).

Now we claim the following three properties:

$$\lim_{n_2 \to \infty} n_2^{1/2} a_{n_2} / A_{n_2} = 0, \tag{2.20}$$

$$\limsup_{n_2 \to \infty} |\psi_{n_2}^*(x)| < \infty \quad \text{for all } -\infty < x < \infty, \tag{2.21}$$

$$\limsup_{n_2 \to \infty} |\varphi_{n_2}^*(y)| < \infty \quad \text{for all } 0 < y < \infty. \tag{2.22}$$

Note first that by (2.15) and (2.16), the relations (2.13) and (2.14) can only occur for some $x > 0$ and $0 < y < 1$, respectively. If (2.13) holds, we first work on the set

$$\Omega_{n_2}(x, t) := \left\{ \frac{k_{n_2}}{n_2} + t \frac{k_{n_2}^{1/2}}{n_2} \le U_{k_{n_2} - [x k_{n_2}^{1/2}], n_2} \right\},$$

where $t \ge x$. On this, similarly as in the proof of Lemma 2.12 in [4], one has

$$\frac{n_2^{1/2} a_{n_2} R_{n_2}^{(2)}}{n_2^{1/2} a_{n_2} \vee A_{n_2}} \le - \frac{n_2^{1/2} a_{n_2}}{n_2^{1/2} a_{n_2} \vee A_{n_2}} \frac{\left[x k_{n_2}^{1/2} \right]}{k_{n_2}^{1/2}} \psi_{n_2}(t). \tag{2.23}$$

Since $\psi_{n_2}(t) \to \infty$, as $n_2 \to \infty$, by (2.13) and since, with $N(0,1)$ standing for a standard normal variable,

$$\lim_{n_2 \to \infty} P\left\{ \Omega_{n_2}(x, t) \right\} = P\left\{ N(0,1) \ge (1 - \alpha)^{1/2} (t + x) \right\} > 0,$$

we see that (2.19) and (2.23) imply (2.20) and (2.21). To get (2.22), note that on the set $\Omega_{n_2}(y) = \{ n_2 U_{1, n_2} < y \}$, we have

$$\frac{n_2^{1/2} a_{n_2} R_{n_2}^{(1)}}{n_2^{1/2} a_{n_2} \vee A_{n_2}} \ge - \frac{n_2^{1/2} a_{n_2}}{n_2^{1/2} a_{n_2} \vee A_{n_2}} \varphi_{n_2}(y). \tag{2.24}$$

Since the right-hand side here is $-\varphi_{n_2}^*(y)$, for all n_2 large enough and since $P\{\Omega_{n_2}(y)\} \to 1 - e^{-y} > 0$, as $n_2 \to \infty$, we see that (2.19) and (2.24) imply (2.22).

If (2.14) holds, then we work first on $\Omega_{n_2}(y)$ to get (2.20) and (2.22), and afterwards on $\Omega_{n_2}(x, t)$ to get (2.21).

Clearly, (2.20), (2.21) and (2.22) imply Conditions 1, 2 and 3 with $a = 0$ along a further subsequence of $\{n_2\}$.

CASE 3. There exists a subsequence $\{n_2\} \subset \{n_1\}$ such that $b_{n_2} = 0$ for all n_2. In this case, the left side of (1.8) is equal in distribution to

$$\frac{n_2}{A_{n_2}}\int_0^{1/n_2} G_{n_2}(u)\,dH(u) + \frac{n_2}{A_{n_2}}\int_{k_{n_2}/n_2}^{U_{k_{n_2},n_2}}\left(G_{n_2}(u) - \frac{k_{n_2}}{n_2}\right)dH(u) + \frac{\mu_{n_2} - C_{n_2}}{A_{n_2}},$$

for all such n_2. If we now denote the first two terms here by $R_{n_2}^{(1)}$ and $R_{n_2}^{(2)}$, respectively, then of course we still have (2.17) for $i = 1$. Since H has no mass on $(1/n_2, k_{n_2}/n_2]$, we have $R_{n_2}^{(2)} = 0$, if $1/n_2 < U_{k_{n_2},n_2} < k_{n_2}/n_2$, and we see that

$$P\left\{|R_{n_2}^{(2)}| < M\right\} \geq P\left\{1/n_2 < U_{k_{n_2},n_2} < k_{n_2}/n_2\right\} \to \tfrac{1}{2}$$

as $n_2 \to \infty$, so that (2.17) is also true for $j = 2$ along $\{n_2\}$. Since (2.18) is still obviously true along $\{n_2\}$ in the present case, we obtain as in Case 2 that $R_{n_2}^{(j)} = O_P(1)$ as $n_2 \to \infty$, $j = 1, 2$. A simplified form of the corresponding argument above now yields (2.21) and (2.22). Thus, again, we obtain Conditions 1, 2 and 3 with $a = 0$ along a subsequence of $\{n_2\}$. The theorem is completely proved. □

In the proof of Corollary 1 we shall require the following.

