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THE CONTACT PROCESS IN A RANDOM ENVIRONMENT

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We show that in one dimension, the contact process in a random environment has an "intermediate phase" in which it survives but does not grow linearly. We conjecture that this does not occur in dimensions $d > 1$.

1. Introduction. We begin by describing the model. Each integer is independently designated as *bad* with probability p and *good* with probability $1 - p$, and these labels are fixed for all time. Into this *random environment*, we introduce a process whose state at time t is $\xi_t \subset \mathbb{Z}$. Points in ξ_t are thought of as occupied by *particles* and with this in mind the dynamics are formulated as follows. (i) Particles are born at unoccupied sites x at a rate equal to the number of occupied neighbors, $|\xi_t \cap \{x - 1, x + 1\}|$. (ii) A particle at x dies at rate Δ if the environment there is bad and at rate δ if the environment there is good.

The idea behind (i) and (ii) is that the reproduction rate of the particles is not affected by the environment, but particles die more rapidly in a hostile environment than in a friendly one. In this picture, an i.i.d. environment is not very realistic and should be replaced by a stationary and ergodic one but, as the reader will see, the i.i.d. model is already quite challenging to analyze.

To indicate what we would like to prove for the contact process, we begin by describing results of Ferreira (1988) for the biased voter model in a random environment (BVMRE). In this model ξ_t is thought of as the set of people in favor of an issue at time t and the system evolves as follows:

$$\text{If } x \notin \xi_t \text{ then } P(x \in \xi_{t+s}|\xi_t) \sim \lambda|\xi_t \cap \{x-1, x+1\}|s \quad \text{as } s \to 0.$$

$$\text{If } x \in \xi_t \text{ then } P(x \notin \xi_{t+s}|\xi_t) \sim \delta_x|\xi_t^c \cap \{x-1, x+1\}|s \quad \text{as } s \to 0.$$

Here δ_x, $x \in \mathbb{Z}$, are i.i.d. random variables, and $f(s) \sim g(s)$ means $f(s)/g(s) \to 1$. Two nice properties of the biased voter model that make it easy to analyze are (i) if $\xi_0^0 = \{0\}$ then for $t \geq 0$ the set ξ_t^0 is either $\varnothing$ or an interval $\{l_t, \dots, r_t\}$, and (ii) when $r_t - l_t > 1$ the boundaries are independent random walks in (slightly different) random environments.

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When $r_t - l_t > 1$, r_t (resp. l_t) jumps one unit to the right (resp. left) at rate λ or one unit to the left (resp. right) at rate δ_x. Let $e = \{\delta_x: x \in \mathbb{Z}\}$ denote a realization of the environment, let P_λ^e denote the law of the process with parameter λ in the environment e and let

$$\Omega_\infty = \{\xi_t \neq \varnothing \text{ for all } t\} = \{l_t \leq r_t \text{ for all } t\}.$$

The results of Solomon (1975) imply:

If $E \log(\lambda/\delta_x) < 0$ then $P_\lambda^e(\Omega_\infty) = 0$ for a.e. environment e.

If $E \log(\lambda/\delta_x) > 0$ then $P_\lambda^e(\Omega_\infty) > 0$ for a.e. environment e.

So the critical value

$$\lambda_c \equiv \inf\{\lambda > 0: P_\lambda^e(\Omega_\infty) > 0 \text{ for a.e. } e\} = \exp(E \log \delta_x). \tag{1}$$

Here and throughout the paper $\equiv$ indicates a definition.

There is a second critical value for the BVMRE. Let ξ_t^* denote the process with $\xi_0^* = (-\infty, 0]$ and let $r_t^* = \sup \xi_t^*$. A simple argument shows [see Durrett (1988), Section 3a] that ξ_t^* and ξ_t^0 can be constructed on the same space with $r_t^* = r_t$ on $\{\xi_t^0 \neq \varnothing\}$. The results of Solomon (1975) imply that

$$r_t^*/t \to \alpha(\lambda) \quad \text{almost surely as } t \to \infty,$$

where $\alpha(\lambda)$ is a constant that satisfies

$$\alpha(\lambda) \begin{cases} > 0 & \text{if } \lambda \in (E\delta_x, \infty), \\ = 0 & \text{if } \lambda \in [1/E\delta_x^{-1}, E\delta_x], \\ < 0 & \text{if } \lambda \in (-\infty, 1/E\delta_x^{-1}]. \end{cases}$$

From the last result we see that

$$\lambda_\alpha \equiv \inf\{\lambda > 0: \alpha(\lambda) > 0\} = E\delta_x. \tag{2}$$

Comparing (2) with (1), we see that if δ_x is random

$$\lambda_\alpha = E\delta_x > \exp(E \log \delta_x) = \lambda_c.$$

In words, the BVMRE has two critical values: one threshold for survival and a higher threshold for linear growth.

The point of this paper is to prove the last statement for the contact process in a random environment (CPRE). To get conditions for sublinear growth, we begin by considering what happens when $\delta = 0$, that is, particles in good environments never die. To state our result in this case, we need some preliminaries about the ordinary contact process (i.e., when all the death rates equal δ). Let ζ_t^0 denote the process with $\zeta_0^0 = \{0\}$, let P_δ be the law of the process when the death rate is δ and let

$$\begin{aligned} \Omega_\infty &= \{\zeta_t^0 \neq \varnothing \text{ for all } t\}, \\ \delta_c &= \sup\{\delta: P_\delta(\Omega_\infty) > 0\}, \\ r_t^0 &= \sup \zeta_t^0, \\ R^0 &= \sup_{t \geq 0} r_t^0. \end{aligned}$$

By considering the state of the process the first time $n \in \xi_t^0$, it is easy to see that

$$P(R^0 \geq n + m | R^0 \geq n) \geq P(R^0 \geq m).$$

If we let $a_n = -\log P(R^0 \geq n)$ then $a_{n+m} \leq a_n + a_m$ and it follows [see, e.g., Durrett (1984), page 1017] that

$$a_n/n \to \inf_{m \geq 1} a_m/m \equiv \gamma_\perp(\delta)$$

and

$$(3) \qquad P(R^0 \geq n) \leq \exp(-\gamma_\perp(\delta) n).$$

It is known that

$$(4) \qquad \gamma_\perp(\delta) > 0 \quad \text{for } \delta > \delta_c.$$

$L_\perp(\delta) \equiv 1/\gamma_\perp(\delta)$ is called the spatial correlation length for the subcritical contact process. For more on this quantity and the proofs of the facts quoted above, see Durrett, Schonmann and Tanaka (1989).

The next result shows that $\gamma_\perp(\delta)$ is important for the study of the CPRE. Let

$$\rho_n = \inf\{t: n \in \xi_t^0\},$$

$$\Omega_\infty = \{\xi_t^0 \neq \varnothing \text{ for all } t\}.$$

THEOREM 1. *Suppose* $\Delta > \delta_c$, $\delta = 0$, *and let* $\mu = \gamma_\perp(\Delta)/\log(1/p)$ *where* $p = P(\delta_x = \Delta)$.

(a) *If* $\mu < 1$ *there is a constant* $c > 0$ *so that* $\rho_n/n \to c$ *a.s. on* Ω_∞.
(b) *If* $\mu \geq 1$ *then* $(\log \rho_n)/\log n \to \mu$ *in probability on* Ω_∞.

Here $X_n \to a$ *in probability on* Ω_∞ *means that for all* $\eta > 0$,

$$P(|X_n - a| > \eta, \Omega_\infty) \to 0$$

and P is the probability law for the CPRE, that is, we do not fix the environment. In other words, on Ω_∞ *the right edge*

$$r_t^0 = \sup \xi_t^0 \approx \begin{cases} t/c & \text{if } \mu < 1, \\ t^{1/\mu} & \text{if } \mu > 1. \end{cases}$$

The key to the proof of Theorem 1 is the following proposition.

PROPOSITION 1. *Consider a modification of the ordinary contact process* $\bar{\zeta}_t$ *with death rate* $\Delta > \delta_c$ *in which* $\bar{\zeta}_0 = \{0\}$ *and the particle at* 0 *never dies. Let* $\sigma_n = \inf\{t: n \in \bar{\zeta}_t\}$. *As* $n \to \infty$,

$$(\log \sigma_n)/n \to \gamma_\perp(\Delta) \quad \textit{in probability}$$

and

$$(\log E\sigma_n)/n \to \gamma_\perp(\Delta).$$

Once this easy result is established, the derivation of Theorem 1 is straightforward. We view the environment as a sequence of bad stretches (maximal intervals of bad sites) with good stretches in between, and look at the time it takes to reach the good sites $0 < x_1, x_2, \dots$ that are at the left edges of good stretches in $[0, \infty)$. Once a good site becomes occupied it never becomes vacant, so if T_k is the time to reach x_k then $T_k - T_{k-1}$, $k \geq 2$, are i.i.d. (on $\{T_1 < \infty\} = \Omega_\infty$). Proposition 1 implies that if $\mu < 1$, $E(T_k - T_{k-1}) < \infty$ and (a) follows from the strong law of large numbers.

