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SYMMETRY GROUPS AND TRANSLATION INVARIANT REPRESENTATIONS OF MARKOV PROCESSES[1]

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The symmetry groups of the potential theory of a Markov process X_t are used to introduce new algebraic and topological structures on the state space and the process. For example, let G be the collection of bijections φ on E which preserve the collection of excessive functions. Assume there is a transitive subgroup H of the symmetry group G such that the only map $\varphi \in H$ fixing a point $e \in E$ is the identity map on E. There is a bijection $\Psi: E \to H$ so that the algebraic structure of H can be carried to E, making E into a group. If there is a left quasi-invariant measure on E, then there is a topology on E making E into a locally compact second countable metric group. There is also a time change $\tau(t)$ of X_t such that $X_{\tau(t)}$ is a translation invariant process on E and $X_{\tau(t)}$ is right-continuous with left limits in the new topology.

1. Introduction. One of the most successful cooperations for probability theory has been the one between Markov processes and potential theory. Many authors consider them to be two aspects of the same theory, but the different outlook and approach of each enriches the study of its counterpart immeasurably. The purpose of this article is to introduce the symmetries of a potential theory as a fundamental tool useful in studying the associated Markov process. Viewed from a general perspective, studying the symmetry group of a potential theory is one method of investigating the underlying geometry of the potential theory, and this is a time-honored theme in mathematics. For the working probabilist, the symmetry group should have some of the structure of the associated Markov process encoded in its algebraic structure, and the algebraic structure thereby becomes a useful adjunct in studying the process. In fact, the symmetry group seems to contain so much information about the process that we feel it will become an addition to the already formidable arsenal of standard Markov process equipment.

There are two types of potential theories which can be associated with a transient strong Markov process (X_t, P^x), namely, the cone $\mathscr{S}$ of excessive functions and the cone Exc of excessive measures. While the first is the classical object of potential theory, the second received much less attention until recently, despite the fact that it was studied by Hunt in his original work. Glover and Mitro [8] formulated a group G consisting of symmetries of $\mathscr{S}$. Roughly speaking, G is defined to be the collection of all bimeasurable bijec-

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tions φ of the state space E of X_t onto itself such that $\mathscr{S} = \{f \circ \varphi: f \in \mathscr{S}\}$. Their original motivation was the search for a method of constructing a large class of Markov functions Φ for X_t or for a time change of X_t. In general, if Φ is a surjection from E onto another state space F which is not an injection, then $\Phi(X_t)$ is not a Markov process. From each subgroup H of G, Glover and Mitro constructed a state space F and a surjection Φ: $E \to F$. They showed (under some mild topological hypotheses) that there is also a time change $\tau(t)$ of X_t such that $\Phi(X_{\tau(t)})$ is a strong Markov process. Few other concrete algorithms for the construction of such Markov functions are available.

Subsequently, Glover [7] formulated the group Sym consisting of symmetries of Exc. Roughly speaking, it consists of all finely continuous bimeasurable bijections φ: $E \to E$ such that Exc $= \{\varphi(\mu): \mu \in \text{Exc}\}$, where $\varphi(\mu)$ is the image of the measure μ under φ. Using appropriate hypotheses, Glover constructed functions Ψ: $E \to F$ such that $\Psi(Z_t)$ is a strong Markov process, where Z_t is a transformation of X_t described in detail in Sections 3 and 7.

These two articles already indicate that a substantial amount of information about X_t is contained in the symmetry groups. We hope to convince the reader that their study is vital since they also contain the independent increment structure of the process. Fortunately, the group structure is often easier to understand than the detailed structure of $\mathscr{S}$ or Exc. For example, one does not need a deep understanding of the collection of positive superharmonic functions on R^d to know that it is invariant under rotations, translations and dilations of R^d, as well as flips about hyperplanes.

Precise definitions for the process (X_t, P^x) and its associated structures are given in Section 2. We have taken virtually all of the structures to be Borel measurable rather than universally measurable to avoid overburdening the text with measure theoretic difficulties and because the structures we discuss are of fundamental interest (even for Brownian motion in R^d).

In Section 3, we begin the study of G and Sym. In particular, we assume that whichever group we are studying is transitive [see (3.7)]; i.e., each point $x \in E$ can be carried to another point $y \in E$ by some element of the group. This insures that the group is large enough for our purposes. Fix a point $e \in E$, and let J be a transitive subgroup of either G or Sym. If we set $J_e = \{\varphi \in J: \varphi(e) = e\}$, then J_e is a subgroup of J and there is a bijection Ψ: $E \to J/J_e = \mathscr{J}$. This bijection allows us to transfer whatever algebraic structure $\mathscr{J}$ possesses to E. In general, $\mathscr{J}$ is a collection of cosets. But if J_e consists only of the identity map (in which case we say J_e is trivial), then $\mathscr{J}$ is a group which is isomorphic to J, so E inherits the group structure of J.

In Section 4, we call X_t a J-translation invariant process if the processes $(\varphi(X_t), P^{\varphi^{-1}(x)})$ and (X_t, P^x) are identical in law for every $x \in E$ and $\varphi \in J$. We define the Λ-increments of X_t with the aid of a measurable selector Λ, and we show that J-translation invariant processes with infinite lifetimes have stationary independent Λ-increments. In (4.5), we show that if G is transitive and 1_E is the potential of a strictly increasing continuous additive functional A_t of X_t, then $X_{\tau(t)}$ is G-translation invariant, where $\tau(t)$ is the right-continuous inverse of A_t. A second result along these lines is the following. If X is a Hunt process with G a transitive group, and if

$$0 < P^x \sum_{0<s<\zeta} e^{-\alpha s} 1_{\{X(s-)\neq X(s)\}} < \infty$$

for some $x \in E$ and $\alpha > 0$, then $X_{\sigma(t)}$ is G-translation invariant, where $\sigma(t)$ is the right-continuous inverse of a strictly increasing continuous additive functional B_t of X_t.

The reader can check that G and Sym are defined algebraically. The maps in the groups need not be homeomorphisms of E [see (3.13)], and the groups themselves come with no natural topology. But there is the remarkable Mackey–Weil theorem which we discuss in Section 5 which can be used to introduce a topology on a group J and, therefore, on the state space E. If some measurability conditions [see (5.1)] are satisfied, and if there is a σ-finite left quasi-invariant measure μ [see (5.2)] on J, then there is a topology on J making J into a locally compact second countable metric group. There is a σ-finite left Haar measure m on J which is equivalent to μ.

A major result of this article is stated in (6.15) for a transient Hunt process on E with H a transitive subgroup of G such that H_e is trivial. If the Mackey–Weil theorem applies, then there is a strictly increasing continuous additive functional A_t of X_t with inverse $\tau(t)$ such that $(X_{\tau(t)}, P^x)$ is an H-translation invariant process on E (which inherits the group structure of H). The potential of $X_{\tau(t)}$ maps functions which are continuous with compact support in the Mackey–Weil topology into functions which are continuous in that topology, and $X_{\tau(t)}$ is right-continuous with left limits in E in the Mackey–Weil topology almost surely. There is a powerful synergy occurring here among the algebraic, potential theoretic and probabilistic structures. Those people interested in Ray–Knight methods for regularizing Markov processes may note that the Mackey–Weil theorem provides an alternate retopologization procedure in this special situation. It has the advantage that it knits together the algebraic and topological structures nicely. Readers interested in stochastic flows may note that the Markov process may be regarded as taking values in H, which is a group of homeomorphisms of E in the new topology. If H happens to be an abelian group, then $X_{\tau(t)}$ simply turns out to be a classical Lévy process on a locally compact abelian group.

In Section 7, we discuss briefly the case when Sub is a transitive subgroup of Sym such that Sub_e is trivial. If there is a σ-finite left quasiinvariant measure and if the potential of 1_E is finite on E, then a transformation of X_t is Sub-translation invariant.

In Section 8, we examine some conditions guaranteeing the existence of left quasiinvariant measures, and Section 9 is devoted to some brief comments about the recurrent case.

2. Markov processes. In this section, we describe the various processes which will be considered in this article. Let E be a Lusin topological state space; that is, E is homeomorphic to a Borel measurable subset of a compact metric space. The Borel field of E will be denoted $\mathscr{E}$. We adjoin an isolated point Δ to E to obtain $E_\Delta = E \cup \{\Delta\}$ and $\mathscr{E}_\Delta = \mathscr{E} \vee \{\Delta\}$. This point will serve as the cemetery for the Markov process after it dies.

Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a right process on $(E, \mathscr{E})$ (see [3], [12]). For convenience, we shall assume that Ω is the space of all maps ω: $[0, \infty) \to E_\Delta$ which are right-continuous and are such that $\omega(t) = \Delta$ if and only if $\omega(t+s) = \Delta$, for every $s > 0$. Set $X_t(\omega) = \omega(t)$, and let $\mathscr{F}_t$ and $\mathscr{F}$ be the appropriate completions of $\mathscr{F}_t^{\,0} = \sigma\{X_s\colon s \le t\}$ and $\mathscr{F}^{\,0} = \sigma\{X_s\colon s \ge 0\}$. For each $t \ge 0$, θ_t: $\Omega \to \Omega$ is the shift operator characterized by $X_s \circ \theta_t = X_{s+t}$. Under the measure P^x, X_t is a time homogeneous strong Markov process with $X_0 = x$ a.s. P^x. If g is an α-excessive function, then $t \to g(X_t)$ is right-continuous a.s. In general, if $\mathscr{G}$ is a σ-algebra, we write $b\mathscr{G}$ (resp., $p\mathscr{G}$) to denote the collection of bounded (resp., positive) $\mathscr{G}$-measurable functions.