LEMMA 2.7. *Let the functions* φ *and* ψ *be as in Conditions* 1 *and* 2 *and consider the constants* $0 \leq b < \infty$, $0 \leq \alpha < 1$ *and* $0 \leq r \leq (1 - \alpha)^{1/2}$. *Assume that* φ *satisfies* (1.4) *and* ψ *satisfies* (1.7). *Then the random variable* $V(\varphi, \psi, b, r, \alpha)$ *is a nondegenerate normal if and only if* $\varphi \equiv 0$, $\psi \equiv 0$ *and* $b > 0$, *in which case* $V(\varphi, \psi, b, r, \alpha)$ *is* $N(0, b^2)$.

PROOF. The sufficiency part is trivial. Suppose that $V(\varphi, \psi, b, r, \alpha) = V_1(\varphi) + V_3(\psi, b, r, \alpha)$, where we use (2.11), is nondegenerate normal. Since $V_1(\varphi)$ and $V_3(\psi, b, r, \alpha)$ are independent, the Cramér characterization forces both to be normal. Theorem 3(i) in [5] says that $V_1(\varphi)$ has an infinitely divisible distribution without a normal component. Hence the only way it can be normal is when it is degenerate, which again by Theorem 3 in [5] implies $\varphi \equiv 0$. Since then in fact $V_1(\varphi) = 0$, $V_3(\psi, b, r, \alpha)$ must be nondegenerate normal. Using now condition (1.7), Proposition 1 from [4] implies that this can happen only if $\psi \equiv 0$ and $b > 0$. □

PROOF OF COROLLARY 1. First we prove the "if" part. Let $\{n_2\}$ be an arbitrary subsequence of $\{n_1\}$. Since Conditions 1, 2 and 3 hold along $\{n_2\}$ with $A_{n_2} \equiv n_2^{1/2} a_{n_2}$, $\varphi \equiv 0$, $\psi \equiv 0$ and $a = 1$, Theorem 1 and Lemma 2.7 imply the existence of a subsequence $\{n_3\}$ such that

$$\left\{\sum_{i=1}^{k_{n_3}} X_{n_3+1-i,n_3} - \mu_{n_3}\right\}/n_3^{1/2}a_{n_3} \to_{\mathscr{D}} N(0,1).$$

This of course implies that the same is true along the original $\{n_1\}$.

Now suppose that (1.10) holds. Let again $\{n_2\} \subset \{n_1\}$ be arbitrary. By Theorem 2 there exists a further subsequence $\{n_3\} \subset \{n_2\}$ such that Conditions 1, 2 and 3 hold along $\{n_3\}$ with $A_{n_3} \equiv A^*_{n_3}$ and appropriate functions φ and ψ satisfying conditions (1.4) and (1.7), respectively, and a constant $0 \le a < \infty$, and the distribution of Z is necessarily that of $V(\varphi, \psi, b, r, \alpha) + c$ with some constant $0 \le b \le a$, $0 \le r \le (1-\alpha)^{1/2}$ and $-\infty < c < \infty$. Thus by Lemma 2.7, $\varphi \equiv 0$, $\psi \equiv 0$ and $b > 0$. Hence $a > 0$, yielding that Conditions 1 and 2 hold along $\{n_3\}$ with $A_{n_3} \equiv n_3^{1/2} a_{n_3}$, $\varphi \equiv 0$ and $\psi \equiv 0$. Since $\{n_2\}$ was arbitrary, the same must be true along the original sequence $\{n_1\}$. □

The proof of Corollary 2 requires some preparations concerning the asymptotic behavior of the functions ψ_n and φ_n. First of all we note that, inverting the results of Section 2.3 in de Haan ([9]):

(2.25) $F \in \Delta(c)$ with $0 < c < \infty$ if and only if $-H(s) = s^{-c}L(s)$, $0 < s < 1$, for some function L slowly varying at zero.

(2.26) $F \in \Delta(c)$ with $-\infty < c < 0$ if and only if $-H(s) = A - s^{-c}L(s)$, $0 < s < 1$, for some function L slowly varying at zero and some finite constant A.

The following lemma is a slight extension of Lemma 2 in [6].

