The Annals of Probability
1991, Vol. 19, No. 2, 525–537

UNIQUENESS FOR SOME DIFFUSIONS WITH DISCONTINUOUS COEFFICIENTS[1]

BY M. CRISTINA CERUTTI, LUIS ESCAURIAZA AND EUGENE B. FABES

University of Minnesota

We show the well-posedness for the martingale problem associated with a strictly elliptic operator with coefficients bounded and continuous on $\mathbb{R}^n$ except for a countable set having at most one cluster point.

0. Introduction. Let L be an elliptic operator defined on $u \in C^2(\mathbb{R}^n)$ by

$$(0.1)\qquad Lu(x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u(x), \quad \text{where } D_{ij} = \frac{\partial^2}{\partial x_i\, \partial x_j},$$

and the matrix $A(x) = (a_{ij}(x))$ is bounded, measurable, symmetric and positive definite. Stroock and Varadhan [7] showed that when the matrix A is continuous and uniformly positive definite there is a unique solution to the martingale problem for L starting at any x in $\mathbb{R}^n$:

For each x in $\mathbb{R}^n$ there is exactly one probability measure P^x on $C([0,\infty), \mathbb{R}^n)$ such that:

1. $P^x(X(0) = x) = 1$.
2. $\varphi(X(t)) - \varphi(X(0)) - \int_0^t L\varphi(X(s))\,ds$ is a P^x-local martingale for all φ in $C^2(\mathbb{R}^n)$.

When A is discontinuous, existence is known to hold regardless of the dimension ([7], Exercise 12.4.3) while uniqueness holds when n is 1 or 2 ([7], Exercises 7.3.3 and 7.3.4). For $n \ge 3$ the argument of Stroock and Varadhan implies uniqueness when

$$\sup_x \sup_{1 \le i, j \le n} |a_{ij}(x) - \delta_{ij}| \le \varepsilon(n),$$

where $\varepsilon(n)$ is a small number depending on dimension. This result can be extended to general continuous coefficients by means of a localization argument.

More recently, Bass and Pardoux [3] have shown uniqueness when $\mathbb{R}^n$ is divided into finitely many polyhedral domains, and the matrix A is positive definite and constant on each polyhedron. Their argument also shows uniqueness when the matrix A is homogeneous of degree 0 and smooth off the origin. In this paper we show uniqueness when the matrix A is bounded, symmetric,

Received October 1989; revised February 1990.
[1]Partially supported by NSF Grant DMS-84-21377-04.
AMS 1980 *subject classifications*. 35B05, 35B10, 35B45, 35C15, 60G12, 60G15, 60G44, 60H10, 60J35.
Key words and phrases. Uniqueness in the Dirichlet problem, Harnack's inequality, maximum principle, martingale problem.

uniformly positive definite and continuous on $\mathbb{R}^n$ except possibly on a countable set with at most one cluster point.

The paper is divided into two sections. In Section 1 we prove a uniqueness result for solutions to the Dirichlet problem associated with L in a smooth domain $D \subset \mathbb{R}^n$,

$$\begin{aligned} Lu &= -f \quad \text{on } D, \\ u &= \varphi \quad \text{on } \partial D, \end{aligned} \tag{0.2}$$

when the matrix A satisfies the latter conditions. Here, the class of uniqueness consists of those functions $u \in C(\overline{D})$ for which there exists a sequence of elliptic operators L^k with smooth coefficients a_{ij}^k converging to a_{ij} on D, so that u is the uniform limit on $\overline{D}$ of the sequence $\{u^k\}$ formed by the solutions to (0.2) with L^k replaced by L. This result was proved by Caffarelli when the matrix A has a finite number of discontinuities and extended by the authors to the infinite case.

In Section 2 we combine this result with the techniques of Bass and Pardoux [3] to show the well-posedness of the martingale problem. The key of the proof is a regularity result satisfied by the solutions to the above Dirichlet problem.

1. Uniqueness to the Dirichlet problem. Throughout this paper we will use the following notation: D, a smooth domain on $\mathbb{R}^n$; ∂D, the boundary of D; $C(\overline{D})$, the set of continuous functions on D; $B_r(x)$, the open ball centered at x of radius r; $W^{2,n}_{\mathrm{loc}}(D)$, the set of functions with two distributional derivatives which are n-integrable on compact subsets of D; $L^p(D)$, the set of p-integrable functions; and $C^\alpha(\overline{D})$, the set of α-Hölder continuous functions on $\overline{D}$.

Let L be the operator in (0.1) where the matrix $A = (a_{ij})$ is bounded, symmetric and for some $\lambda > 0$ satisfies the ellipticity condition

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^n, \tag{1.1}$$

and the entries $a_{ij}(x)$ are continuous functions on $\mathbb{R}^n$ except possibly on a countable set E with at most one cluster point.