Let P_t and U^α denote the semigroup and resolvent of X. We assume throughout this article that X is a Borel right process, by which we mean P_t maps Borel functions into Borel functions. The class of Borel right processes (as described above) constitutes the class of strong Markov processes we consider in this article. We shall need to impose extra regularity conditions from time to time in order to prove a theorem. We describe these hypotheses below, but we emphasize that *these are not blanket hypotheses*. Their use will be prominently advertised in the statements of the individual theorems as needed.

A σ-finite measure λ on $(E, \mathscr{E})$ is called a reference measure for X if $U^\alpha(x, \cdot\,) \ll \lambda$ for every $x \in E$ and $\alpha > 0$.

A Borel right process X is said to be in weak duality with respect to a σ-finite measure λ on $(E, \mathscr{E})$ if there is another Borel right process $\hat{X}$ on $(E, \mathscr{E})$ with semigroup $\hat{P}_t$ and resolvent $\hat{U}^\alpha$ such that

$$\int_E P_t f \cdot g \, d\lambda = \int_E f \cdot \hat{P}_t g \, d\lambda \tag{2.1}$$

for every $f \in p\mathscr{E}$ and $g \in p\mathscr{E}$.

A Borel right process X is said to be in strong duality with respect to a σ-finite measure λ on E if there is another Borel right process $\hat{X}$ on $(E, \mathscr{E})$ such that (2.1) is satisfied and

$$U^\alpha(x, \cdot) \ll \lambda \quad \text{and} \quad \hat{U}^\alpha(x, \cdot) \ll \lambda \tag{2.2}$$

for every $x \in E$, for every $\alpha > 0$. In this case, we write "$(X, \hat{X}, \lambda)$ are in strong duality."

Hypotheses (2.1) and (2.2) are regularity hypotheses about X and are satisfied by large classes of processes. In particular, Lévy processes on R^d satisfy (2.1) and most of them also satisfy (2.2). Weak and strong duality have been investigated extensively, and some of these results will be used later. The reader is referred to ([1], [5], [6]) for discussions of weak and strong duality and the associated potential theories.

Another hypothesis which we assume most of the time is that X is transient. That is, there is a function $q \in b\mathscr{E}$ with $q > 0$ and $Uq < \infty$ on E.

This hypothesis insures that the collection $\mathscr{S}$ of excessive functions is rich enough so that Hunt's balayage theorem holds ([1], [3], [12]). See also Proposition (1.9) in [4] for Hunt's balayage theorem for the class Exc of excessive measures. Recall that $\mathscr{S}$ is the collection of functions $f \in p\mathscr{E}^u$ such that $P_t f \leq f$ for every $t > 0$ and $\lim_{t \to 0} P_t f = f$. Also, Exc is the collection of σ-finite measures ξ on $(E, \mathscr{E})$ such that $\xi P_t \leq \xi$ for every $t > 0$.

Finally, we recall that a Borel right process is called a Hunt process if X is quasi-left-continuous. This means that whenever (T_n) is a sequence of $(\mathscr{F}_t)$-optional times increasing to T, we have $X(T_n)$ converging to $X(T)$ a.s. on $\{T < \infty\}$.

3. Symmetry groups. Let X be a transient Borel right process. The collection $\mathscr{S}$ of excessive functions of X is an indispensable tool in studying X, and it is natural to study the geometry of $\mathscr{S}$ by studying the symmetries of $\mathscr{S}$. The Blumenthal–Getoor–McKean theorem states that $\mathscr{S}$ determines X up to a time change ([1], [2]). Therefore, we say that $\mathscr{S}$ determines the geometric trajectories of X, and the symmetries of $\mathscr{S}$ should be reflected in the symmetries of the geometric trajectories.

Let $\overline{G}$ be the collection of bijections φ: $E \to E$ such that φ and φ^{-1} are $\mathscr{E}/\mathscr{E}$-measurable. For each $\varphi \in \overline{G}$, define $\mathscr{S}_\varphi = \{f \circ \varphi \colon f \in \mathscr{S}\}$.

3.1. Definition. $G = \{\varphi \in \overline{G} \colon \mathscr{S}_\varphi = \mathscr{S}\}$.

If we endow G with the composition operation $(\varphi, \psi) \to \varphi \circ \psi$, then G is a group which we call the symmetry group of $\mathscr{S}$. This group was formulated and studied by Glover and Mitro in [8]. They used subgroups of G to construct Markov functions f: These functions have the property that $f(X_{\tau(t)})$ is a strong Markov process for some time change $\tau(t)$ of X_t.

There is an equivalent formulation of G which is useful in extending the ideas in this article to the case where X is recurrent instead of transient. We treat only the transient case in this article, but we do make some remarks about extending to the recurrent case in Section 9. Let $\varphi \in G$. Since $\mathscr{S}_\varphi = \mathscr{S}$, $\varphi^{-1}(A)$ is finely open whenever $A \in \mathscr{E}$ is finely open, and $g \circ \varphi$ is finely continuous whenever g is continuous on E. Therefore, $g \circ \varphi(X_t)$ is a.s. right-continuous. (In the previous line, we need to extend φ to be a map from E_Δ to E_Δ by setting $\varphi(\Delta) = \Delta$. We make this extension throughout this article.) Since φ is a bijection, $(\varphi(X_t), P^{\varphi^{-1}(x)})$ is a strong Markov process with excessive functions $\mathscr{S}_{\varphi^{-1}} = \mathscr{S}$. Since $h \circ \varphi(X_t)$ is a.s. right-continuous whenever $h \in \mathscr{S}$, $(\varphi(X_t), P^{\varphi^{-1}(x)})$ is a transient Borel right process with the same excessive functions as (X_t, P^x). By the Blumenthal–Getoor–McKean theorem, there is a continuous additive functional A_t^φ of X_t which is strictly increasing and finite on $[0, \zeta)$ such that if $\tau(\varphi, t)$ is the right-continuous inverse of A_t^φ, then $(\varphi(X_t), P^{\varphi^{-1}(x)})$ and $(X_{\tau(\varphi, t)}, P^x)$ are identical in law. Thus we have shown that if $\varphi \in G$, then there is a continuous additive functional A_t^φ with the property in the line above. The reader is invited to complete the proof of the following result which we do not use in this article (but see Section 9).

3.2. PROPOSITION. *G consists of all $\varphi \in \overline{G}$ with the properties:*

(i) *$\varphi(X)$ is a transient Borel right process.*

(ii) *There is a continuous additive functional A_t^φ of X_t which is strictly increasing and finite on $[0, \zeta)$ such that if $\tau(\varphi, t)$ is the right-continuous inverse of A_t^φ, then $(\varphi(X_t), P^{\varphi^{-1}(x)})$ and $(X_{\tau(\varphi, t)}, P^x)$ are identical in law.*

Now we discuss the symmetry group of Exc. What properties of the process should be reflected in the symmetries of Exc? For guidance, we recall the theorem of Getoor and Glover [4] which is of more recent vintage than the Blumenthal–Getoor–McKean theorem. It states that Exc determines X up to the following type of transformation. Let Z be another transient Borel right process on E with resolvent V^α and having the same collection Exc of excessive measures. There are a set $K \in \mathscr{E}$ which is polar for both X and Z and a Borel measurable function c defined on $E - K$ such that $0 < c < \infty$ on $E - K$, and such that c is an excessive function for the restriction of Z to $E - K$. Using c, we h-transform this restriction to obtain a process Z^c on $E - K$. Let T_t be the right-continuous inverse of the continuous additive functional

$$A_t = \int_0^t c^{-1}(Z_s^c)\, ds.$$

Then $Z^c(T_t)$ is identical in law to the restriction of X to $E - K$. This can be expressed analytically by $U(x, dy) = c(x)^{-1}V(x, dy)$ for every $x \in E - K$. Let us call this type of transformation a *link transform by c*; after all, it consists of an h-transform by c linked to a time change using density c^{-1}. Thus, we expect the symmetries of Exc to be reflected in the symmetries common to the collection of link transforms of a particular process.

To define precisely these symmetries, let $\overline{G}_f$ be the collection of bijections $\varphi \in \overline{G}$ such that $\varphi(A)$ and $\varphi^{-1}(A)$ are finely open sets in E whenever A is a finely open set in E. If μ is a measure on $(E, \mathscr{E})$, and if $\varphi \in \overline{G}$, recall that the image $\varphi(\mu)$ of μ under φ is the measure on $(E, \mathscr{E})$ defined by $\varphi(\mu)(B) = \mu(\varphi^{-1}(B))$ for every $B \in \mathscr{E}$. For each $\varphi \in \overline{G}_f$, define $\mathrm{Exc}(\varphi) = \{\varphi(\mu): \mu \in \mathrm{Exc}\}$.

3.3. DEFINITION. $\mathrm{Sym} = \{\varphi \in \overline{G}_f: \mathrm{Exc}(\varphi) = \mathrm{Exc}\}$.

If we endow Sym with the composition operation $(\varphi, \psi) \to \varphi \circ \psi$, then Sym is also a group.