If $\mu > 1$, standard results about coin tossing imply that the largest bad stretch in $[0, n]$ is $\sim \log n/\log(1/p)$, and Proposition 1 implies that the time to cross this bad stretch is $\approx n^\mu$. This provides a lower bound on the time to reach n. The upper bound is proved by computing expected values and by using Chebyshev's inequality.

Theorem 1 provides upper bounds on the rate of growth when $\delta > 0$. To prove that the contact process has two phase transitions, it is enough to show the following theorem.

THEOREM 2. *Suppose $p = P(\delta_x = \Delta) < 1$. There is a $\delta_0(\Delta, p) > 0$ so that if $\delta < \delta_0(\Delta, p)$ then the CPRE survives, that is, for a.e. environment e*:

$$P^e\big(\xi_t^0 \neq \varnothing \text{ for all } t\big) > 0.$$

Here $e = \{\delta_x; x \in \mathbb{Z}\}$ and P^e is the probability law of the contact process in the fixed environment e.

We will now describe the intuition that leads to the proof of Theorem 2, ignoring technicalities. If a single bad site is surrounded by two good stretches of length greater than N_1 (the first of a rapidly increasing sequence of constants to be chosen later), it is harmless and can be ignored. Conversely, good stretches of length less than or equal to N_1 are bad news and are called "1 gaps." Moving to the next level of the construction, we can ignore a "1 gap" that is surrounded on both sides by N_2 consecutive good stretches of length greater than N_1, but two "1 gaps" that are separated by less than N_2 such intervals are bad news and become a "2 gap," etc.

In Section 3 we show that if the N_k are chosen appropriately and the density of bad sites is small (a simple argument reduces the general case to this one), then the fraction of sites in gaps goes to 0 very rapidly as $k \to \infty$. This sets the stage for a multiscale renormalized site construction in Section 4. We prove by induction that the contact process survives long enough in the good region between "k gaps" to tunnel through them.

Combining Theorems 1 and 2 gives information about the "phase diagram" for CPRE that is drawn in Figure 1. The regions A, B, C and D there are open and have boundaries that are solid lines. The process survives in $A \cup B$

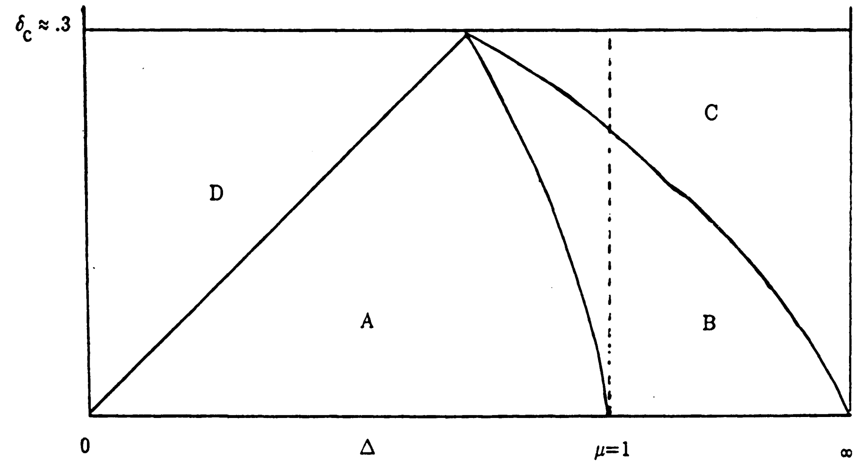


FIG. 1.

and dies in C. It grows linearly in A and has sublinear growth in $B \cup C$. We do not discuss D because there $\Delta < \delta$, that is, the good environment is worse than the bad.

We know that B is not empty because Theorem 1 implies $B \cup C$ contains the region to the right of the dotted line and Theorem 2 tells us that for any $\Delta < \infty$ the system survives for small δ. It is known that the regions A and C meet at the diagonal since $\delta = \Delta$ corresponds to the ordinary contact process. We have drawn the diagram in such a way that A and C do not meet below the diagonal but this is just a guess.

Quantitative information about the boundaries of A, B and C is, as usual, rather sketchy, but one thing is simple to prove. By comparing the CPRE with the BVMRE, one can conclude that if $E(\log \delta_x) > 0$ then the CPRE dies out, and if $E(\delta_x) > 1$ then CPRE has sublinear growth. This observation has been made independently by Liggett (1989) who considered a version of CPRE in which the birth rates are also random and has proved results about the survival and extinction of the contact process in general nonhomogeneous environments that give results for CPRE as special cases.

All of the above discussion has concerned the one-dimensional case. Since random walk in random environment is poorly understood in $d > 1$, we do not expect to make much progress in $d > 1$. However, one thing is easy to show and it seems worthwhile to spell out some conjectures about CPRE in $d = 2$. We begin with the case $\Delta = \infty$, that is, bad sites are never occupied. If the density of good sites $(1 - p) \le p_c(\text{site})$, the critical value for oriented site percolation, then the CPRE dies out for all $\delta > 0$ since all the good sites lie in finite clusters. Conversely, we have the following theorem.

THEOREM 3. *Suppose* $\Delta = \infty$, $1 - p > p_c(\textit{site})$ *and* $\delta < \delta_c(\mathbb{Z})$, *the critical value for the contact process on* $\mathbb{Z}$. *Then the CPRE grows linearly, that is,* *there is a constant* $b > 0$ *so that on* $\Omega_\infty = \{\xi_t^0 \neq \emptyset \text{ for all } t\}$:

$$\liminf_{n\to\infty} \mathit{diameter}(\xi_t^0)/t \geq b \quad a.s.$$

Much more than the last conclusion should be true.

CONJECTURE 1. *Suppose* $\Delta = \infty$, $(1-p) > p_c(\mathit{site})$ *and let* $\delta_c(\mathit{CPRE}) = \sup\{\delta\colon P(\Omega_\infty) > 0\}$. *If* $\delta < \delta_c(\mathit{CPRE})$ *there is a nonrandom convex set D so that if* $C_0 = \{x$ *that can be reached from* 0 *by a path of good sites*$\}$ *and* $\eta > 0$ *then a.s. on* Ω_∞:

$$t(1-\eta)D \cap C_0 \subset \xi_t^0 \subset t(1+\eta)D \quad \text{for large } t.$$

In words, ξ_t^0 grows linearly and has an asymptotic shape. The C_0 is needed for the lower bound since $\xi_t^0 \subset C_0$. For a discussion of similar results in nonrandom environments see Chapters 1, 3 and 11 of Durrett (1988). It should not be too hard to prove the last result for $\delta < \delta_c(\mathbb{Z})$. The challenge is to prove it for $\delta < \delta_c$(CPRE).

In the case $\Delta < \infty$, we believe that the following is true.

CONJECTURE 2. *The conclusion of Theorem* 3 *holds in* $d > 1$ *when* $\Delta < \infty$ *and* $\delta < \delta_c(\mathit{CPRE})$.

Intuitively, in $d > 1$ the process is not forced to go through large regions of bad sites but instead can go around them. Proving Conjecture 2 is likely to be difficult. Only recently have Bezuidenhout and Grimmett (1989) shown that the ordinary contact process grows linearly whenever it survives.

The remainder of the paper is organized as follows: Theorem 1 is proved in Section 2, Theorem 2 is proved in Sections 3 and 4, and Theorem 3 is proved in Section 5. The three proofs are independent of each other and can be read in any order.

2. Results for $\delta = 0$. In this section we will prove Proposition 1 and Theorem 1. Our first step is to construct the process from a *graphical representation*. For each $x, y \in \mathbb{Z}$ with $|x-y| = 1$, let $\{T_n^{(x,y)}\colon n \geq 1\}$ be a Poisson process with rate 1, and let $\{U_n^x\colon n \geq 1\}$ be a Poisson process with rate δ_x (= the death rate at x). At times $T_n^{(x,y)}$ we draw an arrow from x to y to indicate that if x is occupied then y will become occupied (if it is not already). At times U_n^x, we put a δ at x. The effect of a δ is to kill the particle at x (if one is present).