DEFINITION 1. Let $\varphi \in C(\partial D)$, $f \in C_0^\infty(\mathbb{R}^n \setminus \overline{E})$. A function $u \in C(\overline{D})$ will be said to be a good solution to the problem

$$\begin{aligned} Lu &= -f \quad \text{on } D, \\ u &= \varphi \quad \text{on } \partial D, \end{aligned} \tag{1.2}$$

when there exists a sequence of operators $\{L^k\}$ with coefficient matrices A^k satisfying (1.1) and with smooth entries on $\mathbb{R}^n$ so that A^k converges uniformly to A on compact subsets of $\mathbb{R}^n \setminus \overline{E}$, and u is the uniform limit on $\overline{D}$ of the sequence $\{u^k\}$ formed by the strong solutions to the problem (1.2) associated with L^k.

Analogously, if Ω is contained in D (Ω an open set), we will say that a function $u \in C(\overline{\Omega})$ is a good solution to $Lu = -f$ on Ω, when there exists a sequence of operators $\{L^k\}$ satisfying the conditions in Definition 1 and functions $u^k \in C(\overline{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega)$, so that $\{u^k\}$ converges uniformly to u on $\overline{\Omega}$ and $L^k u^k = -f$ on Ω.

Observe that good solutions to (1.2) do exist; for if we regularize the matrix $A(x)$ by means, for instance, of convolution, we obtain a sequence of operators L^k satisfying the conditions in Definition 1. It is well known ([4], Chapter 9) that the corresponding solutions $\{u^k\}$ to (1.2) for L^k satisfy the following uniform estimate:

$$\|u^k\|_{L^\infty(D)} + \|u^k\|_{C^\alpha(\overline{\Omega})} \leq C(\lambda, n, \Omega, D)\left[\|\varphi\|_{L^\infty(\partial D)} + \|f\|_{L^n(D)}\right], \tag{1.3}$$

where $\overline{\Omega}$ is any compact subset of D, and α lies in $(0,1)$, depending on λ and n. Moreover, barrier arguments ([4], Chapter 6) show that the modulus of continuity of u^k at each boundary point is controlled by a constant independent of k, times the sum of the modulus of continuity of φ and the distance to the boundary point. Hence, a subsequence converges uniformly on $\overline{D}$ to a function u in $C(\overline{D})$ with $u = \varphi$ on ∂D. Also, due to the local uniform convergence of $\{A^k\}$ on $\mathbb{R}^n \setminus \overline{E}$, the continuity of A on $\mathbb{R}^n \setminus \overline{E}$ and the L^p-Schauder estimates ([4], Chapter 9), we have that $u \in W^{2,n}_{\text{loc}}(D \setminus \overline{E})$ and $Lu = -f$ almost everywhere on that set.

In probabilistic terms, the solution to problem (1.2) for an operator L with coefficients continuous on a neighborhood of $\overline{D}$ can be represented as

$$u(x) = E^x[\varphi(X(\tau))] + E^x\left[\int_0^\tau f(X(t))\,dt\right] \quad \text{for all } x \text{ in } D,$$

where E^x denotes the expectation with respect to any solution to the martingale problem for L starting at x and τ is the exit time from D. Also, the harmonic measure corresponding to L and evaluated at x in D, $d\omega^x$, is described probabilistically as

$$\int_{\partial D} \varphi(Q)\,d\omega^x(Q) = E^x[\varphi(X(\tau))] \quad \text{for all } \varphi \in C(\partial D).$$

The most important tools used hereafter are the Krylov and Safonov Harnack inequality (HI) ([5]) and the strong maximum principle (SMP) ([4], Chapter 9).

Harnack inequality. Let $u \in W^{2,n}_{\text{loc}}(B_r(x)) \cap C(\overline{B}_r(x))$ satisfy $Lu = 0$ and $u \geq 0$ on $B_r(x)$. Then, for some constant C depending on λ and n,

$$\sup_{B_{r/2}(x)} u \leq C \inf_{B_{r/2}(x)} u. \tag{1.4}$$

Strong maximum principle. Let $u \in W^{2,n}_{\text{loc}}(D)$ satisfy $Lu = 0$ on D. Then u cannot attain a maximum or minimum in D unless it is a constant.

Observe that (1.4) is still satisfied when u is a good solution to $Lu = 0$ and $u \geq 0$ on $B_r(x)$. For if $u \geq 0$ on $B_r(x)$ and $\{u^k\}$ is as in Definition 1, for each

$\varepsilon > 0$ there is a positive integer k_0 so that $u^k + \varepsilon \geq 0$ on $B_r(x)$ for $k \geq k_0$. Since $L^k u^k = 0$ on $B_r(x)$ and $u^k \in W^{2,n}_{\text{loc}}(B_r(x)) \cap C(\overline{B}_r(x))$, we have

$$\sup_{B_{r/2}(x)} (u^k + \varepsilon) \leq C \inf_{B_{r/2}(x)} (u^k + \varepsilon).$$

After letting k tend to ∞ and ε tend to 0, we obtain (1.4) for u.

An easy consequence of HI for good solutions is that the SMP also holds for them; that is, in the last two statements we can replace the condition "$u \in W^{2,n}_{\text{loc}}(\Omega)$" for "$u$ is a good solution to $Lu = 0$ on Ω."

We will use the following form of HI.