Lemma 2.8. *Let L be any function defined on* (0, 1), *bounded on compact subintervals and slowly varying at zero. Let $\{k_n\}$ be a not-necessarily integer-valued sequence satisfying* (1.1) *and $\{l_n\}$ be any sequence of positive numbers such that $k_n/l_n \to \infty$ as $n \to \infty$. Then for any $0 < \nu < \infty$, we have*

$$\lim_{n\to\infty} \left\{ \left(\frac{k_n}{n}\right)^{-\nu} L\left(\frac{k_n}{n}\right) \right\} \Big/ \left\{ \left(\frac{l_n}{n}\right)^{-\nu} L\left(\frac{l_n}{n}\right) \right\} = 0.$$

Proof. The two cases $\alpha > 0$ and $\alpha = 0$ follow from Properties 1 and 2 of Corollary 1.2.1. in [9], respectively. □

Lemma 2.9. *If $F \in \Delta(c)$ for some $c \neq 0$ and* (1.1) *holds, then with the respective L functions from* (2.25) *and* (2.26), *we have the asymptotic equalities*

$$\sigma(1/n, k_n/n) \sim K_c (1/n)^{1/2-c} L(1/n), \qquad \text{if } c > \tfrac{1}{2} \text{ and } \alpha \ge 0;$$

$$\sigma(1/n, k_n/n) \sim \left(\int_{1/n}^{k_n/n} u^{-1} L^2(u)\, du \right)^{1/2}, \quad \text{if } c = \tfrac{1}{2} \text{ and } \alpha = 0;$$

$$\sigma(1/n, k_n/n) \sim K_c (k_n/n)^{1/2-c} L(k_n/n), \quad \text{if } c < \tfrac{1}{2},\ c \neq 0 \text{ and } \alpha = 0,$$

as $n \to \infty$, where

$$K_c = \begin{cases} \left(2c^2/((1-c)(1-2c))\right)^{1/2}, & \text{if } c < \tfrac{1}{2}, c \neq 0, \\ \left(2c/(2c-1)\right)^{1/2}, & \text{if } c > \tfrac{1}{2}. \end{cases}$$

PROOF. The case $c > \frac{1}{2}$ can be inferred from Lemma 1 in [6] and Lemma 2.8. The cases $c = \frac{1}{2}$ and $0 < c < \frac{1}{2}$ are proven in Lemma 6 in [7], and the same proof given there for $0 < c < \frac{1}{2}$ also works for $c < 0$. □

LEMMA 2.10. *Whenever $F \in \Delta(0)$ and* (1.1) *holds with* $\alpha = 0$,

$$\lim_{s \downarrow 0} s^{1/2}\{H(\lambda s) - H(s)\}/\sigma(0,s) = 2^{-1/2}\log\lambda \quad \text{for all } 0 < \lambda < \infty \tag{2.27}$$

and

$$\sigma(0,s) = s^{1/2}l(s) \quad \text{for some function } l \text{ slowly varying at zero.} \tag{2.28}$$

PROOF. Assertion (2.27) follows directly from Lemmas 4 and 6, while (2.28) follows from Lemmas 2 and 6 of Lo [14]. □

LEMMA 2.11. *Assume $F \in \Delta(c)$ for some c and that* (1.1) *holds. If $\alpha > 0$, then with the function ψ_α defined in* (1.13), *for all $x \in R$,*

$$\lim_{n\to\infty} \psi_n(x) = \begin{cases} 0, & \text{if } c > \frac{1}{2}, \text{ or } c = \frac{1}{2} \text{ and } \sigma(0,\alpha) = \infty, \\ \psi_\alpha(x), & \text{if } c < \frac{1}{2}, \text{ or } c = \frac{1}{2} \text{ and } \sigma(0,\alpha) < \infty, \end{cases}$$

while if $\alpha = 0$, then $\psi_n(x) \to 0$ for all $x \in R$ as $n \to \infty$, for any c.

PROOF. The case $\alpha > 0$ follows directly from (1.14) and the definition of ψ_n.

Consider now $\alpha = 0$. In this case we claim that for all $0 < \lambda < \infty$, the following limiting relations hold which clearly imply the second statement:

$$\lim_{n\to\infty} \frac{k_n^{1/2}\{H(\lambda k_n/n) - H(k_n/n)\}}{n^{1/2}a_n} = \begin{cases} K_c^{-1}(1-\lambda^{-c})\operatorname{sign}(c), & \text{if } c < \frac{1}{2}, c \neq 0, \\ 2^{-1/2}\log\lambda, & \text{if } c = 0, \\ 0, & \text{if } c \geq \frac{1}{2}. \end{cases}$$

Here the cases $c < 0$ and $0 < c < \frac{1}{2}$ follow from (2.26) and (2.25) and the third statement of Lemma 2.9, the case $c = \frac{1}{2}$ follows by (2.25), the second statement of Lemma 2.9 and by an application of Lemma 4 in [7], and the case $c > \frac{1}{2}$ follows from (2.25) and the first statement of Lemma 2.9 via an application of Lemma 2.8. Finally, the case $c = 0$ will follow directly from (2.27) of Lemma 2.10 once we can show that when $c = 0$,

$$\sigma(1/n, k_n/n) \sim \sigma(0, k_n/n) \quad \text{as } n \to \infty. \tag{2.29}$$