There are two slight asymmetries between definitions (3.1) and (3.3). First, we know of no nontrivial extension for the group Sym to the case where X is recurrent. Second, a function $\varphi \in G$ automatically has the property that $\varphi(A)$ and $\varphi^{-1}(A)$ are finely open sets whenever A is finely open, so $G \subset \overline{G}_f$. But a function $\varphi \in \overline{G}$ such that $\mathrm{Exc}(\varphi) = \mathrm{Exc}$ need not have these fine continuity properties, and we must postulate them since they are needed to develop much of the theory. For example, if $E = R$, let X be the process described by

$$P^x[X_t = x + t] = 1, \qquad \text{if } x \geq 0,$$
$$P^x[X_t = x - t] = 1, \qquad \text{if } x < 0,$$

and let $\varphi(x) = -x$. Then $\mathrm{Exc}(\varphi) = \mathrm{Exc}$. But while the set $[0, s)$ is finely open for X, $(-s, 0] = \varphi^{-1}([0, s))$ is not finely open. An alternative definition of Sym is given in [7] which weakens the fine continuity hypothesis. Define $\overline{G}_d$ to be the set of maps $\varphi \in \overline{G}$ such that: (i) there is a set $L_\varphi \in \mathscr{E}$ such that L_φ and $\varphi(L_\varphi)$ are polar for X, and (ii) $\varphi^{-1}(A)$ is finely open whenever A is finely open and $A \subset E - \varphi(L_\varphi)$. Then Sym_d is defined to be $\{\varphi \in \overline{G}_d\colon \mathrm{Exc}(\varphi) = \mathrm{Exc}\}$. [This definition would permit $\varphi(x) = -x$ in the example above to be included in Sym_d, even though it is not in Sym.] Under some additional hypotheses, we showed in [7] that each $\varphi \in \mathrm{Sym}_d$ can be adjusted and extended on the Ray space E^R of E so that it becomes a finely continuous bijection on E^R. Similar methods could have been applied in this article to deal with this more general situation, but the cost (measured in added technical complications) seemed too great to warrant it at this point.

It may be useful to formulate some specific examples of these groups, and it is natural to start with Brownian motion in R^d.

3.4. Example. Let X_t be Brownian motion in R^d. Then both G and Sym contain translations, rotations, flips about hyperplanes and dilations of R^d. If Y_t is any time change of X_t, then the symmetry group of the excessive functions for Y is still G, while the symmetry group of the excessive measures for Y will not be the original symmetry group Sym of X_t, in general.

3.5. Example. Let X_t be any Lévy process in R^d. Then G and Sym always contain the translations of R^d, but may contain other transformations, as in the Brownian case.

3.6. Example. Let X_t be the process on R characterized by the semigroup $P_t(x, \cdot) = e^{-t}\varepsilon_x(\cdot)$. Then G consists of all maps $\varphi \in \overline{G}$.

Of course, there are many examples in which G and Sym consist only of the identity map. We call such a group *trivial*.

We often wish to deal only with a subgroup H of G or a subgroup Sub of Sym. In this article, we shall always assume that the particular group we are working with is *transitive*, which means the following.

3.7. Hypothesis. For each pair of points x and y in E, there is a map $\varphi \in H$ (resp., $\varphi \in \mathrm{Sub}$) such that $\varphi(x) = y$.

Several arguments which follow are valid for any transitive group J of bijections on E. Since both H and Sub are such groups, the results we discuss for J apply to both H and Sub. Let us fix, once and for all in this article, a point $e \in E$ to serve as a reference point in E, and let $J_e = \{\varphi \in J\colon \varphi(e) = e\}$. This is a subgroup of J, and we let $\mathscr{J} = J/J_e$ be the collection of left cosets $jJ_e = \{j \circ g\colon g \in J_e\}$. (In this general context, $\circ$ is the group operation in J.)

3.8. Lemma. $jJ_e = \{\varphi \in J\colon \varphi(e) = j(e)\}$.

PROOF. If $\varphi \in jJ_e$, then $\varphi = j \circ g$ for some $g \in J_e$, so $\varphi(e) = j \circ g(e) = j(e)$. Conversely, if $\varphi(e) = j(e)$, then $j^{-1} \circ \varphi(e) = e$, so $j^{-1} \circ \varphi \in J_e$, and we see that $\varphi \in jJ_e$. □

3.9. DEFINITION. For each $x \in E$, let $\Psi(x) = \{j \in J: j(e) = x\}$.

By (3.8), Ψ is a map from E to $\mathscr{J}$, and even more is true.

3.10. LEMMA. *Ψ is a bijection from E to $\mathscr{J}$.*

PROOF. Ψ is clearly injective. On the other hand, if jJ_e is a coset with $j(e) = x$, then $\Psi(x) = jJ_e$, so Ψ is also surjective. □

The bijection $\Psi: E \to \mathscr{J}$ allows us to identify E with $\mathscr{J}$, and we thereby endow E with the structure of a coset space. While this general situation clearly merits detailed study (see Section 4), we shall be particularly interested in this article in the case when J_e is trivial. In this case, $\mathscr{J}$ is isomorphic to J and E inherits a group structure. In particular, if $x, y \in E$, we let $xy = \Psi^{-1}(\Psi(x) \circ \Psi(y))$. Note that E also inherits a group structure in case J is a commutative group, for then $\mathscr{J}$ is also a group, although $\mathscr{J}$ is not isomorphic to J in general.

One more example may serve to illustrate these ideas. Consider the restriction X_t of Brownian motion B_t to $E = R^2 - \{0\}$. Then the symmetry group G for this process X contains the subgroup H generated by the rotations about 0 of E and the dilations of E. This subgroup is transitive, and H_e is trivial, so H can be used to endow E with a group structure. This particular example will be discussed in (9.1).

We conclude this section by comparing our framework to the one commonly used in the theory of topological transformation groups and Lie groups ([9], [10]). There are crucial differences, and that well-developed theory does not apply readily here. We have a group J of finely continuous bijections acting transitively on a Lusin topological space. Since J is defined algebraically, it does not come equipped with an obvious topology, and it may be impossible to equip J with a topology which makes it a topological transformation group on E if we insist on using the original topology of E. Before we describe an example illustrating this statement, let us recall the definitions of a topological group and a topological transformation group.

3.11. DEFINITION. A topological group Γ is a topological space in which for each φ and ψ in Γ, there is a unique product $\varphi\psi$ in Γ with the following properties:

(i) There is a unique element $i \in \Gamma$ such that $i\varphi = \varphi i = \varphi$ for every $\varphi \in \Gamma$.

(ii) For each $\varphi \in \Gamma$, there is an inverse $\varphi^{-1} \in \Gamma$ such that $\varphi\varphi^{-1} = \varphi^{-1}\varphi = i$.

(iii) $\varphi(\psi\gamma) = (\varphi\psi)\gamma$ for all $\varphi, \psi, \gamma \in \Gamma$.
(iv) $\varphi \to \varphi^{-1}$ is continuous in Γ.
(v) $(\varphi, \psi) \to \varphi\psi$ is continuous on $\Gamma \times \Gamma$.

3.12. Definition [10]. A topological transformation group Γ on a Hausdorff topological space M is a topological group such that:

(i) Each element $\varphi \in \Gamma$ is a homeomorphism of M onto M.
(ii) $\varphi(\psi(m)) = (\varphi \circ \psi)(m)$ for every pair $\varphi, \psi \in \Gamma$ and for every $m \in M$.
(iii) $(\varphi, m) \to \varphi(m)$ is continuous on $\Gamma \times M$.

3.13. Example. Let $E = [0, 2\pi)$ with the Euclidean topology so that $d(0, 2\pi - \varepsilon) = 2\pi - \varepsilon$ for $\varepsilon < \pi$. Let X be the transient Borel right process described by $P^x[X_t = (x + t) \text{ modulo } 2\pi] = e^{-t}$, $P^x[X_t = \Delta] = 1 - e^{-t}$. For each $a \in E$, let $\varphi_a(x) = (a + x)$ modulo 2π. Then $H = \{\varphi_a\colon a \in E\}$ is a transitive subgroup of G, but these maps are not homeomorphisms, so H cannot be topologized to become a topological transformation group. However, if we change the topology of E (by identifying 0 and 2π so that E becomes identified with the circle of radius 1 in R^2), then H becomes a group of homeomorphisms of E and can be equipped with a topology making it into a topological transformation group.

4. Translation invariant processes. Let X be a transient Borel right process, and let J be a transitive group of finely continuous bijections on E. We have in mind that J is a subgroup of G or Sym, but we do not require J_e to be trivial in this section. As we described in Section 3, Ψ is a bijection from E to $\mathscr{J} = J/J_e$.

4.1. Definition. X is called J-translation invariant if the processes $(\varphi(X_t), P^{\varphi^{-1}(x)})$ and (X_t, P^x) are identical in law for every $x \in E$ and $\varphi \in J$.

We next give two simple criteria guaranteeing that a time change of X_t is G-translation invariant. But first, we investigate the increments of a J-translation invariant process X_t. If J_e is trivial, then E has a group structure, and it makes sense to talk about increments of the process. If E inherits only a coset structure, then the term increments must be interpreted. We do this with the aid of measurable selectors. Let $[x]$ be the coset in $\mathscr{J}$ consisting of all maps $\varphi \in J$ with $\varphi(e) = x$ [see (3.8)].

4.2. Definition. A map $\Lambda\colon E \to J$ (and written $x \to \Lambda_x$) is called a measurable selector if:

(i) $\Lambda_x^{-1} \in [x]$ for every $x \in E$.
(ii) $(x, y) \to \Lambda_x(y)$ is $\mathscr{E} \times \mathscr{E}$-measurable.
(iii) $(x, y) \to \Lambda_x^{-1}(y)$ is $\mathscr{E} \times \mathscr{E}$-measurable.

4.3. DEFINITION. If Λ is a measurable selector, the Λ-increment of X over the interval $[s, t]$ is the random variable $\Lambda_{X(s)}(X_{t+s})$.