LEMMA 1. *Let $u \in C(\overline{B}_{2r}(x) \setminus B_{r/2}(x))$ be a good solution to $Lu = 0$ and $u \geq 0$ on $\overline{B}_{2r}(x) \setminus B_{r/2}(x)$, $r \leq 1$. Then, for some $\gamma \in (0,1)$ depending on λ and n,*

$$\gamma \sup_{\partial B_r(x)} u \leq \inf_{\partial B_r(x)} u. \tag{1.5}$$

1.1. *Case of finitely many points of discontinuity.* We now assume that the matrix $A(x)$ is continuous on $\mathbb{R}^n$ except possibly at a point, which without loss of generality we consider to be the origin.

THEOREM 1. *Let $u \in C(\overline{D})$ be a good solution to (1.2), where $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, and assume that there exists $v \in C(\overline{D}) \cap W^{2,n}_{\text{loc}}(D \setminus \{0\})$ satisfying*

$$\begin{aligned} Lv &= 0 \quad \text{on } D \setminus \{0\}, \\ v &= 0 \quad \text{on } \partial D, \\ v &> 0 \quad \text{on } D. \end{aligned}$$

Then there exist constants C and $\alpha \in (0,1)$ depending on v, $\|u\|_{L^\infty(D)}$, the distance from the origin to the support of f, λ and n so that

$$|u(x) - u(0)| \leq C|x|^\alpha (v(0) - v(x)) \quad \text{for } x \text{ in } D.$$

PROOF. Since f is supported outside some ball $\overline{B}_r(0)$ contained in D, we have:

(1.6) $u \in W^{2,n}_{\text{loc}}(B_r(0) \setminus \{0\})$ and $Lu = 0$ on $B_r(0) \setminus \{0\}$.

(1.7) By the SMP for good solutions, for each $s \leq r/2$ there exist x_s^1 and x_s^2 with $|x_s^i| = s$ and $u(x_s^1) > u(0)$ and $u(x_s^2) < u(0)$. So by the continuity of u on $\partial B_s(0)$, there exists x_s with $|x_s| = s$ so that $u(x_s) = u(0)$.

(1.8) By the SMP, $v(0) > v(x)$ for all x in D.

Due to (1.8), there is a constant C so that

$$w^{\pm}(x) = \pm(u(x) - u(0)) + C(v(0) - v(x)) \geq 0 \quad \text{on } \partial B_r(0). \tag{1.9}$$

Since $w^{\pm}(0) = 0$, $w^{\pm} \in W^{2,n}_{\text{loc}}(B_r(0) \setminus \{0\})$ and $Lw^{\pm} = 0$ on $B_r(0) \setminus \{0\}$, the SMP implies that (1.9) holds over all $B_r(0)$. By Lemma 1 we have

$$w^{\pm}(x) \geq \gamma w^{\pm}(x_{r/2}) \text{ and } v(0) - v(x_{r/2}) \geq \gamma(v(0) - v(x)) \quad \text{for all } x \text{ in } \partial B_{r/2}(0).$$

These two inequalities imply

$$\pm(u(x) - u(0)) + C(1 - \gamma^2)(v(0) - v(x)) \geq 0 \quad \text{for all } x \text{ in } \partial B_{r/2}(0).$$

Proceeding in this manner, we have

$$\pm(u(x) - u(0)) + C(1 - \gamma^2)^j(v(0) - v(x)) \geq 0 \quad \text{for all } x \text{ in } B_{r/2^j}(0),\ j = 0, 1 \dots,$$

which proves the theorem. □

THEOREM 2. *Problem* (1.2) *is well posed; that is, there exists a unique good solution.*

PROOF. Let $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $\varphi \in C(\partial D)$ and assume that there are two different good solutions u_1, u_2. Set $v = u_1 - u_2$. Without loss of generality we may assume that $v(0) > 0$. Then the SMP and the regularity of the coefficients off the origin show that v satisfies the conditions in Theorem 1. Therefore, we have

$$|u_i(x) - u_i(0)| \leq C|x|^\alpha(v(0) - v(x)) \quad \text{for } x \text{ in } D,\ i = 1, 2.$$

Adding up these two inequalities, we have

$$v(0) - v(x) = o(v(0) - v(x)) \quad \text{as } x \text{ tends to } 0.$$

But this is a contradiction. Hence, v must be identically 0. □

Observe that this argument can be extended to the case when the matrix $A(x)$ is continuous on $\mathbb{R}^n$ except possibly on a finite set of points F. For if u_1, u_2 are two good solutions to (1.2) with $f \in C_0^\infty(\mathbb{R}^n \setminus F)$, setting $v = u_1 - u_2$ we observe that $v \in W^{2,n}_{\text{loc}}(D \setminus F) \cap C(\bar{D})$, $Lv = 0$ on $D \setminus F$ and $v = 0$ on ∂D. If v is not identically 0, say v is nonnegative at some point in D, v should have its maximum at one of the points z in $D \cap F$. By the SMP, $v(z) - v(x)$ would be strictly positive for all x in a punctured ball around z. We can then repeat the argument in Theorem 1 to show that

$$|u_i(x) - u_i(z)| \leq C|x - z|^\alpha(v(z) - v(x)) \quad \text{for } i = 1, 2,$$

and $|x - z|$ small. This would lead us to the same contradiction that we found in the proof of Theorem 2.