But this is immediate from the inequalities $\sigma^2(0, k_n/n) \geq \sigma^2(1/n, k_n/n) \geq \sigma^2(0, k_n/n) - \sigma^2(0, 1/n)$ and (2.28) of Lemma 2.10 via another application of Lemma 2.8. □

LEMMA 2.12. *If $F \in \Delta(c)$ for some c and* (1.1) *holds, then*

$$\lim_{n\to\infty} \varphi_n(y) = \begin{cases} K_c^{-1}(1-y^{-c}) & \text{for all } y > 0, \text{ if } c > \frac{1}{2}, \\ 0 & \text{for all } y > 0, \text{ if } c \le \frac{1}{2}. \end{cases}$$

PROOF. The case $c > \frac{1}{2}$ follows directly from (2.25) and the first statement of Lemma 2.9 for any $0 \le \alpha < 1$.

In the case of $c \le \frac{1}{2}$, we first consider the subcase $\alpha = 0$. Then for $c = \frac{1}{2}$ the assertion of the lemma follows from (2.25), the second statement of Lemma 2.9 and Lemma 4 in [7]. For $c < \frac{1}{2}$, $c \ne 0$, the assertion follows by (2.25), (2.26), the third statement of Lemma 2.9 and an application of Lemma 2.8. When $c = 0$, the assertion follows from (2.29) and the two statements of Lemma 2.10, via one more application of Lemma 2.8.

The subcase $\alpha > 0$ of the case $c \le \frac{1}{2}$ follows from the former subcase when $\alpha = 0$ if we only notice that for $a_n = \sigma(1/n, k_n/n)$ in the denominator of φ_n we have $a_n \ge \sigma(1/n, m_n/n)$ for all large enough n for any sequence $\{m_n\}$ such that $m_n \to \infty$ and $m_n/n \to 0$ as $n \to \infty$. □

PROOF OF COROLLARY 2. The boundary case $c = \frac{1}{2}$ and $\sigma(0, \alpha) = \infty$ follows directly from Lemmas 2.11 and 2.12 combined with Corollary 1.

Next, when $c < \frac{1}{2}$, or $c = \frac{1}{2}$ and $\sigma(0, \alpha) < \infty$, one notes that for r_n in the proof of Lemma 2.3 or in the formulation of Theorem 1 and for r_α in the definition of $M(\alpha)$ we always have $r_n \to r_\alpha$ as $n \to \infty$ along the whole sequence $\{n\}$ of the positive integers. Thus by Lemmas 2.11 and 2.12 and Theorem 1, applied with $A_n \equiv n^{1/2} a_n$, $\psi \equiv \psi_\alpha$, where $\psi_0 \equiv 0$, $\varphi \equiv 0$ and $a = 1$, for any subsequence $\{n_1\} \subset \{n\}$, there exists a further subsequence $\{n_2\} \subset \{n_1\}$ such that the statement of Corollary 2 holds along $\{n_2\}$. Hence it holds along the whole sequence $\{n\}$.

Finally, when $c > \frac{1}{2}$, it can be shown using Lemma 1 in [6] and Lemma 2.8 that for any sequence $\{l_n\}$ of positive numbers such that $l_n \to \infty$ and $k_n/l_n \to \infty$ as $n \to \infty$, we have $\sigma(l_n/n, k_n/n) \sim K_c(l_n/n)^{1/2-c} L(l_n/n)$ as an extension of the first statement of Lemma 2.9, which by (2.25) and a final application of Lemma 2.8 yields

$$\sigma(l_n/n, k_n/n)/\sigma(1/n, k_n/n) \to 0, \quad \text{as } n \to \infty.$$

Thus by Lemmas 2.11 and 2.12 and Theorem 1, applied with $A_n \equiv n^{1/2} a_n$, $\psi \equiv 0$, $\varphi(y) = K_c^{-1}(1 - y^{-c})$ and $b = 0$, we obtain that for every subsequence $\{n_1\} \subset \{n\}$, there exists a further subsequence $\{n_2\} \subset \{n_1\}$ such that the statement of Corollary 2 holds along $\{n_2\}$. This of course gives the same statement along the whole sequence $\{n\}$. □

3. Discussion of the conditions and examples. The next two propositions provide equivalent forms for Conditions 1 and 2, respectively. The first of these forms is a probabilistic interpretation of the respective condition. The second one, following from the first, is a reformulation of the condition in terms of the underlying distribution function F. That the latter reformulations might be possible was suggested to us by a referee.