We say there is a path from (x, s) to (y, t) and write $(x, s) \to (y, t)$ if there is a sequence of times $s_0 = s < s_1 < s_2 < \cdots < s_n < s_{n+1} = t$ and spatial locations $x_0 = x, x_1, \ldots, x_n = y$ so that:

(i) for $i = 1, 2, \ldots, n$ there is an arrow from x_{i-1} to x_i at time s_i;

(ii) the vertical segments $\{x_i\} \times [s_i, s_{i+1}]$, $i = 0, 1, \ldots, n$, do not contain any δ's.

It is sometimes convenient [e.g., in (1) below] to think of the path as being the continuous curve that is the union of the segments in (ii) and the arrows in (i). To define the process starting from an initial configuration A, we let

$$\xi_t^A = \{y\text{: for some } x \in A,\ (x,0) \to (y,t)\}.$$

Since the arrows in a path indicate births, and the absence of δ's indicate that the particles in the path did not die before they gave birth, it is easy to see that the above recipe constructs the contact process. For more details see Section 4a of Durrett (1988) or Chapter 6 of Liggett (1985). In Section 4 we make constant use of the following property.

(1) CROSSING LEMMA. *If paths from* $(x,s) \to (y,t)$ *and* $(x',s') \to (y',t')$ *intersect* (*i.e.*, *the corresponding curves do*), *then the union contains paths from* $(x,s) \to (y',t')$ *and* $(x',s') \to (y,t)$.

Our first result concerns the case $\delta_x \equiv \Delta$. Let

$$\sigma_n = \inf\{t\colon \{0\} \times [0,\infty) \to (n,t)\},$$

where $\{0\} \times [0,\infty) \to (n,t)$ is short for "there is a path from some $(0,s)$ with $s \geq 0$ to (n,t)."

PROPOSITION 1. *If* $\Delta > \delta_c$ *then as* $n \to \infty$,

(a) $$(\log \sigma_n)/n \to \gamma_\perp(\Delta) \quad \textit{in probability},$$

(b) $$(\log E\sigma_n)/n \to \gamma_\perp(\Delta).$$

Here and below, $\gamma_\perp(\Delta)$ and R^0 are the quantities defined in the Introduction.

PROOF. Let $T_1, T_2, \dots$ be the times at which there are arrows from 0 to 1 and let $N(t) = \sup\{k\colon T_k \leq t\}$. If there is a path from $\{0\} \times [0,t]$ to $\{n\} \times [0,t]$, then there is a path from $\{1\} \times \{T_1, \dots, T_{N(t)}\}$ to $\{n\} \times [0,t]$ that lies in $[1,n] \times [0,t]$. If $N(t) \leq 2t$, this probability is smaller than $2t \cdot P(R^0 \geq n-1)$. Since $P(N(t) > 2t) \to 0$ as $t \to \infty$ by the weak law of large numbers, taking $t = n^{-1}\exp(\gamma_\perp(\Delta)n)$ and using (1.3), we can conclude

(2a) $$P\big(\sigma_n \leq n^{-1}\exp(\gamma_\perp(\Delta)n)\big) \to 0,$$

(2b) $$\liminf_{n\to\infty}(\log E\sigma_n)/n \geq \gamma_\perp(\Delta).$$

To get bounds in the other direction, let $\tau^0 = \inf\{t\colon \xi_t^0 = \varnothing\}$. It is known [see Durrett (1984), page 1017] that if $\Delta > \delta_c$ then there is a constant $\gamma_\| > 0$ so that

$$\frac{1}{n}\log P(\tau^0 \geq n) \to -\gamma_\|(\Delta)$$

and

$$P(\tau^0 \geq n) \leq \exp\big(-\gamma_{\|}(\Delta)n\big).$$

The subscripts on our two gammas indicate that they concern the behavior parallel ($\|$) and perpendicular ($\perp$) to the flow of time.

Consider the events $A_j^n = \{(0, (j-1)n^2) \to \{n\} \times [(j-1)n^2, jn^2)\}$. If $\eta > 0$ then for large n,

$$P(A_j^n) = P(A_1^n) \geq P(R^0 \geq n) - P(\tau^0 \geq n^2) \geq \exp(-(1+\eta)\gamma_{\perp}(\Delta)n).$$

Since the A_j^n are independent and $(1 - x^{-1})^{kx} \leq e^{-k}$ for $x > 1$, it follows that

$$\text{(3a)} \qquad P\big(\sigma_n > kn^2 \exp((1+\eta)\gamma_{\perp}(\Delta)n)\big) \leq e^{-k}$$

for large n. Summing over k gives

$$\text{(3b)} \qquad E\sigma_n \leq \left(1 - e^{-1}\right)^{-1} n^2 \exp((1+\eta)\gamma_{\perp}(\Delta)n).$$

Part (a) now follows from (2a) and (3a), (b) follows from (2b) and (3b), and the proof of Proposition 1 is complete. □

Suppose now that the δ_x are i.i.d. and $= \Delta > \delta_c$ or $= 0$ with probabilities p and $1 - p$. Let

$$\rho_n = \inf\{t: n \in \xi_t^0\},$$
$$\Omega_\infty = \{\xi_t^0 \neq \varnothing \text{ for all } t\},$$
$$\mu = \gamma_{\perp}(\Delta)/\log(1/p).$$

THEOREM 1. (a) *If* $\mu < 1$ *there is a constant* c *so that* $\rho_n/n \to c$ *a.s. on* Ω_∞.
(b) *If* $\mu \geq 1$, *then* $(\log \rho_n)/\log n \to \mu$ *in probability on* Ω_∞.

PROOF. If we fix Δ and vary p, μ increases continuously from 0 to ∞, so the result for $\mu = 1$ follows from the ones for $\mu < 1$ and $\mu > 1$. Our second reduction is to argue that it is enough to prove the result when the origin is good. If $\delta_0 = \Delta$, let $l = \sup\{x < 0: \delta_x = 0\}$, $r = \inf\{x > 0: \delta_x = 0\}$ and $\tau = \inf\{t: r \in \xi_t^0\}$. It is easy to see that $\Omega_\infty = \{\tau < \infty\}$, since if the process lives forever, it must occupy l or r and once l is occupied, r will eventually become occupied, since the particle at l will never die. Once r becomes occupied, it will never become vacant. The definition of r assures us that the environment to the right of r is i.i.d. with the original distribution. This shows that the time to reach $n > r$ has the same distribution as $\tau +$ (the time to reach $n - r$ when the origin is good), so we can without loss of generality suppose $\delta_0 = 0$.

Let $b_0 = 0$ and for $i \geq 1$ let

$$a_i = \inf\{m > b_{i-1}: \delta_m = \Delta\},$$
$$b_i = \inf\{m > a_i: \delta_m = 0\},$$
$$L_i = b_i - a_i,$$
$$N(n) = \inf\{i: b_i \geq n\}.$$

Notice that $\delta_m = \Delta$ on $[a_i, b_i)$ and $\delta_m = 0$ on $[b_i, a_{i+1})$, so L_i is the length of the ith "bad stretch," and $N(n)$ is the number of bad stretches that touch the interval $[0, n]$. For $i \geq 1$ let

$$U_i = \inf\{t: a_i - 1 \in \xi_t^0\},$$
$$V_i = \inf\{t: b_i \in \xi_t^0\},$$
$$W_i = V_i - U_i,$$
$$t_k = (\rho_{k+1} - \rho_k)1_{(\delta_k = \delta_{k+1} = 0)}.$$

W_i is the waiting time to cross the ith bad stretch. The t_k are defined so that

$$\sum_{k=0}^{n-1} t_k + \sum_{i=1}^{N(n)-1} W_i \leq \rho_n \leq \sum_{k=0}^{n-1} t_k + \sum_{i=1}^{N(n)} W_i. \tag{4}$$

The t_k, $k \geq 1$, are a one-dependent stationary sequence with $P(t_k = 0) = 1 - (1 - p)^2$ and

$$P(t_k > t) = (1 - p)^2 e^{-t},$$

so it follows from the ergodic theorem that

$$\frac{1}{n} \sum_{k=0}^{n-1} t_k \to (1 - p)^2 \quad \text{a.s.} \tag{5}$$

To attack the other term in (4), we begin by observing that as $n \to \infty$,

$$N(n)/n \to p(1 - p) \quad \text{a.s.}, \tag{6}$$

since there is one interval for each k with $\delta_k = 0$ and $\delta_{k+1} = \Delta$. At this point the argument divides into the two cases.