COROLLARY 1. *The problem* (1.2) *is well posed when the matrix* $A(x)$ *is continuous on* $\mathbb{R}^n$ *except possibly on a finite set of points.*

1.2. *Case of infinitely many points of discontinuity.* We assume that the matrix $A(x)$ is continuous on $\mathbb{R}^n$ except for a countable set of points E which has at most one cluster point. Without loss of generality we may assume that this cluster point is the origin of $\mathbb{R}^n$.

Let D be a smooth domain on $\mathbb{R}^n$ whose closure does not contain the origin. By Corollary 1 the mapping

$$C(\partial D) \to \mathbb{R}, \quad \text{defined by } \varphi \to u(x),$$

is well defined where u is the good solution to

$$\begin{aligned} Lu &= 0, \\ u &= \varphi \quad \text{on } \partial D. \end{aligned}$$

Moreover, the SMP shows that it is a positive linear functional on $C(\partial D)$. Hence, the Riesz representation theorem implies that there exists a unique probability measure $d\omega^x$ on ∂D so that

$$u(x) = \int_{\partial D} \varphi(Q)\, d\omega^x(Q).$$

This measure is called the harmonic measure for L on D at x.

Observe that if u is the good solution to

$$\begin{aligned} Lu &= -f \quad \text{on } D, \\ u &= 0 \qquad \text{on } \partial D, \end{aligned} \tag{1.10}$$

where $f \in C_0^\infty(\mathbb{R}^n \setminus \overline{E})$ and $\{u^k\}$ is a sequence of approximating solutions converging to u, by the Pucci–Aleksandrov inequality ([6] and [1]) we have

$$u^k(x) = \int_D g^k(x,y) f(y)\, dy \quad \text{for } x \text{ in } D, \tag{1.11}$$

where $g^k(x,y)$ is the Green's function for the operator L^k on D and

$$\left[\int_D g^k(x,y)^{n/(n-1)}\, dy\right]^{(n-1)/n} \leq C(\lambda, n, D) \quad \text{for all } x \text{ in } D. \tag{1.12}$$

Inequality (1.12) combined with a diagonalization process implies that there exists a subsequence of indices $\{k_j\}$ so that $g^{k_j}(x,y)$ converges weakly in $L^{n/(n-1)}(D)$ for all x in D to a function $g(x,y)$. From (1.11) we obtain the following representation formula:

$$u(x) = \int_D g(x,y) f(y)\, dy \quad \text{for all } x \text{ in } D.$$

Corollary 1 shows that this process, to generate a Green's function for L on D, determines $g(x, \cdot\,)$ uniquely for all x in D.

On the other hand, any function v in $C(\overline{D})$ which can be represented as

$$v(x) = \int_{\partial D} \varphi(Q)\, d\omega^x(Q) + \int_D g(x,y) f(y)\, dy$$

for some $\varphi \in C(\partial D)$ and $f \in C_0^\infty(\mathbb{R}^n \setminus \overline{E})$ is the unique good solution u to (1.2). To see this, one observes that if $\{L^k\}$ is a sequence of smooth operators as in Definition 1 and u^k is the solution to (1.2) with L replaced by L^k, then

$$u^k(x) = \int_{\partial D} \varphi(Q)\, d\omega_k^x(Q) + \int_D g^k(x,y) f(y)\, dy \quad \text{for } x \text{ in } D,\ k \geq 0,$$

where $d\omega_k^x$ and $g^k(x,y)$ are the harmonic measure and Green's function for L^k on D, respectively. Corollary 1 implies that $d\omega_k^x$ and $g^k(x,y)$ converge weakly to $d\omega^x$ and $g(x,y)$, respectively. These and the fact that u^k converges to u imply that $u = v$.

LEMMA 2. *Let D be a smooth domain whose closure does not contain the origin, and $u_1, u_2 \in C(\overline{D})$ good solutions to $Lu_1 = -f_1$, $Lu_2 = -f_2$, respectively, on D, where f_1 and $f_2 \in C_0^\infty(\mathbb{R}^n \setminus \overline{E})$. Then $u_1 + u_2$ is a good solution to $Lu = -(f_1 + f_2)$ on D.*

PROOF. Let $\{L_i^k\}$ for $i = 1, 2$ be two sequences of operators satisfying the conditions of Definition 1, so that there exist $u_i^k \in C(\overline{D}) \cap W_{\mathrm{loc}}^{2,n}(D)$ satisfying $L_i^k u_i^k = -f_i$ on D and $\{u_i^k\}$ converging uniformly to u_i on $\overline{D}$. We have

$$u_i^k(x) = \int_{\partial D} u_i^k(Q)\, d\omega_{k,i}^x(Q) + \int_D g^{k,i}(x,y) f_i(y)\, dy \quad \text{for } x \text{ in } D,\ k \geq 0,\ i = 1, 2, \tag{1.13}$$

where $d\omega_{k,i}^x$ and $g^{k,i}(x,y)$ are, respectively, the harmonic measure and Green's function for L_i^k on D. From Corollary 1 it follows that $\{d\omega_{k,i}^x\}$ and $\{g^{k,i}(x,y)\}$ converge, respectively, to $d\omega^x$ and $g(x,y)$ in the weak sense, where $d\omega^x$ and $g(x,y)$ are, respectively, the harmonic measure and Green's function for L on D. Letting k tend to ∞ in (1.13) and adding up the two limit equalities, we obtain the representation formula

$$u_1(x) + u_2(x) = \int_{\partial D} (u_1 + u_2)(Q)\, d\omega^x(Q) + \int_D g(x,y)(f_1 + f_2)(y)\, dy \quad \text{for } x \text{ in } D,$$

which proves the lemma. □

THEOREM 3. *Let D be a smooth domain in $\mathbb{R}^n$ which contains the origin, and assume that the origin is the only cluster point of the countable set of discontinuities of the matrix $A(x)$. Then the problem (1.2) is well posed on D.*