For ease of notation we set

$$c_n = -H\left(\frac{1}{n}\right) = Q\left(\left(1 - \frac{1}{n}\right) -\right)$$

and

$$c_n(k_n) = -H\left(\frac{k_n}{n}\right) = Q\left(\left(1 - \frac{k_n}{n}\right) -\right),$$

and for a nondecreasing function h defined on a subset S of the real line R, we define its left-continuous inverse by

$$h^{-1}(x) = \inf\{s \in S: h(s) \geq x\}, \qquad x \in R,$$

where we agree that the infimum of the empty set is $+\infty$. Let Z denote a standard normal random variable and Y denote an exponential random variable with mean 1 and let $\{k_n\}$ be a sequence of integers satisfying (1.1) with $\alpha = 0$.

PROPOSITION 1. *Let ψ be a nondecreasing, left-continuous function on $(-\infty, \infty)$ such that $\psi(0) \leq 0$ and $\psi(0+) \geq 0$. Condition* 1 *holds along $\{n_1\}$ with this ψ if and only if*

$$\text{(3.1)} \qquad \frac{k_{n_1}^{1/2}}{A_{n_1}}\left\{X_{n_1+1-k_{n_1}, n_1} - c_{n_1}(k_{n_1})\right\} \to_{\mathscr{D}} -\psi(Z) \quad \text{as } n_1 \to \infty,$$

which happens if and only if

$$\text{(3.2)} \qquad \frac{1}{k_{n_1}^{1/2}}\left(n_1\left\{1 - F\left(c_{n_1}(k_{n_1}) - xk_{n_1}^{-1/2}A_{n_1}\right)\right\} - k_{n_1}\right) \to \psi^{-1}(x) \quad \text{as } n_1 \to \infty,$$

for every continuity point x of the distribution function of $-\psi(Z)$.

PROPOSITION 2. *Let φ be a nondecreasing left-continuous function on $(0, \infty)$ such that $\varphi(1) \leq 0$ and $\varphi(1+) \geq 0$. Condition* 2 *holds along $\{n_1\}$ with this φ if and only if*

$$\text{(3.3)} \qquad \frac{1}{A_{n_1}}\{X_{n_1, n_1} - c_{n_1}\} \to_{\mathscr{D}} -\varphi(Y) \quad \text{as } n_1 \to \infty,$$

which happens if and only if

$$\text{(3.4)} \qquad n_1\left\{1 - F\left(c_{n_1} - xA_{n_1}\right)\right\} \to \varphi^{-1}(x) \quad \text{as } n_1 \to \infty,$$

for every continuity point x of the distribution function of $-\varphi(Y)$.

PROOF. First we consider Proposition 1. For the uniform (0, 1) order statistics $U_{1,n} \leq \cdots \leq U_{n,n}$ as in (1.3), we introduce

$$W_{k_n,n} = \frac{n}{k_n^{1/2}}\left\{U_{k_n,n} - \frac{k_n}{n}\right\}, \qquad n \geq 1.$$

Note that by (1.3),

$$X_{n+1-k_n,n} =_{\mathscr{D}} Q\big((1 - U_{k_n,n}) -\big) = -H(U_{k_n,n}),$$

and hence, as $n \to \infty$,

$$\frac{k_n^{1/2}}{A_n}\{X_{n+1-k_n,n} - c_n(k_n)\} =_{\mathscr{D}} -\frac{n^{1/2}a_n}{A_n}\psi_n(W_{k_n,n}) + o_P(1).$$

On the other hand,

$$\sup_B \left| P\left\{-\frac{n^{1/2}a_n}{A_n}\psi_n(W_{k_n,n}) \in B\right\} - P\left\{-\frac{n^{1/2}a_n}{A_n}\psi_n(Z) \in B\right\}\right|$$
$$\leq \sup_B |P\{W_{k_n,n} \in B\} - P\{Z \in B\}|,$$

where the supremum is taken over all Borel sets B on the real line, and this upper bound goes to zero as $n \to \infty$ by Proposition 2.10 of Reiss [18], where earlier references concerning this result can also be found. Hence (3.1) holds if and only if

$$\psi_{n_1}^*(Z) = \frac{n_1^{1/2}a_{n_1}}{A_{n_1}}\psi_{n_1}(Z) \to \psi(Z)$$

almost surely as $n_1 \to \infty$, and it is easily checked that this happens if and only if Condition 1 holds along $\{n_1\}$.

The equivalence of (3.1) and (3.2) follows by a standard argument as given in Watts, Rootzén and Leadbetter ([22], page 654).

The proof of the first statement of Proposition 2 is the same as above upon noting that

$$\frac{1}{A_n}\{X_{n,n} - c_n\} =_{\mathscr{D}} -\frac{n^{1/2}a_n}{A_n}\varphi_n(nU_{1,n}) + o_P(1)$$

and, with the supremum taken again over all Borel sets B on the line,

$$\sup_B |P\{nU_{1,n} \in B\} - P\{Y \in B\}| \to 0 \quad \text{as } n \to \infty.$$

The latter follows from Theorem 2.6 of Reiss [18].