CASE 1. $\mu < 1$. In this case $EW_i < \infty$. To prove this, pick $\eta > 0$ so that $\mu(1 + \eta) < 1$ and observe that part (b) of Proposition 1 implies that there is a constant C so that

$$E\sigma_k \leq C \exp((1 + \eta)\gamma_\perp(\Delta)k). \tag{7}$$

Observing that $P(L_i = k) = (1 - p)p^{k-1}$ and conditioning on the length of the interval gives

$$EW_i \leq \sum_{k=1}^{\infty} (1 - p)p^{k-1} C \exp((1 + \eta)\gamma_\perp(\Delta)k) < \infty \tag{8}$$

by the choice of η. With (8) established the rest is routine. The strong law of large numbers implies

$$\frac{1}{m} \sum_{k=1}^{m} W_k \to EW_1 \quad \text{a.s.}, \tag{9}$$

and combining this with (6) yields

$$\frac{1}{n}\sum_{k=1}^{N(n)} W_k \to p(1-p)EW_1. \tag{10}$$

Since the W_k are i.i.d. with $EW_1 < \infty$, an application of the Borel–Cantelli lemma implies $W_m/m \to 0$ a.s. as $m \to \infty$, and it follows from (6) that

$$W_{N(n)}/n \to 0 \quad \text{a.s.} \tag{11}$$

Using (5), (10) and (11) in (4) now gives

$$\rho_n/n \to (1-p)\{(1-p) + pEW_1\} \quad \text{a.s.,}$$

proving part (a).

CASE 2. $\mu > 1$. When $EW_i = \infty$, the situation changes drastically and the largest W_i with $1 \le i < N(n)$ dictates the size of the sum. We begin by observing that an easy argument left to the reader shows

$$(\log k)^{-1} \max_{1 \le j \le k} L_j \to 1/\log(1/p) \quad \text{a.s.} \tag{12}$$

[For a much more general result see Barndorff-Nielsen (1961).] Let

$$M_n = \max_{1 \le j < N(n)} L_j.$$

Letting $k = N_n - 1$ and using (6), which implies $\log(N_n - 1)/\log n \to 1$ a.s., we get

$$M_n/(\log n) \to 1/\log(1/p) \quad \text{a.s.} \tag{13}$$

To get a lower bound on ρ_n, we look at the time to cross the longest interval. If that interval has length greater than or equal to L then (2a) implies that

$$\rho_n \ge L^{-1}\exp(\gamma_\perp(\Delta)L) \tag{14}$$

with high probability. Letting $\eta > 0$ and

$$L = (1-\eta)\log n/\log(1/p)$$

in (14), we have shown

$$\rho_n \ge n^{(1-\eta)\mu}\log(1/p)/(1-\eta)\log n \tag{15}$$

with high probability which proves half of (b).

To get an upper bound on ρ_n, we observe that the first term on the right in (4) is of order n by (5) and hence can be ignored in proving (b). To estimate the second, we observe that $N_n \le n$ and if $\eta > 0$ then by (13) we can assume that all of the first n bad intervals have length at most

$$K_n = (1+\eta)\log n/\log(1/p).$$

Now from part (b) of Proposition 1 there is a constant C so that

$$(16)\qquad E(W_i; L_i \le J) \le \sum_{j=1}^{J} (1-p)p^j C \exp((1+\eta)\gamma_\perp(\Delta)j).$$

The assumption $\mu > 1$ implies $p\exp((1+\eta)\gamma_\perp(\Delta)) > 1$, so

$$(17)\qquad E(W_i; L_i \le J) \le C\{\exp((1+\eta)\gamma_\perp(\Delta) - \log(1/p))\}^J.$$

Here and in what follows, C is a constant whose value is unimportant and will change from line to line. Setting $J = K_n$ and throwing away a term less than 1, we see

$$(18)\qquad E(W_i; L_i \le K_n) \le Cn^{\mu(1+\eta)^2 - 1}.$$

Summing and recalling that $\mu > 1$ and $\eta > 0$,

$$(19)\qquad E\left(\sum_{i=1}^{n} W_i; \max_{1 \le i \le n} L_i \le K_n\right) \le Cn^{\mu(1+\eta)^2}.$$

As remarked above $P(\max_{1 \le i \le n} L_i \le K_n) \to 1$, so the described upper bound follows from Chebyshev's inequality. □

3. Blocks and gaps. In this section we will define the terms "k block" and "k gap" used in the Introduction and give the argument sketched in the Introduction. The first step is to reduce the problem to the case in which the bad sites have small density. Let $\varepsilon > 0$ and pick κ so that $p^\kappa < \varepsilon$. Divide $[0, 1, 2, \dots\}$ into intervals $I_j = \{(j-1)\kappa, \dots, j\kappa - 1\}$, $j \ge 1$, with length κ. If $\delta_n = \delta$ for some $n \in I_j$, we set $e_j^0 = G$ but if $\delta_n = \Delta$ for all $n \in I_j$ we set $e_j^0 = B$. The superscript 0 indicates we are in the 0th step of the construction. All the definitions we will give and the computations we will do are valid (with some minor modifications) for $\kappa > 1$ if one replaces "site" in the arguments below "by interval of length κ." For simplicity, we will only give the proof for the case $\kappa = 1$. In this case $e_n^0 = G$ if $\delta_n = \delta$ and $e_n^0 = B$ if $\delta_n = \Delta$.

Let $T_0^0 = 0$, $Y_0^1 = 0$ and for $n \ge 1$ let

$$T_n^0 = \inf\{m > T_{n-1}^0 : e_m^0 = B\},$$
$$X_n^1 = T_n^0 - T_{n-1}^0 - 1,$$
$$Y_n^1 = 1.$$

For example, if the sequence e_n^0 is

$$\begin{matrix} & & & & & & 7 & 8 & & & & 12 & & & & & & & & 20 \\ G & G & G & G & G & G & B & B & G & G & G & B & G & G & G & G & G & G & G & B \end{matrix}$$

then

$$T_1^0 = 7,\quad T_2^0 = 8,\quad T_3^0 = 12,\quad T_4^0 = 20, \dots,$$
$$X_1^1 = 6,\quad X_2^1 = 0,\quad X_3^1 = 3,\quad X_4^1 = 8, \dots.$$

It follows from the definition that $\{X_n^1: n \geq 1\}$ and $\{Y_n^1: n \geq 1\}$ are i.i.d. sequences. Notice that $X_n^1 = 0$ if there are two consecutive B's. So X_n^1 is the length of the nth block of G's and Y_n^1 is the length of the nth block of B's, with the understanding that two consecutive B's are separated by a block of G's of length 0.

The intervals (T_{n-1}^0, T_n^0) are our "1 blocks." The single bad sites in between are thought of as gaps between the blocks and so are called "1 gaps." If $X_n^1 > N_1$ (the first of a rapidly increasing sequence of constants N_k that will be specified later) we call the nth "1 block" good and set $e_n^1 = G$, otherwise we call it bad and set $e_n^1 = B$. The next step is to combine the good "1 blocks" to make "2 blocks." In order to make the proof work, we need to give our "2 blocks" two tries to get started. Let $T_0^1 = 0$ and for $n \geq 1$ let

$$S_n^1 = \begin{cases} T_{n-1}^1 + 2 & \text{if } e^1(T_{n-1}^1 + 1) = B, \\ T_{n-1}^1 + 1 & \text{if } e^1(T_{n-1}^1 + 1) = G, \end{cases}$$

where we have written $e^1(m)$ for e_m^1 (with $m = T_{n-1}^1 + 1$) to avoid double subscripts. Let

$$T_n^1 = \inf\{m \geq S_n^1 : e_m^1 = B\}$$

and

$$U_n^1 = T_n^1 - S_n^1.$$

For example, if the sequence e_n^1 is

$$\begin{matrix} & & & 4 & 5 & & & & & & & & 13 & 14 & 15 & & & & 19 & & & 22 \\ G & G & G & B & B & G & G & G & G & G & G & G & B & B & B & G & G & G & B & G & G & B \end{matrix}$$

then

$$\begin{array}{lllll} S_1^1 = 1, & S_2^1 = 6, & S_3^1 = 15, & S_4^1 = 16, & S_5^1 = 20, \ldots, \\ T_1^1 = 4, & T_2^1 = 13, & T_3^1 = 15, & T_4^1 = 19, & T_5^1 = 22, \ldots, \\ U_1^1 = 3, & U_2^1 = 7, & U_3^1 = 0, & U_4^1 = 3, & U_5^1 = 2, \ldots. \end{array}$$