PROOF. Then let $u_1, u_2 \in C(\overline{D})$ be two good solutions to (1.2). Set $v = u_1 - u_2$. From Lemma 2 we conclude that v is a good solution to $Lv = 0$ on any open set Ω with $\overline{\Omega} \subset D \setminus \{0\}$. Moreover, $v = 0$ on ∂D. Hence, if v is not identically 0, say v is positive at some point in D, by the SMP applied to v on $D \setminus \{0\}$ we have that $v(0) - v(x)$ is strictly positive for all x in $D \setminus \{0\}$.

Since we are assuming that f is supported outside of a closed ball around 0, say $\overline{B}_r(0)$, both u_1 and u_2 are good solutions to $Lu = 0$ on $B_r(0)$. Setting

$u = u_i$ for $i = 1, 2$, we see that (1.7) still holds in this case, so that, for each $s \leq r/2$, there exists x_s with $|x_s| = s$ and $u(x_s) = u(0)$. As before, there exists C so that

$$(1.14)\quad w^{\pm}(x) = \pm(u(x) - u(0)) + C(v(0) - v(x)) \geq 0 \quad \text{on } \partial Br(0).$$

By Lemma 2, w is a good solution to $Lu = 0$ on $B_r(0) \setminus B_\varepsilon(0)$ for all $\varepsilon < r$. By the SMP we have

$$\inf_{B_r(0)\setminus B_\varepsilon(0)} w^{\pm} \geq \inf_{\partial(B_r(0)\setminus B_\varepsilon(0))} w^{\pm}.$$

Letting ε tend to 0 in the latter inequality, we see that (1.14) also holds over all $B_r(0)$. Since both w and $v(0) - v$ are nonnegative good solutions to $Lu = 0$ on $B_r(0) \setminus B_{r/4}(0)$, from Lemma 1 we get

$$w^{\pm}(x) \geq \gamma w^{\pm}(x_{r/2}) \text{ and } v(0) - v(x_{r/2}) \geq \gamma(v(0) - v(x)) \quad \text{for all } x \text{ in } \partial B_{r/2}(0).$$

Again, this implies that

$$\pm(u(x) - u(0)) + C(1 - \gamma^2)(v(0) - v(x)) \geq 0 \quad \text{for all } x \text{ in } \partial B_{r/2}(0),$$

which together with the SMP for good solutions imply that this last inequality also holds over all $B_{r/2}(0)$. From all these, we see that all the steps carried out in the proof of Theorem 1 can be repeated again in this situation to finally conclude that

$$|u_i(x) - u_i(0)| \leq C|x|^\alpha(v(0) - v(x)) \quad \text{for } |x| \text{ small, } i = 1, 2,$$

which again leads us to the same contradiction as in Theorem 2. □

1.3. *Constructing bad solutions.* In order to establish uniqueness to the martingale problem, we shall also need to study the following process.

Under the same assumptions on the matrix $A(x)$ as in 1.2, let u_ε be the unique good solution to the following problem:

$$(1.15)\quad \begin{aligned} Lu &= 0 \quad \text{for } \varepsilon < |x| < M, \\ u &= 0 \quad \text{on } \partial B_M(0), \qquad 0 < \varepsilon < M, \\ u &= 1 \quad \text{on } \partial B_\varepsilon(0). \end{aligned}$$

In probabilistic terms, u_ε can be represented as

$$u_\varepsilon(x) = P^x(\sigma_\varepsilon < \tau_M),$$

where P^x is any solution to the martingale problem and σ_ε and τ_M are, respectively, the exit times from $\mathbb{R}^n \setminus \overline{B}_\varepsilon(0)$ and $B_M(0)$.

The SMP for good solutions implies that for $|x| \geq \varepsilon_2$, $u_{\varepsilon_2}(x) \geq u_{\varepsilon_1}(x)$ whenever $\varepsilon_1 < \varepsilon_2$. Therefore,

$$(1.16)\quad 0 \leq \lim_{\varepsilon \to 0} u_\varepsilon(x) \equiv v(x) \leq 1$$

is well defined for all x with $0 < |x| \leq M$. As in (1.3) we have that for each

$\delta > 0$ there exist $C = C(\delta, \lambda, n, M)$ and $\alpha \in (0,1)$, $\alpha = \alpha(\lambda, n, M)$, so that

$$\|u_\varepsilon\|_{C^\alpha(B_M(0) \setminus B_\delta(0))} \leq C \quad \text{for } 0 < \varepsilon < \delta.$$

This implies that the above limit is uniform on each of the rings $B_M(0) \setminus B_\delta(0)$ and, therefore, that the limit v is a good solution to $Lu = 0$ on $B_M(0) \setminus B_\delta(0)$ for all $\delta \in (0, M)$.