We believe that this terminology is justified by the special case when J_e is trivial. In that case, E inherits the group structure of J. If we set $\Lambda_x(y) = x^{-1}y$, and if we assume that (4.2ii) and (4.2iii) are satisfied, then Λ is a measurable selector, and $\Lambda_{X(s)}(X_{t+s}) = X_s^{-1}X_{t+s}$, which is the usual increment of a process in a group. [Note that if J is not commutative, then there is another measurable selector $\Lambda_x(y) = yx^{-1}$.]

Let X be a J-translation invariant process and let Λ be a measurable selector. Let f be a bounded positive continuous function on E, and let F be a positive $\mathscr{F}_s$-measurable random variable. Then

$$\begin{aligned} &P^x\big[f\big(\Lambda_{X(s)}(X_{t+s})\big)1_{\{t<\zeta\circ\theta_s\}}F1_{\{s<\zeta\}}\big] \\ &= P^x\big[P^{X(s)}\big[f\big(\Lambda_{X(s)}(X_t(\cdot))\big);t<\zeta\big]F;s<\zeta\big] \\ &= P^x\big[P^{\Lambda_{X(s)}(X_s)}\big[f(X_t);t<\zeta\big]F;s<\zeta\big] \\ &= P^e\big[f(X_t);t<\zeta\big]P^x[F;s<\zeta]. \end{aligned} \tag{4.4}$$

As a special case, assume $\zeta = \infty$ a.s. The previous computation shows that $\Lambda_{X(s)}(X_{t+s})$ is independent of $\mathscr{F}_s$ under P^x and has the same distribution as X_t under P^e. That is, X_t has stationary independent Λ-increments. (In the general case, we obtain $P^e[t+s<\zeta] = P^e[t<\zeta]P^e[s<\zeta]$ by taking $f = F = 1$. This together with the J-translation invariance implies $P_t1 = e^{-\alpha t}$ for some $\alpha \geq 0$. If we set $Q_t = e^{\alpha t}P_t$, then the process associated with the semigroup Q_t is J-translation invariant and has stationary independent Λ-increments.) Because of the remarks in this paragraph, it would be reasonable to call J-translation invariant processes by the name J-Lévy processes.

In Proposition (4.5), we give a simple and common condition guaranteeing that a time change of a transient process is G-translation invariant. Much stronger results are obtained in the succeeding sections when we assume H_e is trivial.

4.5. PROPOSITION. *Let X be a transient Borel right process and assume G is transitive. Assume that 1_E is the potential of a strictly increasing continuous additive functional A_t of X. If $\tau(t)$ is the right-continuous inverse of A_t, then $X_{\tau(t)}$ is G-translation invariant.*

PROOF. Let $Z_t = X_{\tau(t)}$. The right process $(\varphi(Z_t), P^{\varphi^{-1}(x)})$ has excessive functions $\mathscr{S}_{\varphi^{-1}}$. If $\varphi \in G$, then $\mathscr{S}_{\varphi^{-1}} = \mathscr{S}$, and the Blumenthal–Getoor–McKean theorem states there is a strictly increasing continuous additive functional B_t of Z with inverse $\rho(t)$ such that $(\varphi(Z_t), P^{\varphi^{-1}(x)})$ and $(Z_{\rho(t)}, P^x)$ are identical in law. Let V denote the potential of Z. If f is a bounded continuous function on E, then

$$(4.6)\qquad P^{\varphi^{-1}(x)}\int f\circ\varphi(Z_t)\,dt = P^x\int f(Z_{\rho(t)})\,dt,$$

and we obtain $V(f\circ\varphi)(\varphi^{-1}(x)) = V_B f(x)$. Setting $f = 1_E$, we see that $V1_E(x) = U_A 1_E(x) = 1_E(x) = V1_E(\varphi^{-1}(x)) = V_B 1_E(x)$. By uniqueness of potentials, we have $B_t = \min(t,\zeta)$ and $Z_t = Z_{\rho(t)}$. □

For 1_E to be the potential of a strictly increasing continuous additive functional in (4.5), it is necessary that $\zeta < \infty$ a.s. Let us look at another common condition which will work without any finiteness assumption about ζ.

4.7. PROPOSITION. *Let X be a transient Hunt process on E with a transitive symmetry group G. Assume that for some $x \in E$ and $\alpha > 0$,*

$$(4.7')\qquad 0 < P^x \sum_{0<s<\zeta} e^{-\alpha s} 1_{\{X_{s-}\neq X_s\}} < \infty.$$

Then there is a strictly increasing continuous additive functional B_t of X with right-continuous inverse $\sigma(t)$ such that $X_{\sigma(t)}$ is G-translation invariant.

PROOF. Consider the homogeneous random measure

$$\kappa(dt) = \sum_{0<s<\zeta} \varepsilon_s(dt) 1_{\{X_{s-}\neq X_s\}},$$

where ε_s is point mass at s. By hypothesis,

$$0 < P^x \int e^{-\alpha s}\,\kappa(ds) < \infty.$$

Fix $z \in E$ and let $\varphi \in G$ be chosen such that $\varphi^{-1}(z) = x$. Then

$$(4.8)\qquad P^x\int e^{-\alpha s}\,\kappa(ds) = P^{\varphi^{-1}(z)} \sum_{0<s<\zeta} e^{-\alpha s} 1_{\{\varphi(X_{s-})\neq\varphi(X_s)\}},$$

since φ is a bijection. Since $\varphi \in G$, $\varphi(X_s)$ is equivalent in law to a standard process, namely a time change of X_t. Thus $\varphi(X_s)_-$ exists on $(0,\zeta)$. We shall check that the natural processes $\varphi(X_s)_- 1_{(0,\zeta)}(s)$ and $\varphi(X_{s-})1_{(0,\zeta)}(s)$ are indistinguishable. (See Part I of [6] for the natural σ-algebra, the natural section theorem and their relationship with standard processes.) By the natural section theorem ((2.6) in [6]), it suffices to show that $\varphi(X_{T-}) = \varphi(X_T)_-$ a.s. on $\{0 < T < \zeta\}$ for every natural stopping time T. Such a stopping time is characterized (for standard processes) by the fact that $X_{T-} = X_T$ a.s. on $\{0 < T < \zeta\}$ ((5.4) in [6]). Thus $\varphi(X_{T-}) = \varphi(X_T)$ a.s. on $\{0 < T < \zeta\}$. Let (T_n) be an increasing sequence of stopping times with (T_n) strictly increasing on $\{0 < T < \zeta\}$, $\lim_{n\to\infty} T_n = T$ on $\{0 < T < \zeta\}$ and $\lim_{n\to\infty} T_n \geq \zeta$ on $\{T \geq \zeta\}$. Then

$$\varphi(X_T)_- = \lim_{n\to\infty} \varphi(X_{T_n}) \quad \text{a.s. on } \{0 < T < \zeta\}.$$

Since $\varphi(X_t)$ is a standard process with lifetime ζ,

$$\lim_{n\to\infty} \varphi(X_{T_n}) = \varphi(X_T) \quad \text{a.s. on } \{0 < T < \zeta\},$$

and we see that $\varphi(X_{T-}) = \varphi(X_T)_-$ a.s. on $\{0 < T < \zeta\}$.

Thus we may rewrite (4.8) as

$$\begin{aligned} P^{\varphi^{-1}(z)} &\sum_{0<s<\zeta} e^{-\alpha s} 1_{\{\varphi(X_s)_- \neq \varphi(X_s)\}} \\ &= P^z \sum_{0<s<A^\varphi(\zeta)} e^{-\alpha s} 1_{\{X_{\tau(\varphi,s-)} \neq X_{\tau(\varphi,s)}\}} \\ &= P^z \sum_{0<s<\zeta} e^{-\alpha A^\varphi(s)} 1_{\{X_{s-} \neq X_s\}}. \end{aligned}$$

We have shown that for each $z \in E$, there is a strictly positive predictable process Z_t^z such that

$$0 < P^z \int Z_t^z \kappa(dt) < \infty. \tag{4.9}$$

Therefore, the random measure κ is $(\mathscr{P}, P^z)$-locally integrable (in the terminology of Sharpe), so κ has a dual predictable projection $\gamma = \kappa^p$ which is a homogeneous random measure (see [12], (31.5), (31.16)). If we set $B_t = \gamma[0,t]$, then B_t is a continuous additive functional which is finite on $[0,\zeta)$ by virtue of (4.9). Now let $f \in p\mathscr{E}$, and compute

$$\begin{aligned} U_B(f\circ\varphi)(\varphi^{-1}(x)) &= P^{\varphi^{-1}(x)} \int f\circ\varphi(X_s)\, dB_s \\ &= P^{\varphi^{-1}(x)} \sum_{0<s<\zeta} f\circ\varphi(X_{s-}) 1_{\{X_{s-}\neq X_s\}} \\ &= P^{\varphi^{-1}(x)} \sum_{0<s<\zeta} f(\varphi(X_s)_-) 1_{\{\varphi(X_s)_- \neq \varphi(X_s)\}} \\ &= P^x \sum_{0<s<\zeta} f(X_{s-}) 1_{\{X_{s-}\neq X_s\}} \\ &= P^x \int f(X_t)\, dB_t = U_B f(x). \end{aligned}$$

If we let $\sigma(t)$ be the right-continuous inverse of B_t, then the transient processes $(X_{\sigma(t)}, P^x)$ and $(\varphi(X_{\sigma(t)}), P^{\varphi^{-1}(x)})$ have the same potential and so are identical in law. Therefore, $X_{\sigma(t)}$ is G-translation invariant. Finally, we observe that B_t is strictly increasing since its fine support is all of E. For if K is a finely open set in E containing at least one point z, then $\varphi^{-1}(K)$ is finely open and contains $\varphi^{-1}(z)$ for every $\varphi \in G$. If it were the case that $U_B 1_K(z) = 0$, then we would have $U_B 1_{\varphi^{-1}(K)}(\varphi^{-1}(z)) = 0$ for every $\varphi \in G$, and the fine support of B_t would be the null set. This would contradict (4.9). □

5. The Mackey–Weil theorem. Throughout this section, we assume X is a transient Borel right process and J is transitive with J_e trivial. In this case, $\mathscr{J}$ and J are isomorphic and Ψ is a bijection from E to J (see Section 3). We use Ψ to identify E and J, and in particular, Ψ endows E with the group structure of J given by the product $xy = \Psi^{-1}(\Psi(x) \circ \Psi(y))$ whenever $x, y \in E$. This group product notation is useful, but we also find it convenient to use the product in J (which is composition $\circ$) by identifying the point $x \in E$ with the map $\varphi_x = \Psi(x) \in J$. This allows us to refer to image measures of the form $\varphi_x(\mu)$.