S_n^1 and $T_n^1 - 1$ are the indices of the first and last "1 blocks" in the nth "2 block." U_n^1 is the number of "1 blocks" it contains. For $n \geq 1$, we let

$$X_n^2 = X^1(S_n^1) + Y^1(S_n^1) + X^1(S_n^1 + 1) + \cdots + Y^1(T_n^1 - 2) + X^1(T_n^1 - 1),$$

$$Y_n^2 = \begin{cases} Y^1(T_{n-1}^1) + X^1(T_{n-1}^1) + Y^1(T_{n-1}^1 + 1) & \text{if } S_n^1 = T_{n-1}^1 + 1, \\ Y^1(T_{n-1}^1) + X^1(T_{n-1}^1) + Y^1(T_{n-1}^1 + 1) & \\ \quad + X^1(T_{n-1}^1 + 1) + Y^1(T_{n-1}^1 + 2) & \text{if } S_n^1 = T_{n-1}^1 + 2. \end{cases}$$

It follows immediately from the definition that $\{X_n^2: n \geq 1\}$ and $\{Y_n^2: n \geq 1\}$ are i.i.d. To understand the last two definitions, recall that "1 blocks" are intervals of G's that are separated by "1 gaps" of length Y_n^1. X_n^2 then gives the number of sites in the nth "2 block" and Y_n^2 the number of sites in the "2 gaps" between the "2 blocks." Notice that

$$Y_n^2 \le 3 + 2N_1, \tag{1}$$

since $X^1(T_n^1)$ [and $X^1(T_n^1+1)$ if present] are less than or equal to N_1. The careful reader will have noticed that we neglected to define Y_0^2 = the number of sites in the "2 gap" to the left of the first "2 block." Let

$$Y_0^2 = \begin{cases} Y_0^1 & \text{if } S_1^1 = 1, \\ Y_0^1 + X_1^1 + Y_1^1 & \text{if } S_1^1 = 2. \end{cases}$$

The distribution of Y_0^2 is different from that of Y_n^2, $n \ge 1$. Indeed one of the goals of this section is to show that with probability 1, Y_0^k stops growing.

The rest of the construction is the same as the last step. Let $k \ge 2$. If $X_n^k > N_k$ we set $e_n^k = G$ and call the nth "k block" good, otherwise $e_n^k = B$ and we call it bad. We set $T_0^k = 0$ and for $n \ge 1$ define S_n^k, T_n^k and U_n^k by replacing the superscript 1 by k in the earlier definitions. Then X_n^{k+1} and Y_n^{k+1} are defined by replacing superscripts 2 by $k+1$ and 1 by k. The only thing that changes is that if ν_k is the largest possible "k gap" then

$$\nu_{k+1} \le 2N_k + 3\nu_k, \tag{2}$$

where $\nu_1 = 1$, since a "$k+1$ gap" is made up of at most three "k gaps" and two bad "k blocks."

(2) gives upper bounds on the sizes of the gaps. Our next goal is to show that the blocks grow quickly. To do this, we let $\lambda \in (0,1)$, let

$$N_k = \lambda^{-(1.1)^k}$$

and define $\beta_i(k)$ for $i = 1, 2$ by

$$P(X_1^k \le N_k) = \lambda^{\beta_1(k)},$$

$$P(X_1^k \le N_{k+1}) = \lambda^{\beta_2(k)}.$$

Our goal is to show that if ε is chosen so that

$$\beta_1(k) \ge 30(1.3)^k, \qquad \beta_2(k) \ge 20(1.3)^k \tag{$*$}$$

holds for $k = 0$, then $(*)$ will hold for all $k \ge 1$. The next inequality explains why we gave our "$k+1$ blocks" two tries to get started.

$$\beta_1(k+1) \ge 2\beta_2(k) \text{ and hence if } (*) \text{ holds then} \qquad \beta_1(k+1) \ge 40(1.3)^k \ge 30(1.3)^{k+1}. \tag{3}$$

PROOF. (3) follows from the observation that

$$P(X_1^{k+1} \le N_{k+1}) \le P(\max(X_1^k, X_2^k) \le N_{k+1}) = P(X_1^k \le N_{k+1})^2.$$

The equality comes from the fact that the X_i^k are i.i.d. To prove the inequality, we consider three cases:

(i) $\quad X_1^k > N_{k+1},$

(ii) $\quad N_k < X_1^k \le N_{k+1}, \qquad X_2^k > N_{k+1},$

(iii) $\quad X_1^k \le N_k, \qquad X_2^k > N_{k+1}.$

In the first two cases it is clear that $X_1^{k+1} > N_{k+1}$ since X_1^k (resp. X_2^k) is included in the sum of X_1^{k+1}. In the third case it is true thanks to our convention for starting "$k + 1$ blocks." $S_1^k = 2$, so X_2^k is included in the sum for X_1^{k+1}. □

(4) Suppose $(*)$ holds. Then $\beta_2(k+1) \ge 20(1.3)^{k+1}$.

PROOF. We begin by observing the X_j^k are identically distributed and $N_k \ge 1$ so

$$P\left(X_1^{k+1} \le N_{k+2}\right) \le P\left(X_j^k \le N_k \text{ for some } j \le N_{k+2}\right)$$
$$\le N_{k+2}\lambda^{\beta_1(k)} \le \lambda^{\{-(1.1)^{k+2}+30(1.3)^k\}}.$$

This implies

$$\beta_2(k+1) \ge 30(1.3)^k - (1.1)^{k+2}$$
$$\ge 20(1.3)^{k+1} + (1.3)^k\{4 - (1.1)^2\}$$
$$\ge 20(1.3)^{k+1}. \qquad \square$$

(3) and (4) together show that if $(*)$ holds for $k = 0$ it holds for all k. Now

$$(5) \qquad \sum_{k=1}^{\infty} P\left(X_1^k \le N_k\right) = \sum_{k=1}^{\infty} \lambda^{\beta_1(k)} \le \sum_{k=1}^{\infty} \lambda^{30(1.3)^k} < \infty.$$

So by the Borel–Cantelli lemma $X_1^k > N_k$ for $k \ge K(\omega)$ where $K(\omega) < \infty$ a.s. This implies that $S_0^k = 1$, and $Y_0^k = Y_0^{K(\omega)}$ for $k \ge K(\omega)$. That is, the initial "k gap" stops growing after a while. Set $y^* = 1 + Y_0^{K(\omega)}$. Note that y^* is the first site in the first k block for $k \ge K(\omega)$. It is also the first site in some "k block" for $k < K(\omega)$. In the next section we will show that if δ is small the CPRE has probability greater than or equal to $1/2$ of surviving starting with y^* occupied.

4. Renormalization argument. In this section we will complete the proof of Theorem 2. Recall

$$(1) \qquad N_k = \lambda^{-(1.1)^k}.$$

Throughout the section we will suppose

$$(2) \qquad \lambda < 1000^{-100}.$$

This ridiculously small value of λ is needed for (14) below. To help the reader find his or her way through the forest of details, the section is divided into five subsections.

4.1. *Preliminaries on the sizes of blocks and gaps.* Before getting involved in probabilistic details, we will do some arithmetic. By (3.2) all the "k gaps" are smaller than ν_k where $\nu_1 = 1$ and

$$\nu_k \le 2N_{k-1} + 3\nu_{k-1} \quad \text{for } k \ge 2.$$

(3) LEMMA. *For all $k \ge 1$, $\nu_k \le 3N_{k-1}$ and $N_{k-1} \le N_k/9$, so $\nu_k \le N_k/3$.*

PROOF. The third inequality is a consequence of the first two. Iterating the bound above and using $\nu_1 = 1 < 2 < 2N_0$ to neaten things up gives

$$\nu_k \le \sum_{j=1}^{k} 2 \cdot 3^{j-1} N_{k-j}.$$

To estimate the sum, we observe $N_k = (n_{k-1})^{1.1}$, so for $k \ge 1$,

$$N_{k-1}/N_k = (N_{k-1})^{-0.1} \le \lambda^{0.1} < 1/9$$

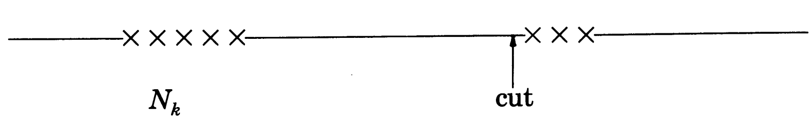
when $\lambda < 9^{-10}$. Consequently,

$$\nu_k \le N_{k-1} \sum_{j=1}^{\infty} 2 \cdot 3^{j-1} \cdot 9^{-(j-1)} = 3N_{k-1}. \qquad \square$$

In the last section we described a procedure that started with "1 blocks" = intervals of good sites and then inductively combined "k blocks" to get "$k+1$ blocks." For the argument below it is inconvenient if the blocks are too long so our next step is to show:

(4) LEMMA. *By breaking the "k blocks" into pieces that begin and end with "$(k-1)$ blocks" we can without loss of generality suppose that the length of each "k block" is in $[N_k, 3N_k]$.*

PROOF. We proceed by induction. The conclusion is clearly true when $k = 1$ since any string of G's with length in $[jN_1, (j+1)N_1)$ can be broken into $j-1$ strings of length N_1 and 1 with length in $[N_1, 2N_1)$. When $k > 1$ and we have a "k block" of length greater than $3N_k$, we make our first cut at the first right endpoint of a "$k-1$ block" that we encounter after N_k. In the worst possible case the point N_k lies in a "$(k-1)$ gap" (marked by $\times\times\times\times$'s in the diagram below).