LEMMA 3. *Under the same assumptions on the matrix $A(x)$, either v is identically 0 or $v \in C(\overline{B}_M(0))$ with $0 < v(x) < v(0)$ for $0 < |x| < M$.*

PROOF. If for some z with $0 < |z| < M$ we have $v(z) = 0$, the SMP applied to v on $B_M(0) \setminus \{0\}$ would imply that v is identically 0. Otherwise, we have $0 < v(x) \leq 1$ for all x in $B_M(0) \setminus \{0\}$. Let β denote the $\limsup_{x \to 0} v(x)$. The SMP implies that $0 < v(x) \leq \beta$ on $B_M(0) \setminus \{0\}$. Otherwise, we could find a sequence $\{x_i\}$ converging to 0 with $|x_i| > |x_{i+1}|$ and $\beta < v(x_i) < v(x_{i+1})$ for all $i > 0$, which contradicts the definition of β. At the same time, we can find $\{y_j\}$ converging to 0 with $|y_j| > |y_{j+1}|$ and $v(y_j)$ converging to β. By the SMP we have

$$\max_{|y_{j+1}| \leq |x| \leq y_j|} (\beta - v(x)) \leq \max_{|x| = |y_j| \text{ or } |y_{j+1}|} (\beta - v(x)).$$

Also, from Lemma 1 we conclude that

$$\max_{|x| = |y_j| \text{ or } |y_{j+1}|} (\beta - v(x)) \leq \gamma^{-1} \max\big(\beta - v(y_j), \beta - v(y_{j+1})\big),$$

which proves the lemma. □

As a simple consequence of what has been done so far, we can state the following lemma.

LEMMA 4. *Under the same assumptions on the matrix $A(x)$ let $u \in C(\overline{B}_M(0))$ be the good solution to the problem*

$$Lu = -h \quad \text{on } B_M(0),$$
$$u = 0 \quad \text{on } \partial B_M(0),$$

where $h \in C_0^\infty(\mathbb{R}^n \setminus \overline{E})$. Let v denote the limit of the process defined in (1.16). *If v is not identically* 0, *there exist constants C and $\alpha \in (0,1)$ depending on v, u and λ, n, respectively, so that*

$$|u(x) - u(0)| \leq C|x|^\alpha (v(0) - v(x)) \quad \text{for } |x| \text{ small}.$$

REMARK. The reason for the title of this subsection is that when the function v is not identically 0, what we find is a function which is a good solution to $Lv = 0$ off the origin with $v = 0$ on ∂D. But this function is a "bad solution" because it does not satisfy the SMP in $B_M(0)$. Examples of operators for which a bad solution exists are, for instance,

$$L_\alpha = \sum_{i,j=1}^{n} \left[\alpha\delta_{ij} + (1 - n\alpha)\frac{x_i x_j}{|x|^2} \right] D_{ij}, \qquad 0 < \alpha < \frac{1}{2(n-1)}.$$

The function $u(x) = 1 - |x|^\beta$, $\beta = [1 - 2(n-1)\alpha]/[1 - (n-1)\alpha]$ ([6]), is a bad solution for L on $B_1(0)$. If $d\omega_\varepsilon^x$ denotes the harmonic measure for L on $B_1(0)/B_\varepsilon(0)$, we have

$$u(x) = \int_{\partial B_\varepsilon(0)} u(Q)\, d\omega_\varepsilon^x(Q) = (1 - \varepsilon^\beta) u_\varepsilon(x).$$

Letting ε tend to 0, we conclude that $v(x) = u(x)$.

2. Uniqueness to the martingale problem. In what follows we will use the following notation: $\Omega = C([0,\infty), \mathbb{R}^n)$, $X(t)\colon \Omega \to \mathbb{R}^n$ the position of the path ω at time t, $\mathscr{M} = \sigma(X(t)/t \ge 0)$ is the σ-algebra generated by $X(t)$ for $t \ge 0$ and for a stopping time τ, we set $\mathscr{M}_\tau = \sigma(X(t \wedge \tau)/t \ge 0)$. We will consider the stopping times

$$\begin{aligned} \sigma_\varepsilon &= \inf\{t \ge 0/X(t) \in \overline{B}_\varepsilon(0)\}, \\ \sigma_0 &= \inf\{t \ge 0/|X(t)| = 0\}, \\ \tau_\varepsilon &= \inf\{t \ge 0/|X(t)| \ge \varepsilon\}. \end{aligned}$$

THEOREM 4. *Let L be the operator in* (0.1), *where the matrix* $A(x)$ *satisfies* (1.1) *and is continuous on* $\mathbb{R}^n$ *except possibly on a countable set E with at most one cluster point. Then the martingale problem for L is well posed.*

PROOF. As before we start with the simplest case.

2.1. *Case of one point discontinuity.* We assume that $A(x)$ is continuous on $\mathbb{R}^n \setminus \{0\}$.