The equivalence of (3.3) and (3.4) is an easy exercise well known in extreme value theory. □

We note that since ψ and φ can only be constants if they are zero, the limits in (3.1) and (3.3) are nondegenerate if and only if $\psi \equiv 0$ and $\varphi \equiv 0$, respectively. If $\psi \equiv 0$, then we have (3.2) with

$$\psi^{-1}(x) = \begin{cases} -\infty, & \text{if } x < 0, \\ \infty, & \text{if } x > 0, \end{cases}$$

and if $\varphi \equiv 0$, then we have (3.4) with

$$\varphi^{-1}(x) = \begin{cases} 0, & \text{if } x < 0, \\ \infty, & \text{if } x > 0. \end{cases}$$

Our first example shows that limiting distributions in Theorem 1 can arise along subsequences of $\{n\}$ with $\varphi \not\equiv 0$, $\psi \not\equiv 0$ and $b > 0$.

EXAMPLE 1. Let $1 \le k_n \le n$ be integers such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$, let $\{n_1, n_2, \dots\}$ be a subsequence of $\{n\}$ and $\{d_{n_j}, j \ge 1\}$ be a sequence of arbitrary positive numbers. For j large enough to make $k_{nj}/n_j < \frac{1}{2}$, consider

$$H(s) := \begin{cases} H\left(\dfrac{k_{n_j}}{n_j}\right), & \text{if } \dfrac{k_{n_j}}{n_j} - \dfrac{k_{n_j}^{1/2}}{4n_j} < s \le 2\dfrac{k_{n_j}}{n_j}, \\ H\left(\dfrac{k_{n_j}}{n_j}\right) - d_{n_j}, & \text{if } \dfrac{1}{2n_j} < s \le \dfrac{k_{n_j}}{n_j} - \dfrac{k_{n_j}^{1/2}}{4n_j}, \\ H\left(\dfrac{1}{k_{n_j}} +\right) - k_{n_j}^{1/2} d_{n_j}, & \text{if } \dfrac{1}{n_j k_{n_{j+1}}} < s \le \dfrac{1}{2n_j}. \end{cases}$$

Clearly, a quantile function Q with this corresponding H function exists as long as

$$\frac{2k_{n_{j+1}}}{n_{j+1}} \le \frac{1}{n_j k_{n_{j+1}}}, \quad \text{that is,} \quad k_{n_{j+1}}^2 \le \frac{1}{2}\frac{n_{j+1}}{n_j},$$

for all large enough j. (This is the case, for example, if $n_j = [2^{j \log j}]$ and $k_{n_j} = [2^{3^{-1} \log j}]$.) Noting the asymptotic equality

$$\begin{aligned} \sigma^2\left(\frac{1}{n_j}, \frac{k_{n_j}}{n_j}\right) &= \left\{\left(\frac{k_{n_j}}{n_j} - \frac{k_{n_j}^{1/2}}{4n_j}\right) - \left(\frac{k_{n_j}}{n_j} - \frac{k_{n_j}^{1/2}}{4n_j}\right)^2\right\} d_{n_j}^2 \\ &\sim \frac{k_{n_j}}{n_j} d_{n_j}^2 \quad \text{as } j \to \infty, \end{aligned}$$

it is easy to show that

$$\lim_{j\to\infty} \psi_{n_j}(x) = \psi^*(x) := \begin{cases} -1, & \text{if } -\infty < x \le -\frac{1}{4}, \\ 0, & \text{if } -\frac{1}{4} < x < \infty, \end{cases}$$

$$\lim_{j\to\infty} \varphi_{n_j}(y) = \varphi^*(y) := \begin{cases} -1, & \text{if } 0 < y \le \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} < y < \infty, \end{cases}$$

and that

$$\lim_{j\to\infty} \sigma\left(l_{n_j}/n_j, k_{n_j}/n_j\right)/\sigma\left(1/n_j, k_{n_j}/n_j\right) = 1,$$

for any sequence $\{l_{n_j}\}$ such that $l_{n_j} \to \infty$ and $l_{n_j}/k_{n_j} \to 0$ as $j \to \infty$. Moreover, $r_{n_j} \to 1$ as $j \to \infty$. Thus we have

$$k_{n_j}^{-1/2} d_{n_j}^{-1} \left\{ \sum_{i=1}^{k_{n_j}} X_{n_j+1-i, n_j} - \mu_{n_j} \right\} \to_{\mathscr{D}} V(\varphi^*, \psi^*, 1, 1, 0), \quad \text{as } j \to \infty,$$

where $V(\varphi^*, \psi^*, 1, 1, 0) = N(\frac{1}{2}) + Z_1 + \min(Z_1 + \frac{1}{4}, 0)$ and, by simple computation, $\mu_{n_j} = -k_{n_j} H(k_{n_j}/n_j) + d_{n_j}(k_{n_j} - k_{n_j}^{1/2}/4)$.