Even in this case, the piece to the left of the cut is

$$\le N_k + \nu_{k-1} + 3N_{k-1} \le N_k + N_{k-1}/3 + 3N_{k-1} \le N_k(37/27)$$

by (3). The "$(k-1)$ gap" after the cut becomes a "k gap," and we are left with a "k block" of length at least $(3 - 38/27)N_k > N_k$. □

4.2. *The three inequalities that are the key to the proof of Theorem* 2. Let

$$M_k = \alpha^{-6N(k-1)}, \tag{5}$$

where $N(k-1)$ is an alternate way of writing N_{k-1} to avoid subscripted superscripts and $\alpha > 0$ satisfies

$$\alpha \le e^{-2\Delta}(1 - e^{-1}), \quad \alpha < e^{-21/2} \quad \text{and} \quad \alpha < \alpha_0, \tag{6}$$

where α_0 is a constant we will introduce at the end of this subsection. Let $[a, b]$ and $[c, d]$ be "k blocks" separated by a "k gap" (b, c). Let

$$C_k = \{\{a\} \times \{0\} \to \{b\} \times [0, 3N_k M_{k-1}] \text{ in } [a, b] \times \mathbb{R}\},$$
$$D_k = \{[a, b] \times \{0\} \to [a, b] \times \{2M_k\} \text{ in } [a, b] \times \mathbb{R}\},$$
$$E_{k+1} = \{\{a\} \times [0, M_k] \to \{d\} \times [0, M_k] \text{ in } [a, d] \times \mathbb{R}\},$$

where $S \to T$ in $[a, b] \times \mathbb{R}$ means there is a path in the graphical representation (described in Section 2) from some point in S to some point in T that stays in $[a, b] \times \mathbb{R}$. We will explain the rationale behind the choice of these events later. To see the relative sizes of the boxes involved, note

$$M_k/M_{k-1} = \alpha^{-6\{N(k-1)-N(k-2)\}} \ge \alpha^{-5N(k-1)} \tag{7}$$

by (3), and $N_k = N_{k-1}^{1.1}$ by (1), so $3N_k M_{k-1}$ is much smaller than M_k.

Let $\pi_1 = 1/4$ and $\pi_j = 36\alpha^{N(j-2)/72}$ for $j \ge 2$. We will prove by induction that

$$(*) \qquad \begin{aligned} p_k &\equiv P(C_k) \ge \prod_{j=1}^{k} (1 - \pi_j), \\ q_k &\equiv 1 - P(D_k) \le \alpha^{N(k-1)}, \\ r_k &\equiv 1 - P(E_k) \le \alpha^{N(k-1)}. \end{aligned}$$

The $\equiv$ must not be taken too literally. The exact probability of the events in question depends on the lengths of the "k blocks" and "k gaps" involved and on the lengths of the "j blocks" and "j gaps" they contain. What we are proving, of course, are bounds that hold whenever all "j blocks" have length in $[N_j, 3N_j]$.

Before proving $(*)$, we will indicate why it implies Theorem 2. We begin by noting that the assumptions that $\lambda < 1000^{-100}$ and $\alpha < 1/27$ are more than enough to conclude:

(8) LEMMA. $\Sigma_{j=2}^{\infty}\pi_j < 1/8$ *and hence* $\Pi_{j=1}^{k}(1 - \pi_j) \ge 1 - \Sigma_{j=1}^{k}\pi_j \ge 5/8$ *for all* k.

PROOF. $N(i+1) = N(i)^{1.1} \geq \lambda^{-0.1}N(i)$ for all $i \geq 0$, so our choice of λ implies $N(i+1)/72 \geq N(i)/72 + 1$. Using this and the definition of π_j,

$$\sum_{j=2}^{\infty} \pi_j \leq 36\alpha^{N(0)/72} \sum_{j=2}^{\infty} \alpha^{(j-2)} \leq 72 \cdot 2^{-1000^{99}},$$

since $\alpha < 1/2$, $N(0) = \lambda^{-1}$ and $72 < 100$. The right-hand side is much smaller than $1/8$ so the proof of (8) is complete. □

Applying $P(C_k) \geq 5/8$ to the "k block" $[a, b]$ containing the point y^* identified at the end of the last section, we conclude that with probability greater than or equal to $5/8$ there is a path from $\{y^*\} \times \{0\}$ to $\{b\} \times [0, 3N_k M_{k-1}]$. (Recall $a = y^*$.) Now $\alpha < 1/27$ and $N(0) = \lambda^{-1} > 1000^{100}$ are more than enough [with $(*)$] to imply $P(D_k^c) \leq 1/8$. As we observed after (7), $3N_k M_{k-1}$ is much smaller than M_k for all k, so the crossing lemma implies that on $C_k \cap D_k$, the contact process starting with y^* occupied survives up to M_k. $P(C_k \cap D_k) \geq 1/2$, so $(*)$ implies the conclusion of Theorem 2.

To prove $(*)$, we will use C_k and D_k to produce E_{k+1} and then we use D_k and E_{k+1} to produce D_{k+1} and C_{k+1}. To get the induction started, we observe that if $\delta = 0$ and α_0 is chosen small enough then $P(C_1) \geq 7/8$ for $\alpha < \alpha_0$ and $P(D_1) = 1$. So if δ is small then the inequalities for $P(C_1)$ and $P(D_1)$ hold. This is the only place where δ appears in the proof.

4.3. *Estimating* $P(E_{k+1})$. To cross the "k gap" (b, c), we observe that if $l = c - b$ then:

(9) LEMMA. $P((b,0) \to \{c\} \times [l-1, l]) \geq \alpha^l$.

PROOF. The event in question occurs if for $0 \leq m < l$ we have

$$A_m = \Big\{\text{there is birth from } m \text{ to } m+1 \text{ during } [k, k+1) \text{ and no death at } m \text{ during } \big[(m-1)^+, m+1\big)\Big\}.$$

[The $(m-1)^+ = \max\{m-1, 0\}$ is to take care of $m = 0$.] The bound follows by noting that the A_m are independent and have $P(A_m) \geq \alpha$ by (6). □

Note: For ease of later reference we will call the event used to get the lower bound in (9) "drilling through (b, c)." Observe that $l \leq \nu_k$ by (3.2).

The probability in (9) is small but we compensate for that by giving ourselves lots of opportunities. See Figure 2. We look for crossings

$$(a, 7jN_k M_{k-1}) \to \{b\} \times [7jN_k M_{k-1}, (7j+3)N_k M_{k-1}]$$

for $0 \leq j \leq M_k/7N_k M_{k-1}$. ($j$ is an integer.) These events are C_k's and by induction and (8) have probability greater than or equal to $1/2$. At the end of

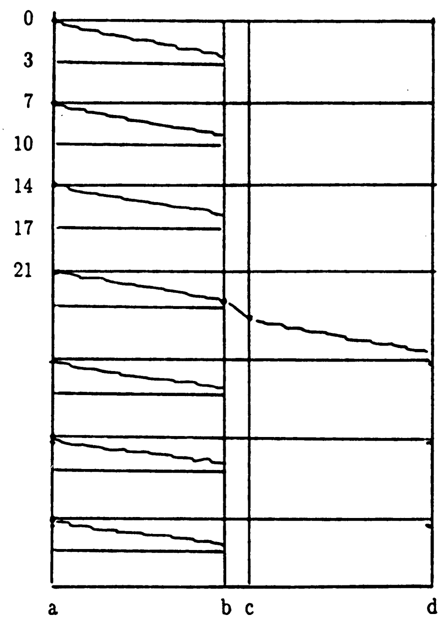


FIG. 2.

each successful crossing we try to drill through (b, c). If the drilling succeeds, we reach c at a time

$$\leq (7j + 3)N_k M_{k-1} + \nu_k \leq (7j + 4)N_k M_{k-1}$$

by (3). At the end of a successful drilling there is a probability greater than or equal to $1/2$ of a crossing from c to d before time $(7j + 7)N_k M_{k-1}$. Combining the three estimates we have a success probability of at least

$$\alpha^l/4 \geq \alpha^{\nu_k}/4$$

on each attempt.