5.1. HYPOTHESIS. Throughout this section, we assume $(x, y) \to xy$ and $(x, y) \to x^{-1}y$ are $\mathscr{E} \times \mathscr{E}$-measurable.

Now we focus our attention on the existence and consequences of a left Haar measure on $(E, \mathscr{E})$.

5.2. DEFINITION. If μ is a measure on $(E, \mathscr{E})$ and $x \in E$, μ^x is the measure on $(E, \mathscr{E})$ defined by $\mu^x(A) = \mu(xA)$ for every $A \in \mathscr{E}$. A σ-finite measure μ on $(E, \mathscr{E})$ is said to be left quasi-invariant if $\mu^x \ll \mu$ for every $x \in E$. A σ-finite measure m on $(E, \mathscr{E})$ is said to be a left Haar measure if $m^x = m$ for every $x \in E$.

The definitions of right quasi-invariant and right Haar measures are analogous, but we do not need them in this article. A σ-finite measure ν on $(E, \mathscr{E})$ can be transferred to a measure $\Psi(\nu)$ on the group J, and it will be useful to translate the notions in (5.2) into the notation of J. It is easy to check that

$$\mu^z(f) = \int f(z^{-1}x)\mu(dx) = \int f\big(\Psi^{-1}(\varphi_z^{-1} \circ \varphi_x)\big)\mu(dx).$$

Since $\varphi_z^{-1} \circ \varphi_x(e) = \varphi_z^{-1}(x)$, $\varphi_z^{-1} \circ \varphi_x = \Psi(\varphi_z^{-1}(x))$. Therefore,

$$\mu^z(f) = \int f(\varphi_z^{-1}(x))\mu(dx) = \varphi_z^{-1}(\mu)(f).$$

Thus, μ is left quasi-invariant if and only if μ is σ-finite and $\varphi(\mu) \ll \mu$ for every $\varphi \in J$, and m is a left Haar measure if and only if m is σ-finite and $\varphi(m) = m$ for every $\varphi \in J$.

The existence of a left Haar measure on $(E, \mathscr{E})$ has profound implications for the structure of E and X. Weil's converse to Haar's theorem states that if there is a left Haar measure, then the underlying group E is locally compact in the correct topology [13]. The Mackey–Weil theorem is a strengthening of this already remarkable result, and we need its full power later. See [11] for a proof of this result.

5.3. THEOREM. *Let $(E, \mathscr{E})$ be a Lusin topological space with a group structure satisfying* (5.1). *Assume there is a σ-finite left quasi-invariant measure μ on $(E, \mathscr{E})$. Then there exists a topology on E making E into a* *locally compact second countable metric group such that*:

(i) *The Borel σ-algebra of the topology is $\mathscr{E}$.*
(ii) *μ and the left Haar measure have the same null sets.*

It is worth considering two elementary examples here. For the first, recall (3.13) in which $E = [0, 2\pi)$ with the Euclidean topology. It has a natural group structure given by $xy = (x + y)$ modulo 2π. The Mackey–Weil topology identifies 0 and 2π so that E becomes the standard compact rotation group on the circle. For the second example, consider the rational numbers $Q \subset R$ with its natural additive structure. For a left quasi-invariant measure, we can take $m = \sum \varepsilon_s$, where the sum runs over all $s \in Q$. In fact, m is a σ-finite left Haar measure and the Mackey–Weil topology is the discrete topology on Q.

With this theorem in hand, our task later is to find conditions which imply the existence of left quasi-invariant measures on $(E, \mathscr{E})$. For the moment, we shall use this result to study E and X. There are two cases to consider, the first being when $J = H$ is a subgroup of G and the second being when $J = \text{Sub}$ is a subgroup of Sym. While these two cases are similar in spirit (being duals of each other), the detailed proofs are different, so we separate them. The first case is discussed in Section 6, and the second is discussed in Section 7.

6. The case $J = H$. Throughout this section, X is a transient Borel right process. [Eventually, we strengthen this hypothesis slightly so that the final result (6.15) is stated only for a Hunt process X.] We also assume that H is a transitive subgroup of G with H_e trivial. As discussed before, the group structure of H can be transferred to E with the aid of the bijection Ψ. This section is devoted to exploring the implications of the following assumptions.

6.1. HYPOTHESIS. The group structure satisfies (5.1) and there is a σ-finite left quasi-invariant measure on $(E, \mathscr{E})$.

By (5.3), there is a σ-finite left Haar measure m on $(E, \mathscr{E})$. We refer to the topology yielded by (5.3) as the MW topology, and we are going to abandon the original topology on E. Therefore, throughout this section, when we refer to a topological property of a set (for example, "A is open and relatively compact"), the statement is to be interpreted in the MW topology. If we wish to refer to the original topology on E, we shall always add the phrase "in the original topology of E".

Choose a function $q \in \mathscr{E}$ with $0 < q \leq 1$ such that $Uq \leq 1$. Define

$$B_t^1 = \int_0^t q(X_s)\, ds$$

and let $\sigma(t)$ be the right-continuous inverse of B_t^1. If we set $Y_t = X_{\sigma(t)}$, then $Y = (\Omega, \mathscr{F}, \mathscr{F}_{\sigma(t)}, Y_t, \theta_{\sigma(t)}, P^x)$ is a transient Borel right process with potential V satisfying $V1_E \leq 1$. Let K be a fixed nonempty set in E which is open and relatively compact and which satisfies $V1_K(e) > 0$. If $\varphi \in H$, the processes $(\varphi(Y_t), P^{\varphi^{-1}(x)})$ and (Y_t, P^x) have the same excessive functions, so there is a strictly increasing continuous additive functional A_t^φ of Y_t with right-continuous inverse $\tau(\varphi, t)$ such that $(\varphi(Y_t), P^{\varphi^{-1}(x)})$ and $(Y_{\tau(\varphi,t)}, P^x)$ are identical in law. Thus, for $g \in p\mathscr{E}$, we have

$$P^{\varphi^{-1}(x)}\int g \circ \varphi(Y_t) 1_K(Y_t)\, dt = P^x \int g\big(Y_{\tau(\varphi,t)}\big) 1_K \circ \varphi^{-1}\big(Y_{\tau(\varphi,t)}\big)\, dt$$

$$= P^x \int g(Y_t) 1_K \circ \varphi^{-1}(Y_t)\, dA_t^\varphi.$$

In order to proceed, we need to know that A_t^φ can be made jointly measurable.

6.2. LEMMA. *There is a process B_t^φ such that*:

(i) *For each φ, B_t^φ and A_t^φ are indistinguishable.*
(ii) $(t, x, \omega) \to B_t^{\Psi(x)}(\omega)$ *is $\mathscr{B}(R^+) \times \mathscr{E} \times \mathscr{F}^0$-measurable.*

PROOF. For each pair $(x, \varphi) \in E \times H$, define a measure $L((x, \varphi), d\omega)$ on $(\Omega, \mathscr{F}^0)$ by setting $L((x, \varphi), F) = P^x[A_\infty^\varphi \cdot F]$ for every $F \in p\mathscr{F}^0$. Assume for the moment that we have shown that $(x, z) \to L((x, \varphi_z), F)$ is $\mathscr{E} \times \mathscr{E}$-measurable. Doob's lemma [12] then yields a density $C(x, z, \omega) \in \mathscr{E} \times \mathscr{E} \times \mathscr{F}^0$ such that $L((x, \varphi_z), F) = P^x[C(x, z, \cdot\,)F(\cdot)]$ for every $F \in p\mathscr{F}^0$. If we set $C_\infty^z(\omega) = C(X_0(\omega), z, \omega)$, then C_∞^z is $\mathscr{E} \times \mathscr{F}^0$-measurable and $C_\infty^z = A_\infty^{\Psi(z)}$ a.s. Furthermore, if we define

$$C_t^z = C_\infty^z - C_\infty^z \circ \theta_t,$$

then $C_t^z = A_t^{\Psi(z)}$ a.s., and $(z, \omega) \to C_t^z(\omega)$ is $\mathscr{E} \times \mathscr{F}^0$-measurable. Define

$$B_t^{\Psi(z)}(\omega) = \liminf_{s \downarrow t\ s \in Q} C_s^z(\omega).$$

Then $t \to B_t^{\Psi(z)}$ is continuous a.s., $B_t^{\Psi(z)}$ and $A_t^{\Psi(z)}$ are indistinguishable and $(t, x, \omega) \to B_t^{\Psi(x)}(\omega)$ is $\mathscr{B}(R^+) \times \mathscr{E} \times \mathscr{F}^0$-measurable.

