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ON AN INDEPENDENCE CRITERION FOR MULTIPLE WIENER INTEGRALS[1]

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Üstünel and Zakai have recently obtained a necessary and sufficient condition for two multiple Wiener integrals with respect to the same Brownian motion to be independent. In the present note, the sufficiency of their condition is shown to be a simple consequence of the classical relationship between multiple Wiener integrals and Hermite polynomials. The original proofs use Malliavin calculus.

In a sequence of recent papers, [2]–[4], Üstünel and Zakai have obtained an interesting necessary and sufficient condition for two multiple Wiener integrals $I_p(f)$ and $I_q(g)$ with respect to the same Brownian motion on [0, 1] to be independent. More precisely, assuming f and g to be symmetric, they show that $I_p(f)$ and $I_q(g)$ are independent iff $f \otimes_1 g = 0$ a.e., where

$$(f \otimes_1 g)(s_1, \dots, s_{p-1}, t_1, \dots, t_{q-1}) = \int_0^1 f(s_1, \dots, s_{p-1}, u) g(t_1, \dots, t_{q-1}, u)\, du,$$

and where a.e. refers to the $(p+q-2)$-dimensional Lebesgue measure. Note in particular that the statement reduces for $p = q = 1$ to the well-known elementary fact that two jointly Gaussian random variables are independent iff they are uncorrelated.

The necessity of the condition is proved in the paper [4] by elementary arguments which also suggest its sufficiency. However, a rigorous proof of the sufficiency assertion seems to require different methods, and here the authors rely on the quite sophisticated tools of Malliavin calculus. Our present aim is to point out how the sufficiency part may also be obtained as a simple consequence of the classical relationship, due to Itô [1], between multiple Wiener integrals and Hermite polynomials. In spite of this fact, the original argument clearly remains interesting, because of the authors' development of powerful new techniques to handle the notion of independence in a multivariate context.

To state the result of Itô, define the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2/2} \left(\frac{d}{dx}\right)^n e^{-x^2/2}, \qquad n = 0, 1, 2, \dots,$$

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so that $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$, and so on. Let B be the common Brownian motion on $[0,1]$, and let $\varphi_1, \varphi_2, \dots$ be any orthonormal elements in $L^2([0,1])$. Then Itô shows that

$$(1) \qquad I_p\left(\varphi_1^{\otimes p_1} \otimes \cdots \otimes \varphi_m^{\otimes p_m}\right) = \prod_{k=1}^{m} H_{p_k}\left(\int \varphi_k \, dB\right),$$

whenever $p_1 + \cdots + p_m = p$. Here $f \otimes g$ denotes the tensor product of f and g, while $f^{\otimes n}$ is the corresponding nth power. Moreover, $\int f dB$ denotes the Wiener integral of a function $f \in L^2([0,1])$ with respect to B.

To prove that $I_p(f)$ and $I_q(g)$ are independent whenever $f \in L^2([0,1]^p)$ and $g \in L^2([0,1]^q)$ are symmetric with $f \otimes_1 g = 0$ a.e., let H_f denote the closed subspace of $L^2([0,1])$ spanned by all functions

$$f_A(t) = \int \cdots \int_A f(s_1, \dots, s_{p-1}, t) \, ds_1 \cdots ds_{p-1}, \qquad t \in [0,1],$$

with A an arbitrary Borel set in $[0,1]^{p-1}$, and define H_g similarly in terms of g. From the condition $f \otimes_1 g = 0$ a.e. and Fubini's theorem, it is seen that H_f and H_g are orthogonal. Hence if $\xi = \int \varphi \, dB$ and $\eta = \int \psi \, dB$ with $\varphi \in H_f$ and $\psi \in H_g$, we get by Itô's formula $E\xi\eta = \int \varphi\psi = 0$, and since ξ and η are jointly centered Gaussian, they must be independent.

Let us now introduce some complete orthonormal systems (CONS) $\varphi_1, \varphi_2, \dots$ in H_f and $\psi_1, \psi_2, \dots$ in H_g, and write $\xi_i = \int \varphi_i \, dB$ and $\eta_j = \int \psi_j \, dB$ for all i and j. Then any finite linear combinations $\sum a_i \xi_i$ and $\sum b_j \eta_j$ are independent, so by the Cramér–Wold theorem the entire sequences (ξ_i) and (η_j) are independent. Since the tensor products $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_p}$ form a CONS in $H_f^{\otimes p}$, it is easily seen that f admits an orthogonal expansion

$$(2) \qquad f = \sum \cdots \sum c_{i_1, \dots, i_p} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_p},$$

and then (1) yields a corresponding expansion of $I_p(f)$ into orthogonal polynomials in $\xi_1, \xi_2, \dots$. Similarly, $I_q(g)$ may be expanded into polynomials in $\eta_1, \eta_2, \dots$, and since the two sequences are independent, the asserted independence between $I_p(f)$ and $I_q(g)$ follows.

As noted by Üstünel and Zakai, their independence criterion extends immediately to arbitrary families of multiple Wiener integrals. Thus any two collections $\{I_{p_\alpha}(f_\alpha)\}$ and $\{I_{q_\beta}(g_\beta)\}$ are independent, iff $I_{p_\alpha}(f_\alpha)$ is independent of $I_{q_\beta}(g_\beta)$ for any α and β. From this, it follows in turn that pairwise independence between the integrals $I_{p_\alpha}(f_\alpha)$ implies that they are all independent. It is interesting to notice that the same conclusions are obtainable by a simple modification of our present proof.

It seems unlikely that similar methods could be used to prove the necessity of the Üstünel–Zakai condition. Indeed, already for $p = q = 2$, the statement reduces to the nonobvious fact that, whenever the sequences $\xi_1, \xi_2, \dots$ and $\eta_1, \eta_2, \dots$ are i.i.d. $N(0,1)$ and jointly Gaussian, while $a_1, a_2, \dots$ and $b_1, b_2, \dots$ are nonzero constants with $\sum a_i^2 < \infty$ and $\sum b_j^2 < \infty$, the sequences (ξ_i) and (η_j) are independent provided that independence holds between the sums

$\sum a_i(\xi_i^2 - 1)$ and $\sum b_j(\eta_j^2 - 1)$. In higher dimensions, the complexity of the corresponding conditions becomes frightening, due to the fact that diagonalization in (2) is no longer available.

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