We have arranged things so that the successive attempts are independent. To count the number of attempts, $\mu(k)$, we observe

$$(10) \qquad \mu(k) \equiv M_k/7N_k M_{k-1} \geq \alpha^{-5N(k-1)}/7N_k$$

by (7). [Here and in what follows we write $N(k-1)$ for N_{k-1} to avoid subscripted superscripts.] The successive attempts are independent so

$$r_{k+1} = 1 - P(E_{k+1}) \leq (1 - \alpha^{\nu_k}/4)^{\mu(k)} \leq \exp(-\mu(k)\alpha^{\nu_k}/4),$$

since $1 - x \leq e^{-x}$. The width of the "k gap" $\nu_k \leq 3N_{k-1}$ by (3), so using (10) now gives

$$r_{k+1} \leq \exp\left(-\alpha^{-2N(k-1)}/28N_k\right).$$

To neaten things up, observe that $\exp(y) \geq y^3/3!$ for $y > 0$, and $N_k = (N_{k-1})^{1/1}$, so

$$r_{k+1} \leq \exp\left(-8N(k-1)^{1.9} \log(1/\alpha)^3/(28 \cdot 6)\right)$$
$$\leq \exp\left(-N(k)\log(1/\alpha)\left\{\log(1/\alpha)^2/(7 \cdot 3)\right\}\right) \leq \alpha^{N(k)},$$

since our choice of α in (6) implies $\log(1/\alpha)^2 \geq 21$. This proves the third inequality in $(*)$.

4.4. *Estimation of $P(C_{k+1})$ and $P(D_{k+1})$ by a renormalized site construction.* To begin, we observe that a "$(k+1)$ block" by assumption has length in $[N_{k+1}, 3N_{k+1}]$ so it is composed of L "k blocks," where

$$N_{k+1}/(3N_k + \nu_k) \leq L \leq 3N_{k+1}/N_k.$$

$N_{k+1} = (N_k)^{1.1}$ and $\nu_k \leq N_k/3$ by (3), so

$$(0.3)N_k^{0.1} \leq L \leq 3N_k^{0.1}. \tag{11}$$

Suppose that the "k blocks" that make up our "$(k+1)$ block" are (listed from left to right) $[a_1, b_1], \dots, [a_L, b_L]$. ($a_1 = a$, $a_L = b$.) Let

$$\mathscr{L}_1 = \{(i,j): 1 \leq i \leq L,\ j \geq 1,\ i+j \text{ is even}\}.$$

We say that $(i,j) \in \mathscr{L}_1$ is open if

$$[a_i, b_i] \times \{(j-1)M_k\} \to [a_i, b_i] \times \{(j+1)M_k\},$$
$$\{b_i\} \times [jM_k, (j+1)M_k] \to \{a_{i-1}\} \times [jM_k, (j+1)M_k],$$
$$\{a_i\} \times [jM_k, (j+1)M_k] \to \{b_{i+1}\} \times [jM_k, (j+1)M_k].$$

When $i = 1$ (resp. $i = L$) we do not require the second (resp. third) path to exist since these events involve boxes that are undefined. See Figure 3 where the paths for $(i,j) = (1,1)$ and $(2,2)$ are drawn as solid lines.

The events are defined in such a way that

(12a) the probability that a site is closed, ρ, satisfies $\rho \leq q_k + 2r_{k+1} \leq 3\alpha^{N(k-1)}$ by induction;

(12b) if $|i - i'| + |j - j'| > 2$ then the events "(i,j) is open" and "(i',j') is open" are independent;

(12c) if there is a path from $(i,1)$ to (j,n) on $\mathscr{L}_1$, that is, a sequence of open sites $(i_1, 1), \dots, (i_n, n)$ in $\mathscr{L}_1$ with $i_1 = i$, $i_n = j$, and $|i_m - i_{m-1}| = 1$ for $2 \leq m \leq n$, then there is path in the contact process from $[a_i, b_i] \times \{0\}$ to $[a_j, b_j] \times \{(n+1)M_k\}$ in $[a,b] \times \mathbb{R}$.

In the terminology of Section 9 of Durrett (1984), we have just demonstrated that the CPRE dominates one dependent oriented percolation on $\mathscr{L}_1$.

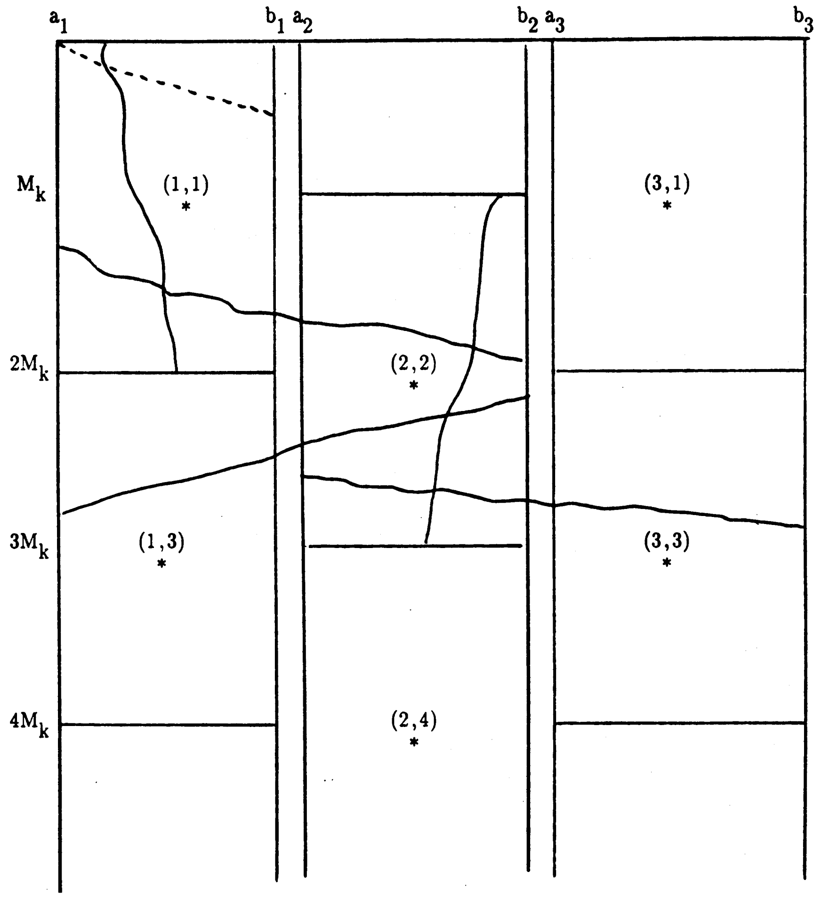


FIG. 3.

The desired bound on $P(D_{k+1})$ now follows from:

(13) LEMMA. *If $\rho < 6^{-36}$ then the probability that there is no path from $\{1,\dots,L\}\times\{1\}$ to $\{1,\dots,L\}\times K$ on $\mathscr{L}_1$ is at most $2K(3\rho^{1/36})^L$.*

The proof of (13) is a standard "contour argument." To avoid interrupting the flow of ideas, it is postponed to the end of the section. (12a) tells us that

$$\rho \le 3\alpha^{N(k-1)} \le 3(2^{-1000^{100}}) < 6^{-36}.$$

Taking $K = M_{k+1} (\ge 2M_{k+1}/M_k)$ in (13) gives

$$q_{k+1} \le 2\cdot 3^{37L/36}\left(\alpha^{N(k-1)}\right)^{L/36} M_{k+1} \le 2\cdot 3^{4N(k)^{0.1}}\left(\alpha^{N(k-1)N(k)^{0.1}/120}\right)\alpha^{-6N(k)},$$

since $(0.3)N_k^{0.1} \le L \le 3N_k^{0.1}$ by (11). Now

$$N(k-1)N(k)^{0.1} = N(k)^{1/1.1}N(k)^{0.1} = N(k)^{1.11/1.1}$$

and

$$(14) \qquad N(k)^{0.01/1.1} \geq \lambda^{-0.01} \geq 1000 \geq 8 \cdot 120$$

by our assumption on λ. So $N(k-1)N(k)^{0.1}/120 - 6N(k) \geq 2N(k)$ and we get

$$q_{k+1} \leq 2 \cdot 81^{N(k)^{0.1}} \alpha^{2N(k)} \leq \alpha^{N(k)},$$

since $\alpha \leq 1/81$. We have thus proved the second inequality in $(*)$.