So all that remains to complete the proof of this lemma is to verify that $(x, z) \to P^x[A_\infty^{\Psi(z)} \cdot F]$ is $\mathscr{E} \times \mathscr{E}$-measurable whenever $F \in p\mathscr{F}^0$. Let $\mathscr{P}$ denote the σ-algebra on $R^+ \times \Omega$ generated by the left-continuous processes adapted to $(\mathscr{F}_t^0)$. There is a process $Z_t \in \mathscr{P}$ which is indistinguishable from the predictable projection of F ([12], page 209).

Since

$$P^x\big[A_\infty^{\Psi(z)} \cdot F\big] = P^x \int Z_t\, dA_t^{\Psi(z)},$$

it is enough to show that

$$(x, z) \to P^x \int Z_t\, dA_t^{\Psi(z)}$$

is $\mathscr{E} \times \mathscr{E}$-measurable whenever Z_t is a bounded positive left-continuous $(\mathscr{F}_t^0)$-adapted process. Since these processes can be approximated by indicators of stochastic intervals, it suffices to show that

$$(6.3)\qquad (x,z) \to P^x \int 1_{(S,T]}(t)\, dA_t^{\Psi(z)} = P^x\left[A_T^{\Psi(z)} - A_S^{\Psi(z)}\right]$$

is measurable whenever $S \leq T$ are $(\mathscr{F}_t^0)$-optional times. Let $\zeta^Y = \inf\{t: Y_t = \Delta\}$ and let $l(x) = P^x[\zeta^Y] = V1_E(x)$. Note that $l \leq 1$ and l is $\mathscr{E}$-measurable. We can compute the potential of A_t^φ in terms of l:

$$P^x[A_\infty^\varphi] = P^x \int 1_E(Y_{\tau(\varphi,t)})\, dt = P^{\varphi^{-1}(x)} \int 1_E(\varphi(Y_t))\, dt$$

$$= P^{\varphi^{-1}(x)}[\zeta^Y] = l(\varphi^{-1}(x)).$$

Therefore, the expression on the right side of (6.3) is $P^x[l \circ \varphi_z^{-1}(X_S) - l \circ \varphi_z^{-1}(X_T)]$. Since l is $\mathscr{E}$-measurable and $(z,x) \to \varphi_z^{-1}(x)$ is continuous on $E \times E$, $l \circ \varphi_z^{-1}(x)$ is $\mathscr{E} \times \mathscr{E}$-measurable. Thus $(z,x) \to P^x[l \circ \varphi_z^{-1}(X_S) - l \circ \varphi_z^{-1}(X_T)]$ is $\mathscr{E} \times \mathscr{E}$-measurable. □

We need the previous measurability result to define a diffuse homogeneous random measure κ_K. For each $z \in E$, set

$$\kappa_K^z(ds) = 1_K \circ \varphi_z^{-1}(Y_s)\, dB_s^{\Psi(z)}$$

and define

$$\kappa_K(ds) = \int \kappa_K^z(ds)\, m(dz).$$

Each κ_K^z is a diffuse homogeneous random measure since the distribution function of κ_K^z is a continuous additive functional. It follows that κ_K is a diffuse homogeneous random measure. A major task in this section is to show that its distribution function exists and is finite on $[0, \zeta^Y)$ a.s.

Let D be a left invariant metric on E [10] and let $K_n = \{x \in E: D(x,e) < n\}$ be the open relatively compact ball of radius n around e. Define the diffuse homogeneous random measures

$$\kappa_n^z(ds) = \kappa_{K(n)}^z(ds),$$

$$\kappa_n(ds) = \int \kappa_n^z(ds)\, m(dz).$$

6.4. LEMMA. *There are* (i) *a sequence* (a_n) *of strictly positive constants and* (ii) *a function* $h > 0$ *on* E *which is continuous* (*in the MW topology*) *such that if*

$$\kappa(ds) = \sum_{n=1}^{\infty} a_n \kappa_n(ds),$$

then $P^x \int h(Y_s) \kappa(ds) \leq 1$ *for every* $x \in E$.

PROOF. For L, any relatively compact open set in E, define the random variables

$$(1_L * \kappa_n^z)_\infty = \int_0^\infty 1_L(Y_s)\kappa_n^z(ds),$$

$$(1_L * \kappa_n)_\infty = \int_0^\infty 1_L(Y_s)\kappa_n(ds).$$

Note that $(1_L * \kappa_n^z)_\infty$ can be nonzero only if $\varphi_z(K_n) \cap L \neq \varnothing$. Since $S_n^L = \{z\colon \varphi_z(K_n) \cap L \neq \varnothing\}$ is relatively compact, we have $m(S_n^L) < \infty$ and

$$\begin{aligned} P^x[(1_L * \kappa_n)_\infty] &= \int_{S_n^L} P^x[(1_L * \kappa_n^z)_\infty]\, m(dz) \\ &\le \int_{S_n^L} P^x[\kappa_n^z[0,\infty)]\, m(dz) \\ &\le \int_{S_n^L} l(\varphi_z^{-1}(x))\, m(dz) \le m(S_n^L) < \infty. \end{aligned}$$

(Recall $l \le 1$.) Now define $S_n^p = \{z\colon \varphi_z(K_n) \cap K_p \neq \varnothing\}$. Let us check that $S_n^p \subset K_{2n}$ for $2n > p$. Suppose $z \in K_{2n}^c$ and $x \in K_n$. Then $D(zx, e) = D(x, z^{-1})$. But z^{-1} is also in K_{2n}^c, so $D(x, z^{-1}) > n$. Therefore, $zx \notin K_p$. Thus $m(S_n^p) \le m(K_{2n})$ for $2n > p$. Choose a sequence (a_n) of strictly positive constants such that $\sum a_n m(K_{2n}) < \infty$. Set $\kappa = \sum a_n \kappa_n$. Then for every $x \in E$,

$$\begin{aligned} P^x\left[\left(1_{K(p)} * \kappa\right)_\infty\right] &= \sum_n a_n \int P^x\left[\left(1_{K(p)} * \kappa_n\right)_\infty\right] \\ &\le \sum_n a_n m(S_n^p) < \infty. \end{aligned}$$

Let $b_k = \sum a_n m(S_n^k)$. Let h_k be any continuous function on E such that (i) $0 \le h_k \le 1$, (ii) $h_k = 1$ on K_{k-1} and (iii) $h_k = 0$ on K_k^c. If we set

$$h = \sum_k \frac{1}{b_k 2^k} h_k,$$

then h is the desired function. □

It follows from (6.4) that

$$C_t = \int_0^t h(Y_s)\kappa(ds)$$

is a continuous additive functional which is finite on $[0, \zeta^Y)$, and in fact, $P^x[C_\infty] \le 1$ for every $x \in E$.

6.5. DEFINITION. Let $V_\kappa(x, dy)$ be the kernel defined by

$$V_\kappa f(x) = P^x \int f(Y_s)\kappa(ds),$$

whenever $f \in p\mathscr{E}$.

Our next result shows that V_κ is H-translation invariant.

6.6 LEMMA. *For every $f \in p\mathscr{E}$ and $\psi \in H$, $V_\kappa(f \circ \psi)(\psi^{-1}(x)) = V_\kappa f(x)$.*

PROOF. If $f \in p\mathscr{E}$ and $\psi \in H$, then

$$\begin{aligned}
&V_\kappa(f \circ \psi)(\psi^{-1}(x)) \\
&\quad = P^{\psi^{-1}(x)} \int f \circ \psi(Y_s) \kappa(ds) \\
&\quad = \sum a_n \int P^{\psi^{-1}(x)} \int f \circ \psi(Y_s) 1_{K(n)} \circ \varphi_z^{-1}(Y_s)\, dB_s^{\Psi(z)}\, m(dz) \\
&\quad = \sum a_n \int P^{\varphi_z^{-1} \circ \psi^{-1}(x)} \int f \circ \psi \circ \varphi_z(Y_s) 1_{K(n)}(Y_s)\, ds\, m(dz) \\
&\quad = \sum a_n \int P^{\varphi_z^{-1}(x)} \int f \circ \varphi_z(Y_s) 1_{K(n)}(Y_s)\, ds\, m(dz) \\
&\quad = V_\kappa f(x),
\end{aligned}$$

since $\psi \circ \varphi_z = \varphi_{\psi(z)}$ and $\psi(m) = m$. □

6.7. COROLLARY. (i) *If f is continuous with compact support on E, then $V_\kappa f(x)$ is continuous on E.*
(ii) $V_\kappa(f \circ \varphi_x)(e) = V_\kappa f(x)$.