Since the coefficients of L are nice off the origin, it is well known ([7]) that if P^x is a solution starting at x different from the origin, the restriction of P^x to $\mathscr{M}_{\sigma_0}$ is uniquely determined. Moreover, for $\varepsilon < |x| < M$ and $h \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, we have

$$(2.1)\quad P^x(\sigma_\varepsilon < \tau_M) = u_\varepsilon(x) \quad \text{and} \quad E^x\left[\int_0^{\tau_M \wedge \sigma_\varepsilon} h(X(t))\, dt\right] = -w_\varepsilon(x),$$

where u_ε is as in (1.15) [in this case the good solution to (1.15) trivially coincides with the classical strong solution obtained from the L^p-Schauder theory] and w_ε is the good solution to the problem (which in this case coincides again with the strong solution)

$$(2.2)\quad \begin{aligned} Lw_\varepsilon &= h \quad \text{on } B_M(0)/B_\varepsilon(0), \\ w_\varepsilon &= 0 \quad \text{on } \partial B_M(0)/B_\varepsilon(0). \end{aligned}$$

As is proved in [7], it suffices to show uniqueness for P^x, where $\{P^x/x \in \mathbb{R}^n\}$ is a family of solutions that forms a strong Markov process. Following the argument of Bass and Pardoux [3], it suffices to show that for such a family, the operators

$$R(h)(x) = E^x\left[\int_0^{\tau_M} h(X(t))\,dt\right] \quad \text{for } h \in C_0^\infty(\mathbb{R}^n \setminus \{0\}),\ x \text{ in } B_M(0)$$

are uniquely determined for all $M > 0$.

By the strong Markov property, if x lies in $B_M(0) \setminus \{0\}$ we have

$$\begin{aligned} E^x\left[\int_0^{\tau_M} h(X(t))\,dt\right] &= E^x\left[\int_0^{\tau_M \wedge \sigma_0} h(X(t))\,dt\right] \\ &\quad + P^x(\sigma_0 < \tau_M) E^0\left[\int_0^{\tau_M} h(X(t))\,dt\right]. \end{aligned} \tag{2.3}$$

Since σ_ε decreases to σ_0 as ε tends to 0 we have

$$P^x(\sigma_0 < \tau_M) = \lim_{\varepsilon \to 0} P^x(\sigma_\varepsilon < \tau_M) = \lim_{\varepsilon \to 0} u_\varepsilon(x) = v(x),$$

where v is given in Lemma 3. Analogously,

$$E^x\left[\int_0^{\tau_M \wedge \sigma_0} h(X(t))\,dt\right] = \lim_{\varepsilon \to 0} E^x\left[\int_0^{\tau_M \wedge \sigma_\varepsilon} h(X(t))\,dt\right] = -\lim_{\varepsilon \to 0} w_\varepsilon(x),$$

where w_ε is defined in (2.2). By similar arguments to those in Section 1, the latter process has a bounded limit $w \in C(\overline{B}_M(0) \setminus \{0\})$, which is a good solution to $Lu = h$ on $B_M(0) \setminus B_\delta(0)$ for all δ in $(0, M)$ [in this particular case $w \in W^{2,n}_{\text{loc}}(B_M(0) \setminus \{0\})$ and $w = 0$ on $\partial B_M(0)$]. Since h is supported outside some ball $B_r(0)$, by subtracting from w the good solution to

$$\begin{aligned} Lu &= 0 \quad \text{on } B_r(0), \\ u &= w \quad \text{on } \partial B_r(0), \end{aligned}$$

we find a function which is bounded, continuous on $\overline{B}_r(0) \setminus \{0\}$, vanishing on $\partial B_r(0)$ and a good solution to $Lu = 0$ on $B_r(0) \setminus B_\delta(0)$ for δ in $(0, r)$. The argument in Lemma 3 shows that this function can be extended continuously to the origin. This clearly shows that the same is true for w.

From (2.3) we see that for x in $B_M(0) \setminus \{0\}$,

$$E^x\left[\int_0^{\tau_M} h(X(t))\,dt\right] = -w(x) + v(x) E^0\left[\int_0^{\tau_M} h(X(t))\,dt\right];$$

so $R(h)(x)$ is determined for x different from 0 by $R(h)(0)$.

By the strong Markov property and for $\varepsilon \le r$, we have

$$\begin{aligned} &E^0\left[\int_0^{\tau_M} h(X(t))\,dt\right] \\ &\quad = E^0\left[E^{P^{X(\tau_\varepsilon)}}\left[\int_0^{\tau_M} h(X(t))\,dt\right]\right] \\ &\quad = -E^0[w(X(\tau_\varepsilon))] + E^0[v(X(\tau_\varepsilon))] E^0\left[\int_0^{\tau_M} h(X(t))\,dt\right]. \end{aligned}$$

Hence,

$$E^0\left[\int_0^{\tau_M} h(X(t))\,dt\right] = -\frac{E^0[w(X(\tau_\varepsilon))]}{E^0[1-v(X(\tau_\varepsilon))]} \quad \text{for all } \varepsilon \le r. \tag{2.4}$$

Now, there are two possible cases:

(a) The function v is identically 0. If we let ε tend to 0 in (2.4), we obtain

$$E^0\left[\int_0^{\tau_M} h(X(t))\,dt\right] = -w(0)$$

and w is a uniquely determined function.