To estimate $p_{k+1} = P(C_{k+1})$, we let

$$\mathscr{L}_2 = \{(i,j): i, j \geq 1 \text{ and } i + j \text{ is even}\}$$

and use a second result about oriented percolation:

(15) LEMMA. *If* $\rho < 6^{-72}$ *and* $L \leq K/2$ *then*

$$P((1,1) \to \{L\} \times [0,K] \text{ on } \mathscr{L}_2) \geq 1 - 12\rho^{1/72}.$$

Again the proof is postponed to the end of the section. Now if

$$\{a_1\} \times \{0\} \to \{b_1\} \times [0, 3N_k M_{k-1}] \text{ in } [a_1, b_1] \times \mathbb{R}$$

(i.e., C_k occurs in the first "k block" $[a_1, b_1]$; see the dotted line in Figure 3) and we have

$$F_{k+1} = \{\text{there is a path from } (1,1) \text{ to } [L,\infty) \times \{3N_{k+1}\} \text{ on } \mathscr{L}_2\},$$

then the crossing lemma implies C_{k+1} occurs in the "$(k+1)$ block" $[a, b]$. To check this, recall that C_{k+1} is the event

$$\{a\} \times \{0\} \to \{b\} \times [0, 3N_{k+1}M_k] \text{ in } [a, b] \times \mathbb{R}$$

and the site (i, j) in $\mathscr{L}_2$ corresponds to the space time box $[a_i, b_i] \times [(j-1)M_k, (j+1)M_k]$ in the graphical representation of the contact process.

C_k and E_{k+1} are increasing events on the graphical representation, so it follows from Harris's inequality [see Durrett (1988), page 129, and use the construction in Section 5c to extend the result to continuous time] that

$$p_{k+1} \geq p_k P(E_{k+1}).$$

(12a) tells us

$$\rho \leq 3\alpha^{N(k-1)} \leq 3(2^{-1000^{100}}) < 6^{-72}.$$

Now $L \leq 3N_{k+1}/N_k \leq N_{k+1}/2$ by (4) and the definition of N_k, so using (15) we have

$$P(E_{k+1}) \geq 1 - 12(3\alpha^{N(k-1)})^{1/72}.$$

Since $3^{1/72} \leq 3$ this proves the first inequality in $(*)$ and completes the induction step.

4.5. *Estimates for one-dependent percolation.* To complete the proof, we need to establish (13) and (15). We use the notation of Section 10 of Durrett (1984) which the reader can consult for more details.

PROOF OF (13). Let $\mathscr{C}$ be the set of points that can be reached by a path from $\{1,\dots,L\}\times\{1\}$. Let $D=\{(a,b): |a|+|b|\le 1\}$ and

$$W=\bigcup_{(i,j)\in\mathscr{C}}(i,j)+D.$$

D is for diamond and W is for wet region. The boundary of the unbounded component of $[1,L]\times[0,\infty)-W$ is denoted by Γ and called the contour associated with $\mathscr{C}$. We orient the boundary of D in a counterclockwise fashion and use the orientation this induces on Γ. If there is no path from $\{1,\dots,L\}\times\{1\}$ to $\{1,\dots,L\}\times\{K\}$ then there is a contour starting on $\{L\}\times\{1,2,\dots,K\}$ and going to $\{1\}\times\{1,2,\dots,K\}$. The number of starting points for the contour is less than or equal to K, and once the starting point is chosen there are less than or equal to 3^n contours of length n.

The next thing to argue is that each contour has probability less than or equal to $\rho^{n/36}$. To do this, we call segments of Γ that decrease (resp. increase) the x coordinate segments of type 1 (resp. type 2), and let n_i be the number of segments of type i. It is easy to see that

$$n_1+n_2=n \quad \text{and} \quad n_1-n_2=L,$$

so $n_1\ge n/2$. For reasons indicated in Section 10 of Durrett (1984), there must be greater than or equal to $n_1/2$ closed sites for the contour to exist and there is a subset of size greater than or equal to $n_1/18$ that are independent, so a contour of length n has probability less than or equal to $\rho^{n/36}$. The shortest possible contour has length L, so if $\rho<6^{-36}$ the event in (12) has probability at most

$$\sum_{n=L}^{\infty}K3^n\rho^{n/36}=K\left(3\rho^{1/36}\right)^L/\left(1-3\rho^{1/36}\right)\le 2K\left(3\rho^{1/36}\right)^L. \qquad \square$$

PROOF OF (15). Let $\mathscr{C}$ be the set of points on $\mathscr{L}_2$ that can be reached from $(1,1)$. Define D, W and Γ, and orient Γ as in the last proof. If F_{k+1} does not occur then there is either

(A) a contour from $(2,1)$ to $\{1\}\times[2,\infty)$ or
(B) a contour from $(2,1)$ to some point (m,K) with $m<L$.

In case (A) there is no percolation starting from $(1,1)$ in $\mathscr{L}_2$. If we let n_1 and n_2 be the number of segments of types 1 and 2 as in the last proof, then

$$n_1+n_2=n \quad \text{and} \quad n_1-n_2=1,$$

so $n_1\ge n/2$. Repeating the proof of (13) shows that if $\rho<6^{-36}$ then

$$P(A)\le\sum_{n=1}^{\infty}3^n\rho^{n/36}\le 2\left(3\rho^{1/36}\right).$$

In case (B) the shortest contour has length K and the number of segments of types 1 and 2 satisfies

$$n_1+n_2=n \quad \text{and} \quad n_2-n_1\le L.$$

Subtracting the last two relations

$$2n_1 \geq n - L \geq (K/2) + n - K,$$

and summing the by now familiar series shows that if $\rho < 6^{-72}$,

$$P(B) \leq \sum_{n=K}^{\infty} 3^n \rho^{\{(K/2)+n-K\}/36} \leq 2\left(3\rho^{1/72}\right)^K.$$

Using the fact that $K \geq 1$ and adding the bounds gives (15). □

5. Proof of Theorem 3. Very few new ideas are needed for the proof and the result is not exciting so we just sketch the proof, assuming the reader is familiar with the techniques of percolation in two dimensions. All the facts we use can be found in Section 6b of Durrett (1988). Let $\varepsilon > 0$. If L is large the probability of a horizontal crossing of

$$B \equiv (-3L, 3L) \times (-L, L)$$

[i.e., from $\{3L\} \times (-L, L)$ to $\{3L\} \times (-L, L)$] by good sites is greater than $1 - \varepsilon$. Call $(m, n) \in \mathbb{Z}^2$ open if there is a horizontal crossing of $(2Lm, 2Ln) + B$ by open sites and also a vertical crossing of $(2Lm, 2Ln) + B'$ where B' is a 90° rotation of B. If ε is small then with probability 1 there is a sequence of open sites $(m_0, n_0), (m_1, n_1), (m_2, n_2) \dots$ with $(m_{k+1}, n_{k+1}) - (m_k, n_k) \in \{(1,1)(-1,1)\}$, for $k \geq 0$. By using alternatively the vertical and horizontal crossings of the translates of B' and B associated with these sites, we can construct an infinite self-avoiding path Π through the good sites and a one-to-one mapping φ from $\{0, 1, 2, \dots\}$ into Π so that $\varphi(0) \in (m_0, n_0) + B$ and for each k there is a $j_k \leq 12L^2(k + 1)$ so that $\varphi(j_k) \in (m_k, n_k) + B$. If some point in Π becomes occupied and the resulting contact processes on Π does not die out, then comparison with the contact process on $\{0, 1, 2, \dots\}$ (which has the same critical value as the contact process on $\mathbb{Z}$) and use of the linear growth of that process shows that the conclusion of Theorem 3 holds. For the necessary facts about the contact process on $\{0, 1, \dots\}$, see Durrett and Griffeath (1983).

To complete the proof, we use a "restart argument." We wait until a site on Π becomes occupied and then look at the contact process on Π it generates; that is, we ignore infections to or from Π^c. If the contact process on Π does not die out we are done. If it dies out at time σ, we wait until a site on Π becomes occupied and try again. To see that Π will be occupied infinitely many times on $\{\xi_t^0 \neq \varnothing$ for all $t\}$ we observe that in supercritical site percolation [see, e.g., Section 6b of Durrett (1988)] the origin is surrounded by infinitely many circuits of good sites and each time a circuit is crossed there is positive probability that the contact process will spread along the circuit and occupy Π.

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