PROOF. (i) follows immediately from (ii) and the fact that $|f| \le rh$ for some $r > 0$. Formula (ii) follows from (6.6) by taking $\psi = \varphi_x$. □

6.8. LEMMA. *The fine support of C_t is E.*

PROOF. Let L be a relatively compact finely open set for Y_t. Since $h > 0$, it is enough to show that $V_\kappa 1_L(x) > 0$ for every $x \in L$. But

$$\begin{aligned}
V_\kappa 1_L(x) &= \sum a_n \int P^x \int 1_L(Y_s) 1_{K(n)} \circ \varphi_z^{-1}(Y_s)\, dB_s^{\Psi(z)}\, m(dz) \\
&= \sum a_n \int P^{\varphi_z^{-1}(x)} \int 1_{\varphi_z^{-1}(L)}(Y_s) 1_{K(n)}(Y_s)\, ds\, m(dz).
\end{aligned}$$

For each $\varphi \in H$, $\varphi^{-1}(L)$ is relatively compact and finely open and is contained in $K(N_\varphi)$ for some N_φ. Since $\varphi^{-1}(x) \in \varphi^{-1}(L)$,

$$P^{\varphi^{-1}(x)} \int 1_{\varphi^{-1}(L)}(Y_s) 1_{K(N_\varphi)}(Y_s)\, ds > 0,$$

and it follows that $V_\kappa 1_L(x) > 0$. □

Thus, C_t is a strictly increasing continuous additive functional which is finite on $[0, \zeta^Y)$ a.s. We would like to show that

$$\kappa[0, t] = \int_0^t h^{-1}(Y_s)\, dC_s$$

is finite on $[0, \zeta^Y)$ a.s. We do not know much yet about the behavior of Y_t in the MW topology. Can Y_t rush out to infinity before ζ^Y, in which case $h^{-1}(Y_t)$ blows up? We show later that this cannot happen if X_t is a Hunt process in the original topology of E. In that case, it will turn out that $t \to h^{-1}(Y_t)$ is bounded on $[0, s]$ for every $s < \zeta^Y$. Our approach involves examining the behavior of Y_t in the MW topology by using techniques akin to those in Ray theory ([3], [12]).

Let $\mathscr{C}_K$ denote the collection of functions $f: E \to R$ which are continuous with compact support (in the MW topology). Since C_t is a strictly increasing continuous additive functional with $V_C 1 \le 1$, there is a countable set $\mathscr{D} \subset \mathscr{C}_K$ such that $\{V_C f: f \in \mathscr{D}\}$ separates points in E. If we let $\mathscr{D}_K = \{fh: f \in \mathscr{D}\}$, then we have that $\{V_\kappa g: g \in \mathscr{D}_K\}$ separates points in E since $h * \kappa = C$. Enumerate the functions (g_j) in $\mathscr{D}_K$ and let $M_j = \sup\{|V_\kappa g_j|(x): x \in E\}$. Let I be the compact space $\prod[-M_j, M_j]$. If $p = (p_j)$ and $n = (n_j)$ are two points in I, we define the distance between p and n to be

$$D(p,n) = \sum 2^{-j} \frac{|p_j - n_j|}{1 + |p_j - n_j|}.$$

D is a metric on I which is compatible with its product topology. Define an injection $\Xi: E \to I$ by $\Xi(x) = (V_\kappa g_j(x))_{j=1}^\infty$. Let ρ be a metric on E defined by $\rho(x, y) = D(\Xi(x), \Xi(y))$. Then Ξ is an isometry of (E, ρ) onto $(\Xi(E), D)$, so Ξ extends to be an isometry $\overline{\Xi}$ from the completion $\overline{E}$ of E onto $\overline{\Xi(E)}$, the closure of $\Xi(E)$ in I.

6.9. Lemma. *h extends to be a continuous function $\overline{h}$ on $\overline{E}$ with $\overline{h}(\overline{E} - E) = 0$.*

Proof. Recall from the proof of (6.4) that

$$h = \sum_{k=1}^\infty \frac{1}{b_k 2^k} h_k,$$

where h_k is supported by $\overline{K}_k$, the closure of K_k in the MW topology. Let us show that $\overline{K}_k$ is closed in $(\overline{E}, \rho)$, also. Let $(x_n) \subset \overline{K}_k$ be a Cauchy sequence in $(\overline{E}, \rho)$. Since $\overline{K}_k$ is compact, there is a subsequence $(x_{n(p)})$ such that $(x_{n(p)})$ converges to a point $x \in \overline{K}_k$ in the MW topology. Therefore, since each function $V_\kappa g_j$ is continuous, $V_\kappa g_j(x_{n(p)})$ converges to $V_\kappa g_j(x)$ for each j and we conclude that $(x_{n(p)})$ converges to x in $(\overline{E}, \rho)$. It follows that (x_n) converges to $x \in \overline{K}_k$, so $\overline{K}_k$ is closed in $(\overline{E}, \rho)$. Let $\mathscr{C}(\overline{K}_k)$ be the restrictions of the functions $V_\kappa g_j$ to $\overline{K}_k$. Since $\mathscr{C}(\overline{K}_k)$ separates points and does not vanish at any point in $\overline{K}_k$, the uniform closure of the smallest algebra containing $\mathscr{C}(\overline{K}_k)$ consists of all continuous functions (in the MW topology) on $\overline{K}_k$. Thus the two topologies agree on every compact set. We see that h_k extends to a continuous function $\overline{h}_k$ by setting $\overline{h}_k = 0$ on $\overline{E} - E$. If we set

$$\overline{h} = \sum_{k=1}^\infty \frac{1}{b_k 2^k} \overline{h}_k,$$

then the series converges uniformly, so $\bar{h}$ is continuous on $(\bar{E}, \rho)$ and $\bar{h}(\bar{E} - E) = 0$. □

We can transfer the process Y_t to $\Xi(E)$ by setting $Z_t = \Xi(Y_t)$. Then Z_t is right-continuous on $\Xi(E)$ with left limits in $\overline{\Xi(E)}$ since $V_\kappa g_j(Y_t)$ is right-continuous with left limits a.s. for each j. Now we must assume X is a Hunt process for our next result.

6.10. Lemma. *Assume X_t is a Hunt process in the original topology on E. If $T = \inf\{t: Z_{t-} \notin \Xi(E)\}$, then $T \geq \zeta^Y$ a.s.*

In the statement above, Z_{t-} refers to the left limit of Z_t in the topology of $(\overline{\Xi(E)}, D)$.

Proof. We must show that $V_\kappa g_j(Y_t)_-$ and $V_\kappa g_j(Y_{t-})$ are indistinguishable as processes on $[0, \zeta^Y)$. Note that $Y_{\zeta-}$ may not exist since Y_t is a time change of the Hunt process X_t. It suffices to show that $V_\kappa g_j(X_t)_-$ and $V_\kappa g_j(X_{t-})$ are indistinguishable on $[0, \infty)$. Let S be a finite $(\mathscr{F}_t)$-predictable time. Then

$$P^x[V_\kappa g_j(X_S)_-] = P^x[V_C(g_j/h)(X_S)_-]. \tag{6.11}$$

Let (S_n) be a sequence of finite $(\mathscr{F}_t)$-optional times announcing S. Then we can rewrite (6.11) as

$$\begin{aligned}\lim_{n\to\infty} P^x[V_C(g_j/h)(X_{S(n)})] &= \lim_{n\to\infty} P^x\big[V_C(g_j/h)\big(Y\big(B^1_{S(n)}\big)\big)\big]\\ &= \lim_{n\to\infty} P^x\int_{B^1_{S(n)}}^\infty (g_j/h)(Y_s)\,dC_s\\ &= P^x\int_{B^1_S}^\infty (g_j/h)(Y_s)\,dC_s\\ &= P^x[V_C(g_j/h)(X_S)].\end{aligned}$$

Since S is predictable and X is a Hunt process, $X_S = X_{S-}$ a.s. on $\{S < \infty\}$, and we obtain

$$P^x[V_\kappa g_j(X_S)_-] = P^x[V_\kappa g_j(X_{S-})]. \tag{6.12}$$

Since the processes $V_\kappa g_j(X_t)_-$ and $V_\kappa g_j(X_{t-})$ are both $(\mathscr{F}_t)$-predictable, (6.12) and the section theorem show that they are indistinguishable. Thus if R is any $(\mathscr{F}_{\sigma(t)})$-optional time,

$$Z_{R-} = \big(V_\kappa g_j(Y_R)_-\big)_{j=1}^\infty = \big(V_\kappa g_j(Y_{R-})\big)_{j=1}^\infty \quad \text{a.s. on } \{0 < R < \zeta^Y\}.$$

Since $Y_{R-} = X_{\sigma(R)-} \in E$ a.s. on $\{0 < R < \zeta^Y\}$, $Z_{R-} \in \Xi(E)$ a.s. on $\{0 < R < \zeta^Y\}$, and we see that the $(\mathscr{F}_{\sigma(t)})$-optional set $\{0 < t < \zeta^Y: Z_{t-} \notin \Xi(E)\}$ is evanescent, so $T \geq \zeta^Y$ a.s. □

6.13. Corollary. *Assume that X_t is a Hunt process in the original topology of E. Then Y_t is right-continuous with left limits a.s. on $[0, \zeta^Y)$ in the MW topology.*

PROOF. Recall that (K_k) is a sequence of open sets increasing to E such that $\overline{K}_k$ is compact in the MW topology and in $(\overline{E}, \rho)$. We observed in the proof of (6.9) that these two topologies agree on $\overline{K}_k$ for every k. But Y_t is right-continuous with left limits in $(\overline{E}, \rho)$, and if we let $R = \inf\{t: Y_{t-} \notin E\} = \inf\{t: Y_t = \infty\}$ (the left limit being taken in the MW topology and ∞ being the point at infinity in the one-point compactification of E), then (6.10) yields $R \geq \zeta^Y$ a.s. Therefore, Y_t is right-continuous with left limits a.s. in the MW topology. □

All of this work culminates in the following resolution.