We emphasize that the numbers $d_{n_j} > 0$ determining the jump sizes of the quantile function Q are arbitrary in the previous example. Therefore, they can be chosen so that the underlying distribution has moments of arbitrarily high order.

The second example relates the convergence in distribution of extreme sums along subsequences to that of the whole sum.

EXAMPLE 2. Suppose that $F(0) = 0$ and there exist a subsequence $\{n_1, n_2, \dots\} \subset \{n\}$ and constants $A_{n_j} > 0$ and B_{n_j} such that

$$A_{n_j}^{-1} \left\{ \sum_{i=1}^{n_j} X_i - B_{n_j} \right\} \to_{\mathscr{D}} W \quad \text{as } j \to \infty, \tag{3.5}$$

where W is an infinitely divisible random variable with a nondegenerate nonnormal component. It is shown in [5] that we can choose

$$B_{n_j} = n_j \int_0^{1-1/n_j} Q(s)\, ds.$$

Note that necessarily $\mathrm{Var}(X) = \infty$, so that we must have

$$\lim_{j \to \infty} A_{n_j}/n_j^{1/2} = \infty. \tag{3.6}$$

By Theorem 5 in [4], for each $0 < \beta < 1$,

$$\sum_{i=1}^{[\beta n]} X_{i,n} - n \int_0^{[\beta n]/n} Q(s)\, ds = O_P(n^{1/2}) \quad \text{as } n \to \infty.$$

Thus by (3.6), for each $0 < \beta < 1$,

$$A_{n_j}^{-1} \left\{ \sum_{i=[\beta n_j]+1}^{n_j} X_{i, n_j} - n_j \int_{[\beta n_j]/n_j}^{1-1/n_j} Q(s)\, ds \right\} \to_{\mathscr{D}} W \quad \text{as } j \to \infty.$$

Hence by a simple diagonal selection procedure we can find a sequence $\{k_{n_j}\}$ such that $k_{n_j} \to \infty$ and $k_{n_j}/n_j \to 0$ and

$$A_{n_j}^{-1} \left\{ \sum_{i=1}^{k_{n_j}} X_{n_j+1-i, n_j} - n_j \int_{1-k_{n_j}/n_j}^{1-1/n_j} Q(s)\, ds \right\} \to_{\mathscr{D}} W \quad \text{as } j \to \infty.$$

Our last example connects Conditions 1, 2 and 3 with the notion of stochastic compactness for sums and maxima and provides a relatively general situation when the scaling factor A_n is the same for extreme sums, whole sums and maxima. We say that F is stochastically compact, written $F \in SC$, if there are sequences $A_n > 0$ and B_n such that for every subsequence $\{m_j\} \subset \{n\}$, there exists a further subsequence $\{n_j\} \subset \{m_j\}$ such that (3.5) holds with a nondegenerate W. We call $\{X_{n,n}\}$ stochastically compact if there exists a sequence $C_n > 0$ such that for every subsequence $\{n'\} \subset \{n\}$, there exists a further subsequence $\{n''\} \subset \{n'\}$ such that $X_{n'',n''}/C_{n''}$ converges in distribution to a nondegenerate random variable as $n'' \to \infty$.

EXAMPLE 3. Assume that $F(0) = 0$ and F is not in the domain of partial attraction of a normal law. Corollary 12 in [5] says that $F \in SC$ if and only if $X_{n,n}$ is stochastically compact. (This was proved earlier by de Haan and Resnick [10].) In this case one can choose, according to the same corollary, $C_n = A_n = a(n) = n^{1/2}\sigma(1/n, 1-(1/n))$, where $\sigma(\cdot,\cdot)$ is defined following (1.3). Also, for such an F it is readily inferred from Corollary 10 in [5] that $F \in SC$ if and only if

$$\limsup_{s \downarrow 0} s^{1/2}Q(1-\lambda s)/\sigma(s, 1-s) < \infty \quad \text{for all } 0 < \lambda < \infty.$$

Combining these two facts with the simple observation that $a(n) \geq n^{1/2}a_n$, $n \geq 1$, for any sequence $\{k_n\}$ satisfying (1.1) with $\alpha = 0$ or $\alpha > 0$, we easily see that whenever such an F is in SC, each subsequence of $\{n\}$ contains a further subsequence along which Conditions 1, 2 and 3 are simultaneously satisfied with $A_n = a(n)$ for some ψ, φ and $0 \leq a < \infty$. In fact the stochastic compactness of the maxima forces $\varphi \not\equiv 0$.