(b) The function v satisfies $0 < v < 1$ on $B_M(0) \setminus \{0\}$. From (2.4) we have

$$E^0\left[\int_0^{\tau_M} h(X(t))\,dt\right] = -\lim_{\varepsilon\to 0}\frac{E^0[w(X(\tau_\varepsilon))]}{E^0[1-v(X(\tau_\varepsilon))]}. \tag{2.5}$$

If $v(0) < 1$, we conclude that

$$E^0\left[\int_0^{\tau_M} h(X(t))\,dt\right] = -\frac{w(0)}{1-v(0)}. \tag{2.6}$$

Otherwise, we must have

$$\lim_{\varepsilon\to 0} E^0[w(X(\tau_\varepsilon))] = w(0) = 0.$$

Let then u denote the good solution to the problem

$$Lu = h \quad \text{on } B_M(0),$$
$$u = 0 \quad \text{on } \partial B_M(0).$$

Applying the SMP to $w - u + u(0)v$ on $B_M(0) \setminus \{0\}$, we conclude that this function is identically 0. Plugging this into (2.5), we get

$$E^0\left[\int_0^{\tau_M} h(X(t))\,dt\right] = -\lim_{\varepsilon\to 0}\frac{E^0[u(X(\tau_\varepsilon)) - u(0)]}{E^0[1-v(X(\tau_\varepsilon))]} - u(0).$$

From Lemma 4 or Theorem 1 we conclude that the above limit is 0. Hence,

$$E^0\left[\int_0^{\tau_M} h(X(t))\,dt\right] = -u(0) \tag{2.7}$$

and the right-hand sides of (2.6) and (2.7) are values of uniquely determined functions. This proves the theorem in this case.

2.2. *Case of infinitely many points.* We assume the conditions in Theorem 4 and without loss of generality we may assume that the cluster point of the set E is the origin.

From what we have proved and [7], it follows again that the restriction to $\mathscr{M}_{\sigma_0}$ of any solution P^x starting at x different from 0 is uniquely determined. Moreover, it is well known ([7]) that if we regularize the matrix $A(x)$ by convolution and $\{P_k^x\}$ denotes the solutions starting at x for the operators $\{L^k\}$ obtained in this way, that some subsequence of $\{P_k^x\}$ (which we denote with the same indices) converges weakly to a solution Q^x of the martingale problem for L starting at x. From the definition of good solution and the weak convergence, we obtain

$$Q^x(\sigma_\varepsilon < \tau_M) = \lim_{k \to 0} P_k^x(\sigma_\varepsilon < \tau_M) = u_\varepsilon(x),$$

$$E^x\left[\int_0^{\tau_M \wedge \sigma_\varepsilon} h(X(t))\,dt\right] = \lim_{k \to 0} E_k^x\left[\int_0^{\tau_M \wedge \sigma_\varepsilon} h(X(t))\,dt\right] = -w_\varepsilon(x),$$

where u_ε and w_ε are, respectively, the good solutions to (1.15) and (2.2), $h \in C_0^\infty(\mathbb{R}^n/\overline{E})$ and E^x and E_k^x denote the expectations with respect to Q^x and P_k^x, respectively. But $\{\sigma_\varepsilon < \tau_M\}$ and $\int_0^{\tau_M \wedge \sigma_\varepsilon} h(X(t))\,dt$ are $\mathscr{M}_{\sigma_0}$-measurable. Therefore, (2.1) also holds for any solution P^x starting at x different from 0.

From these and Lemma 4 it is clear that the argument in 2.1 can be applied again with the only difference that, in this case, the functions u_ε and w_ε are no longer the classical strong solutions to (1.15) and (2.2), respectively, on $B_M(0) \setminus B_\varepsilon(0)$, but they are the good solutions to those problems. Hence, whenever we used HI or SMP in 2.1, we have to replace them by the HI or SMP for good solutions. This proves Theorem 4. □

REFERENCES

[1] Aleksandrov, A. D. (1963). Conditions of uniqueness and estimations of a solution of Dirichlet's problem. *Vestnik Leningrad Univ. Mat. Mekh. Astronom.* **18** 5–29.

[2] Bass, R. F. (1988). Uniqueness of solutions to the Dirichlet problem for operators with homogeneous coefficients. Preprint.

[3] Bass, R. F. and Pardoux, E. (1987). Uniqueness for diffusions with piecewise constant coefficients. *Probab. Theory Related Fields* **76** 557–572.

[4] Gilbarg, D. and Trudinger, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*, 2nd ed. Springer, New York.

[5] Krilov, N. V. and Safanov, M. V. (1979). An estimate of the probability that a diffusion process hits a set of positive measure. *Soviet Math. Dokl.* **20** 253–255.

[6] Pucci, C. (1966). Limitazioni per soluzioni di equazioni ellittiche. *Ann. Mat. Pura Appl.* (*4*) **74** 15–30.

[7] Stroock, D. W. and Varadhan, S. R. S. (1979). *Multidimensional Diffusion Processes.* Springer, New York.

School of Mathematics
127 Vincent Hall
University of Minnesota
206 Church Street, S.E.
Minneapolis, Minnesota 55455