6.14. PROPOSITION. *$\kappa[0, t]$ is a continuous additive functional of Y_t which is strictly increasing and finite on $[0, \zeta^Y)$ a.s.*

PROOF. Recall that

$$\kappa[0,t] = \int_0^t h^{-1}(Y_s)\, dC_s,$$

where C_t is a strictly increasing continuous additive functional with potential $P^x[C_\infty] \leq 1$. Since $h > 0$ is continuous on E and Y_s is right-continuous with left limits in E, $h^{-1}(Y_s)$ is bounded on $[0, t]$ whenever $t < \zeta^Y$. Therefore, $\kappa[0, t] < \infty$ a.s. on $[0, \zeta^Y)$. □

Let $\pi(t)$ be the right-continuous inverse of $\kappa[0, t]$, and define $W_t = Y_{\pi(t)}$. Then W_t is a transient right process with H-translation invariant potential V_κ [see (6.6) and (6.7)]. Let $\varphi \in H$, and let W^φ be the transient right process $(\varphi(W_t), P^{\varphi^{-1}(x)})$. Note that the potential of W^φ is $V^\varphi f(x) = V_\kappa(f \circ \varphi)(\varphi^{-1}(x)) = V_\kappa f(x)$. Therefore, W and W^φ are transient processes with the same potentials and are therefore identical in law. This allows us to summarize in the main result of this section.

6.15. THEOREM. *Let X be a transient Hunt process on E with H a transitive subgroup of G and H_e trivial. If (6.1) holds, then there is a continuous additive functional N_t of X_t which is strictly increasing and finite on $[0, \zeta)$ such that if $\gamma(t)$ is the right-continuous inverse of N_t, then $(X_{\gamma(t)}, P^x)$ is an H-translation invariant process.*

PROOF. It is easy to check that $N_t = \kappa[0, B_t^1]$ is the desired continuous additive functional. □

7. The case J = Sub and $U1_E < \infty$. Now we turn to a special case of the dual problem where J = Sub is transitive and Sub_e is trivial. As before, X is a transient Borel right process, and the group structure of Sub is transferred to E using the bijection Ψ.

7.1. HYPOTHESIS. The group structure satisfies (5.1) and there is a σ-finite left quasi-invariant measure on $(E, \mathscr{E})$.

By (5.3), there is a σ-finite left Haar measure m on $(E, \mathscr{E})$. In this article, we do not discuss the general case of Sub, but we can indicate quickly a special case.

7.2. THEOREM. (i) *Assume $U1_E < \infty$ on E. The potential*

$$W(x, dy) = U1_E(x)^{-1}U(x, dy)$$

is the potential of a Sub-translation invariant process Y_t.

(ii) *If X_t is a Hunt process, then Y_t is right-continuous with left limits in E in the MW topology.*

As we noted in Section 3, Y_t is the $U1_E$-link transform of the process X_t. The proof of Theorem 7.2(i) is really a corollary of the proof of (1.4) in [7]. There, we showed that $Wg = W(g \circ \varphi) \circ \varphi^{-1}$ from which it followed that $W^\alpha g = W^\alpha(g \circ \varphi) \circ \varphi^{-1}$ for every $g \in p\mathscr{E}$ and for every $\varphi \in$ Sub. This analytic equality says that Y_t is Sub-translation invariant. For Theorem 7.2(ii), note that W maps bounded continuous functions on E into bounded continuous functions. An argument analogous to the one in (6.10) yields Theorem 7.2(ii).

8. Existence of left quasi-invariant measures. We now come to the problem of existence of left quasi-invariant (LQI) measures on $(E, \mathscr{E})$. We shall discuss various conditions which guarantee their existence, but the list is not exhaustive.

The simplest and most common situation is undoubtedly the one in which one can recognize that H or Sub is a locally compact topological transformation group when equipped with an appropriate topology. In fact, these can often be recognized as Lie groups. The group of translations of R^d is one such example, as is the group of rotations and dilations of $R^d - \{0\}$. A locally compact topological group has a σ-finite left Haar measure m, and m is LQI. Another common situation occurs for Sym and Sub.

8.1. PROPOSITION. *Assume Sub is transitive and Sub_e is trivial. If $(X, \hat{X}, \lambda)$ are in strong duality, then λ is LQI for Sub.*

PROOF. If $\varphi \in$ Sub, then Exc$(\varphi) =$ Exc. Since $\lambda \in$ Exc, $\varphi(\lambda) = \hat{g} \cdot \lambda$ for some coexcessive function $\hat{g}$. Therefore, $\varphi(\lambda) \ll \lambda$ for every $\varphi \in$ Sub. □

Before reading the proof of (8.2), recall (4.5).

8.2. PROPOSITION. *Assume that H is transitive and H_e is trivial. If $(X, \hat{X}, \lambda)$ are in strong duality and if 1_E is the potential of a strictly increasing continuous additive functional A_t of X, then the Revuz measure of A_t is LQI for H.*

PROOF. If we let $\tau(t)$ be the right-continuous inverse of A_t, then $X_{\tau(t)}$ is G-translation invariant by (4.5). There is a dual continuous additive functional $\hat{A}_t$ of $\hat{X}$ with Revuz measure μ and right-continuous inverse $\sigma(t)$ such that $(X_{\tau(t)}, \hat{X}_{\sigma(t)}, \mu)$ are in strong duality. Let $u(x,y)$ be the potential density. If $f \in p\mathscr{E}$, then

$$P^x \int f(X_{\tau(t)})\, dt = \int u(x,y) f(y) \mu(dy),$$

while

$$P^{\varphi^{-1}(x)} \int f \circ \varphi(X_{\tau(t)})\, dt = \int u(\varphi^{-1}(x), \varphi^{-1}(y)) f(y) \varphi(\mu)(dy).$$

Since the two left-hand sides in the previous lines are equal, we have that for each $x \in E$, the measures $u(x,y)\mu(dy)$ and $u(\varphi^{-1}(x), \varphi^{-1}(y))\varphi(\mu)(dy)$ are identical. It follows that $\varphi(\mu)$ is equivalent to μ. $\square$

An analogous argument (which we omit) yields the following result. Recall (4.7).

8.3. PROPOSITION. *Assume the following*: (i) X *is a Hunt process*; (ii) $(X, \hat{X}, \lambda)$ *are in strong duality*; (iii) H *is transitive*; (iv) H_e *is trivial*; *and* (v) *hypothesis* (4.7′) *holds. Then the Revuz measure* μ *of the continuous additive functional* B_t *constructed in the proof of* (4.7) *is LQI for* H.

Here is a nice potential theoretic condition for Sub. Recall that a measure $\xi \in$ Exc can be decomposed uniquely into the sum of a measure potential μU and a harmonic piece η. Let Pot denote the collection of measure potentials and let Har denote the collection of harmonic measures of X. We say Har is one dimensional if Har contains a nonzero measure and if any two nonzero measures in Har are scalar multiples of one another.

8.4. PROPOSITION. *Assume Sub is transitive and* Sub_e *is trivial. Suppose Har is one dimensional. Then* $\eta \in$ *Har is LQI for Sub.*

PROOF. Let $\varphi \in$ Sub. Since $U(h \circ \varphi)(\varphi^{-1}(x)) = c_\varphi^{-1}(x) Uh(x)$, we have

$$\varphi(\varphi^{-1}(\mu)U) = \int c_\varphi^{-1}(x) U(x, dy) \mu(dx),$$

and this implies that $\mathrm{Pot}(\varphi) = \{\varphi(\mu)\colon \mu \in \mathrm{Pot}\} = \mathrm{Pot}$. Since $\mathrm{Exc}(\varphi) = \mathrm{Exc}$, we have $\mathrm{Har}(\varphi) = \{\varphi(\mu)\colon \mu \in \mathrm{Har}\} = \mathrm{Har}$. Therefore, if $\eta \in$ Har, $\varphi(\eta) \ll \eta$ by the one dimensionality of Har. $\square$

9. The nontransient case. Some of the discussion in this article extends without very much change to the recurrent case. In this case, G should be taken to consist of all $\varphi \in \overline{G}$ satisfying Proposition 3.2(ii). This eliminates the dependence on the collection of excessive functions, which is too small to be of use in the nontransient case. One does need to be a bit careful in using the Blumenthal–Getoor–McKean theorem in this case. If we have two transient processes, we need only check that they have the same hitting distributions on sets in E in order that they be time changes of one another. In general, without specific transience assumptions, one needs to have identical hitting distributions on sets in E_Δ to conclude that they are time changes of one another. For example, if E consists of just one point $\{x\}$, the two processes $P^x[X_t = x] = 1$ for every $t > 0$ and $Q^x[Y_t = x] = e^{-t}$ for every $t > 0$, have the same hitting distribution on E, but not on E_Δ, and are not time changes of one another.

However, the discussion in Section 4 applies to the recurrent case, and in particular, (4.7) is true for a recurrent Hunt process. The results in Section 6 depend heavily on the transience assumption. The reader can check that some of the results in Section 8 are also true in the recurrent case.

We end with one computation illustrating some of the ideas discussed.

9.1. EXAMPLE. Let $B_t = (X_t, Y_t)$ be Brownian motion in $R^2 - \{0\}$. Then the subgroup H of G generated by the dilations and rotations in $R^2 - \{0\}$ is transitive and H_e is trivial. H is isomorphic to the direct product of the circle group C with the group $D = (0, \infty)$, where the group operation in D is multiplication. The left Haar measure for H is the product of Lebesgue measure on C and the measure dr/r on D. If we write B_t in polar coordinates (Θ_t, R_t), then (Θ_t, R_t) is a Markov process with speed measure $r\,dr\,d\theta$. If we time change it by the inverse $\tau(t)$ of the continuous additive functional

$$A_t = \int_0^t R_t^{-2}\,dt,$$

then $(\Theta_{\tau(t)}, R_{\tau(t)})$ is H-translation invariant.

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