Acknowledgments. The authors thank the Bolyai Institute, University of Szeged, the Laboratoire de Statistique Théorique et Appliquée, Université de Paris VI, the Mathematisches Forschungsinstitut, Oberwolfach and the Department of Mathematical Sciences, University of Delaware, for bringing them together for collaborative work. They also appreciate the insightful comments of the referee.

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

[2] CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L. and MASON, D. M. (1986). Weighted empirical and quantile processes. *Ann. Probab.* **14** 31–85.

[3] CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L. and MASON, D. M. (1986). Normal and stable convergence of integral functions of the empirical distribution function. *Ann. Probab.* **14** 86–118.

[4] CSÖRGŐ, S., HAEUSLER, E. and MASON, D. M. (1988). The asymptotic distribution of trimmed sums. *Ann. Probab.* **16** 672–699.

[5] CSÖRGŐ, S., HAEUSLER, E. and MASON, D. M. (1988). A probabilistic approach to the asymptotic distribution of sums of independent, identically distributed random variables. *Adv. in Appl. Math.* **9** 259–333.

[6] CSÖRGŐ, S., HORVÁTH, L. and MASON, D. M. (1986). What portion of the sample makes a partial sum asymptotically stable or normal? *Probab. Theory Related Fields* **72** 1–16.

[7] CSÖRGŐ, S. and MASON, D. M. (1986). The asymptotic distribution of sums of extreme values from a regularly varying distribution. *Ann. Probab.* **14** 974–983.

[8] CSÖRGŐ, S. and MASON, D. M. (1987). Approximations of weighted empirical processes with applications to extreme, trimmed and self-normalized sums. In *Proc. First World Congress Bernoulli Soc., Tashkent, USSR, 1986* **2** 811–819. VNU Science, Utrecht.

[9] DE HAAN, L. (1970). *On Regular Variation and its Application to the Weak Convergence of Sample Extremes*. Math. Centre Tract 32, Mathematisch Centrum, Amsterdam.

[10] DE HAAN, L. and RESNICK, S. I. (1984). Stochastic compactness and point processes. *J. Austral. Math. Soc. Ser. A* **37** 307–316.

[11] FERGUSON, T. S. and KLASS, M. J. (1972). A representation of independent increment processes without Gaussian components. *Ann. Math. Statist.* **43** 1634–1643.

[12] HAHN, M. G., MASON, D. M. and WEINER, D. C., eds. (1990). *Sums, Trimmed Sums, and Extremes*. Birkhäuser, Basel.

[13] LEPAGE, R., WOODROOFE, M. and ZINN, J. (1981). Convergence to a stable distribution via order statistics. *Ann. Probab.* **9** 624–632.

[14] LO, G. S. (1989). A note on the asymptotic normality of sums of extreme values. *J. Statist. Plann. Inference* **22** 127–136.

[15] MASON, D. M. (1985). The asymptotic distribution of generalized Rényi statistics. *Acta Sci. Math.* (*Szeged*) **48** 315–323.

[16] MASON, D. M. and SHORACK, G. R. (1988). Necessary and sufficient conditions for asymptotic normality of L-statistics. Report 124, Dept. Statistics, Univ. Washington.

[17] MEJZLER, D. G. (1949). On a theorem of B. V. Gnedenko. *Sb. Trudov Inst. Mat. Akad. Nauk Ukrain. SSR* **12** 31–35. (In Russian.)

[18] REISS, R.-D. (1981). Uniform approximation to distributions of extreme order statistics. *Adv. in Appl. Probab.* **13** 533–547.

[19] ROSSBERG, H.-J. (1967). Über das asymptotische Verhalten der Rand-und Zentralglieder einer Variationsreihe (II). *Publ. Math. Debrecen* **14** 83–90.

[20] SHORACK, G. R. and WELLNER, J. A. (1986). *Empirical Processes with Applications to Statistics*. Wiley, New York.

[21] STIGLER, S. M. (1973). The asymptotic distribution of the trimmed mean. *Ann. Statist.* **1** 472–477.

[22] WATTS, V., ROOTZÉN, H. and LEADBETTER, M. R. (1982). On limiting distributions of intermediate order statistics from stationary sequences. *Ann. Probab.* **10** 653–662.

SÁNDOR CSÖRGŐ
BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
H-6720 SZEGED
HUNGARY

ERICH HAEUSLER
MATHEMATICAL INSTITUTE
UNIVERSITY OF MUNICH
THERESIENSTRASSE 39
D-8000 MUNICH 2
GERMANY

DAVID M. MASON
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELAWARE
NEWARK, DELAWARE 19716